Managing Interest Rate Volatility Risk: *Key Rate Vega*

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As a market convention, interest rate volatility is measured by a swaption volatility surface. The volatility surface is a set of Black volatilities derived from quoted at-the-money swaption prices over a range of tenors and expiration dates. The volatility surface is similar to the swap curve in that it is stochastic, changing continually to reflect market expectations and the buy and sell order flows for interest rate volatilities. And this surface is an important practical input, often as important as the swap curve, to the valuation of many interest rate contingent claims.

The volatility surface is a subject of many previous studies. Heidari and Wu [2003] use principal component analysis to show that the volatility surface has three orthogonal movements, independent of the principal movements of the yield curve. They establish the stochastic nature of the volatility surface. Collin-Dufresne and Goldstein [2002] use interest rate straddles to provide empirical evidence of "unspanned stochastic volatility" showing that interest rate derivatives cannot be dynamically hedged or replicated by bonds alone (with no embedded options), because of the significant presence of volatility risk. Thus, there is a need to hedge the vega risks for any dynamic replication.

Another strand of empirical studies relates the volatility surface to the implied volatility functions of arbitrage-free interest rate models and has shown that the implied volatilities estimated are indeed stochastic (Amin and Morton [1994] and De Jong, Driessen, and Pelsser [2001]). Further, Amin and Ng [1997] have shown that implied volatilities have informational content in predicting future interest rate volatilities. These studies show that the vega measure should be used to manage volatility risk in investment, hedging, and risk reporting as an integral part of market risks on trading floors, and for portfolio management and enterprise risk management.

To manage the risk of an interest rate contingent claim, practitioners need both duration and vega measures which indicate an instrument's sensitivities to the shift in the swap curve and the volatility surface, respectively. To manage interest rate risk, practitioners use duration sensitivity along the yield curve which is called key rate duration (see Ho [1992]). However, to date, there are significant challenges in determining the vega buckets for interest rate derivatives. One challenge is the computational intensity required in determining the measurement. Pietersz and Pelsser [2003] use the Brace, Gatarek, and Musiela [1997] interest rate model to determine the vega buckets confined to swaptions on the anti-diagonal buckets (where the sum of expiry and tenor is 31 years) of the volatility surface. The process requires 1 to 5.8 million scenario paths. But, Heidari and Wu [2003] show that volatility risk is not confined to the anti-diagonal buckets of the volatility surface, but to the entire surface that generates the independent movements. Therefore, to extend the key rate duration measures—approximately 11 sensitivity measures—to the volatility surface vega measures—approximately 120 numbers—poses a practical problem.

This article proposes a practical construction of key rate vegas that can be used to effectively hedge stochastic volatilities. The approach exploits a property of the generalized Ho-Lee model. The appendix provides a detailed description of the generalized Ho-Lee model and an even more detailed description is provided in Ho and Lee [2007]. In particular, Ho and Mudavanhu [2007] show that the stochastic movements of the volatility surface can be captured by the implied volatility function of the one-factor, or extended to *n*-factor, generalized Ho-Lee model. Therefore, any interest rate contingent claim can be valued by the swap curve and implied volatility curve, and hedging can effectively be implemented by controlling for only the movements of the swap curve and the implied volatility curve, without taking the entire volatility surface into account. Based on this finding, we can determine the sensitivities of an interest rate contingent claim along the generalized Ho-Lee model's implied volatility function and can construct key rate vegas, analogous to the construction of key rate durations.

This article is organized in the following manner. We first describe the construction of key rate vegas and their properties. We then study the key rate vegas of swaptions. Finally, we use key rate vegas and key rate durations to illustrate a hedging strategy. In particular, we use swaps and swaptions to hedge an interest-only mortgage strip. Practical extensions of key rate vegas are then provided.

KEY RATE VEGA

Key rate vegas can be constructed in a way that is analogous to key rate durations. We first assume that the valuation model is a one-factor generalized Ho-Lee model. For a particular evaluation date, the arbitrage-free interest rate model is used to determine the implied volatility function using a volatility surface. The interest rate model is calibrated to the volatility surface by fitting a function—the implied volatility function—as the volatility input, such that the sum of squared differences between the model prices and the observed at-the-money swaption prices is minimized.

Then, the implied volatility function is shifted at particular terms. For illustration, the shift of the implied volatility function is depicted in Exhibit 1. The first key rate shift is a shift at the nearby term and the shift declines linearly to the second key rate. The second key rate shift is the shift of the second key rate, and it declines linearly in both directions of the key rate shift. Other key rates are defined analogously. The last key rate shift is the shift of the last key rate term and the shift declines linearly to the key rate on the left-hand side and stays constant on the right-hand side.

The shifts are then added to the initial implied volatility function as depicted in Exhibit 2.

Then, the key rate vega is defined as the proportional change of the security value per unit shift. Let the key rate vega for the *i*-th term be KRV(i), so that we have

$$\mathrm{KRV}(i) = (V(+) - V(-))/(2 \cdot V \cdot \Delta) \qquad (1)$$

where V(+), V(–), and V are the values of the security with a positive, negative, and no key rate shift, respectively. The symbol Δ represents the key rate shift. The vector of KRV(i) (i = 1, 2,..., n) measures the sensitivities of the security along the implied volatility function.

By construction, key rate vegas have two useful properties. Because key rate vegas are derived from the Taylor expansion of the valuation function of the implied volatility function, the key rate vega of a portfolio of interest rate contingent claims is the weighted average of the key rate vegas of the interest rate contingent claims, where the weights are the portfolio weights.

EXHIBIT 1
Implied Volatility Function Shift of Key Rate Vega

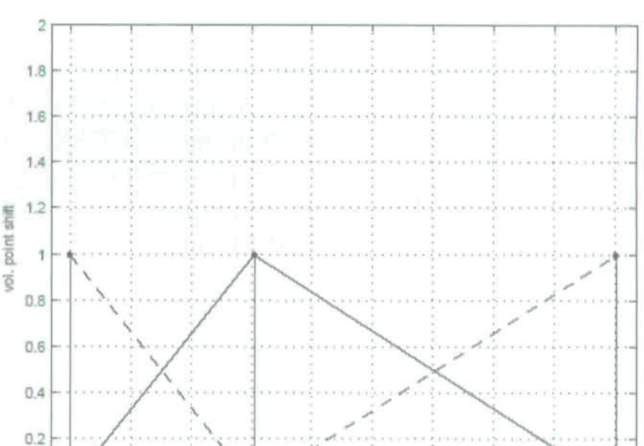
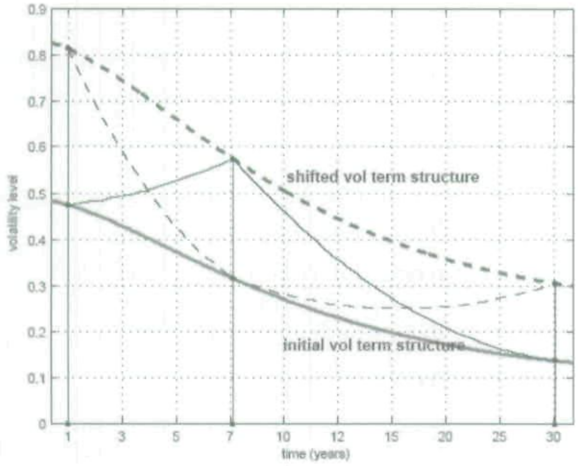


EXHIBIT 2
Shifted Implied Volatility Function Using Key Rate Vega



Specifically, let a portfolio value V be

$$V = \sum V(j) \tag{2}$$

where $V(j)$ is the value of the position in the j-th interest rate contingent claim of the portfolio. Let $KRV(i|j)$ and $KRV(i)$ be the i-th key rate vega of the j-th contingent claim and the portfolio, respectively. Then, we have

$$KRV(i) = \sum (V(j)/V)\, KRV(i|j) \quad \text{for each } i = 1, 2, \ldots, n \tag{3}$$

The second property of the key-rate vega is that the sum of the key rate vegas must equal the parallel shift of the implied volatility function. This property follows from the construction of the key rate vegas; that is, when all the key rates are shifted, the implied volatility function has effectively made a parallel shift.

KEY RATE VEGAS OF SWAPTIONS

This section studies the key rate vegas of swaptions to gain insight into the vega risk profiles of interest rate contingent claims as measured by the key rate vegas. For the sake of discussion in this section, we assume that key rate vegas are determined for a fixed set of terms which are 1, 2, 3, 5, 7, 10 and 15 years. We begin with an investigation of one year at-the-money swaptions, which are swaptions that have a tenor of one year over the range of the option maturities. The results are presented in Exhibit 3.

The results can be explained intuitively. The NX1 swaptions are at-the-money. Therefore, their strike rates are the forward one-year rates. First, note that the key rate vegas are approximately zero before the option expiration. This result shows that any change in volatilities, as measured along the implied volatility function before the time to expiration of the option, has minimal impact on the option value. Obviously, the change of volatility after the termination of the swaption also has no impact on the swaption value. As a result, key rate vegas are significant when they have terms between the expiration and the termination of the option.

In this sense, the NX1 swaptions on the volatility surface are analogous to the zero-coupon bonds on the yield curve. Zero-coupon bonds have zero key rate durations except at the bond's maturity. That is, a zero-coupon bond is only sensitive to the change of the key rate equal to that of the bond's maturity. Analogously, one-year swaptions are only sensitive to the key rate vegas at the option expiration.

In this simulation, while we use the NX1 swaptions in these calculations, the results can be generalized appropriately to any NXk swaptions, where k can be one month, three months, or any time period.

To gain further insight into the volatility sensitivities of the at-the-money swaptions, we next simulate the key rate vegas of anti-diagonal swaptions. The results are presented in Exhibit 4.

These results can also be explained intuitively. Consider the at-the-money swaption expiring in one year with a 10-year tenor. The option value is dependent on the 10-year rate volatility over a one-year horizon. But the 10-year rate volatility is dependent on the one-year forward rate volatilities. Therefore, the swaption has significant key rate vegas over 10 years. The results show that anti-diagonal swaptions have vega risks from the expiration date to the termination date of the swaptions.

As noted earlier, the sum of the key rate vegas must represent a parallel movement of the implied volatility function. The impact of this parallel movement of the implied volatility function should be similar to one Black volatility-point shift for the swaptions on the volatility surface. Now, we can investigate the change of the volatility surface to the key rate vegas. Consider Exhibit 5, where V and V' are the values of the swaptions with and without one volatility-point shift (1%), respectively. Black Vols are

EXHIBIT 3
Key Rate Vegas of One-Year Swaptions

	Key Rate Vegas							Sum of KRVs
	1yr	2yr	3yr	5yr	7yr	10yr	15yr	
1 × 1	2.66	2.76	0.00	0.00	0.00	0.00	0.00	5.41
2 × 1	-0.02	2.52	2.30	0.00	0.00	0.00	0.00	4.81
3 × 1	-0.01	-0.03	3.59	1.04	0.00	0.00	0.00	4.59
4 × 1	-0.01	-0.02	1.31	3.35	0.00	0.00	0.00	4.63
5 × 1	-0.01	-0.02	-0.05	3.88	0.96	0.00	0.00	4.77
6 × 1	0.00	-0.01	-0.03	1.52	3.63	0.00	0.00	5.11
7 × 1	0.00	0.00	-0.01	-0.04	4.70	0.62	0.00	5.27
8 × 1	0.00	-0.01	-0.02	-0.06	3.19	2.65	0.00	5.75
9 × 1	0.00	0.00	-0.01	-0.03	1.44	4.49	0.00	5.89
10 × 1	0.00	0.00	-0.01	-0.03	-0.07	6.15	0.33	6.37

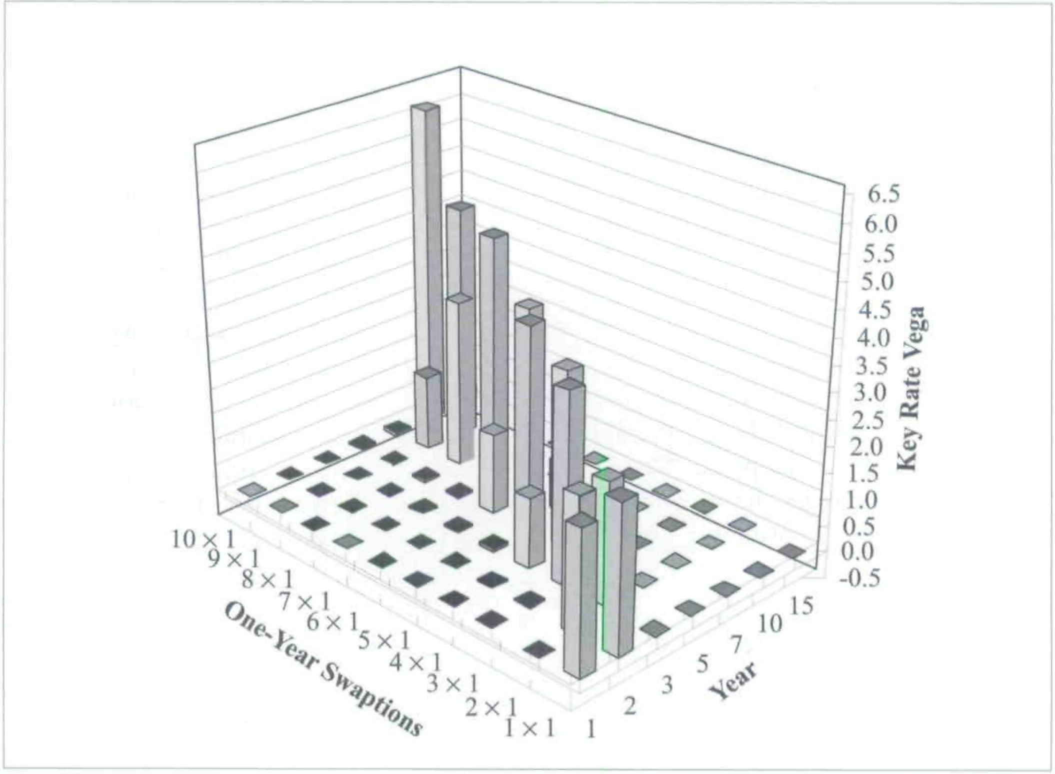


EXHIBIT 4
Key Rate Vegas of Anti-Diagonal Swaptions

	Key Rate Vegas							Sum of
	1yr	2yr	3yr	5yr	7yr	10yr	15yr	KRVs
1 × 10	0.56	0.65	0.85	1.00	1.10	0.87	0.02	5.05
2 × 9	0.00	0.57	0.96	1.13	1.24	0.98	0.02	4.90
3 × 8	-0.01	-0.03	0.97	1.34	1.46	1.14	0.03	4.89
4 × 7	-0.01	-0.02	0.31	1.54	1.78	1.39	0.03	5.03
5 × 6	0.00	0.00	-0.02	1.23	2.19	1.74	0.04	5.17
6 × 5	0.00	0.00	-0.02	0.44	2. 68	2.24	0.05	5.40
7 × 4	0.00	-0.01	-0.03	-0.07	2.66	3.02	0.07	5.62
8 × 3	0.00	-0.01	-0.03	-0.06	1.72	4.15	0.10	5.86
9 × 2	0.00	0.00	-0.01	-0.03	0.77	5.19	0.15	6.07
10 × 1	0.00	0.00	-0.01	-0.03	-0.07	6.15	0.33	6.37

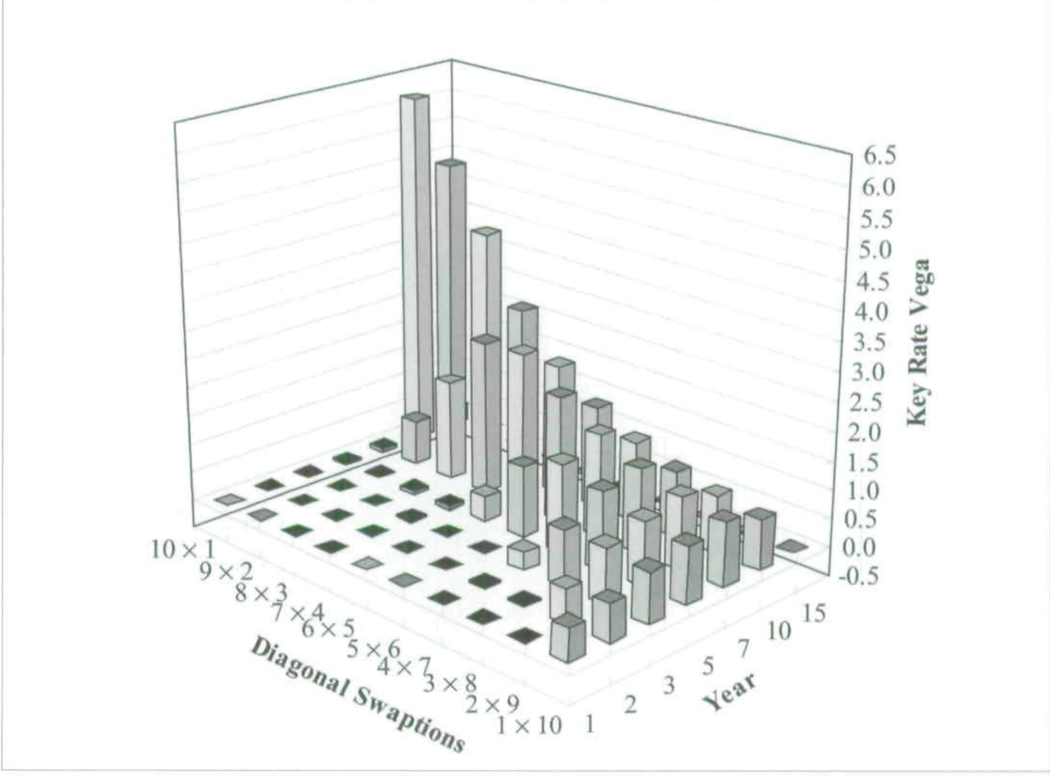


EXHIBIT 5

Difference Between the Sum of Key Rate Vegas and Black Volatility Point of One-Year Swaptions

V	Black Vol	V'	**Vol Point**	**Differ**
0.34	0.19	0.36	5.31	0.11
0.51	0.21	0.54	4.73	0.07
0.61	0.21	0.64	4.60	-0.01
0.64	0.20	0.67	4.85	-0.22
0.66	0.20	0.69	4.97	-0.20
0.60	0.17	0.63	5.72	-0.61
0.61	0.17	0.65	5.67	-0.40
0.56	0.15	0.59	6.36	-0.61
0.62	0.17	0.65	5.76	0.13
0.55	0.15	0.59	6.47	-0.10

the Black volatilities of the swaptions on the evaluation date. A Vol Point is the percentage change in the value of the swaption for one Black volatility-point change. The Diff is the difference between the Vol Point and the sum of all key rate vegas of the swaption.

Exhibits 5 and 6 show that the Black volatility-point vega measures are similar to the sum of the key rate vegas for both one-year swaptions and anti-diagonal swaptions.

The results show that the differences of the two measures are similar. Therefore, we may interpret key rate vegas as the decomposition of the Vol Point vega risk of a swaption along key rate terms. Indeed, using Equation (A-3), which specifies the implied volatility function, we can determine the volatilities of a one-period rate along the forward curve. It can be shown that these volatilities are similar to the Black volatilities of NX1 swaptions, as one would expect. But the implied volatility function does more. It extracts information from other swaption prices to specify volatilities along other rate scenarios.

EXHIBIT 6

Difference Between the Sum of Key Rate Vegas and Volatility Point of Anti-Diagonal Swaptions

V	Black Vol	V'	**Vol Point**	**Differ**
2.98	0.20	3.13	4.92	0.13
3.50	0.19	3.68	5.16	-0.26
3.69	0.19	3.88	5.17	-0.28
3.48	0.18	3.67	5.38	-0.35
3.17	0.18	3.35	5.49	-0.32
2.69	0.17	2.85	5.74	-0.34
2.21	0.17	2.34	5.86	-0.24
1.66	0.16	1.76	6.09	-0.23
1.11	0.16	1.18	6.26	-0.19
0.55	0.15	0.59	6.47	-0.10

IO HEDGING WITH KEY RATE DURATIONS AND KEY RATE VEGAS

This section presents the results of hedging an interest-only (IO) mortgage strip using swaps and swaptions by matching both key rate durations and key rate vegas. The hedge is held over a two-month period from September 29 to November 30, 2006, and uses the swap curves and volatility surfaces of those dates.

We use the strip IO to illustrate key rate vega hedging for two reasons. First, the strip IOs have significant embedded options. When mortgage-backed securities experience significant prepayment, much of the interest income from the mortgage-backed securities is not available to strip IO investors. The strip IOs have sold significant prepayment options to the mortgagors and, as a result, the key rate vegas are important to these financial instruments.

The second reason for using a strip IO for this illustration is that strip IOs are important components of mortgage servicing fees. Many banks hold mortgage servicing fees on their balance sheet, and this balance sheet item has become particularly important in recent years. Managing the market risks of mortgage serving fees is a concern to many banks in their balance sheet management. Therefore, the key rate vega hedging strategy should be a useful tool for many banks, mortgage conduits, and mortgage companies.

The strip IO in this hedging strategy is based on a 30-year GNMA with a 6.20% gross coupon and 5.70% net coupon. The weighted-average remaining life is 336 months, which is a relatively new bond. The key rate durations and key rate vegas of the IO are provided in Exhibit 7 and are based on seven key rate points—1, 2, 3, 5, 7, 10 and 15 years.

The duration of the IO is –8.4337 years. Consistent with the behavior of an IO, a fall in interest rates would lead to a decrease in value as a result of the increase in prepayments, due to falling rates, which reduces the cash flows to the IO holders. The IO is most sensitive to a rate shift in the 7-year term because the key rate duration is highest at the 7-year term of –4.0862 years. That is, a drop of 1%

EXHIBIT 7

Key Rate Duration and Key Rate Vega of an IO

	KRD & KRV							Sum
Year	1	2	3	5	7	10	15	
KRD	2.2502	1.5946	(3.0067)	(3.0525)	(4.0862)	(3.0026)	0.8695	(8.4337)
KRV	0.0023	0.0060	0.0101	(0.0013)	(0.0282)	(0.0536)	(0.0254)	(0.0900)

in the 7-year key rate means the IO value would fall 4.0862%, isolating from the convexity and other higher-order terms. This result can be explained by the effect of the prepayment model, where the conditional prepayment rate depends on the movement of the 30-year fixed-rate mortgage rate, which is related to the long-term Treasury rate.

The vega, which is the sum of the key rate vegas, is –0.09. That is, for a 1% increase in the implied volatility function, the IO value drops by 0.09%. The vega sensitivities are pronounced in the 7- to 15-year range, suggesting that the embedded options in the IOs are long dated, as one would expect for an IO based on a 30-year fixed-rate mortgage.

The hedging strategy begins with construction of a portfolio of swaptions to hedge the key rate vegas of the IO. The hedging instruments are the NX1 at-the-money swaptions. For illustrative purposes, these hedging instruments are used because, as mentioned earlier, the NX1 swaptions identify the volatilities of the forward rates and they represent the volatility risk drivers of the volatility surface. In practice, the choice of hedging swaptions should be based on their relative value and liquidity. The key rate vegas are calculated for all these swaptions (Exhibit 8).

Let the notional amount of the i-th swaption be N(i), the j-th key rate vega of the i-th swaption be KRV (j | i), and the value of the i-th swaption be v(i) based on a $100 notional amount. Then, the portfolio of swaptions that would hedge the key rate vegas of the IO, based on the $100 notional value, is determined by the N(i), i = 1,..., 7, where N(i) is determined by the following equation:

$$KRV(j) = \sum N(i)\, v(i)\, KRV(j \mid i) \quad \text{for each } j = 1,\ldots,7 \tag{4}$$

Equation (4) is derived from Equation (3), applied to swaptions.

The portfolio, composed of the IO net of the hedging portfolio of NX1 swaptions, can then be hedged by standard methodology with swaps using key rate durations. Specifically, note that the DV01 (delta value per 1 bp) at the i-th term is related to the i-th term key rate duration, KRD(i), by

EXHIBIT 8

Portfolio of Swaptions for Key Rate Vega Hedging

ID	Type	Type	Pay Freq	Fixed Rate(%)	Index	Notional($)
1	1×1	Pay Fixed	Semiannual	4.99	LIB6M	(9.42)
2	2×1	Pay Fixed	Semiannual	4.94	LIB6M	(9.90)
3	3×1	Pay Fixed	Semiannual	5.04	LIB6M	(8.81)
4	4×1	Pay Fixed	Semiannual	5.14	LIB6M	(4.27)
5	6×1	Pay Fixed	Semiannual	5.23	LIB6M	18.97
6	9×1	Pay Fixed	Semiannual	6.00	LIB6M	69.80
7	14×1	Pay Fixed	Semiannual	6.00	LIB6M	35.68

EXHIBIT 9

Portfolio of Swaps for Key Rate Duration Hedging

ID	Tenor(year)	Type	Pay Freq	Fixed Rate(%)	Index	Notional($)
1	1	Pay Fixed	Quarterly	5.36	LIB3M	54.91
2	2	Pay Fixed	Quarterly	5.10	LIB3M	23.45
3	3	Pay Fixed	Quarterly	5.00	LIB3M	(20.12)
4	5	Pay Fixed	Quarterly	5.00	LIB3M	(3.09)
5	7	Pay Fixed	Quarterly	5.10	LIB3M	(7.15)
6	10	Pay Fixed	Quarterly	5.12	LIB3M	(22.73)
7	15	Pay Fixed	Quarterly	5.20	LIB3M	(1.29)

$$\mathrm{DV01(i)} = \mathrm{V} \cdot \mathrm{KRD(i)} \cdot 0.0001 \tag{5}$$

where V is the value of the position. The hedging vanilla swaps are given in Exhibit 9. The DV01 of each swap is calculated. The notional amount for each swap—to match the DV01 of the combined portfolio of the IO and swaption notional amounts—is then determined. The last column of Exhibit 9 presents the results.

HEDGING EFFECTIVENESS OF THE REPLICATING PORTFOLIO

The IO position and the replicating portfolio are held over the period from September 29 to November 30, 2006. During this period, the yield curve shifts quite significantly, dropping approximately 20 bps beyond 7 years. Volatility also falls over the same period. The swap curves and implied volatility functions are given in Exhibit 10.

The hedging results are provided in Exhibit 11. The results show that the hedging strategy is effective. The IO value falls by 1.31% over the two-month time horizon as swap rates fall. However, the portfolio with the swaptions and swaps, which neutralized both the key rate durations and key rate vega risks, has risen 0.15%. The result is that the hedging portfolio contributed $0.34 returns on a $23.52 base value by hedging the shift in the swap curve and the implied volatility function. The swaptions contributed $0.10 in value. The hedging results, using only swaps, are presented in Exhibit 12 for comparison.

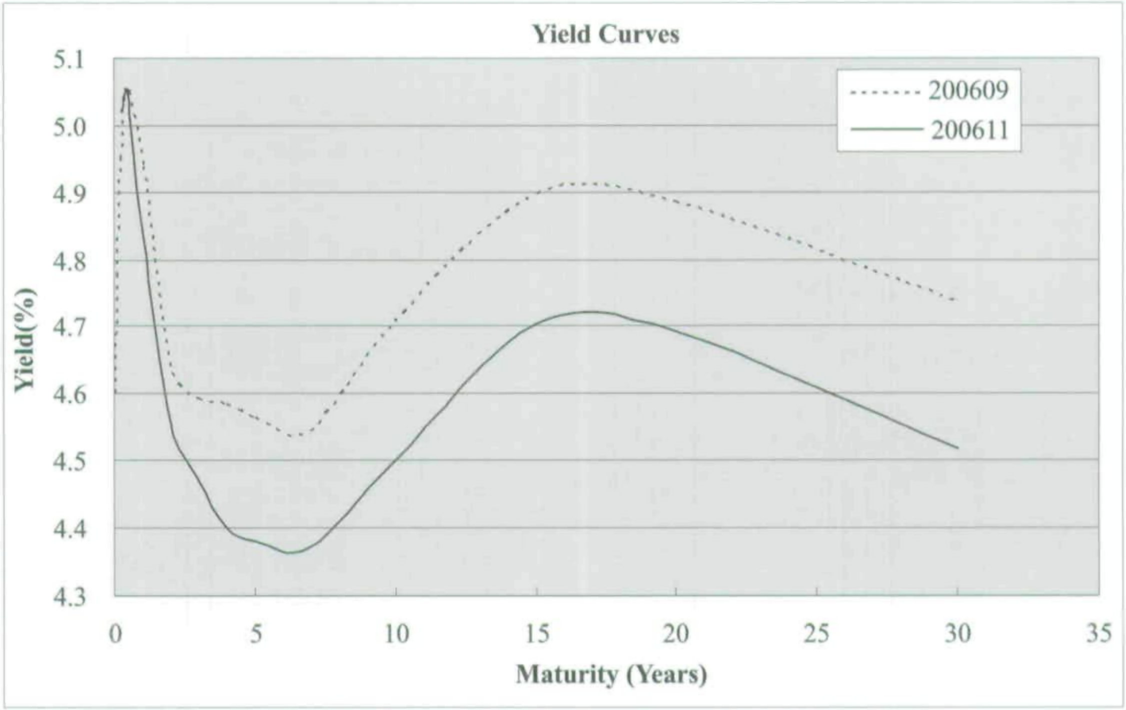
The purpose of this section is not to provide an empirical test on the effectiveness of hedging using key rate vega, but to illustrate a methodology using key rate vega to hedge an interest rate contingent claim. In particular, it shows how both swaps and swaptions have to be used to neutralize the stochastic nature of the swap curve and the volatility surface.

Furthermore, this section emphasizes the basic ideas behind the key rate vega hedging methodology and not specific implementation of the hedging strategy. For example, as happens often in practice, some shorter-dated swaptions may be used and the key terms may include three months and six months, instead of starting with a one-year swaption. The hedging portfolios of swaps and swaptions do not have to share the same set of terms. An optimal hedge portfolio, in practice, must concurrently consider the goals of the hedging strategy, characteristics of the underlying instruments, market conditions, and directional views. A discussion of this point is beyond the scope of this section.

While this article has described key rate vegas for the one-factor generalized Ho-Lee model, the construction can be directly extended to any n-factor model. This is because an n-factor generalized Ho-Lee model is a direct extension of the one-factor model, with n orthogonal implied volatility functions. Therefore, for an n-factor model, n sets of key rate vegas can be specified. To the extent that a multi-factor interest rate model is more appropriate for the interest rate contingent claim, then a corresponding set of key rate vegas can be used. It is well

EXHIBIT 10

Swap Curve and Implied Volatility Function



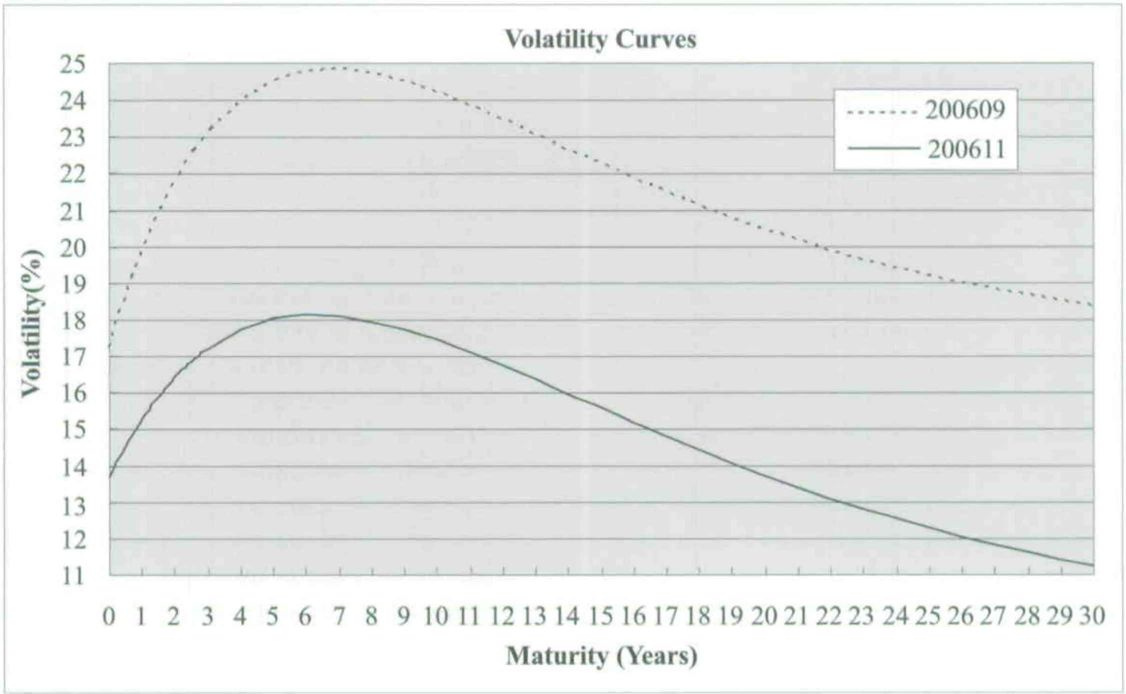


EXHIBIT 11
Hedging Results with Key Rate Duration and Key Rate Vega

	Total Portfolio (IO & Hedging Portfolio)	IO	Hedging Portfolio (Swaptions & Swaps & Cash)
2006-09-29	23.52	23.52	0.00
2006-11-30	23.56	23.22	0.34
Differ(%)	**0.15**	**(1.31)**	-------

EXHIBIT 12
Portfolio of Swaps for Key Rate Duration Hedging Comparison

ID	Tenor(year)	Type	Pay Freq	Fixed Rate(%)	Index	Notional($)
1	1	Pay Fixed	Quarterly	5.36	LIB3 M	63.25
2	2	Pay Fixed	Quarterly	5.10	LIB3 M	23.89
3	3	Pay Fixed	Quarterly	5.00	LIB3 M	(23.91)
4	5	Pay Fixed	Quarterly	5.00	LIB3 M	(15.57)
5	7	Pay Fixed	Quarterly	5.10	LIB3 M	(17.80)
6	10	Pay Fixed	Quarterly	5.12	LIB3 M	(11.69)
7	15	Pay Fixed	Quarterly	5.20	LIB3 M	2.72

	Total Portfolio (IO & Hedging Portfolio)	IO	Hedging Portfolio (Swaps & Cash)
2006-09-29	23.52	23.52	0.00
2006-11-30	23.56	23.22	0.44
Differ(%)	**0.17**	**(1.31)**	-------

documented in the literature that a two-factor model can explain most of the movements of the yield curve.

CONCLUSION

This article proposes a volatility risk measure, an extension of the vega measure, to measure risks along the implied volatility function. The measure is derived from the generalized Ho-Lee model. The risk measure is called the key rate vega. We simulate key rate vegas for a set of at-the-money swaptions. The results provide insights into the behavior of the swaptions in a stochastic volatility environment.

The results also have important implications for hedging interest rate contingent claims. This article suggests that when the volatility surface is stochastic, delta hedging of interest rate contingent claims with vanilla swaps may not be effective because of the vega effect. This article shows that both swaps and swaptions should be used even for dynamic hedging. In hedging volatility risk, we cannot simply use one vega measure. For example, we cannot use a short-dated option to hedge the volatility risk of a long-dated option. To manage volatility risk appropriately, we need to measure the value sensitivity of an option to the change in the implied volatility function at the key rate points on the curve.

APPENDIX

The Two-Factor Generalized Ho-Lee Model

Let $P_{i,j}^{n}(T)$ be the price of a T-year bond at time n, at state (i, j). Then the bond price is specified by combining two one-factor models. Specifically, we have

$$P_{i,j}^{n}(T)=\frac{P(n+T)}{P(n)}\prod_{k=1}^{n}\left(\frac{1+\delta_{0,1}^{k-1}(n-k)}{1+\delta_{0,1}^{k-1}(n-k+T)}\right)\prod_{k=1}^{n}\left(\frac{1+\delta_{0,2}^{k-1}(n-k)}{1+\delta_{0,2}^{k-1}(n-k+T)}\right)\prod_{k=0}^{i-1}\delta_{k,1}^{n-1}(T)\prod_{k=0}^{j-1}\delta_{k,2}^{n-1}(T) \tag{A-1}$$

where

$$\begin{aligned}\delta_{i,1}^{n}(T)&=\delta_{i,1}^{n}\delta_{i,1}^{n+1}(T-1)\left(\frac{1+\delta_{i+1,1}^{n+1}(T-1)}{1+\delta_{i,1}^{n+1}(T-1)}\right)\\ \delta_{i,2}^{n}(T)&=\delta_{i,2}^{n}\delta_{i,2}^{n+1}(T-1)\left(\frac{1+\delta_{i+1,2}^{n+1}(T-1)}{1+\delta_{i,2}^{n+1}(T-1)}\right)\end{aligned} \tag{A-2}$$

and the one-period forward volatilities are given by definition as

$$\begin{aligned}\delta_{i,1}^{m}(1)=\delta_{i,1}^{m}&=\exp\left(-2\cdot\sigma_1(m)\min\left(R_{i,1}^{m},R\right)\Delta t^{3/2}\right)\\ \delta_{i,2}^{m}(1)=\delta_{i,2}^{m}&=\exp\left(-2\cdot\sigma_2(m)\min\left(R_{i,2}^{m},R\right)\Delta t^{3/2}\right)\end{aligned} \tag{A-3}$$

where the function $\sigma_j(n)=(a+bn)\exp(-cn)+d$ is specified by the parameters a, b, c, and d, which can be obtained from the calibration to the market price of the swaption.

Using the direct extension, we can specify the one-period rates for the two-factor model for any future period m and state i, and $R_{i,1}^{m}$ and $R_{i,2}^{m}$ are defined by

$$\begin{aligned}R_{i,1}^{m}\Delta t&=-\log\left(\frac{P(m+1)}{P(m)}\right)-\sum_{k=1}^{m}\log\left(\frac{1+\delta_{0,1}^{k-1}(m-k)}{1+\delta_{0,1}^{k-1}(m-k+1)}\right)-\sum_{k=0}^{i-1}\log\left(\delta_{k,1}^{m-1}(1)\right)\\ R_{i,2}^{m}\Delta t&=-\log\left(\frac{P(m+1)}{P(m)}\right)-\sum_{k=1}^{m}\log\left(\frac{1+\delta_{0,2}^{k-1}(m-k)}{1+\delta_{0,2}^{k-1}(m-k+1)}\right)-\sum_{k=0}^{i-1}\log\left(\delta_{k,2}^{m-1}(1)\right)\end{aligned} \tag{A-4}$$

ENDNOTE

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