On Pricing CDOs with Meixner Distributions

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The subprime meltdown that started in early 2007 resulted in a loss of startling scale.[1] The misuse of leveraging instruments such as CDOs is one of many possible reasons of the meltdown. Therefore, there is a compelling need for pricing models that can realistically reflect risk inherent in the market.

Early Collateralized Debt Obligation (CDO) pricing models are based on Gaussian distributions due to their tractability. However, these models exhibit implied correlation smiles analogous to volatility smiles of Black and Scholes' [1973] model. These models also underestimate the prices of senior tranches. The standard models fail to capture two important features: tail dependence and aymmetric dependence structure. Hull and White [2004] address the tail dependence issue by employing a double-*t* distribution which has fat tails.

Asymmetric dependence, a more challenging issue, refers to a phenomenon that firm defaults are more correlated during the market's downturn (see, e.g., Ang and Chen [2002], de Servigny and Renault [2002], and Das et al. [2006]). A possible reason for asymmetric dependence is counterparty risks as explained in Jarrow and Yu [2001]. Another possible reason is a contagion effect, as explained in Collin-Dufresne et al. [2003].

Ideally, we want a model to be able to capture both tail dependence and asymmetric dependence structure. Burtshell et al. [2007b] address these issues by allowing the correlation parameter to be stochastic. Another strand of literature follows the development in option pricing, e.g., Madan et al. [1998], Carr and Wu [2004], Huang and Wu [2004], by employing Lévy distributions to capture stylized features of underlying asset values which exhibit skewness and excess kurtosis.

In general, adopting Lévy distributions comes at the expense of tractability: most Lévy distributions do not have analytical density functions. Examples of few that do, however, include variance gamma, normal inverse Gaussian, and Meixner distributions. Moosbrucker [2006a,b] explores the use of a variance gamma distribution and its variation in CDO pricing, while a normal inverse Gaussian has been used by Kalemanova et al. [2007] and Brunlid [2006]. As an alternative, we believe a Meixner distribution is an attractive choice because of its implementation simplicity. Moreover, the impact of adjusting model parameters on skewness and kurtosis seems more transparent than others.

This study provides a framework for incorporating Meixner distributions into the copula-based and structural approaches in modeling credit risk. The article then compares the performances of Meixner-based models to those of the more traditional ones in pricing the CDOs. In addition, it also investigates the characteristics of the risk measures based on expected discounted loss and expected shortfall for each model.

Using the data of CDX NA IG series 5 and 6, we find that, in terms of the mean absolute pricing errors (MAPEs), Meixner-based models outperform standard models in all cases. As expected, the results show that under the Meixner distribution the credit shock correlations in the lower tail are more dependent than those in the upper tail. The results also show that the risk measures derived from the Meixner copula model, the Meixner structural model, and the double-*t* copula model are more reasonable than those of the Gaussian copula and Brownian structural models.

A REVIEW OF PREVIOUS STUDIES

In this section we briefly review two strands of literature on CDO pricing; the copula-based approach and the structural form approach.

Copula Models

Li [2000] proposes the default (survival) time copula model, now the industry's standard model. Under this framework, marginal default probabilities are implied from the defaultable instruments such as bonds, then they are linked together using the Gaussian copula. The framework is later extended to other copulas such as Student-*t* and Archimedean (e.g., Schönbucher [2000]; Laurent and Gregory [2003]; Galiani [2003]; Elizalde [2005]). Since the credit risk tends to move together during extreme events, Hull and White [2004] apply the one-factor double-*t* copula model to price CDOs and n^{th}-to-default credit default swaps. Their double-*t* model improves the pricing accuracy since it takes into account the tail dependence in a pool of credit risk.

Burtschell et al. [2007a] compares the following copula models: Gaussian, stochastic correlation, Student-*t*, double-*t*, Clayton and Marshall-Olkin. Student-*t* and Clayton copula models give similar results to the Gaussian copula model whereas the Marshall-Olkin copula model results in overly fattening the tail of the loss distributions. The double-*t* model produces tails that are fatter than the Gaussian copula's but are thinner than the Marshall-Olkin copula's, resulting in a better fit to market quotes. The stochastic correlation copula model, on the other hand, can match skewness as observed in the market data.

In recent years, Lévy-based models have become widely used in pricing CDOs due to their flexibility. Kalemanova et al. [2007] apply a form of a Lévy distribution called normal inverse Gaussian with the assumption of a large homogeneous portfolio. Moosbrucker [2006a,b] further extends this approach to a variance gamma copula model and other copula models. Brunlid [2006] compares a number of hyperbolic Lévy copula models including the normal inverse Gaussian, variance gamma, and skewed Student-*t* copula models. The results suggest that the distribution of credit risk should have negative skewness and high kurtosis. Albrecher et al. [2007] suggest that, in general, one can use any Lévy processes and their associated distributions, e.g., shifted gamma, shifted inverse Gaussian, variance gamma, normal inverse Gaussian, Meixner, for the one-factor copula model.

Correlated-Structural Models

Hull, Predescu, and White [2005] propose a CDO pricing methodology based on the first passage model of Black and Cox [1976]. Under this framework, the firm's asset value is assumed to follow a correlated geometric Brownian motion. The model incorporates empirical evidence that default correlations are positively dependent on the default rates whereas the recovery rates are negatively dependent on the default rates. Their study finds that a model with stochastic correlation substantially improves the accuracy of the model. The model with stochastic recovery rate also helps, but less so.

Similar to the copula framework, Lévy-based models have been used with a structural framework. Luciano and Schoutens [2005] assume that the market factor component follows a gamma process whereas the idiosyncratic component follows a Brownian motion, resulting in a variance gamma process. Moosbrucker [2006b] further extends the framework to various cases: correlated gamma processes with independent Brownian motions, identical gamma processes with dependent Brownian motions, and sums of two independent variance gamma processes. The study also compares these structural models with copula-based models. In general the variance gamma structural model fits the market quotes better than the Gaussian and double-*t* copula models. In addition, Baxter [2006] applies a number of Lévy structural models such as catastrophe gamma, variance gamma, gamma, Brownian gamma structural models to single-name and multi-name credit portfolios and concludes that the simple gamma is effective in pricing credit derivatives.

In this study we employ a Meixner distribution introduced by Schoutens and Teugels [1998] and Schoutens [2002] because of its ability to fit assets' returns and its implementation's simplicity. In addition, the impact of adjusting models' parameters on the skewness and kurtosis of the distribution is also more transparent. The framework to extend the traditional credit risk models with the Meixner distribution is presented in the next section.

METHODOLOGY AND DATA

This study investigates two groups of models. The first group comprises a Gaussian copula model proposed by Li [2000], a double-*t* copula model by Hull and White [2004], and a Meixner copula model obtained from replacing a Gaussian distribution in a Gaussian copula by a Meixner distribution. The second group includes the structural model proposed by Hull, Predescu, and White [2005] and a correlated Meixner structural model. This section provides background on Meixner distributions and how they can be incorporated into copula-based and structural form approaches.

Meixner Distributions/Processes. A Meixner distribution is a special case of a Lévy distribution which is infinitely divisible. For each positive integer n, the Meixner characteristic function $\phi(u)$ is the n-th power of a Meixner characteristic function:

$$\phi(u;\, \alpha,\, \beta,\, \delta,\, \mu) = [\phi(u;\, \alpha,\, \beta,\, \delta/n,\, \mu/n)]^n$$

where $\alpha > 0$, $-\pi < \beta < \pi$, $\delta < 0$, and $\mu \in \mathbb{R}$. Hence, it is stable under convolution, i.e., the distribution function of the sum of independent Meixner random variables is still a Meixner distribution.

A Meixner distribution is one of few Lévy distributions that has an analytical probability density function:

$$f_{Meixner}(x;\, \alpha,\, \beta,\, \delta,\, \mu) = \frac{(2\cos(\beta/2))^{2\delta}}{2\alpha\pi\, \Gamma(2\delta)} \exp\frac{\beta(x-\mu)}{\alpha} \left| \Gamma\left(\delta + \frac{i(x-\mu)}{\alpha}\right)\right|^2, x \in \mathbb{R}$$

The moments of this distribution is shown in the second column of Exhibit 1. Distributions used in the copula-based model need to have zero mean and unit variance. For the Meixner distribution, this is achieved by setting $\delta = \frac{2\cos^2(\beta/2)}{\alpha^2}$ and $\mu = -\frac{\sin(\beta)}{\alpha}$. The resulting moments are shown in the last column of Exhibit 1. The two remaining free parameters α and β reflect skewness and kurtosis.

A Meixner process $X = \{X_t,\, t \geq 0\}$ is a continuous-time stochastic process with the following properties:

1. $X_0 = 0$,
2. X_t has independent increments with distribution

$$X_{t+\Delta t} - X_t \sim \text{Meixner}(\alpha,\, \beta,\, \delta\Delta t,\, \mu\Delta t)$$

The distribution of X_t follows Meixner $(\alpha, \beta, \delta t, \mu t)$, and the distribution of $\frac{X_t}{\alpha}$ follows Meixner $(1, \beta, \delta t, \frac{\mu}{\alpha} t)$. This observation is used to simplify the simulation of Meixner structural model later. For more details on the Meixner distributions and processes, see Schoutens [2002, 2003] and Albrecher et al. [2007].

Exhibit 2 illustrates and compares the dependence structures of Gaussian, double-*t*, and Meixner copulas. They are bi-variate plots of 3,000 sample points with different degrees of dependency—ρ = 0.2 and 0.9. For the Gaussian copula, the shape of the bi-variate dependent structure is elliptical. For the double-*t* copula with the degree of freedom 3, the points clustered more densely

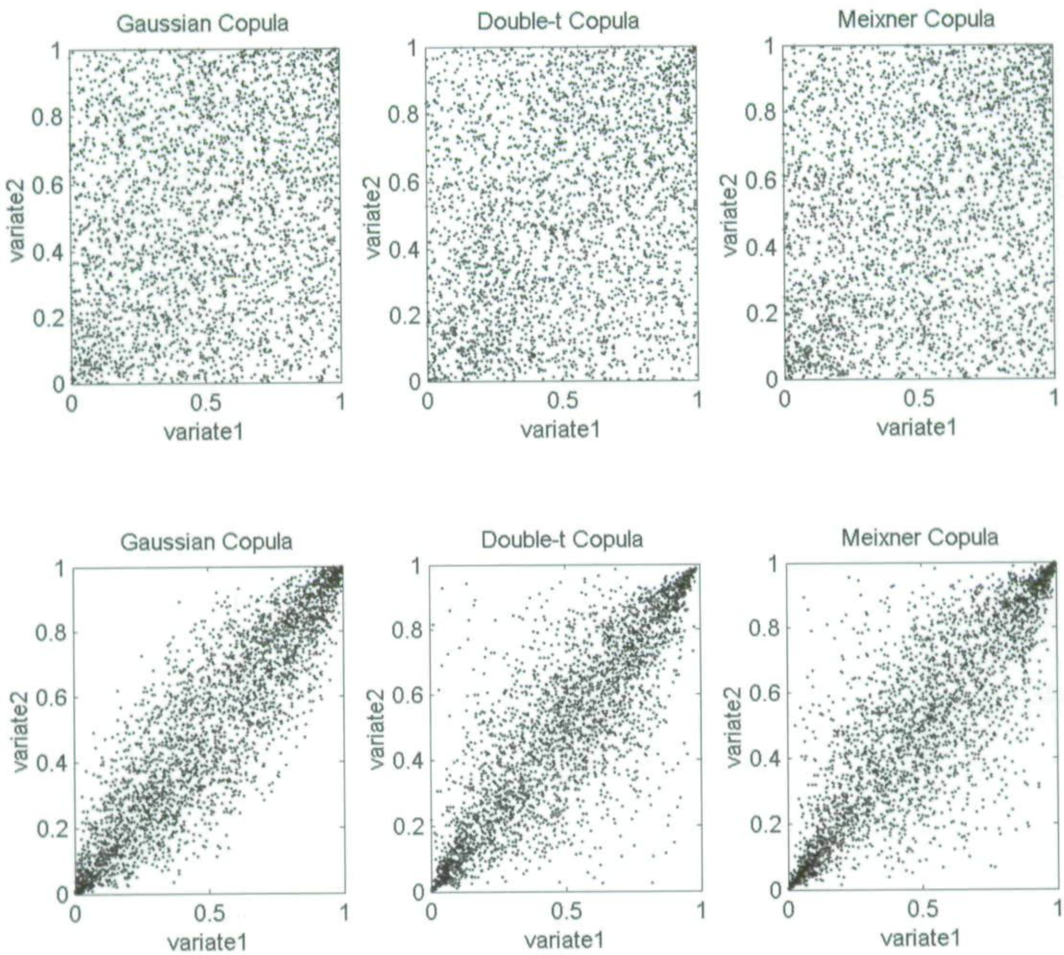
EXHIBIT 1

The Moments of the Meixner Distribution

	Meixner $(\alpha,\beta,\delta,\mu)$	Meixner $(\alpha,\beta,\frac{2\cos^2(\beta/2)}{\alpha^2},-\frac{\sin(\beta)}{\alpha})$
mean	$\mu + \alpha\delta\ \tan(\beta/2)$	0
variance	$\frac{1}{2}\alpha^2\delta\,(\cos^{-2}(\beta/2))$	1
skewness	$\sin(\beta/2)\sqrt{2/\delta}$	$\alpha\tan(\beta/2)$
kurtosis	$3 + (2-\cos(\beta))/\delta$	$3 + (\frac{3-2\cos^2(\beta/2)}{2\cos^2(\beta/2)})\alpha^2$

EXHIBIT 2

The Dependence Structure of a Gaussian Copula, a Double-*t* Copula with the Degree of Freedom 3, and a Meixner Copula with Parameters $\alpha = 2.6753$, $\beta = -0.70036$



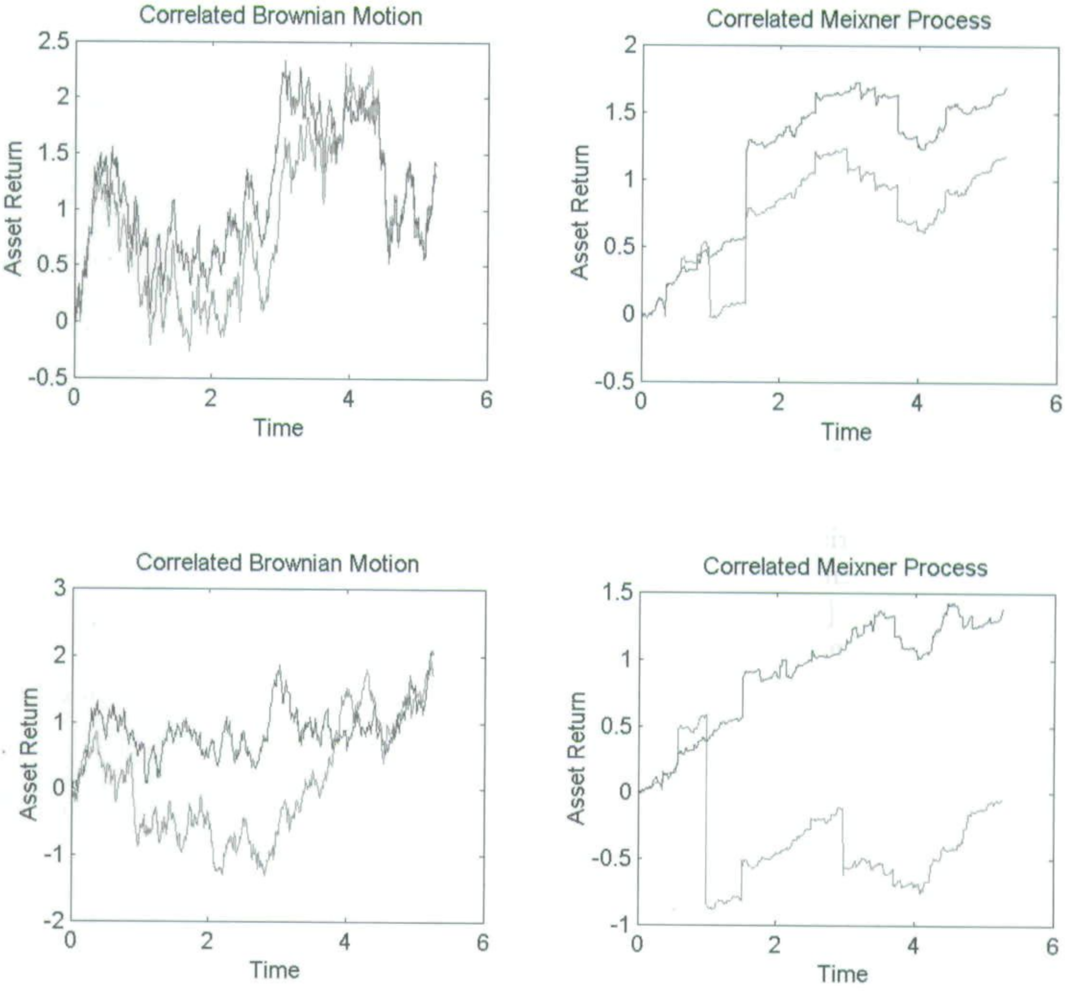
at both tails than the Gaussian copula. For the Meixner copula with $\alpha = 2.6753$, $\beta = -0.70036$, the points clustered even more at both tails. Moreover, because β is negative, the probability density function of the Meixner copula has a fatter left tail than right tail, and the lower tail of the Meixner copula is, in turn, is more clustered than the upper tail.

Exhibit 3 shows the sample paths simulated from correlated standard Brownian motions and the correlated Meixner processes with parameters $\alpha = 1$, $\beta = -1.0472$ and $\delta = 0.0500$ with different degrees of dependency—$\rho = 0.2$ and 0.9. In a correlated Brownian structural model, both the market factor and the idiosyncratic factor have no jumps; whereas those factors of a Meixner structural model are pure jump processes.

CDO pricing framework. Let t_i denote the time at the end of an arbitrary period i, and $P(t_i)$ the corresponding notional principal of one of the tranches. For the premium leg of a CDO, the premium paid at time t_i is an annualized fixed proportion s of the sum of the expected outstanding notional principal and the accrued premium. Following Hull and White [2006], we assume that the defaults, if any, occur at the middle of each period. The value of the premium leg is then:

EXHIBIT 3

The Sample Paths of Correlated Standard Brownian Motions and Correlated Meixner Processes with Parameters ($\alpha = 1$, $\beta = -1.0472$, $\delta = 0.0500$)



$$\text{premium leg} = s\sum_{i=1}^{n}(t_i - t_{i-1})\, E[P(t_i)]e^{-rt_i} + s\sum_{i=1}^{n}\left(\frac{t_i - t_{i-1}}{2}\right)\{E[P(t_{i-1})] - E[P(t_i)]\}e^{-r(\frac{t_i+t_{i-1}}{2})}$$

The value of the default leg at time t_i is the sum of discounted change of the expected notional principal of the tranche from the previous period:

$$\text{default leg} = \sum_{i=1}^{n}\{E[P(t_{i-1})] - E[P(t_i)]\}e^{-rt_i}$$

So, s is chosen to equate the value of the two legs:

$$s = \frac{\Sigma_{i=1}^{n}\{E[P(t_{i-1})] - E[P(t_i)]\}e^{-rt_i}}{\left(\begin{array}{c}\Sigma_{i=1}^{n}(t_i - t_{i-1})E[P(t_i)]e^{-rt_i} + \\ \Sigma_{i=1}^{n}(\frac{t_i - t_{i-1}}{2})\{E[P(t_{i-1})] - E[P(t_i)]\}e^{-r(\frac{t_i+t_{i-1}}{2})}\end{array}\right)} \quad (1)$$

To estimate $E[P(t_i)]$, denote the lower and upper loss limits of any tranche by K_L and K_U, respectively. Let Z_{t_i} stands for the *portfolio loss rate* at time t_i, representing the percentage of the cumulative loss in the *portfolio* value at time t_i. Given Z_{t_i}, the value of the outstanding notional principal of the tranche is:

$$P(t_i;Z_{t_i}) = P(0)\times min\left(1, max\left(0, \frac{\left(K_U - Z_{t_i}\right)}{\left(K_U - K_L\right)}\right)\right) \quad (2)$$

where $P(0)$ is the initial notional principal of the tranche.

The expected outstanding notional principal of the tranche at the future time t_i is then:

$$E[P(t_i)] = \int_{z=0}^{z=1} P(t_i;z)dF_{t_i}(z) \approx \sum_z P(t_i;z)\cdot Pr(Z=z) \quad (3)$$

where F_{t_i} is the cumulative distribution function of the portfolio loss rate Z_{t_i}. See Laurent and Gregory [2003]; Elizalde [2005]; Hull and White [2006] for more details.

Under the following common assumptions,

- the underlying portfolio is homogeneous,
- the recovery rate R of each firm is constant at 40% (see Varma and Cantor [2005]),
- the default correlation ρ is constant over time,
- the weights of firms in the portfolio are equal and are constant over time,

Z_{t_i} can be expressed as follow:

$$Z_{t_i} = \frac{n_{t_i}}{N}\times(1-R) \quad (4)$$

where n_{t_i} is the number of default firms by time t_i, and N is the number of firms in the portfolio. The portfolio loss rate distribution function F_{t_i} can be derived from the distribution function of the number of defaults.'

All models fall under the above framework but differ on the specification of portfolio loss rate distributions. Next we derive the portfolio loss rate distributions under the Meixner distribution.

Meixner copula model. Let $1_{l,t_i,\{default\}}$ be an indicator function of the default event of firm l by time t_i. Then the number of defaulted firms is $n_{t_i} = \Sigma_{l=1}^{N} 1_{l,t_i,\{default\}}$. Denote individual default probabilities at time t_i by p_{t_i}, and assuming that defaults occur *independently*, the probability that k out of N firms default has a binomial distribution:

$$Pr(n_{t_i} = k) = \binom{N}{k} p_{t_i}^k (1-p_{t_i})^{N-k} \quad (5)$$

$$Pr(n_{t_i} \le k) = \sum_{l=1}^{k} \binom{N}{l} p_{t_i}^l (1-p_{t_i})^{N-l} \quad (6)$$

As a result, the portfolio loss distribution becomes

$$F_{t_i}(z) = Pr\left(n_{t_i} \le \left\lfloor \frac{zN}{(1-R)} \right\rfloor\right) = \sum_{l=1}^{\lfloor zN/(1-R)\rfloor} \binom{N}{l} p_{t_i}^l (1-p_{t_i})^{N-l} \quad (7)$$

where

$$p_{t_i} = 1 - e^{-\lambda t_i}$$

and λ is a hazard rate.

To model correlated defaults, the proxy for the default of firm j, X_j, can be decomposed into the common factor M and the firm-specific factor Y_j:

$$X_j = \sqrt{\rho}M + \sqrt{1-\rho}Y_j$$

where ρ is the degree of default dependency.

Given the common factor M, the probability of default $p_{t_i|m}$ is conditionally independent:

$$p_{t_i|m} = F_Y\left(\frac{F_{M+Y}^{-1}\left(p_{t_i}\right) - \sqrt{\rho}m}{\sqrt{1-\rho}}\right) \quad (8)$$

Hence, the portfolio loss distribution under all possible factor values becomes:

$$F_{t_i}(z) = \int_{m=-\infty}^{m=+\infty} \sum_{l=1}^{\lfloor zN/(1-R)\rfloor} \binom{N}{l} p_{t_i|m}^l (1-p_{t_i|m})^{N-l} dF_M(m) \quad (9)$$

where F_M, F_Y, and F_{M+Y} are the cumulative probability distributions of M, Y, and X, respectively. A Meixner copula model is obtained by substituting F_M, F_Y, and F_{M+Y} with a Meixner distribution that has zero mean and unit variance.

Correlated-Meixner structural model. Assuming that the firm's asset value follows an exponential of a Lévy process, under the risk-neutral setting, the asset value follows

$$V_j(t) = V(0)exp((r_f - \theta)t + \mathbb{Z}_j(t)) \tag{10}$$

where $V_j(t)$ is the value of firm j at time t
r_f is the risk-free rate of return
θ is the mean-correcting parameter that makes the discounted firm value process a martingale
$\mathbb{Z}_j$ is a Lévy process

To simplify the simulation, we write the process $\mathbb{Z}_j(t)$ as $\alpha X_j(t)$, where α is a scaling factor, and $X_j(t)$ is a scaled Lévy process of $\mathbb{Z}_j(t)$ corresponding to a proxy of asset returns.

Following Black and Cox [1976], the default condition is:

$$V_j(t) \leq K \exp(\gamma t)$$

where γ is a growth parameter of the default barrier. The default condition rewritten in term of X_j is:

$$X_j(t) \leq a + bt$$

where:

$$a = \frac{lnK - lnV(0)}{\alpha}, \; b = \frac{\gamma - (r_f - \theta)}{\alpha}$$

Thus, the default time of firm j is

$$\tau_j = \inf\{t \geq 0; X_j(t) \leq a + bt\}$$

The scaled process X_j is assumed to follow a correlated Meixner process with parameters $\alpha = 1$, β, δ, and a degree of dependency ρ:

$$\begin{aligned} X_j(t+dt) &= X_j(t) + dX_j(t) \\ dX_j(t) &= \sqrt{\rho}dM(t) + \sqrt{1-\rho}dY_j(t) \end{aligned} \tag{11}$$

M and Y_j are independent Meixner processes with $M(0) = Y_j(0) = 0$. Their increments are distributed as Meixner (α, β, δdt, $-\alpha\delta dt\tan(\beta/2)$). The sample paths of X_j are obtained by sampling the paths of the market factor M and the firm-specific factor Y_j. For the simulation purpose, we need to know a and b, which we can infer directly from credit default swap (CDS) quotes. Thus, it is not necessary to know θ.

Data

The study tests the performance of each model by pricing the CDOs written on CDX NA IG. The CDX NA IG is a credit default swap index. It consists of the 125 most actively traded CDSs of investment-graded firms in North America, equally weighted, divided into five standard tranches: 0–3%, 3–7%, 7–10%, 10–15% and 15–30%. One can buy or sell each tranche separately, called a single tranche trade. Every six months, a new index series consisting of the most actively traded CDSs is launched.

The CDO data are from the GFI credit derivative data through Reuters, while the data on CDX NA IG index and US Zero Curves are extracted from Datastream. We use all available data, as of October 16, 2006, of CDX NA IG Series 5 and Series 6 covering both the on-the-run period and the off-the-run period. The on-the-run period of Series 5 is from September 20, 2005, to March 19, 2006, while the on-the-run period of Series 6 is from March 20, 2006, to September 19, 2006.[2] Similar to Hull and White [2006], only the days where all tranches have both bid and ask quotes are considered.

RESULTS

In this section, we discuss the model calibration and present the parameters estimated from the samples. The performances of the Meixner-based models are then examined and compared with those of the traditional models. Finally, the risk measures of the CDOs from each model are presented and discussed.

The results in this section depends on MAPE, which is calculated as follows

$$MAPE = \frac{\sum_{i=1}^{n} APE_{t_i}}{n}$$

where $APE_{t_i} = \sum_{tranche2}^{tranche5} \left| ModelSpread_{t_i} - MarketSpread_{t_i} \right|$,
n is the number of periods until maturity

Calibration for Copula Models

There are three elements to be calibrated from the market data: a hazard rate of each firm, the correlation parameter, and copula parameters.

Under the homogeneity assumption, the hazard rate λ of a firm can be backed out from the CDX NA IG index. Specifically, the hazard rate λ is implied by equating the following expression for s with a market quote:

$$s = \frac{(1-R)\Sigma_{i=1}^{n}\{E[P(t_{i-1})] - E[P(t_i)]\}e^{-rt_i}}{\left(\begin{array}{c}\Sigma_{i=1}^{n}(t_i - t_{i-1})E[P(t_i)]e^{-rt_i} + \\ \Sigma_{i=1}^{n}(\frac{t_i - t_{i-1}}{2})\{E[P(t_{i-1})] - E[P(t_i)]\}e^{-r(\frac{t_i + t_{i-1}}{2})}\end{array}\right)} \qquad (12)$$

where $E[P(t_i)] = 1 - p_{t_i} = e^{-\lambda t_i}$.

To calibrate for each series, we fix the copula parameters; α, β for the Meixner copula, and the degree of freedom for the double-t copula. For specific values of copula parameters, the daily correlation parameter ρ is calibrated from the equity tranche by Equations (1), (2), (3), (8), (9), and the corresponding MAPE can be computed. Changing the values of copula parameters yields new values of ρ and MAPE. The calibrated values of the copula parameters and ρ are the ones that minimize MAPE.

Calibration for Correlated Structural Models

For the structural models, instead of implying the hazard rate, we back out the default barriers from the CDX NA IG index. Then the correlation parameter and the distribution's parameters can be estimated in the same fashion as the copula models.

For the correlated Brownian structural model, the parameters $a = \frac{\ln K - \ln V(0)}{\sigma}$ and $b = \frac{\gamma - (r_f - \frac{\sigma^2}{2})}{\sigma}$ are estimated by using 5- and 10-year CDS spreads and Equation (12) where $E[P(t_i)] = 1 - p(0,t_i)$. $p(0,t_i)$ is the probability of default obtained from the first passage time's equation (see Harrison [1990]), i.e., $p(0,t_i) = \Phi(\frac{a+bt_i}{\sqrt{t_i}}) + exp(-2ab)\Phi(\frac{a-bt_i}{\sqrt{t_i}})$.[3]

For the correlated Meixner structural model, the parameters $a = \frac{\ln K - \ln V(0)}{\alpha}$ and $b = \frac{\gamma - (r_f - \theta)}{\alpha}$ are similarly estimated as in the correlated Brownian structural model. In addition, we have to estimate two more parameters which are the β and δ of the Meixner process. Unlike the Brownian structural model, the first passage time distribution of the Meixner process has no closed form, making calibration more difficult. To estimate a and b, we have to assume the value of β and δ first and then simulate until we obtain the barrier parameters that equate the CDS prices to the market quotes. Note that in all cases we use a time step of one quarter ($\Delta t = 0.25$).

For example, consider CDX NA IG series 6 with maturity of 5 and 10 years. We sample the paths that follow Meixner process $(1, \beta, \delta, -\delta\ \tan(\beta/2))$ over time horizon of 10 years, then we find $p(0,t_i)$, the probability of first hitting the barrier $a + bt_i$ by time t_i. Next, the 5YS6 and 10YS6 CDS premium are computed by Equation (12). The barrier parameters (a, b) are adjusted until these CDS premiums match the market quotes. For 7YS6, we use the data from 5YS6 and 7YS6 to find the suitable barrier. As in the case of the copula models, the set of parameters ρ, β and δ are chosen to minimize the overall mean absolute pricing error. The estimated parameters are presented in Exhibit 4. The

EXHIBIT 4

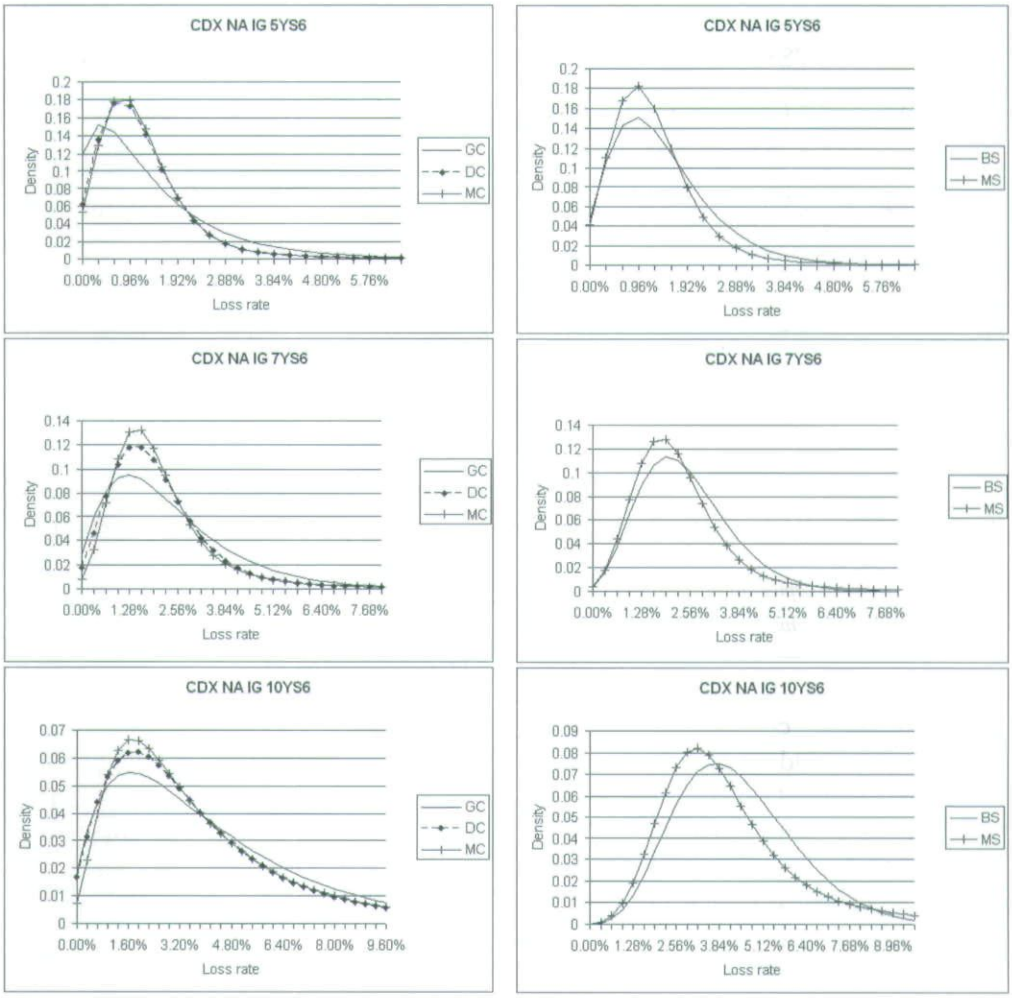
Summary of Parameters Calibrated from the Market Quotes of CDOs on the CDX NA IG Index

		CDX NA IG					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
DC	df	3	3	3	3	6	6
MC	α	3.2225	2.6753	1.6657	1.3724	0.0180	0.0132
	β	-0.6911	-0.7004	-1.9726	-1.8162	-3.0644	-3.0949
MS	β	-0.6435	-1.0472	-0.7499*	-0.8261*	-1.3694*	-1.3694*
	δ	0.0400	0.0500	0.0268	0.0280	0.0400	0.0400

DC = double-t copula; MC = Meixner copula; MS = Meixner structural.

**The numbers shown are obtained by smoothing the paths of the Market factor in order to speed up the simulation.*

EXHIBIT 5
The Portfolio Loss Distribution of CDX NA IG



degree of freedom in the double-t copula model that minimizes MAPE is 3 for all time series except 10YS5 and 10YS6 whose degree of freedom are 6. For the Meixner copula model, the optimal parameters (α, β) vary across the time series: (2.6753,–0.7004) for 5YS6, (1.3724,–1.8162) for 7YS6, and (0.0132,–3.0949) for 10YS6. The parameters for the correlated Meixner structural model, (β, δ) are (–1.0472, 0.0500) for 5YS6, (–0.8261, 0.0280) for 7YS6, and (–1.3694, 0.0400) for 10YS6. It can be seen that all the β's of the Meixner-based models are negative. Note that the correlated Meixner structural's parameters of 10YS5 and 10YS6 may not be at a global optimum. Even so, it is sufficient for the correlated Meixner structural model to outperform other models.

Comparison of Models

It is informative to compare the portfolio loss distributions from each model. Exhibit 5 shows the loss distributions on the issued date (23 Mar 2006) of the CDX NA IG 5YS6, 7YS6, and 10YS6.

In addition, the mean absolute pricing error (MAPE) of each model is examined. Exhibit 6 reports the MAPEs of each model for all series, from which we can draw the following conclusion. For every series, the models based on Meixner distribution (MC, MS) have the lowest MAPE. Models based on double-t distribution provide the next best performance, while the Gaussian-based models yield the worst performance.

To confirm the performances of the models, we employ a paired Z-test as in Houweling and Vorst [2005].[4]

Exhibit 6

The Mean Absolute Pricing Errors (MAPEs) and Their Standard Deviations

	CDX NA IG					
	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
GC	102.65	99.35	168.31	152.61	200.15	185.40
	(33.70)	(19.18)	(54.17)	(26.00)	(48.29)	(24.01)
DC	25.53	24.36	59.98	39.98	158.47	158.15
	(17.32)	(9.91)	(31.19)	(16.13)	(33.22)	(43.64)
MC	21.93	13.74	43.76	25.94	154.66	151.11
	(13.76)	(8.02)	(24.66)	(13.00)	(31.69)	(42.81)
BS	84.68	78.84	156.72	146.60	255.14	235.55
	(40.73)	(37.36)	(47.89)	(73.78)	(73.07)	(44.50)
MS	22.12	10.55	53.86*	25.81*	98.53*	72.97*
	(10.64)	(6.34)	(34.05)	(17.13)	(58.65)	(32.97)

GC = Gaussian copula; DC = double-t copula; MC = Meixner copula; BS = correlated Brownian structural; MS = Meixner structural.
**The numbers shown are obtained by smoothing the paths of the market factor.*

The results reported in Exhibit 7 confirm that for copula models, the Meixner copula yields the best fit, followed by the double-*t* copula and the Gaussian copula. This suggests that the model, by taking into account the tail dependence of each asset in the baskets, can improve the capability to capture the capital structure of CDOs. A more flexible model that can handle the asymmetric tail dependence will further increase the performance. The parameters β of the Meixner copula model that minimize the MAPE are negative for all data samples, confirming that the assets in the portfolio are more correlated in the lower tail than in the upper tail. We conclude that, for CDO valuation, considering only the symmetric tail dependence is not sufficient since firms tends to default together during the downturns but are more independent during the upturns.

For the structural models, the correlated Meixner structural model significantly outperforms the standard structural model in all cases. For 5YS6, the *t*-stat is 36.44 while 7YS6's is 13.30, and 10YS6's is 14.89. The *t*-stats in series 5 for 5Y, 7Y, and 10Y are 16.86, 18.91, and 14.98, respectively. This suggests that the assumption that firm value follows a Brownian motion may not be entirely appropriate. The firm value can have rare large moves governed by jumps in addition to the frequent small moves governed by diffusion. This study assumes that the firm value is driven wholly by jumps of infinite activity and infinite variation, which can capture both the rare events and the extremely frequent small moves. The result shows that modeling the asset return as a Meixner process is more realistic. Furthermore, the optimal parameters have negative β's for all cases, implying that a firm is more likely to suffer from large down-side jumps than to benefit from the large up-side jumps. The large down-side jumps can be either from the market factor or the firm-specific factor, or both. When it is from market factor, the default probabilities of all firms increase tremendously.

Risk Measures

To provide further insight concerning the risk of CDOs, we calculate and compare the expected discount losses (EDL) and expected shortfall (ES) based on each model. Specifically, the EDL is calculated from:

$$EDL = default\ leg = \sum_{i=1}^{n} \{E[P(t_{i-1})] - E[P(t_i)]\} e^{-rt_i} \quad (13)$$

The expected shortfall ES_j of tranche j can be calculated as:

EXHIBIT 7
The Results of the Paired Z-Tests

		CDX NA IG					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
GC vs MC	Mean Diff.	80.72	85.61	124.55	126.68	45.49	34.29
	S.D. of Diff.	(34.65)	(24.12)	(59.81)	(25.92)	(37.24)	(42.76)
	t-stats	26.36	37.73	22.62	51.26	13.10	8.06
DC vs MC	Mean Diff.	3.60	10.63	16.22	14.04	3.81	7.05
	S.D. of Diff.	(16.22)	(14.09)	(26.65)	(11.83)	(9.22)	(3.17)
	t-stats	2.51	8.02	6.61	12.45	4.43	22.35
GC vs DC	Mean Diff.	77.12	74.98	108.33	112.63	41.68	27.25
	S.D. of Diff.	(20.93)	(12.26)	(36.59)	(15.84)	(38.29)	(43.00)
	t-stats	41.68	65.01	32.16	74.60	11.67	6.37
BS vs MS	Mean Diff.	62.57	65.10	94.82	103.80	140.70	128.29
	S.D. of Diff.	(41.99)	(18.99)	(54.48)	(81.85)	(100.72)	(86.61)
	t-stats	16.86	36.44	18.91	13.30	14.98	14.89

GC = Gaussian copula; DC = double-t copula; MC = Meixner copula; BS = correlated Brownian structural; MS = Meixner structural.
Note: The number of observations of 5YS5 is 128 while 5YS6's is 113 observations; 7YS5's, 7YS6's, 10YS5's, and 10YS6's are 118, 110, 115, and 101, respectively. All numbers are in basis point except the t-stats that is in unit of one.

$$ES_j = \frac{\int_{1-\alpha}^{1} VaR_j(x)dx}{\alpha} \tag{14}$$

VaR_j, the credit VaR of the tranche j, is calculated as:

$$VaR_j(1-\alpha) = P_j(0) \times min\left(1, max\left(0, \frac{K_{U_j} - F_p^{-1}(1-\alpha)}{K_{U_j} - K_{L_j}}\right)\right) \tag{15}$$

where $P_j(0)$ is the total value of the single tranche j, F_p is the distribution of the portfolio loss rate. At a confidence level $(1-\alpha)\%$, it is related to VaR_p, the maximum potential loss of the portfolio, as follows:

$$F_p\left(\frac{VaR_p}{P(0)}\right) = 1-\alpha$$

Exhibit 8 shows the risk measures of each model by tranche. For the equity tranche (0–3% tranche), the measures obtained from all models are not much different. For the mezzanine tranches (tranche 3%–7% and tranche 7%–10%), there is no conclusive pattern for the risk measures. For the senior tranches (tranche 10%–15% and tranche 15%–30%), however, the differences in magnitude of ES and EDL obtained from each model are quite drastic. Among the copula-based model, the Gaussian copula model yields the smallest estimates of ES and EDL, while the double-*t* copula model yields the largest estimates, and the Meixner copula models are in between. For the structural models, the correlated Brownian structural model also yields lower estimates than the correlated Meixner structural model.

We also compute the ratio of ES to EDL, which indicates the size of loss in an extreme case relative to an average loss. The ratio increases from roughly 2 for the equity tranche to as high as 25 for the senior tranches. Since EDL is also the premium computed from the

EXHIBIT 8

The Risk Measures for Various Tranches of CDX NA IG 5YS6

	tranche 0%-3%			tranche 3%-7%			tranche 7%-10%			tranche 10%-15%			tranche 15%-30%		
	EDL	ES	ES/EDL	EDL	ES	ES/EDL	EDL	ES	ES/EDL	EDL	ES	ES/EDL	EDL	ES	ES/EDL
GC	0.4518	1.0000	2.2135	0.0727	0.8985	12.355	0.0109	0.2692	24.706	0.0018	0.0439	24.945	0.0001	0.0015	25.178
DC	0.4532	1.0000	2.2065	0.0402	0.7200	17.913	0.0118	0.2819	23.869	0.0070	0.1659	23.685	0.0036	0.0843	23.499
MC	0.4554	1.0000	2.1959	0.0369	0.6733	18.231	0.0086	0.2099	24.370	0.0041	0.0993	24.396	0.0015	0.0363	24.513
BS	0.4744	1.0000	2.1081	0.0618	0.7837	12.681	0.0042	0.1081	25.791	0.0003	0.0084	25.799	0.0000	0.0002	25.988
MS	0.4633	1.0000	2.1583	0.0355	0.6477	18.247	0.0077	0.1908	24.895	0.0035	0.0855	24.692	0.0013	0.0321	24.257

EDL = expected discount loss; ES = expected shortfall; GC = Gaussian copula; DC = double-t copula; MC = Meixner copula; BS = correlated Brownian structural; MS = Meixner structural.

models, the ratio of 25 for the most senior tranche, as compared to 2 for the equity tranche, suggests that *in the extreme event the premium for the most senior tranche may not be commensurate to the risk assumed.*

CONCLUSION

This article provides a framework to price CDOs based on a Meixner distribution. A Meixner distribution has properties such as fat tails and skewness, and its associated process has a jump component and is relatively simpler to implement than other Lévy processes. It is shown that the Meixner distribution can be applied to both copula-based and structural form approaches. A Meixner distribution can be used to construct the Meixner copula, which has asymmetric tail dependence; whereas in the correlated structural model, the asset returns are assumed to follow a correlated Meixner process. Using the historical prices of CDOs on the CDX NA IG, the performances of the proposed models are examined and compared to those of standard models such as Gaussian copula model, double-*t* copula model, and correlated Brownian motion structural model. The results show that the Meixner-based models outperform the standard models in terms of the mean absolute pricing errors. For example, for series 5YS6, the mean absolute pricing error of the Meixner copula model is 13.74 bps, which is 10.62 bps less than that of the double-*t* copula model and 85.61 bps less than that of the Gaussian copula model. Similarly, the pricing error of the the correlated Meixner structural model is 10.55, which is 68.29 bps less than that of the correlated Brownian structural model. Hence, an asymmetric dependence structure can improve the pricing performance.

Based on the calibration results, the skewness parameters (β) of the Meixner copula model are negative for all six cases. Thus, the dependence structure is such that the co-movements in the lower tail are more extreme than in the upper tail. Correlated Meixner structural models also have negative β in all six cases, suggesting that firm value has a greater likelihood of large downside jumps than large upside jumps. This is reasonable because the default seems more correlated during the downturns and less correlated in normal situations.

We also investigate the risk measurements based on the proposed models. In general it is found that the risk measures obtained from Gaussian-based models, in comparison to other models, are quite high for the earlier tranches and fall sharply to unrealistically low level for more senior tranches. The double-*t* copula model, by including tail dependence, provides more realistic risk measures. Between the double-*t* copula model and Meixner-based models, the risk measures of the latter tend to be smaller.

In summary, in addition to tail dependence, asymmetry is also an important property that should be considered when valuing credit dependent securities. Imposing a negative skewness in the distribution of market factor provides more dependence in the lower tail of its associated copula and also more likelihood of common large downside jumps in its associated process. These extensions are more realistic and can improve the performance of CDO pricing models. Moreover, the risk measures have more validity because they are based on the portfolio distribution calibrated from real-world market quotes.

ENDNOTES

The authors wish to thank Seksan Kiatsupaibul and seminar participants at the 57th Annual Meeting of the Midwest Finance Association. We are especially grateful for detailed

comments from Jean Helwege and Sira Suchintabandid which greatly improved the article. Kridsda Nimmanunta acknowledges the support by Chulalongkorn University Graduate Scholarship to Commemorate the 72nd Anniversary of His Majesty King Bhumibol Adulyadej. Any remaining errors or omissions are the responsibility of the authors.

[1]As of February 2008, the total loss at 30 of the world's largest banks and securities firms compiled by Bloomberg had reached $163 billions.

[2]We also include off-the-run data since we are concerned that the number of data points might not be sufficient. However, including off-the-run data does not affect the conclusion.

[3]In this study it is not necessary to find the asset volatility σ and the relative difference between the default barrier and the asset value $K/V(0)$; the values of a and b are sufficient to price CDOs.

[4]Specifically, the Z-test is expressed as $Z_{A,B} = \sqrt{n}\frac{\bar{d}_{A,B}}{s_{A,B}}$ where $d_{A,B,t_i} = APE_{A,t_i} - APE_{B,t_i}$, $\bar{d}_{A,B}$ is the sample mean of d_{A,B,t_i}, $s_{A,B}$ is the sample standard deviation of d_{A,B,t_i}, and n is the sample size.

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