Generalized Ho-Lee Model: *A Multi-Factor State-Time Dependent Implied Volatility Function Approach*

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This article presents an arbitrage-free binomial lattice interest rate model. The model uses a state- and time-dependent implied volatility function. This function captures from the swaption volatility surface not just the perceived interest rate volatility, but the interest rate distributional form in terms of a mix of lognormal and normal distributions and the correlations of the rates. It is a multi-factor model and can capture much of the yield curve's upward- or downward-sloping movements.

While arbitrage-free interest rate modeling has been a subject of much research, the current interest rate models fail to use the at-the-money swaption prices over a range of tenors and expiration dates to capture the market-implied correlations and the distributional forms of interest rates. For example, the Ho-Lee and Hull-White models assume that short-term rates follow a normal process. The Black-Derman-Toy [1988] model and the Brace-Gatarek-Musiela [1997] model assume that the rate follows a lognormal process. By way of contrast, this article presents a model that provides a lognormal interest rate process when the simulated interest rates are low and normal when the simulated rates are high. The "threshold rate," where the lognormal distribution switches to the normal distribution and visa versa, is implied from the market prices.

The implied volatility function consists of multiple functions that represent the independent movements of interest rates. These functions are also specified by the swaption prices. Therefore, the model can capture the implied correlations of the interest rates estimated from the market prices. By way of contrast, the string models and the Brace-Gatarek-Musiela [1997] model use historical correlations of interest rates. This approach cannot capture changing views of the market regarding interest rate correlations and, therefore, cannot provide accurate pricing.

This article also provides empirical tests of the model based on 70 at-the-money swaption prices on each date and monthly data from June 30, 2003, to June 30, 2004. There are four main results: 1) the model can be calibrated to the swaption prices for each date, using only six parameters, with an average error ranging from 1%–3%, and average errors vary with the dates; 2) the two-factor model provides significantly higher explanatory power than the one-factor model based on Davidson and Mackinnon's C test; 3) the calibrated interest rate movements, in general, behave more like a normal model than a lognormal model; and 4) the implied "instantaneous" principal movements of the yield curve are consistent with those estimated from the historical data which suggests that market prices do incorporate the appropriate correlation of interest rates. The simulated yield curve can be upward or downward sloping, and the movement exhibits mean reversion. All these behaviors are consistent with historical experience.

The article begins with the assumptions of the model, the specification of the model, and the algorithm in valuing a bond. Then, we present the extensions of the model to multi-factors and the model of a T-period maturity discount bond. The next section then reports the empirical results from calibrating the interest rate model to the swaption prices. We then describe the behavior of interest rate movements according to the interest rate model. Finally, we discuss the implications of the model.

DESCRIPTIONS OF THE MODEL

The model is an extension of the Ho-Lee [1986] model. The Ho-Lee model assumes the forward volatility of interest rate movements to be a constant, independent of state and time. The 1986 article noted that the methodology can be extended to incorporate state- and time-dependent volatilities.[1] In this article, we provide such a model.

Model Assumptions

For clarity of exposition, we begin with a one-factor model which we will generalize to a multi-factor model in the following section. This is a discrete time model where all agents buy and sell securities without any market friction. We use a recombining binomial lattice model to specify interest rate uncertainties, with the up-state and down-state having an equal risk-neutral probability of 0.5. A node on the binomial lattice is represented by (n, i) where n denotes the time and i the state, and where $0 \leq n \leq N$ and $0 \leq i \leq n$, for some large integer N, is the time horizon of the model.

$P(T)$ denotes the price of a zero-coupon bond with a face value of \$1 at time T for $0 \leq T \leq N$, observed at (0, 0). This set of prices determines the discount function $P(\cdot)$ at the initial time. All the agents in the market can observe the discount function. All securities are perfectly divisible and trading has no short selling constraints. These bonds are traded at all the nodes of the binomial lattice, and their prices are denoted by $P_i^n(T)$ which is the price at state i at time n with a remaining maturity of T periods. Therefore, $P_i^n(\cdot)$ is the discount function at node (n, i), and by definition $P_i^n(0) = 1$ for all i and n.

The term structure movement is arbitrage-free if there is no arbitrage opportunity for holding any portfolio of discount bonds at each node point of the binomial lattice over any one period. Such is the case if the expected risk-neutral return of holding a discount bond of any maturity T over any one period is the prevailing one-period risk-free return. Further, the model value of a T-period discount bond equals the observed bond price $P(T)$.

To describe the interest rate model, it suffices to describe the one-period discount bond $P_i^n(1)$, denoted by P_i^n with $0 \leq n \leq N$ and $0 \leq i \leq n$, for simplicity. This is because, given these one-period discount bonds, we can use the backward substitution approach to price any interest rate contingent claim given the cash flows C_i^n assigned to each node (n, i), where $0 \leq n \leq N$ and $0 \leq i \leq n$. Specifically, let V_i^n be the value of the interest rate contingent claim at time n and state i, and then the value of the contingent claim is given by the terminal condition

$$V_i^N = C_i^N \quad \text{for } 0 \leq i \leq N \tag{1}$$

and the recursive equation

$$V_i^n = \frac{1}{2} P_i^n (V_{i+1}^{n+1} + V_i^{n+1}) + C_i^n$$
$$\text{for all } 0 \leq \mathrm{n} \leq \mathrm{N} - 1 \text{ and } 0 \leq \mathrm{i} \leq \mathrm{n} \tag{2}$$

The value of the contingent claim is given by $V = V_0^0$.

The Model

The building blocks of the binomial model are the binomial volatilities δ_i^n, where $0 \leq i \leq n$. δ_i^n is the proportional decrease in the one-period bond value from state i to state $i + 1$ at time n. Without loss of generality, we assume that the bond price decreases and the bond yield increases, with state i, and hence $\delta_i^n < 1$. When $\delta_i^n = 1$, by definition there is no risk at the binomial node with respect to the up-state and down-state outcomes as shown for

$$\delta_i^n = \frac{P_{i+1}^{n+1}}{P_i^{n+1}} \quad \text{for all } n,\ 0 \leq n \leq N-1 \text{ and } 0 \leq i \leq n \tag{3}$$

However, several requirements are imposed on these binomial volatilities.

Condition 1. *The state-dependent volatility requirement asserts that the proportional change in the one-period bond price is proportional to the interest rate level when rates are low and constant when rates are high.*

To specify Condition 1, we begin with the definition of the one-period yield,

$$R_i^n = -\log P_i^n / \Delta t \tag{4}$$

where Δt is the time interval of one period. For example, if one binomial period (the step size of the lattice) is one month, then Δt is 1/12.

According to Equation (3), we have

$$\log \delta_i^n = -\log P_i^{n+1} + \log P_{i+1}^{n+1} \tag{5}$$

Substituting Equation (4) into (5), we have

$$\log \delta_i^n = (R_i^{n+1} - R_{i+1}^{n+1})\Delta t \tag{6}$$

By the market convention, interest rate volatilities are the standard deviations of proportional change in the annualized yields of bonds. Let σ_i^n be the annualized volatility at time n and state i, noting that the difference of the two binomial outcomes is two standard deviations. We then have

$$R_{i+1}^{n+1} - R_i^{n+1} = 2\sigma_i^n R_i^n \sqrt{\Delta t} \tag{7}$$

Substituting $R_{i+1}^{n+1} - R_i^{n+1}$ of Equation (7) into (6), and simplifying, we derive the relationship between the binomial volatilities and the market convention of interest rate volatilities as

$$\delta_i^n = \exp(-2\sigma_i^n R_i^n \Delta t^{3/2}) \tag{8}$$

Equation (8) presents the one-to-one relationship between δ_i^n and $\sigma_i^n R_i^n$. Let R be some fixed interest rate level, which we call the threshold rate, independent of both n and i. We now assume that, on the one hand, the interest rate movement is lognormal when $R_i^n < R$, and therefore σ_i^n is a function of n, denoted by $\sigma(n)$ and not state i. On the other hand, we assume that the interest rate movement becomes normal when $R_i^n > R$, evolving continuously from the lognormal process to the normal process. And therefore, $\sigma_i^n R_i^n$ is independent of state i, and equals $\sigma(n)R$. This motivates the following specification of σ_i^n:

$$\sigma_i^n R_i^n = \sigma(n)\min(R_i^n, R) \tag{9}$$

where $\sigma(n)$ is some continuous function of time n, which can be interpreted as the term structure of volatilities.[2] Substituting Equation (9) into Equation (8), we have the following specification of the forward volatilities:

$$\delta_i^n = \exp(-2\sigma(n)\min(R_i^n, R)\Delta t^{3/2}) \tag{10}$$

We further assume that the function $\sigma(n)$ is specified by parameters σ_0, σ_∞, α_0, α_1, α_∞:

$$\sigma(n) = (\sigma_0 - \sigma_\infty + \alpha_0 n)\exp(-\alpha_\infty n) + \alpha_1 n + \sigma_\infty \tag{11}$$

The parameters σ_0, σ_∞, α_0, α_1, α_∞ can be interpreted as follows: The parameter σ_0 is the short-rate volatility over the first period. Within the model construction, the short rate is the one-period rate. The expression $\alpha_1 n + \sigma_\infty$ is approximately the short-rate forward volatility at time n, if we assume n is sufficiently large and the exponential decay term becomes small. The expression $\alpha_0 + \alpha_1$ and parameter α_∞ are the short-term and long-term slopes of the term structure of volatilities. When these parameters are significantly positive, the short-term slope ensures the term structure of volatility slopes upward and the long-term slope ensures that the volatility curve decays rapidly.

The parameter α_∞ is the proportional decrease of volatility per year. The steeper the decay, the stronger is the mean reversion. For this reason, we use the notation α. This notation is consistent with the literature where α is often used to denote the speed of mean reversion of interest rate movements.

The specification of the term structure of volatilities is motivated by the observed market volatility curve. The volatility curve tends to decay exponentially with a term linearly related to time and with, at times, a hump in the short-to-intermediate term. Equations (10) and (11) are important in specifying the arbitrage-free interest rate model. Equation (10) ensures interest rates are non-negative and non-explosive, and Equation (11) ensures mean-reversion behavior.

Condition 2. *The arbitrage-free condition.*

The arbitrage-free yield-curve movement condition applies to all bonds with different maturities T. We, therefore, need to consider the binomial volatilities with another dimension T, $\delta_i^n(T)$. Specifically,

$$\delta_i^n(T) = \frac{P_{i+1}^{n+1}(T)}{P_i^{n+1}(T)}, \quad 0 \le T \tag{12}$$

That is, the binomial volatility is the proportion of the T-year bond at $(i+1)$-th state to the i-th state at time $n+1$. Note that $\delta_i^n(0) = 1$ because the one-period bond has no uncertainties over one period. The volatility for the one-period bond is $\delta_i^n(1) = \delta_i^n$, which is given numbers for the time being.

We will show later that the arbitrage-free condition requires that

$$\delta_i^n(T) = \delta_i^n \delta_i^{n+1}(T-1)\left(\frac{1+\delta_{i+1}^{n+1}(T-1)}{1+\delta_i^{n+1}(T-1)}\right) \tag{13}$$

Equation (13) defines the relationships of the binomial volatilities of T-year bonds and is important to the construction of the arbitrage-free rate model.

Note that if the binomial volatilities are independent of states, but dependent on time, then Condition 2 implies that

$$\delta^n(T) = \delta^n \delta^{n+1} \ldots \delta^{n+T-1} \tag{14}$$

When the binomial volatilities are both state and time independent, then

$$\delta^{n+1}(T) = \delta^T \tag{15}$$

where T is the power and not a superscript.

Equations (14) and (15) are used in the models described in Ho and Lee [2004] and Ho and Lee [1986], respectively. Therefore, Equation (13) shows that this article's model is a generalization of the previous models.

Given the above conditions, we will also show that the bond pricing model for the one-period bond price at node (n, i) is

$$P_i^n = \frac{P(n+1)}{P(n)} \prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n-k))}{(1+\delta_0^{k-1}(n-k+1))} \prod_{j=0}^{i-1} \delta_j^{n-1} \tag{16}$$

Equation (16) is the specification of the arbitrage-free interest rate model and is the key result presented in this article. Heath, Jarrow, and Morton (HJM) [1992] provides an arbitrage-free interest rate model for a given term structure of volatilities in a continuous-time framework. In the same spirit, we show that as long as the binomial volatilities satisfy Equation (13), then the arbitrage-free interest rate model is given by Equation (16). Note that our model requires the recombining of the interest rate movements to maintain computational tractability, whereas the HJM model does not require the simulated interest rates to recombine. Therefore, our model provides an additional analytical structure. When the binomial step size tends to zero, our discrete time model would converge to a continuous time model. But there is no analogous procedure in converting a continuous model to a recombining binomial lattice model. As a result, an HJM approach does not have a procedure to derive the model presented in this article.

Equations (13) and (16) specify the interest rate model from the term structure of volatilities. In this article, we also provide a specific interest rate model, whose term structure of volatilities is given by Equations (4), (10) and (11). Thus the set of difference Equations (4), (10), (11), (13), and (16) completes the description of the interest rate movement model.

A Recursive Algorithm

Before we show that Equations (13) and (16) ensure that the model will satisfy the arbitrage-free condition, we describe a recursive algorithm to generate the one-period bond prices, P_i^n.

To construct arbitrage-free one-period bond prices, we assume that we are given the initial discount function $P(T)$, the threshold rate R, and the term structure of volatilities $\sigma(n)$. The recursive algorithm begins with $n = 0$ and $I = 0$. According to Equation (10), we can solve for δ_0^0 given $R_0^0 = -\log P_0^0(1)/\Delta t$, where $P_0^0(1) = P(1)$. Note that

Equation (13) is automatically satisfied because we assume that $\delta_i^n(0) = 1$, as there is no uncertainty initially.

Now, we proceed with the recursive procedure. Let $n = 1$ and Equation (16) becomes

$$P_0^1 = \frac{P(2)}{P(1)}\frac{(1+\delta_0^0(0))}{(1+\delta_0^0(1))} = \frac{P(2)}{P(1)}\frac{2}{(1+\delta_0^0)} \tag{17}$$

$$P_1^1 = P_0^1\delta_0^0 \tag{18}$$

Now, we substitute the values P_0^1 and P_1^1 in Equation (4) to derive the forward volatilities for period $n = 1$,

$$\delta_i^1 = \exp\left(-2\sigma(1)\min(R_i^1, R)\Delta t^{3/2}\right) \quad \text{for } i = 0,1 \tag{19}$$

We repeat this procedure for period 2, $n = 2$. Using Equation (13), we can now derive the binomial volatilities for term $T = 2$,

$$\delta_i^n(2) = \delta_i^n\delta_i^{n+1}\left(\frac{1+\delta_{i+1}^{n+1}}{1+\delta_i^{n+1}}\right) \quad \text{for } n = 0,1 \quad \text{and } 0 \le i \le n \tag{20}$$

We now use Equation (16) to determine the one-period bond prices,

$$P_0^2 = \frac{P(3)}{P(2)}\frac{(1+\delta_0^0)}{(1+\delta_0^0(2))}\frac{2}{(1+\delta_0^1)} \tag{21}$$

$$P_1^2 = P_0^2\delta_0^1 \tag{22}$$

$$P_2^2 = P_0^2\delta_0^1\delta_1^1 \tag{23}$$

Using these one-period bond prices and Equation (10), we can determine the one-period binomial volatilities δ_i^2, for $i = 0, 1, 2$.

This completes the specifications for $n = 2$, and we proceed to period 3, $n = 3$. Once again, we begin with Equation (13) to determine the forward volatilities of terms greater than 1,

$$\delta_i^n(3) = \delta_i^n\delta_i^{n+1}(2)\left(\frac{1+\delta_{i+1}^{n+1}(2)}{1+\delta_i^{n+1}(2)}\right) \tag{24}$$

Then, apply Equation (16) to determine the one-period bond price for state 0,

$$P_0^3 = \frac{P(4)}{P(3)}\frac{(1+\delta_0^0(2))}{(1+\delta_0^0(3))}\frac{(1+\delta_0^1)}{(1+\delta_0^1(2))}\frac{2}{(1+\delta_0^2)} \tag{25}$$

We now repeatedly apply Equation (16) to determine the one-period bond prices for other states,

$$P_1^3 = P_0^3\delta_0^2,\ P_2^3 = P_0^3\delta_0^2\delta_1^2,\ P_3^3 = P_0^3\delta_0^2\delta_1^2\delta_2^2 \tag{26}$$

From these one-period bond prices, use Equation (10) to determine the one-period forward volatilities.

We can describe the recursive procedure more generally. The building blocks of bond valuation are the binomial volatilities, $\delta_i^n(T)$, for $n = 0,1, \ldots, m$; $i = 0, \ldots, n$; $T = 0, \ldots, m-n$. The recursive procedure begins with a pyramid of the values $\delta_i^n(T)$, with the top being $\delta_0^0(m)$ and the base defined by δ_0^{m-1}, δ_{m-1}^{m-1}, and δ_0^0. We say that the pyramid has dimension m. Then, the recursive procedure has four steps to increase the dimension of the pyramid from m to $m + 1$:

Step 1: Derivation of the one-period bond price for state 0 and time m.

By assumptions of the recursive algorithm, we take the following forward volatilities $\delta_0^0(m)$, $\delta_0^1(m-1)$, $\delta_0^0(m-2)$, … , $\delta_0^{m-2}(2)$, $\delta_0^0(m-1)$, $\delta_0^1(m-2)$, $\delta_0^2(m-3)$, … , $\delta_0^{m-2}(1)$, and $\delta_0^{m-1}(1)$, $\delta_1^{m-1}(1)$, $\delta_2^{m-1}(1)$, … , $\delta_{m-1}^{m-1}(1)$ as given. Now, apply Equation (16) to derive the one-period discount factor at time m and i-th state P_0^m.

Step 2: Derivation of the one-period discount factors for states 1, … , m for time m.

We use Equation (16) repeatedly to calculate P_i^m for $i = 1, \ldots, m$ using the one-period binomial volatilities, δ_0^{m-1}, δ_1^{m-1}, … , δ_{m-1}^{m-1}. Specifically, we use the following equation,

$$P_i^m = P_0^m\prod_{j=0}^{i-1}\delta_j^{m-1} \quad \text{for } i = 1, 2, \ldots, m \tag{27}$$

Step 3: Derivation of the one-period binomial volatilities for time m and states $0, \dots, m$.

First, we use Equation (4) to derive R_i^m from P_i^n. Then, we use Equation (10), $\delta_i^m(1) = \delta_i^m = \exp(-2 \cdot \sigma(m) \min(R_i^m, R)\Delta t^{3/2})$ to calculate δ_i^m for $i = 0, \dots, m$.

This step extends the base of the pyramid of the binomial volatilities by one step, from dimension $m - 1$ to m.

Step 4: Derivation of the higher order of the binomial volatilities.

We use Equation (13) to derive an *additional layer on the face of the pyramid*. Given $\delta_i^m(1)$ for $i = 0, \dots, m$, we can derive $\delta_i^{m-1}(2)$ for $i = 0, \dots, m - 1$, using Equation (13). We continue this recursive method until we reach $\delta_0^0(m + 1)$. This step enables us to add a layer to the face of the pyramid. Starting from the top, this layer is defined by $\delta_0^0(m + 1)$, then the two binomial volatilities, $\delta_0^1(m)$, $\delta_1^1(m)$, and then the three binomial volatilities, $\delta_0^2(m-1)$, $\delta_1^2(m - 1)$, $\delta_2^2(m - 1)$, until we reach the base of the pyramid. This completes the recursive algorithm in that we have increased the dimension of the pyramid from m to $m + 1$.

These four steps of the recursive procedure specify the model using a constructive prescription.

EXTENSIONS OF THE MODEL

Determining the T Period Bond Value at Node (n, i), $P_i^n(T)$

Thus far we have derived one-period bond prices at any node (n, i). According to Equation (16), we can recursively determine the T-period bond price $P_i^n(T)$ at any node (n, i) with any maturity T. The bond price is specified by the binomial volatilities $\delta_i^n(T)$ and is

$$P_i^n(T) = \frac{P(n+T)}{P(n)} \prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n-k))}{(1+\delta_0^{k-1}(n-k+T))} \prod_{j=0}^{i-1} \delta_j^{n-1}(T) \quad (28)$$

$\frac{P(n+T)}{P(n)}$ is the forward price of the T-year bond, n-period hence, based on the initial discount function $P(\cdot)$. $\prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n-k))}{(1+\delta_0^{k-1}(n-k+T))}$ is the convexity term that adjusts for the convexity of the bonds of maturity T such that all bonds satisfy the arbitrage-free condition whereby the risk-neutral expected returns of all bonds equal the one-period rate.

$\prod_{j=0}^{i-1} \delta_j^{n-1}(T)$ specifies the proportional change in value of a T-year bond relative to the forward bond price at state i.

Note that, for $T = 1$, Equation (28) reduces to the pricing of a one-period bond, Equation (16).

To illustrate the valuation of a T-period bond, we consider Equation (28) for $n = 2$ and 3.

For $n = 2$, we need first to derive the forward volatilities, or a pyramid with dimension $T + 1$.

Then, the T-period bonds at time $n = 2$ are given by

$$P_0^2(T) = \frac{P(T+2)}{P(2)} \frac{(1+\delta_0^0)}{(1+\delta_0^0(T+1))} \frac{2}{(1+\delta_0^1(T))} \quad (29)$$

$$P_1^2(T) = P_0^2(T)\delta_0^1(T) \quad (30)$$

$$P_2^2(T) = P_0^2(T)\delta_0^1(T)\delta_1^1(T) \quad (31)$$

And for $n = 3$, we need the pyramid of forward volatilities of dimension $T + 2$, and the bond prices are given by

$$P_0^3(T) = \frac{P(T+3)}{P(3)} \frac{(1+\delta_0^0(2))}{(1+\delta_0^0(T+2))} \times \frac{(1+\delta_0^1)}{(1+\delta_0^1(T+1))} \frac{2}{(1+\delta_0^2(T))} \quad (32)$$

$$\begin{aligned} P_1^3(T) &= P_0^3(T)\delta_0^2(T) \\ P_2^3(T) &= P_0^3(T)\delta_0^2(T)\delta_1^2(T) \\ P_3^3(T) &= P_0^3(T)\delta_0^2(T)\delta_1^2(T)\delta_2^2(T) \end{aligned} \quad (33)$$

The m-Factor Model

The one-factor model can be extended to a two-factor model in a straightforward manner. We assume that there are two independent factors. Each factor is specified by a binomial lattice. The two lattices are combined into a multinomial model, with 0.25 risk-neutral probability for each of the four outcomes of each step.

The intuitive idea is based on the early observation that the yield of a bond is the sum of three parts: the forward rate, the convexity adjustment, and the stochastic movement. A two-factor model is the combined movement of two independent factors in i and j directions. If we project the movement in one direction and focus only on the i movements, then we should have a factor arbitrage-free movement. Similarly, if we focus on the j direction, the movement is also arbitrage free. The co-movement is the sum of the two arbitrage-free movements. Given this intuitive motivation, the two-factor arbitrage-free model is given below. Let $P^n_{i,j}(T)$ be the price of a T-year bond at time n, at state (i, j). Then, the bond price is specified by combining two one-factor models. Specifically, we have

$$P^n_{i,j}(T) = \frac{P(n+T)}{P(n)} \prod_{k=1}^{n} \frac{(1+\delta_{0,1}^{k-1}(n-k))}{(1+\delta_{0,1}^{k-1}(n-k+T))} \times \frac{(1+\delta_{0,2}^{k-1}(n-k))}{(1+\delta_{0,2}^{k-1}(n-k+T))} \prod_{k=0}^{i-1} \delta_{k,1}^{n-1}(T) \prod_{k=0}^{j-1} \delta_{k,2}^{n-1}(T) \tag{34}$$

where

$$\delta^n_{i,1}(T) = \delta^n_{i,1}\delta^{n+1}_{i,1}(T-1)\left(\frac{1+\delta^{n+1}_{i+1,1}(T-1)}{1+\delta^{n+1}_{i,1}(T-1)}\right)$$

$$\delta^n_{i,2}(T) = \delta^n_{i,2}\delta^{n+1}_{i,2}(T-1)\left(\frac{1+\delta^{n+1}_{i+1,2}(T-1)}{1+\delta^{n+1}_{i,2}(T-1)}\right) \tag{35a}$$

and the one-period forward volatilities are given by $\delta^m_{i,1}(1) = \delta^m_{i,1}$, by definition of $\delta^m_{i,1}$.

By extending the specification of δ^m_i to the two-factor model, we have $\delta^m_{i,1} = \exp(-2 \cdot \sigma_1(m)\min(R^m_{i,1}, R)\Delta t^{3/2})$.

Similarly, we can define $\delta^m_{i,2}$ for the other factor, and we have

$$\delta^m_{i,2}(1) = \delta^m_{i,2} = \exp(-2 \cdot \sigma_2(m)\min(R^m_{i,2}, R)\Delta t^{3/2}) \tag{35b}$$

Using the direct extension, we can specify the one-period rates for the two-factor model for any future period m and state i, and $R^m_{i,1}$ and $R^m_{i,2}$ are defined by

$$R^m_{i,1}\Delta t = -\log\left(\frac{P(n+1)}{P(n)}\right) + \sum_{k=0}^{n-1}\log\left(\frac{(1+\delta_{0,1}^{k-1}(n-k))}{(1+\delta_{0,1}^{k-1}(n-k+T))}\right) + \sum_{k=0}^{i-1}\delta^{n-1}_{k,1}(T)$$

and

$$R^m_{i,2}\Delta t = -\log\left(\frac{P(n+1)}{P(n)}\right) + \sum_{k=0}^{n-1}\log\left(\frac{(1+\delta_{0,2}^{k-1}(n-k))}{(1+\delta_{0,2}^{k-1}(n-k+T))}\right) + \sum_{k=0}^{i-1}\delta^{n-1}_{k,2}(T) \tag{35c}$$

For clarity of exposition, we have presented the two-factor model. The generalization of the two-factor model to the multifactor model is clear. For an m-factor model, the bond price is represented by $P^n_{i_1 i_2, \ldots, i_m}(T)$.

The model shows that the price of a T-period bond at any state and time n has three components as previously described. First, $\frac{P(n+T)}{P(n)}$ represents the forward price based on the initial yield curve. Second, $C(n,T)$ is the convexity term given by

$$C(n,T) = \prod_{k=1}^{n} \frac{(1+\delta_{0,1}^{k-1}(n-k))\ldots(1+\delta_{0,m}^{k-1}(n-k))}{(1+\delta_{0,1}^{k-1}(n-k+T))\ldots(1+\delta_{0,m}^{k-1}(n-k+T))} \tag{36}$$

This term adjusts the bond price such that the additional returns derived from the convexity of the bondare countered by this adjustment to assure the arbitrage-free condition. Third, the stochastic movement term $\Lambda(i_1, \ldots, i_m)$ is given by

$$\Lambda(i_1,\ldots,i_m) = \prod_{k=0}^{i_1-1}\delta^{n-1}_{k,1}(T)\ldots\prod_{k=0}^{i_m-1}\delta^{n-1}_{k,m}(T) \tag{37}$$

Then the m-factor bond price is given by

$$P^n_{i_1,i_2,\ldots,i_m}(T) = \frac{P(n+T)}{P(n)} C(n,T)\Lambda(i_1,\ldots,i_m) \tag{38}$$

The Arbitrage-Free Interest Rate Movement Condition

We now need to show that the model in Equation (10) satisfies the two conditions of arbitrage-free interest rate movements. Specifically, we require the following:

Condition 1: *Consistency with the one-period discount factor.*

$$P^n_{i_1,i_2,\ldots,i_m}(T) = 2^{-m} P^n_{i_1,i_2,\ldots,i_m} \cdot \sum_{k_1=0}^{1} \cdots \sum_{k_m=0}^{1} P^{n+1}_{i_1+k_1,\ldots,i_m+k_m}(T-1) \quad (39)$$

This condition requires that all bonds have a risk-neutral expected return over one period at each node to be the risk-free rate, a condition consistent with Harrison and Kreps [1979].

Condition 2: *Consistency with the discount function.*

If $P^n_{i_1,i_2,\ldots,i_m}(0) = 1$ for a given n, and for all $0 \le i_1,\ldots, i_m \le n$, Equation (11) would imply that $P^0_{0,\ldots,0}(n) = P(n)$, the given n-period initial discount bond price. The following proposition provides the proof.

Proposition 1: *The Model for Bond Prices.*

We consider a two-factor model in this proof, and the results can be generalized to higher dimension. Let $P(T)$ be the initial discount function. The price of a T-period zero-coupon bond with \$1 principal at time n and states i and j are given by Equation (38),

$$\begin{aligned} P^n_{i,j}(T) &= \frac{P(n+T)}{P(n)} \prod_{k=1}^{n} \frac{(1+\delta^{k-1}_{0,1}(n-k))}{(1+\delta^{k-1}_{0,1}(n-k+T))} \\ &\times \frac{(1+\delta^{k-1}_{0,2}(n-k))}{(1+\delta^{k-1}_{0,2}(n-k+T))} \prod_{k=0}^{i-1} \delta^{n-1}_{k,1}(T) \prod_{k=0}^{j-1} \delta^{n-1}_{k,2}(T) \end{aligned} \quad (40)$$

where the binomial volatilities are related by Equation (35),

$$\delta^n_{i,1}(T) = \delta^n_{i,1}\delta^{n+1}_{i,1}(T-1)\left(\frac{1+\delta^{n+1}_{i+1,1}(T-1)}{1+\delta^{n+1}_{i,1}(T-1)}\right)$$

$$\delta^n_{i,2}(T) = \delta^n_{i,2}\delta^{n+1}_{i,2}(T-1)\left(\frac{1+\delta^{n+1}_{i+1,2}(T-1)}{1+\delta^{n+1}_{i,2}(T-1)}\right) \quad (41)$$

where $\delta^n_{i,1}(0) = 1$, $\delta^n_{i,2}(0) = 1$, $\delta^n_{i,1}(1) = \delta^n_{i,1}$, and $\delta^n_{i,2}(1) = \delta^n_{i,2}$, which are given.

The T-year bond price $P^n_{i,j}(T)$ satisfies the arbitrage-free condition of Equation (39) so that

$$P^n_{i,j}(T+1) = 2^{-2} P^n_{i,j} \cdot \sum_{i=0}^{1}\sum_{j=0}^{1} P^{n+1}_{i,j}(T) \quad (42)$$

Proof: *We begin with the one-factor model in the proof. According to the pricing model, Equation (28), we have*

$$\begin{aligned} P^n_i(T) &= \frac{P(n+T)}{P(n)} \frac{(1+\delta^0_0(n-1))}{(1+\delta^0_0(n-1+T))} \\ &\times \frac{(1+\delta^1_0(n-2)}{(1+\delta^1_0(n-2+T))} \cdots \frac{(1+\delta^{n-1}_0(0))}{(1+\delta^{n-1}_0(T))} \\ &\times \delta^{n-1}_0(T) \cdots \delta^{n-1}_{i-1}(T) \end{aligned} \quad (43)$$

We need to show that the arbitrage-free condition holds, that is,

$$P^n_i(T) = \frac{1}{2} P^n_i (P^{n+1}_i(T-1) + P^{n+1}_{i+1}(T-1)) \quad (44)$$

Now, we can express

$$\begin{aligned} P^n_i &= \frac{P(n+1)}{P(n)} \frac{(1+\delta^0_0(n-1))}{(1+\delta^0_0(n))} \frac{(1+\delta^1_0(n-2))}{(1+\delta^1_0(n-1))} \\ &\cdots \frac{(1+\delta^{n-1}_0(0))}{(1+\delta^{n-1}_0(1))} \delta^{n-1}_0 \cdots \delta^{n-1}_{i-1} \end{aligned} \quad (45)$$

$$\begin{aligned} P^{n+1}_i(T-1) &= \frac{P(n+T)}{P(n+1)} \frac{(1+\delta^0_0(n))}{(1+\delta^0_0(n-1+T))} \\ &\times \frac{(1+\delta^1_0(n-1)}{(1+\delta^1_0(n-2+T))} \cdots \frac{(1+\delta^n_0(0))}{(1+\delta^n_0(T-1))} \\ &\times \delta^n_0(T-1) \cdots \delta^n_{i-1}(T-1) \end{aligned} \quad (46)$$

$$P^{n+1}_{i+1}(T-1) = P^{n+1}_i(T-1)\delta^n_i(T-1) \quad (47)$$

By a direct computation, using Equations (43), (45), (46), and (47), Equation (44) can be derived. We note that, by assumption, Equation (41) for the one-factor model gives

$$\frac{\delta_k^n(T)}{\delta_k^{n+1}(T-1)\delta_k^n} = \frac{(1+\delta_{k+1}^{n+1}(T-1))}{(1+\delta_k^{n+1}(T-1))} \quad \text{for } k=0, \ \ldots, \mathrm{i} \tag{48}$$

Similarly, we can show that the arbitrage-free condition holds for higher dimension.

To illustrate, we first consider $n = 1$. We have, from Equation (40) with $i = j = 0$,

$$P_{0,0}^1(T) = \frac{P(T+1)}{P(1)} \frac{2}{(1+\delta_{0,1}^0(T))} \frac{2}{(1+\delta_{0,2}^0(T))} \tag{49}$$

Now we need to show that given Equation (49), the arbitrage-free condition holds, letting $n = 0$ in Equation (11),

$$P_{0,0}^0(T+1) = 2^{-2} P_{0,0}^0 \cdot \sum_{i=0}^{1}\sum_{j=0}^{1} P_{i,j}^1(T) \tag{50}$$

Note that $P_0^0(1) = P(1)$ and $P_0^0(T) = P(T)$. Using Equation (40), the right side of Equation (50) becomes

$$\frac{1}{4}P(1)\cdot\frac{P(T+1)}{P(1)}\frac{2}{(1+\delta_{0,1}^0(T))}\frac{2}{(1+\delta_{0,2}^0(T))} \times\left[1+\delta_{0,1}^0(T)+\delta_{0,2}^0(T)+\delta_{0,1}^0(T)\cdot\delta_{0,2}^0(T)\right] \tag{51}$$

In simplifying the above expression, it is reduced to $P(T)$. Hence, we have shown that the right side equals the left side of Equation (50).

Next, we consider the case $n = 2$. The arbitrage-free condition, Equation (50), becomes

$$P_{0,0}^1(T+1) = 2^{-2} P_{0,0}^1 \cdot \sum_{i=0}^{1}\sum_{j=0}^{1} P_{i,j}^2(T) \tag{52}$$

By using Equation (40) for the specific node points and maturity, we have the following Equations (53) to (58):

$$P_{0,0}^1(T+1) = \frac{P(T+2)}{P(1)}\frac{2}{(1+\delta_{0,1}^0(T+1))}\frac{2}{(1+\delta_{0,2}^0(T+1))} \tag{53}$$

$$P_{0,0}^1 = \frac{P(2)}{P(1)}\frac{2}{(1+\delta_{0,1}^0)}\frac{2}{(1+\delta_{0,2}^0)} \tag{54}$$

$$P_{0,0}^2(T) = \frac{P(2+T)}{P(2)}\frac{(1+\delta_{0,1}^0)}{(1+\delta_{0,1}^0(1+T))}\frac{2}{(1+\delta_{0,1}^1(T))} \times\frac{(1+\delta_{0,2}^0)}{(1+\delta_{0,2}^0(1+T))}\frac{2}{(1+\delta_{0,2}^1(T))} \tag{55}$$

$$P_{1,0}^2(T) = P_{0,0}^2(T)\delta_{0,1}^1(T) \tag{56}$$

$$P_{0,1}^2(T) = P_{0,0}^2(T)\delta_{0,2}^1(T) \tag{57}$$

$$P_{1,1}^2(T) = P_{0,0}^2(T)\delta_{0,1}^1(T)\delta_{0,2}^1(T) \tag{58}$$

Substituting these equations in Equation (52), and simplifying, we show that the arbitrage-free condition of Equation (52) is satisfied.

A more general proof is given in Appendix A.

Q.E.D.

Corollary: *The model prices of the zero-coupon bonds are consistent with the initial observed discount function P(T).*

Proof: *In the special base, for $n = i_1 = i_2 = \ldots = i_m = 0$, according to Equation (10), $P_{0,0,\ldots,0}^0(T) = P(T)$.*

Q.E.D.

THE MODEL CALIBRATION RESULTS

Data

We now apply the model specified by Equations (4), (10), (11), (13), and (16) to the observed swaption prices in the market. Garban International, a broker in the swaption market, provided the data. The sample period is from June 30, 2003, through June 30, 2004, using monthly data. The swaptions expire in 1, 2, 3, 4, 5, 7, and

10 years. The option gives the holder the right to enter into a swap at a specified fixed rate with tenor 1, 2,..., 10 years. The fixed rates are set such that the swaptions are at-the-money at the time of pricing. The prices of the swaptions are quoted in volatilities (%) based on the Black model for swaptions. There are 70 observations for each date. The swaption prices for June 30, 2004, are presented in Appendix B. The key interest rates—the swap spot rates—for this date are explained below:

Years	0.25	0.5	1	2	3	4	5
Yields	1.61%	1.94%	2.47%	3.15%	3.68%	4.13%	4.38%
Years	6	7	8	9	10	15	20
Yields	4.64%	4.85%	5.03%	5.04%	5.14%	5.82%	6.05%

Calibration Procedure

For each date, we determine the term structure of volatilities such that the model prices fit the observed swaption prices. To calibrate the one-factor model, we use a non-linear search procedure in determining the parameters of the function $\sigma(n)$. According to Equation (11), these parameters are σ_0, σ_∞, α_0, α_1, α_∞ such that the sum of squared proportional errors between the observed and model prices of the benchmark securities is minimized.

Specifically, P_j^0 and P_j^t are the observed and theoretical swaption prices, respectively, for $j = 1, \ldots, 70$. We define the objective function to be

$$J(\sigma_0,\sigma_1,\sigma_\infty,\sigma_0,\sigma_\infty)=\sum_{j=1}^{70}\left(\frac{P_j^o-P_j^t}{P_j^o}\right)^2 \quad (59)$$

We then use a non-linear optimization procedure to determine the minimized objective function J denoted by J^* by changing the choice variables σ_0, σ_∞, α_0, α_1, α_∞. For the two-factor model, we have six choice variables, σ_0, σ_∞, α_0, α_1, α_∞, for the first factor and a constant σ for the second factor in Equation (35b). The mean error of the estimation, given that there are 70 observed prices for each date, is defined as

$$e=\sqrt{J^*}/70 \quad (60)$$

We assume that threshold rate R of Equation (10) is 3%. We also consider the alternative model, assuming R to be 9%. We have tested the model with different threshold rates, and we have found that the model performs best at 3%. For this reason, we use the 3% model as a benchmark.

The binomial lattice model is based on a one-month step size. To value a 10-year swaption with the option expiring in 10 years, our lattice needs to incorporate a 20-year term structure, or 240 monthly steps. Therefore, we use a yield curve extending to 20 years, based on the swap rate.

The Empirical Model and the Davidson and Mackinnon C Test

To test the explanatory power of the two-factor model, we use the Davidson and Mackinnon C test. Let P_j^0 be the j-th swaption observed price, $j = 1, \ldots, 70$, on a particular date. Let $\Phi_i(X_i, \beta)$ be the proposed model, where X_i is a vector of observations which are the spot rates and the swaption prices; β is a vector of parameters to be estimated, and ε_{0i} is an error term $N(0,\theta)$. Let $\Gamma_i(Z_i, \gamma)$ be the alternative model, where Z_i is a vector of observations on the exogenous variables; γ is a vector of parameters to be estimated; and ε_{1i} is an error term $N(0,\theta^*)$.

Let the null hypothesis be

$$H_0: P_j^0=\Phi_j(X_j,\beta)+P_j^0\varepsilon_{0j} \quad (61)$$

and the alternative hypothesis be

$$H_1: P_j^0=\Gamma_j(Z_j,\gamma)+P_j^0\varepsilon_{1i} \quad (62)$$

The C test assumes that the observed swaption prices can be explained by a convex combination of the two models specified as

$$P_j^0=(1-\alpha)\Phi_j+\alpha\Gamma_j+P_j^0\varepsilon_j \quad (63)$$

where ε_j is assumed to be normal with zero mean.

Rearranging, we get

$$(P_j^0-\Phi_j)/P_j^0=\alpha((P_j^0-\Phi_j)/P_j^0-(P_j^0-\Gamma_j)/P_j^0)+\varepsilon_j \quad (64)$$

Equation (64) says that if the alternative model Γ_j can provide better explanatory power to the model Φ_j (in terms of minimizing the sum of squared proportional errors), then α should be significantly different from 0. This hypothesis can be tested using a conventional asymptotic t-test.

The Two-Factor Model versus the One-Factor Model

We compare the estimation errors of the one-factor model, Equations (13) and (16), and the alternative two-factor model, based on Equations (35) and (36). In both cases, we use 3% for the threshold rate R. The mean percentage errors of the models are reported in columns A and B in Exhibit 1. We then apply the C test from Equation (64) to the two models for each date. The coefficients, t-statistics, and R-square are reported in columns C, D, and E.

Based on the mean errors reported in Exhibit 1, the empirical results support the validity of both the two-factor and one-factor models. The models have reasonably small errors in the calibration. The results are particularly reasonable when we consider that the two-factor and one-factor models use only six and five parameters, respectively, for the calibration. Further, the models are calibrating to some long-dated swaptions where the swaption market is less liquid. The price quotes are subject to the bid-ask spread of the market and quotes are dependent on the size of the transaction. These considerations can result in a price quoted in a range of 2%.

The t-statistics (column D) and the adjusted R-squared (column E) of the C test support the hypothesis that the two-factor model provides higher explanatory power than the one-factor model for all the dates, with the R-squared exceeding 80% for some dates. The average percentage errors over the sample period for the two-factor and one-factor model are 2.30% and 2.97%, respectively, showing that the two-factor model provides a lower average error.

The model provides further insight into the explanatory power of the two-factor model. Equation (64) can be rearranged as follows:

$$\frac{(P_j^0 - \Gamma_j)}{P_j^0} = \frac{(\alpha - 1)}{\alpha}\frac{(P_j^0 - \Phi_j)}{P_j^0} + \frac{1}{\alpha}\varepsilon_j$$

Since the coefficients α are estimated between approximately 2.0 and 3.2, the empirical model shows

EXHIBIT 1

Comparison of the One-Factor and Two-Factor Models

	One- and Two-Factor Model of Mean Errors		C test		
	A	B	C	D	E
Dates	2-factor model*	1-factor model*	Coefficients α	t-statistics	R-square(%)
06/30/03	3.51%	4.34%	1.969	7.636	46
07/30/03	3.73%	4.42%	2.776	8.119	49
08/29/03	2.28%	2.93%	3.048	13.19	72
09/30/03	3.49%	4.28%	3.215	11.02	64
10/30/03	2.60%	3.34%	2.973	12.95	71
11/28/03	1.50%	2.20%	1.981	13.08	71
12/31/03	1.83%	2.65%	2.623	19.92	85
01/31/04	1.72%	2.49%	2.42	16.52	80
02/27/04	1.65%	2.33%	2.626	16.96	81
03/31/04	2.22%	3.11%	2.944	21.6	87
05/31/04	1.03%	1.44%	1.989	11.34	65
06/30/04	2.09%	2.14%	1.877	2.062	6
Average	2.30%	2.97%	2.537	12.866	65

**Note that the threshold rate is 3%.*

that the two-factor model percentage error for each swaption is between 0.5 and 0.69 of the error of the one-factor model plus a random error term. The errors may be small as suggested by the high t-statistics and R-squared values, depending on the sample date.

The Normal Model versus the Lognormal Model

The Ho-Lee and Hull-White models assume that the market perceives interest rate movements to be a normal model. In contrast, the Black-Derman-Toy and Black-Karasinski models assume that the interest rate movements are approximated by a lognormal model. Which assumption is more empirically correct based on the sample swaption prices? We use our two-factor model to provide some insight into this question. Specifically, we compare the model using a threshold rate of 3% to the alternative model using a higher threshold rate of 9%. The former model would support that the interest rate movement is better explained by a normal model, while the latter would support a lognormal interest rate movement. The results are reported in Exhibit 2. Columns A and B provide the mean error of each date for the two-factor model with the threshold rates 3% and 9%, respectively. For the C test, Columns C, D, and E provide the coefficients, test statistics, and the adjusted R-squared, respectively.

The results show that the model with a lower threshold rate (3%) generally has lower mean errors than the model with a higher threshold rate (9%), except for the date June 30, 2004. The difference is also shown to be significant by the t-statistics. Therefore, the empirical results suggest that the model is more akin to the normal model. Note that R can be a choice parameter in the calibration procedure. Our research has shown that, in general, the optimized threshold rate is around 3% for our sample period.

MODEL RESULTS

The Yields of Zero-Coupon Bonds

We determine here the yield of a zero-coupon bond. For clarity of the exposition, we present the results for the one-factor model. The extension of the one-factor model to the m-factor model is straightforward. The yields

EXHIBIT 2

Comparison of a Normal-like Model and a Lognormal-like Model

	Two-Factor Model Mean Errors		C test		
	A	B	C	D	E
Dates	3% threshold rate	9% threshold rate	Coefficients α	t-statistics	R-square(%)
06/30/03	3.51%	3.79%	1.289	3.494	15
07/30/03	3.73%	4.08%	2.558	4.888	26
08/29/03	2.28%	2.67%	2.542	7.136	42
09/30/03	3.49%	3.84%	2.129	4.74	25
10/30/03	2.60%	2.97%	1.184	4.697	24
11/28/03	1.50%	1.86%	1.067	6.132	35
12/31/03	1.83%	2.25%	1.919	7.348	44
01/31/04	1.72%	2.04%	1.733	6.15	35
02/27/04	1.65%	1.89%	1.88	5.507	31
03/31/04	2.22%	2.62%	2.048	6.172	36
05/31/04	1.03%	1.36%	1.576	8.259	50
06/30/04	2.09%	1.48%	0.0635	0.5661	0
Average	2.30%	2.57%	1.666	5.424	30

are then used to depict interest rate movements based on the calibrated interest rate model.

By definition, the yield of a T-period bond at node (n, i) is given by

$$R_i^n(T) = -\frac{1}{T\Delta t}\log P_i^n(T) \quad (65)$$

Using Equation (65) and Equation (34), we can derive the yield of the T-year bond at each node (n, i). It is given as

$$R_i^n(T)T\Delta t = -\log\frac{P(n+T)}{P(n)} - \sum_{k=1}^{n}\log\frac{(1+\delta_0^{k-1}(n-k))}{(1+\delta_0^{k-1}(n-k+T))} - \sum_{j=0}^{i-1}\log\delta_j^{n-1}(T) \quad (66)$$

In particular, the yield for the one-period bond is

$$R_i^n\Delta t = -\log\frac{P(n+1)}{P(n)} - \sum_{k=1}^{n}\log\left(\frac{(1+\delta_0^k(n-k))}{(1+\delta_0^k(n-k+1))}\right) - \sum_{j=0}^{i-1}\log\delta_j^{n-1} \quad (67)$$

For illustration, the one-period yield for $n = 1$ is

$$R_0^1\Delta t = -\log\frac{P(2)}{P(1)} - \log\frac{2}{(1+\delta_0^0)} \quad (68)$$

$$R_1^1\Delta t = -\log\frac{P(2)}{P(1)} - \log\frac{2}{(1+\delta_0^0)} - \log\delta_0^0 \quad (69)$$

The model is used to calculate the interest rates at each node point of the lattices. They will be depicted later in the article.

Analysis of Interest Rate Movements

Given these empirical results, we now proceed to study the simulated yield curve movements based on the two-factor model with a 3% threshold rate using Equation (69). We provide three main observations: a) the simulated interest rates exhibit mean reversion, and no negative and unreasonably high interest rates; b) the simulated instantaneous principal movements of the yield curve are consistent with the historical movements; and c) the simulated yield curves can be inverted.

Interest rate distribution. Exhibit 3 depicts the one-factor lattice model. It shows that the model has almost no negative interest rates and minimal extremely high interest rates. There are some negative interest rates with negligible absolute value because of the discrete time approximation of the lognormal process. Footnote 2 provides the explanation of negative rates more precisely. Also note that the lattice tends not to expand with time indicating the mean reversion behavior of the interest rate model.

We should also note that extremely high and low interest rates are unlikely outcomes based on risk-neutral probabilities. When we simulate the scenario paths in the lattices, most of the paths, in fact, appear in the middle part of the lattice. This result is demonstrated in Exhibit 4. The results of randomly simulating 1,000 scenarios from the two-factor model show that all paths of one-month rates are bounded from zero and most paths lie between 0% and 10% of the interest rate levels.

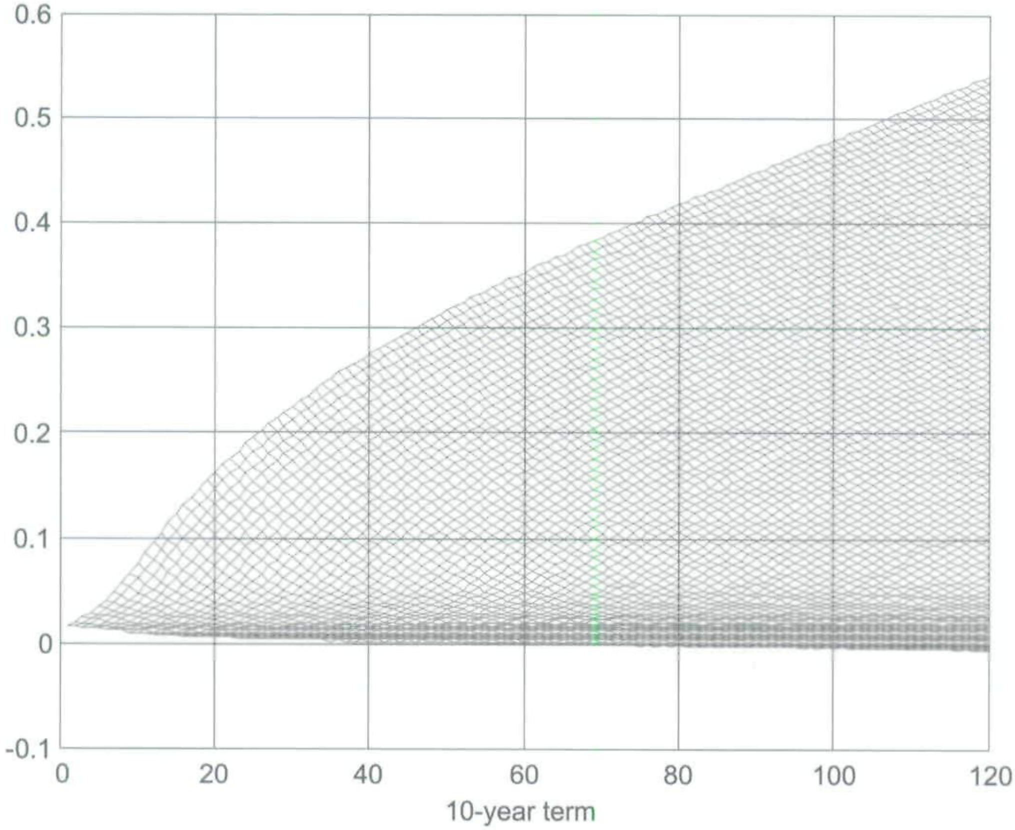
Term structure of volatilities and co-movements of the yield curve. We now consider the term structure of volatilities of the two-factor model. The estimated function of the steepening movement $\sigma(n)$ of Equation (11) for each date is presented in Exhibit 5.

The results show that the steepening movement requires the function to decline monotonically, except on July 30, 2003, and November 28, 2003, in this sample period, inducing mean reversion in the process. This is consistent with the observed market term structure of volatilities where it may rise in the short-term segment of the curve.

Next, we use the model to construct the *instantaneous* variance-covariance of the interest rates. For each specific date, we calibrate the model to the observed swaption prices. Then, based on the model, we consider the four possible spot yield curves at the nodes one step from the initial point. The model assumes that the risk-neutral probabilities are 0.25 for each outcome. We can calculate the variance-covariance matrix of the interest rates for terms 1, 2, … , 10 years, based on the proportional change in the yields of the bonds. Using this matrix we can determine the principal components of the yield curve movements. These principal components are called *implied*

EXHIBIT 3

One-Month Interest Rate on One-Factor Model



yield curve movements. Since these implied yield curve movements are based on a two-factor model calibrated from the observed swaption prices, we only consider the first two movements.

The results show that the implied principal components maintain a similar shape over the sample period. A representative steepening movement based on June 30, 2004, is presented on the upper panel of Exhibit 6.

We use historical monthly yield curve data from February 1998 to August 2004. Using the proportional changes of the yield curve over the one-month period, we determine the variance-covariance matrix of the interest rates. The steepening movement of the principal components is depicted in the lower panel of Exhibit 6. The upper and lower panels of Exhibit 6 show that the implied steepening movement is similar to the historical steepening movement.

The results show that the implied principal movements of the yield curve, based on the observed swaption prices, are similar to those estimated from the historical data. Furthermore, the amount of movement explained by parallel movements over the observation period is also consistent with that estimated from historical data. Exhibit 6 presents the percentage of yield curve movements explained by implied parallel movement of the yield curve over the sample period.

Inverted Yield Curve. While our results suggest that the parallel movement of the yield curve is dominant, the simulated yield curves do exhibit inversion, with the short-rate level higher than the long-rate level, due to the steepening yield curve movements.

We consider a five-year (60-month) time horizon. Based on the $60 * 60$ possible outcomes, 15-year rates range from 1.47%–23.44%. With the steepening movement, 2-year rates vary over a larger range of 1.13%–26.11%. Indeed, if the spread, defined as the 15-year rate minus the 2-year rate, has a positive value, the yield curve is inverted. We see that inverted yield curves are exhibited at many nodes. The results show that the yield curve is inverted when state i equals or exceeds 42.

EXHIBIT 4

Randomly Simulated Paths of the Two-Factor Model

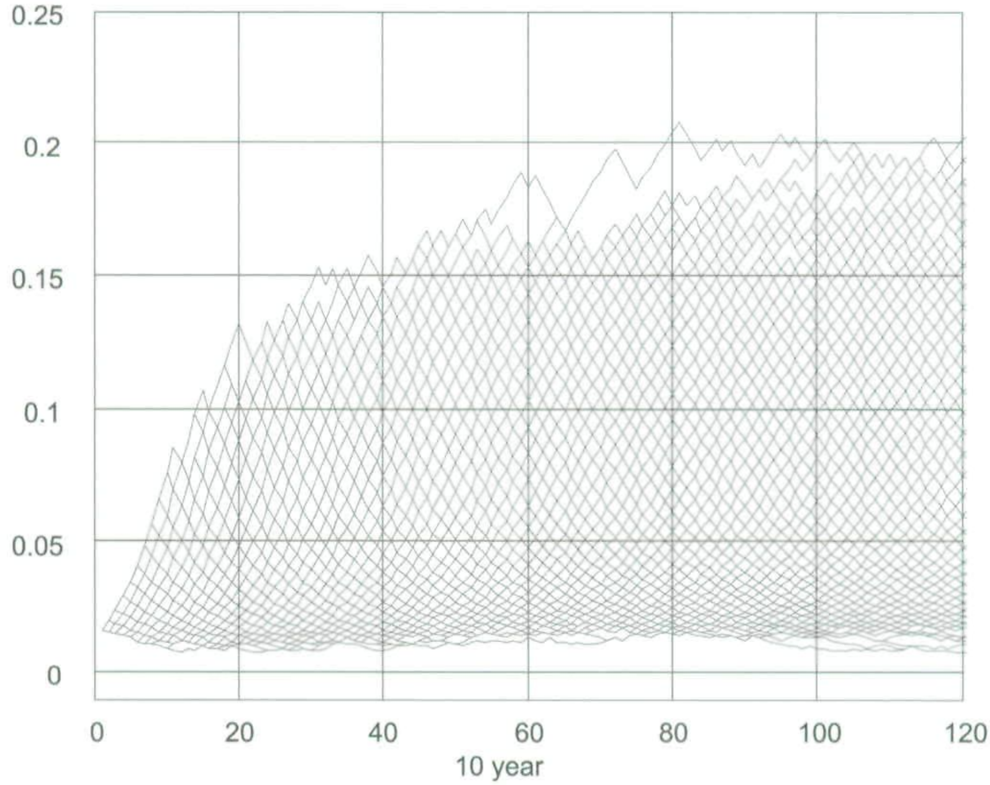


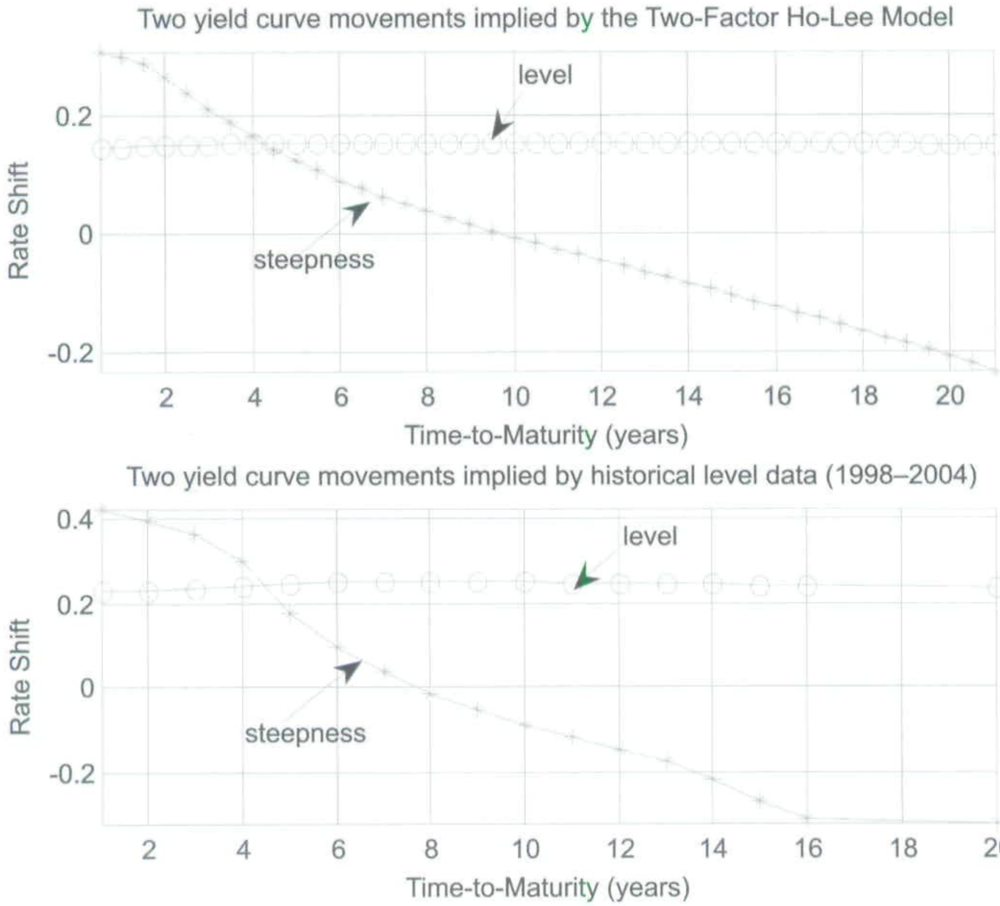
EXHIBIT 5

Estimated Volatilities of the Steepening Movement Factor (June 30, 2003–June 30, 2004)

Year	0.25	0.5	1	2	3	4	5	6	7	8	9	10	15
6/30/03	56.1%	55.2%	53.4%	49.9%	46.5%	43.1%	39.9%	36.7%	33.7%	30.7%	27.8%	24.9%	11.8%
7/30/03	38.4%	40.7%	44.1%	47.1%	46.8%	44.7%	41.7%	38.4%	35.1%	31.9%	28.9%	26.2%	15.3%
8/29/03	65.7%	63.5%	59.5%	53.0%	47.7%	43.4%	39.7%	36.4%	33.4%	30.5%	27.9%	25.3%	12.9%
9/30/03	72.9%	70.0%	64.7%	56.0%	49.3%	44.0%	39.7%	36.0%	32.8%	29.8%	27.1%	24.5%	11.8%
10/30/03	59.8%	58.7%	56.5%	52.3%	48.3%	44.5%	40.8%	37.3%	34.0%	30.8%	27.7%	24.7%	11.4%
11/28/03	47.7%	48.8%	50.1%	50.0%	47.9%	44.9%	41.5%	38.0%	34.7%	31.5%	28.6%	25.8%	13.9%
12/31/03	61.6%	59.1%	55.1%	49.3%	45.2%	41.8%	38.7%	35.7%	32.8%	29.9%	27.0%	24.0%	9.5%
1/30/04	62.8%	60.0%	55.3%	48.8%	44.4%	40.9%	37.8%	35.0%	32.3%	29.6%	26.9%	24.3%	11.2%
2/27/04	60.4%	59.0%	56.3%	51.7%	47.8%	44.5%	41.5%	38.9%	36.4%	34.0%	31.8%	29.7%	19.5%
3/31/04	63.2%	61.2%	57.5%	51.5%	46.6%	42.5%	38.9%	35.6%	32.5%	29.6%	26.8%	24.0%	10.5%
5/31/04	51.6%	50.9%	49.4%	46.6%	43.9%	41.4%	38.9%	36.6%	34.3%	32.2%	30.1%	28.1%	18.8%
6/30/04	42.1%	42.6%	43.0%	42.0%	40.1%	38.0%	36.1%	34.4%	32.9%	31.7%	30.6%	29.6%	25.5%

EXHIBIT 6

Two Yield Curve Movements Implied by the Two-Factor Model



IMPLICATIONS AND CONCLUSIONS

This article presents an arbitrage-free binomial interest rate model which has the following four desirable attributes: 1) an arbitrage-free model that takes the yield curve as given; 2) a recombining lattice model for accurate pricing of American and some Asian options and that can be used to determine stratified sampling (See Ho [1992] for a construction of stratified sampling using a recombining lattice.); 3) a multifactor model to capture yield-curve shape movements; and 4) a state-time dependent implied volatility function that ensures mean reversion, a market-determined mix of the lognormal and normal distributions, and the implied correlations of interest rates, estimated from market swaption prices.

We have also presented empirical evidence to support the validity of the model. Further, we have shown that the simulated yield curves based on the model are consistent with the behavior of the yield curve historically. A more extensive empirical test of the model over three currencies using a five-year period of monthly data is also presented in Ho and Mudavanhu [2007]. The model is

EXHIBIT 7

Percentage of Movements Explained by Parallel Yield Curve Shifts (Ho-Lee Model)

6/30/03	7/30/03	8/29/03	9/30/03	10/30/03	11/28/03	12/31/03	1/31/04	2/27/04	3/31/04	5/31/04	6/30/04
99.08%	99.51%	98.99%	98.78%	99.02%	99.32%	98.86%	98.93%	99.56%	98.88%	99.66%	99.99%

EXHIBIT 8

Spread between Long Rate (15-year maturity) and Short Rate (2-year maturity)

	j = 60	54	48	42	36	30	24	18	12	6	0
i = 60	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267	-0.0267
54	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198	-0.0198
48	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129	-0.0129
42	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061	-0.0061
36	0.00084	0.00084	0.00084	0.00084	0.00084	0.00084	0.0009	0.0008	0.0009	0.0009	0.0008
30	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078
24	0.0149	0.0148	0.0149	0.0149	0.0149	0.0149	0.0149	0.0149	0.0148	0.0149	0.0149
18	0.0136	0.0136	0.0136	0.0135	0.0136	0.01361	0.01354	0.01357	0.0136	0.01358	0.01358
12	0.0078	0.0078	0.0079	0.00789	0.00782	0.00785	0.00785	0.00785	0.00785	0.00788	0.00785
6	0.0045	0.0044	0.0044	0.00446	0.00443	0.00443	0.00443	0.00443	0.00443	0.00439	0.0045
0	0.0034	0.0034	0.00346	0.00345	0.00345	0.00345	0.00345	0.00345	0.00341	0.0035	0.0034

shown to be robust and the movements of the implied volatility functions are studied.

The article shows that a state-time dependent implied-volatility function may have significant informational content extracted from swaption prices. It suggests that swaption prices may reflect a more complex relationship between instantaneous volatilities and the interest rate level, and this relationship is stochastic in historical time.

APPENDIX A

Proof of Proposition 1

The discount function with the time-to-maturity T at time n and state i

$$P_i^n(T) = \frac{P(n+T)}{P(n)} \prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n-k))}{(1+\delta_0^{k-1}(n-k+T))} \prod_{j=0}^{i-1} \delta_j^{n-1}(T) \qquad \text{(A-1)}$$

The discount function with the time-to-maturity 1 at time n and state i

$$P_i^n(1) = \frac{P(n+1)}{P(n)} \prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n-k))}{(1+\delta_0^{k-1}(n-k+1))} \prod_{j=0}^{i-1} \delta_j^{n-1}(1) \qquad \text{(A-2)}$$

Divide Equation (A-1) by Equation (A-2) to get $\frac{P_i^n(n+1)}{P_i^n(1)}$

$$\frac{P_i^n(n+1)}{P_i^n(1)} = \frac{P(n+T)}{P(n+1)} \prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n-k+1))}{(1+\delta_0^{k-1}(n-k+T))} \frac{\prod_{j=0}^{i-1} \delta_j^{n-1}(T)}{\prod_{j=0}^{i-1} \delta_j^{n-1}(1)} \qquad \text{(A-3)}$$

The discount function with the time-to-maturity T–1 at time n + 1and state i

$$P_i^{n+1}(T-1) = \frac{P(n+1+T-1)}{P(n+1)} \prod_{k=1}^{n+1} \frac{(1+\delta_0^{k-1}(n+1-k))}{(1+\delta_0^{k-1}(n+1-k+T-1))} \times \prod_{j=0}^{i-1} \delta_j^n(T-1) \qquad \text{(A-4)}$$

$$= \frac{P(n+1+T-1)}{P(n+1)} \prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n+1-k))}{(1+\delta_0^{k-1}(n+1-k+T-1)} \cdot \frac{(1+\delta_0^n(n+1-n-1))}{(1+\delta_0^n(n+1-n-1+T-1))} \prod_{j=0}^{i-1} \delta_j^n(T-1)$$

$$= \frac{P(n+T)}{P(n+1)} \prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n+1-k))}{(1+\delta_0^{k-1}(n-k+T))} \cdot \frac{(1+\delta_0^n(0))}{(1+\delta_0^n(T-1))} \prod_{j=0}^{i-1} \delta_j^n(T-1))$$

$$= \frac{P(n+T)}{P(n+1)} \prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n+1-k))}{(1+\delta_0^{k-1}(n-k+T))} \cdot \frac{2}{(1+\delta_0^n(T-1))} \prod_{j=0}^{i-1} \frac{\delta_j^{n-1}(T)}{\delta_j^{n-1}} \frac{(1+\delta_j^n(T-1))}{(1+\delta_{j+1}^n(T))}$$

$$\because \delta_j^{n-1}(T) = \delta_j^{n-1} \delta_j^n(T-1) \frac{(1+\delta_{j+1}^n(T-1))}{(1+\delta_j^n(T-1))} \text{ fxrom Equation (6)}$$

$$= \frac{P(n+T)}{P(n+1)} \prod_{k=1}^{n} \frac{(1+\delta_0^{k-1}(n+1-k))}{(1+\delta_0^{k-1}(n-k+T))} \prod_{j=0}^{i-1} \frac{\delta_j^{n-1}(T)}{\delta_j^{n-1}} \cdot \frac{2}{(1+\delta_i^n(T-1))} \qquad \text{(A-5)}$$

Comparing Equation (A-5) with Equation (A-3), we see that

$$P_i^{n+1}(T-1)=\frac{P_i^n(T)}{P_i^n(1)}\frac{2}{(1+\delta_i^n(T-1))} \tag{A-6}$$

Similarly, we can show that

$$P_{i+1}^{n+1}(T-1)=\frac{P_i^n(T)}{P_i^n(1)}\frac{2\,\delta_i^n(T-1)}{(1+\delta_i^n(T-1))} \tag{A-7}$$

Plugging Equation (A-6) and Equation (A-7) into Equation (44), we can see that Equation (44) holds as required.

$$P_i^n(T)=\frac{1}{2}P_i^n(P_i^{n+1}(T-1)+P_{i+1}^{n+1}(T-1)) \tag{44}$$

For the two-factor model, we can show that

$$P_{i+1,j+1}^{n+1}(T-1)=\frac{P_{i,j}^n(T)}{P_{i,j}^n(1)}\frac{2\delta_{i,1}^n(T-1)}{(1+\delta_{i,1}^n(T-1))}\frac{2\delta_{j,2}^n(T-1)}{(1+\delta_{j,2}^n(T-1))} \tag{A-8}$$

$$P_{i+1,j}^{n+1}(T-1)=\frac{P_{i,j}^n(T)}{P_{i,j}^n(1)}\frac{2\delta_{i,1}^n(T-1)}{(1+\delta_{i,1}^n(T-1))}\frac{2}{(1+\delta_{j,2}^n(T-1))} \tag{A-9}$$

$$P_{i,j+1}^{n+1}(T-1)=\frac{P_{i,j}^n(T)}{P_{i,j}^n(1)}\frac{2}{(1+\delta_{i,1}^n(T-1))}\frac{2\delta_{j,2}^n(T-1)}{(1+\delta_{j,2}^n(T-1))} \tag{A-10}$$

$$P_{i,j}^{n+1}(T-1)=\frac{P_{i,j}^n(T)}{P_{i,j}^n(1)}\frac{2}{(1+\delta_{i,1}^n(T-1))}\frac{2}{(1+\delta_{j,2}^n(T-1))} \tag{A-11}$$

From Equation (A-8), (A-9), (A-10) and (A-11), we can see that Equation (A-12) holds.

$$\begin{aligned}P_{i,j}^n(T)=\frac{1}{4}P_{i,j}^n(&P_{i+1,j+1}^{n+1}(T-1)+P_{i+1,j}^{n+1}(T-1)+P_{i,j+1}^{n+1}(T-1)\\&+P_{i,j}^{n+1}(T-1))\end{aligned} \tag{A-12}$$

We can show that the m-factor model satisfies the no-arbitrage condition in the same way.

APPENDIX B

Swaption Black Volatilities on June 30, 2004

Option Term (years)	Swap Tenor(years)									
	1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr
1yr	30.95	28.31	25.99	24.24	22.65	21.54	20.83	20.00	19.20	18.55
2yr	24.94	23.37	21.95	21.18	20.05	19.27	18.87	18.30	17.70	17.16
3yr	22.15	20.74	20.06	18.97	18.25	17.67	17.16	16.90	16.50	16.07
4yr	20.38	19.41	18.36	17.67	17.30	16.83	16.38	15.90	15.60	15.18
5yr	18.78	18.03	17.34	16.73	16.35	15.92	15.51	15.10	14.80	14.43
7yr	16.50	15.64	15.13	14.85	14.35	14.04	13.47	13.50	13.20	12.73
10yr	13.47	12.97	12.56	12.16	11.77	11.81	11.60	11.40	11.20	10.99

ENDNOTES

We would like to thank Inho Kim at Korea Bond Pricing Company in assisting us deriving the model and Blessing Mudavanhu at Merrill Lynch and Company in assisting us with the research.

[1]The article noted in the footnote, "In general, h(T), h*(T), and π can be dependent on n and i and a more general class of AR models can be determined." And the footnote referred to the general framework presented in the working paper by Thomas Ho and Sang Bin Lee, "Term Structure Move-

Appendix C

Percentage Errors of Swaption Estimated Prices

Swaption Error (% Diff) June 30, 04				
Option Term	Swap Tenor	Two factor beta9	Two factor beta3	One factor beta3
1	1	(0.85)	(1.80)	(2.57)
2	1	1.15	0.99	0.66
3	1	1.80	2.84	2.95
4	1	1.05	1.59	2.07
5	1	0.25	1.01	1.76
7	1	0.28	1.19	1.94
10	1	1.44	2.48	3.21
1	2	(0.08)	(0.13)	0.63
2	2	(0.62)	(1.13)	(0.38)
3	2	(0.31)	(0.94)	(0.22)
4	2	(1.79)	(2.98)	(3.38)
5	2	(0.29)	0.35	0.20
7	2	(1.39)	(1.65)	(1.70)
10	2	0.31	0.39	0.39
1	3	(0.70)	(0.29)	(0.28)
2	3	(0.61)	0.09	0.10
3	3	(0.59)	(1.12)	(1.08)
4	3	(0.41)	(1.37)	(1.33)
5	3	(0.24)	(1.21)	(1.19)
7	3	0.03	(0.92)	(0.93)
10	3	1.07	3.41	3.03
1	4	(1.24)	(1.18)	(1.41)
2	4	0.28	0.73	0.58
3	4	(1.24)	(0.37)	(0.48)
4	4	(1.37)	(0.13)	(0.23)
5	4	(2.76)	(3.09)	(3.17)
7	4	(2.56)	(3.37)	(3.46)
10	4	(0.67)	(1.42)	(1.54)
1	5	0.19	(0.45)	(0.60)
2	5	0.77	0.18	(0.00)
3	5	1.33	(1.48)	(1.62)
4	5	1.55	0.76	0.66
5	5	(0.12)	0.11	0.03
7	5	(0.40)	0.54	0.46
10	5	(0.89)	(1.87)	(1.92)
1	6	(0.56)	(1.96)	(2.04)
2	6	0.09	(1.07)	(1.19)
3	6	0.37	(0.57)	(0.72)
4	6	1.68	0.88	0.70
5	6	2.16	1.41	1.20
7	6	2.61	3.97	3.89
10	6	1.81	3.81	3.75
1	7	1.16	3.71	3.65
2	7	(1.09)	(1.27)	(1.31)
3	7	(0.30)	(1.08)	(1.14)
4	7	0.40	(0.12)	(0.20)
5	7	1.05	0.78	0.66
7	7	1.61	1.49	1.35
10	7	2.75	2.74	2.58
1	8	3.36	3.38	3.20
2	8	1.80	5.98	5.98
3	8	(2.93)	(5.89)	(5.86)
4	8	(2.44)	(5.60)	(5.60)
5	8	(0.59)	(2.54)	(2.55)
7	8	(0.61)	(1.72)	(1.74)
10	8	0.52	(0.03)	(0.05)
1	9	(0.30)	(0.53)	(0.55)
2	9	3.15	3.05	3.06
3	9	2.86	1.93	1.97
4	9	1.40	(0.44)	(0.37)
5	9	(1.53)	1.31	1.45
7	9	(2.37)	0.98	1.18
10	9	(2.44)	1.11	1.39
1	10	(2.46)	1.14	1.38
2	10	(2.52)	1.06	1.19
3	10	(0.56)	1.34	1.40
4	10	(0.32)	0.05	0.05
5	10	0.10	(0.89)	(0.95)
7	10	0.57	(1.68)	(1.78)
10	10	0.96	(2.45)	(2.59)

ments and Interest Rates Contingent Claims Pricing," Salomon Brothers Center Series, New York University, 1986.

[2]Since this is a discrete time model, the interest rates can still become negative as a result of the discrete time approximation, even for some small volatilities when the rates are low. Equation (4) cannot ensure that δ_i^n are always bounded by one. For implementation, we use $\delta_i^n = \exp(-2\sigma(n)\max(\min(R_i^n, R)\,\Delta t^{3/2}, \varepsilon))$ for some small ε, say, 0.0001 or 1 bp.

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