Less Risk for Strong Returns in Bond Portfolios

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The variance of price changes is one obvious measure of bond portfolio risk. For one example, bond mutual fund managers, whose performance is frequently measured in holding period return performance over monthly, quarterly, or annual periods, should seek good measures of conditional variance of price changes (V_p). Of course, mutual fund shareholders worry about the variance of their wealth invested in bond funds.

As another example, consider bank regulators and managers computing value at risk (VAR) measures for bond portfolios where VAR is computed by utilizing confidence intervals and variance of bond values. VAR is computed in dollar value where a sizeable error in variance (standard deviation) of price changes obviously leads to a proportionate sizeable error in VAR.

The purpose of this research is to analyze the relation between V_p and the level of interest rates. Here, we note that conventional wisdom and textbooks commonly suggest that price variance is negatively related to level of yield because duration *decreases* with yield.[1] However, we note that yield volatility tends to *increase* with level of yields while, again, duration *decreases* with the level of yields.[2] Furthermore, the relation is complicated by the fact that convexity *decreases* with the level of yields. When is the variance/level of yields relation positive and when is it negative? Our results show that the answer depends on bond maturity and the time varying conditional volatility of yields. Generally, price variance decreases with yield *only* for longer maturities. Furthermore, this longer maturity result may be more of a convexity effect than a duration effect.

Whether the relation is positive or negative has, obviously, important implications. Consider a bond portfolio manager who models expected return and risk where risk is measured by variance of bond value. Assume the portfolio manager is expecting a significant decrease in bond yields for the next period. In a basic expected return and risk formulation, this means a large positive expected holding period return.[3] Such a large gain is obviously appealing, but it is even more so if V_p (risk) is positively related to the level of yields such that the risk of the portfolio is expected to decrease in the face of lower yields and strong positive expected returns. Clearly, a measure of (expected) performance will be strong and such bonds are very attractive. On the other hand, if V_p is negatively related to interest rates, then the expected decrease in yields increases the expected variance of the portfolio and the expected decrease in level of yield is not as attractive because the risk-adjusted measure of (expected) bond performance may decline with greater expected risk.

For a strategic application, consider a (balanced) mutual fund with roughly equal proportions of equities and bonds. The mutual fund aggressively advertises itself as one that (by holding both equities and bonds) can reduce risk by wisely reallocating between equities and bonds. At the same point in time as the above bond scenario, assume the manager expects 1) equity returns to be similar to expected bond returns, and 2) a dramatic increase in volatility for equities. Should the manager allocate a dramatically higher percentage of the portfolio to bonds to reduce overall portfolio risk? The answer very likely depends on the expected change in bond risk. That is, will bond risk (V_P) increase or decrease with the expected decline in level of yields?[4]

The next section describes the alternative models for V_P. The first order model, V_{P1}, suggests that variance depends only on duration and yield volatility. In contrast, the second order model, V_{P2}, shows that price variance also depends on convexity, expected change in yield, and the fourth moment of yield changes. In the following section, we supply the mathematics for determining whether V_P is positively or negatively related to the level of yields. Subsequently, we use duration, convexity, expected yield changes, and estimates of yield volatility to empirically map how V_P is related to level of yields for both zero coupon Treasury bonds and par coupon Treasury bonds.

ALTERNATIVE MODELS FOR VARIANCE OF PRICE CHANGES

The First-Order Model

It is well known that the first order (Model 1) approximation to change in bond price, $\Delta P_t = P_t - P_{t-1}$, is given by

$$\Delta P_t = -d_{t-1}\, \Delta y_t P_{t-1} \tag{1}$$

where y_t is the *realized* yield at time t and $\Delta y_t = y_t - y_{t-1}$. Below we allow for *forecasted* Δy_t to be $z_{t-1,}$ where the realized error in the forecast is ε_t.[5] Assume the bonds initially sell at par so that P_{t-1} is assumed constant at 1,000. (This is consistent with our use of constant maturity yields in our time series analysis.) Here, d_{t-1} is the modified duration that we hereafter refer to as duration. The conditional variance of price changes in the first order model is, thus,

$$V_{P1} = \mathrm{Var}(\Delta P_t \,|\, F_{t-1}) = d_{t-1}^2\, \sigma_t^2\, P_{t-1}^2 \tag{2}$$

where σ_t^2 is the variance of yields for a specific maturity. Greater d_{t-1} and σ_t^2 obviously increase V_{P1}. However, as yields change, d_{t-1} *decreases* with the level of yields while, in contrast, σ_t^2 frequently *increases* with the level of yields.

The Second Order Model

The well known second order (Model 2) approximation to the change in bond price is given by

$$\Delta P_t = \left\{ -d_{t-1}\, \Delta y_t + \frac{1}{2} C_{t-1}\, (\Delta y_t)^2 \right\} P_{t-1} \tag{3}$$

where C_{t-1} is convexity. Thus, a more accurate second order model for variance is given by Pappu, Lakshmivarahan, and Stock [2006], as

$$V_{P2} = \left\{ \frac{1}{2} C_{t-1}^2\, \sigma_t^4 + g_{t-1}^2\, \sigma_t^2 \right\} P_{t-1}^2 \tag{4}$$

In contrast to V_{P1}, V_{P2} also depends on the fourth moment of yield changes, σ_t^4, and g_{t-1} where g_{t-1} is defined as $C_{t-1}\, z_{t-1} - d_{t-1}$. If y_t follows a random walk such that z_{t-1} is zero, V_{P2} assumes a simpler structure given by

$$V_{P2} = \left\{ \frac{1}{2} C_{t-1}^2\, \sigma_t^4 + d_{t-1}^2\, \sigma_t^2 \right\} P_{t-1}^2 \tag{5}$$

ESTIMATES OF PARAMETERS FOR V_{P1} AND V_{P2}

In order to estimate, analyze, and compare V_{P1} and V_{P2}, one must estimate z_{t-1} and σ_t^2. Many studies have estimated the system for changes in yields and the associated volatility of yields. Given our purposes, we choose Koutmos [1998] as it most clearly and succinctly 1) estimates the sensitivity of volatility to yield for multiple maturities, and 2) focuses on producing a simple and illustrative time varying volatility of yield, σ_t^2, dependent on the level of yields (y_{t-1}). Using weekly data, Koutmos [1998] finds a very strong relationship between the level of yields and volatility of yields. More specifically, his system for weekly yield changes and volatility of yield changes for different maturities is

$$\Delta y_t = \alpha + \beta y_{t-1} + \varepsilon_t \tag{6}$$

Here *expected (forecasted)* change in yield, z_{t-1}, is $\alpha + \beta y_{t-1}$. The conditional variance of yield changes is

$$\sigma_t^2 = \psi^2 \, y_{t-1}^{2\gamma} \tag{7}$$

Exhibit 1 shows that the estimated values of ψ and γ are 0.0345 and 0.7975, respectively, for a three-month maturity. Other ψ and γ values for different maturities are also reported in Exhibit 1. Of course, these parameters relate the volatility of rates to the level of rates where one would intuitively expect rates to be more volatile at higher levels because of scale reasons. Cochrane [1999] maintains that bond market volatility is higher (lower) when interest rates are higher (lower). Binomial models of forward rates such as Kalotay, Williams, and Fabozzi [1993] formulate volatility of forward rates as dependent on the level of interest rates. Consistent with the above logic, Engle and Patton [2001] maintain that interest rates are less volatile at lower levels because interest rates cannot become negative.[6]

Exhibit 1 shows the Koutmos [1998] estimates of all parameters for three-month, six-month, one-year, and three-year maturities. We obtain constant maturity yields from the St. Louis Federal Reserve for the same time period as his study, and use the estimations given in Exhibit 1 to compute z_{t-1} (expected Δy_t) and σ_t^2.

ANALYSIS OF V_{P1} AND V_{P2} AS A FUNCTION OF LEVEL OF YIELDS (y_{t-1})

Analysis of Model 1

The sign and magnitude of the impact of level of yield (y_{t-1}) upon Model 1 variance of price changes very simply depends on the comparative strength of a) changes in σ_t^2 due to changes in level of yield, and b) changes in duration due to changes in level of yield. It is obvious from above that σ_t^2 tends to *increase* with level of yield. In contrast, it is well known that duration *decreases* with level of yields. See Appendix A for the details of these relationships.

The relation of variance with respect to yield is thus

$$\frac{d[V_{P1}]_t}{d\,y_{t-1}} = P_{t-1}^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right] d_{t-1}^2 + 2P_{t-1}^2\, \sigma_t^2\, d_{t-1} \left[\frac{\partial d_{t-1}}{\partial y_{t-1}}\right]$$

Model 1 Relation = Part A + Part B

We refer to the first term on the right side as Part A where Part A is clearly positive when ψ and γ are posi-

EXHIBIT 1

System of Weekly Changes in Treasury Yields drawn from Koutmos [1998]

$$\Delta y_t = \alpha + \beta r_{t-1} + \varepsilon_t$$

$$\sigma_t^2 = \psi^2 \, r_{t-1}^{2\gamma}$$

	3-Mo.	6-Mo.	1-Yr.	3-Yr.
α	0.0061	0.0063	0.0093	0.0132
β	-0.0019	-0.0012	-0.0018	-0.0028
Ψ	0.0345	0.0307	0.0452	0.0831
γ	0.7975	0.8573	0.6245	0.3209

tive. Similarly, we refer to the second term as Part B which is clearly negative as the derivative of duration with respect to yield is negative. If ψ and γ are strong enough, the positive impact of Part A will dominate the negative impact of Part B. Focusing on γ, one may develop the first order threshold for Part A to dominate, as given in Appendix B1 (zero coupon) and B2 (par coupon). For a zero coupon bond, the critical γ is merely $y_{t-1}/(1+y_{t-1})$. We later show that V_{P1} is a reasonably accurate measure of yield sensitivity for our shortest maturities so that this γ analysis is then quite useful.

Analysis of Model 2

The relation of V_{P2} to level of yields is much more complex, and we divide it into Parts A, B, and C. For purposes of exposition, we divide the bracketed pieces of Part C into C1 and C2.

$$\frac{d[V_{P2}]_t}{dy_{t-1}} = \underbrace{P_{t-1}^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right] g_{t-1}^2}_{\text{Part A}} + \underbrace{2P_{t-1}^2 \sigma_t^2 g_{t-1} \left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{\text{Part B}}$$

$$\underbrace{+ \frac{P_{t-1}^2}{2} \left\{ \underbrace{\left[2C_{t-1} \frac{\partial C_{t-1}}{\partial y_{t-1}}\right] (\sigma_t^2)^2}_{\text{C1}} + \underbrace{2C_{t-1}^2 \sigma_t^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right]}_{\text{C2}} \right\}}_{\text{Part C}}$$

Model 2 Relation = Part A + Part B + Part C

See Appendix C for definitions of convexity and details of Model 2. As in Model 1, Part A is always positive as volatility of interest rates rises with the level of yields. Part B of Model 2 includes duration (as g_{t-1} depends on duration) but is only roughly similar to Part B of Model 1. That is, Part B is complex and may, theoretically, either be positive or negative, as g_{t-1} and the derivative of g_{t-1} with respect to yield are not sign definite. (See Appendix C.) The complexity of part B is at least partially due to the dependence of both g_{t-1} and the derivative of g_{t-1} upon z_{t-1} (expected change in yield) where z_{t-1} can, of course, be positive or negative. The derivative of z_{t-1} with respect to yield, which is also part of the derivative of g_{t-1}, depends upon the lagged level coefficient (β) which is typically negative in a mean reverting context. Our empirical analysis finds that Part B is usually negative, consistent with duration declining with level of yield, but is sometimes positive.

Part C represents nonlinear interactive aspects of the relation. As the derivative of convexity with respect to yield is negative, the first part of the bracketed term of Part C (C1) is negative while, on the other hand, the second term in brackets (C2) is always positive. Thus, the sign of part C is case specific and depends on factors such as maturity. Intuitively, we expect Part C to play a larger role for longer maturities. This is because convexity and its derivative with respect to yield increase very strongly as maturity increases.

EMPIRICAL RESULTS

Computation of the Time Series Relation of V_{P1} with Respect to Yield

The empirical results for computation of the Model 1 relation are quite simple. See Exhibits 2A, 3A, 4A, and 5A for the four different maturities. Part A is always positive, Part B is always negative, and the sum is always positive, thus suggesting price variance always increases with level of yields for these maturities. Accordingly, if yields decrease and capital gains occur, conditional price variance declines. Thus, stronger returns are appealingly accompanied by lower risk. However, especially for longer maturities, the Model 1 relation is not as accurate as Model 2, because convexity effects are much stronger for longer maturities.

Computation of the Time Series Relation of V_{P2} with Respect to Yield

We now discuss the computation of the derivative of V_{P2} with respect to yield for different maturities of Treasury bonds in Exhibits 2B, 3B, 4B, and 5B. Of course, the sign and magnitude of the relationship for V_{P2} will be similar to that for V_{P1} if Parts A and B are similar across the two models, and Part C of V_{P2} is small. The sign of Part C may potentially determine whether a positive relation for the first order model is enhanced, reduced, or reversed. We begin with shorter maturities and then discuss longer maturities.

EXHIBIT 2A

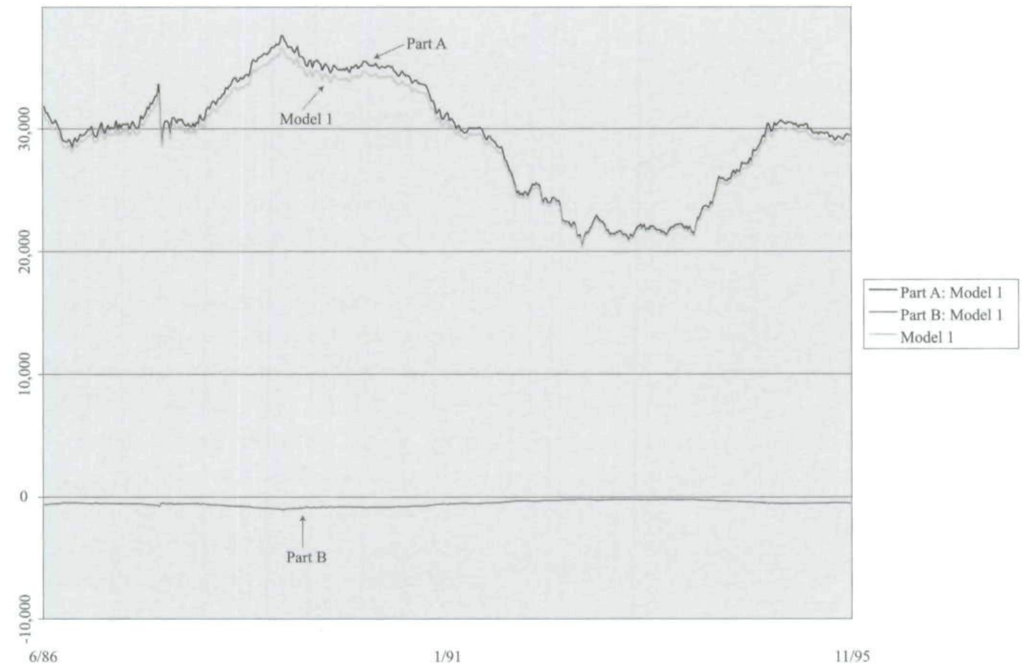
Parts of Model 1 Derivative: Derivative of Variance with Respect to Yield

(Three-month maturity)

$$\frac{d[V_{P1}]_t}{dy_{t-1}} = P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]d_{t-1}^2 + 2P_{t-1}^2\sigma_t^2 d_{t-1}\left[\frac{\partial d_{t-1}}{\partial y_{t-1}}\right]$$

Model 1 Relation = Part A + Part B

This exhibit depicts the different parts of the sensitivity of price variance to level of yield for Model 1. In this case, the total relation is always positive.



Three-Month and Six-Month Results for V_{P2}

For the three- and six-month maturities (Exhibits 2B and 3B), we find that Parts A and B are both very similar to the first order model. This is due to g_{t-1} typically being quite similar to d_{t-1}, because C_{t-1} times z_{t-1} (part of g_{t-1}) is small. (Convexity is relatively small for short maturities but grows rapidly for longer maturities.) As Part C2 has greater magnitude than Part C1, Part C is *positive*, thus marginally *increasing* the (positive) sensitivity of variance to yield compared to the V_{P1} case. However, the impact of Part C is small in comparison to Parts A and B. To illustrate the weak effect of Part C for short maturities, we find that it averages only about 3% of the V_{P1} derivative for the three-month case. Thus, our plots show that the V_{P1} and V_{P2} derivatives are quite similar for these two short maturities.

One-Year Results for V_{P2}

The one-year maturity results (Exhibit 4B) are quite complex and different from shorter maturities in three ways. First, consider Part B. Instead of being negative throughout as in Model 1, it is positive for a while. This is because the derivative of g_{t-1} with respect to y_{t-1}, while usually positive, becomes strongly negative for a time, thus making Part B positive.[7] Note that g_{t-1} is negative throughout. Second, consider Part C. Unlike the shorter maturities, it is now *negative*. Third, consider that the Model 2 total relation sign is usually negative but becomes positive for a brief time. Thus, the one-year maturity is a cross-over intermediate maturity where the impact of yield upon price variance is sometimes quite weak and not even sign definite. Therefore, estimating the sign of the change in risk of a one-year maturity, due to changes in

Exhibit 2B

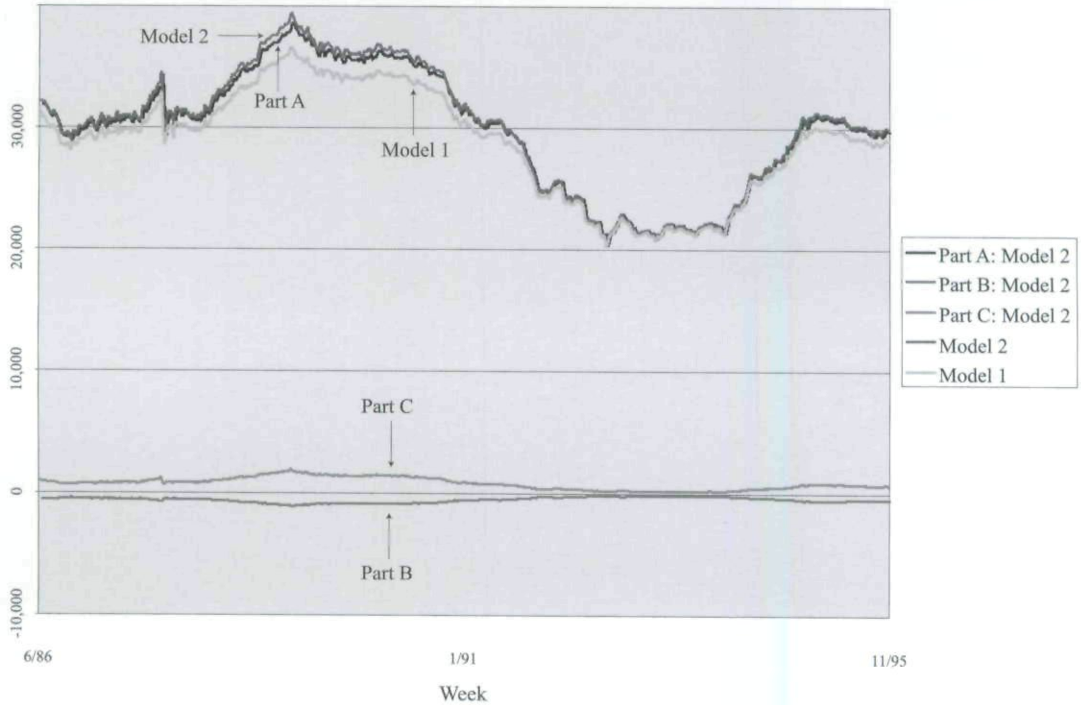
Parts of Model 2 Derivative: Derivative of Variance with Respect to Yield

(Three-month maturity)

$$\frac{d[V_{P2}]_t}{dy_{t-1}} = \underbrace{P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]g_{t-1}^2}_{\text{Part A}} + \underbrace{2P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{\text{Part B}} + \underbrace{\frac{P_{t-1}^2}{2}\left\{\underbrace{\left[2C_{t-1}\frac{\partial C_{t-1}}{\partial y_{t-1}}\right](\sigma_t^2)^2}_{\text{C1}} + \underbrace{2C_{t-1}^2\sigma_t^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]}_{\text{C2}}\right\}}_{\text{Part C}}$$

Model 2 Relation = Part A + Part B + Part C

This exhibit depicts the different parts of the sensitivity of price variance to level of yield for Model 2. Part C is unique to Model 2 but not very large for shorter maturities. In this case, the total relation is always positive and the total relation is close to that of Model 1.



level of yield, may be particularly difficult and time specific. Additionally, whatever the sign, the magnitude of the relation to level of yield may sometimes be quite small.

Three-Year Results for V_{P2}

Now consider the three-year maturity second order case in Exhibit 5B, where Part A is positive and quite similar to the first order three-year result. Part B is always negative, similar to the three-year Model 1 result (but clearly different from the one-year second order result directly above).

Part C is negative throughout and helps make the three-year second order total relation negative. That is, higher levels of interest rates always *reduce* the price variance. The greater impact of Part C for longer maturities is largely due to the stronger (negative) derivative of convexity with respect to yield (see Appendix C). (In fact, the three-month convexity derivative is quite small and practically indistinguishable from zero.) For the three-year case, the Part C magnitude alone averages about 3.86 fold as great as the total V_{P1} derivative.

Finally, it is noteworthy that, unlike the other maturities, Part A of Model 2 and Part B of Model 2 for the three-year are quite similar in magnitude for much, but not all, of Exhibit 5B and their opposing signs then approximately neutralize each other. Thus, Part C of the three-year second order model, totally absent in Model 1,

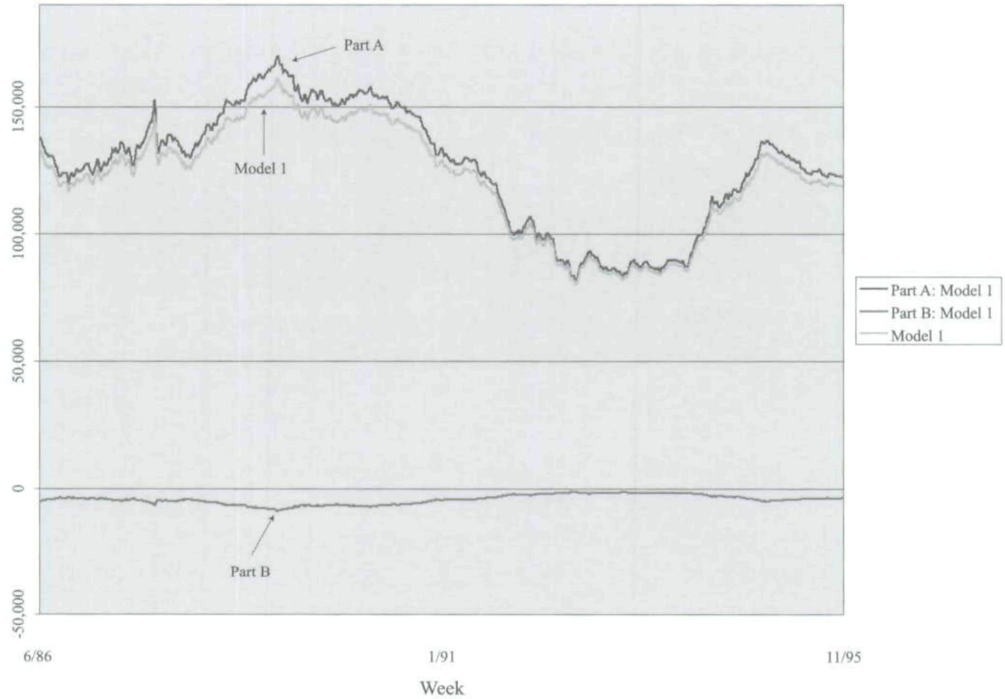
EXHIBIT 3A
Parts of Model 1 Derivative: Derivative of Variance with Respect to Yield

(Six-month maturity)

$$\frac{d[V_{P1}]_t}{dy_{t-1}} = P_{t-1}^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right] d_{t-1}^2 + 2P_{t-1}^2 \sigma_t^2 d_{t-1} \left[\frac{\partial d_{t-1}}{\partial y_{t-1}}\right]$$

Model 1 Relation = Part A + Part B

This Exhibit depicts the different parts of the sensitivity of price variance to level of yield for Model 1. In this case, the total relation is always positive.



largely determines the sign and magnitude of the relation of price variance with respect to yield. In summary, Part C in Exhibit 5B is very similar to the pattern and magnitude of the Model 2 total relation (derivative) for much of the graph.

Summary of Parts of Model 2 Total Relation

Exhibit 6 summarizes the signs for the parts of the second order relation (Model 2) according to the alternative maturities. Part A is always positive for all maturities and Part B is negative except for some positive values in the one-year case. Again, this Part B behavior for one-year maturities makes the sign of the (total) second order relation positive at times. Part C is always weakly *positive* for shorter maturities (three and six month) but always strongly *negative* for longer maturities. In summary, the sign of the second order relation, last column of Exhibit 6, is always positive for the shorter maturities, variable for the one-year case, and always negative for the longest maturity (three years).

Of course, it is important for bond portfolio managers to know that the more accurate relation of V_{P2} to yields is negative when the less accurate first order relation *misleadingly* indicates a positive relation for longer maturities. In such a case, portfolio risk will increase, not decrease, as yields decrease. Assuming decreasing yields, an astute portfolio manager with longer maturities should expect strong returns but, unfortunately, *increased* risk. At the same time, lower yields for short maturities may well be accompanied by *decreasing* risk. This scenario encourages a portfolio manager to perhaps prefer shorter maturities under such conditions.[8]

EXHIBIT 3B

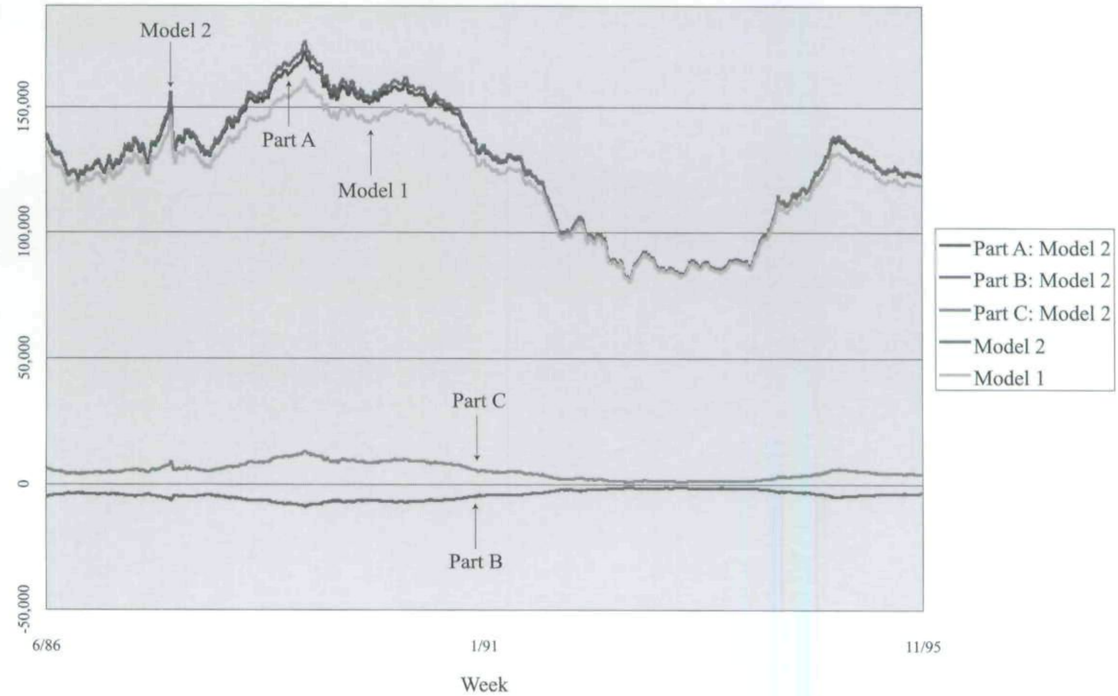
Parts of Model 2 Derivative: Derivative of Variance with Respect to Yield

(Six-month maturity)

$$\frac{d[V_{P2}]_t}{dy_{t-1}} = \underbrace{P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]g_{t-1}^2}_{\text{Part A}} + \underbrace{2P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{\text{Part B}} + \underbrace{\frac{P_{t-1}^2}{2}\left\{\underbrace{\left[2C_{t-1}\frac{\partial C_{t-1}}{\partial y_{t-1}}\right](\sigma_t^2)^2}_{\text{C1}} + \underbrace{2C_{t-1}^2\sigma_t^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]}_{\text{C2}}\right\}}_{\text{Part C}}$$

Model 2 Relation = Part A + Part B + Part C

This exhibit depicts the different parts of the sensitivity of price variance to level of yield for Model 2. Part C is unique to Model 2 but not very large for shorter maturities. In this case, the total relation is always positive and the total relation is close to that of Model 1.



Furthermore, we emphasize that patterns of sensitivity to yield can be dramatically different in V_{P2} compared to V_{P1}. That is, for a three-year maturity, the (misleading) *maximum* V_{P1} derivative occurs about the same time as the (more accurate) V_{P2} *minimum* derivative occurs. Thus a very large error is realized if the simpler first order model is used.

CONCLUSION

Bond portfolio managers need good measures of bond price variance (risk) where the variance can depend on many things. We focus on one factor, level of yields, that is obviously very easy to obtain and strongly related to return. Duration is *negatively* related to the level of yields. Furthermore, higher order models include the fact that convexity is *negatively* related to the level of yields. However, the theory of interest rate processes has commonly maintained that yield volatility is *positively* related to level of yields. The net effect needs to be determined so that portfolio managers can develop informed strategies.

We use both first and second order time varying models of price variance where the first order is much shorter and, thus, easier to implement. That is, the first order variance merely depends on yield volatility and duration. After estimating time varying yield volatility, we find that first order price variance is always *positively* related to yields for all maturities. Therefore, using the first order

EXHIBIT 4A

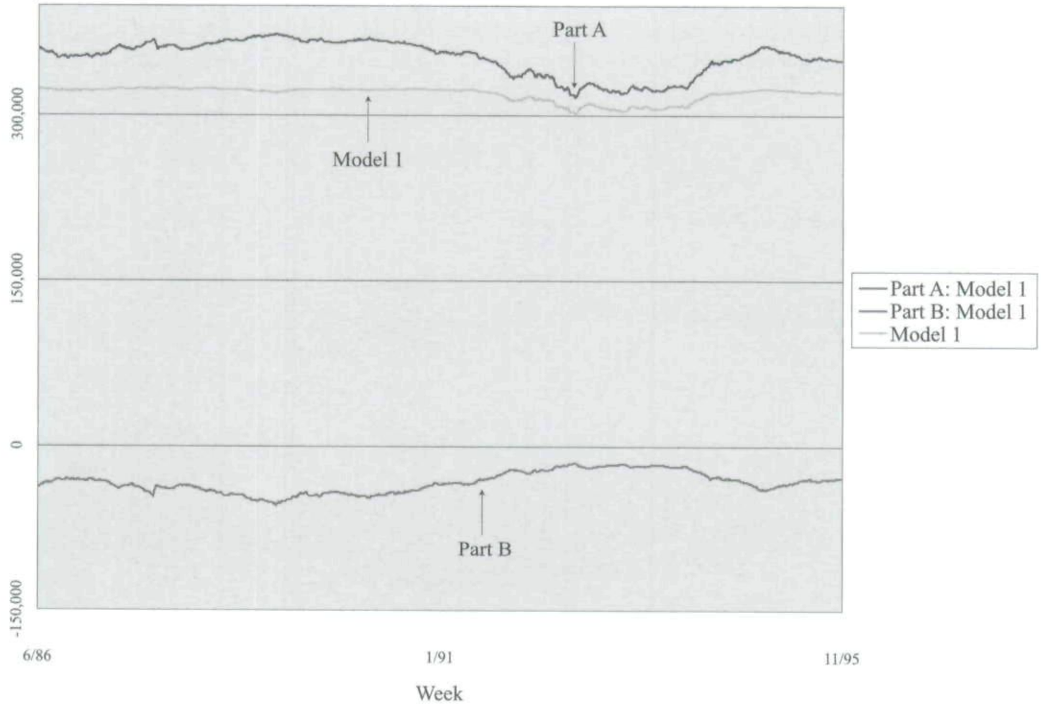
Parts of Model 1 Derivative: Derivative of Variance with Respect to Yield

(One-year maturity)

$$\frac{d[V_{P1}]_t}{dy_{t-1}} = P_{t-1}^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right] d_{t-1}^2 + 2P_{t-1}^2 \sigma_t^2 d_{t-1} \left[\frac{\partial d_{t-1}}{\partial y_{t-1}}\right]$$

Model 1 Relation = Part A + Part B

This exhibit depicts the different parts of the sensitivity of price variance to level of yield for Model 1. In this case, the total relation is always positive. However, Part B is much stronger than for shorter maturities.



model, a portfolio manager would estimate that expectations of decreasing yields not only produce greater expected portfolio values (returns) but also *decrease* portfolio risk.

Many portfolio managers could easily use first order models, but using such models in developing investment strategies can lead to very large mistakes in estimating changes in portfolio risk, especially for longer maturities. We find that a second order model, compared to a first order model, frequently gives dramatically different relationships between price variance and level of yields. For example, the second order relation for a one-year maturity can be either positive or negative while a first order model for the one-year maturity always gives a positive relationship. For a three-year maturity, the second order relationship between level of yields and variance is always *negative* while the first order relation is always *positive*. Thus, for three-year maturities, an expected decrease in yields which increases portfolio value may well significantly *increase* portfolio risk instead of significantly decreasing risk (as a first order model suggests). Of course, the risk profile of three-year maturities may then be less attractive than shorter maturities.

EXHIBIT 4B

Parts of Model 2 Derivative: Derivative of Variance with Respect to Yield

(One-year maturity)

$$\frac{d[V_{P2}]_t}{dy_{t-1}} = \underbrace{P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]g_{t-1}^2}_{\text{Part A}} + \underbrace{2P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{\text{Part B}} + \underbrace{\frac{P_{t-1}^2}{2}\left\{\underbrace{\left[2C_{t-1}\frac{\partial C_{t-1}}{\partial y_{t-1}}\right](\sigma_t^2)^2}_{\text{C1}} + \underbrace{2C_{t-1}^2\,\sigma_t^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]}_{\text{C2}}\right\}}_{\text{Part C}}$$

Model 2 Relation = Part A + Part B + Part C

This exhibit depicts the different parts of the sensitivity of price variance to level of yield for Model 2. Part C is very sizeable compared to Parts A and B. Also, Part C is unique to Model 2 (absent from Model 1). In this case, the Model 2 total relation is sometimes positive and but usually negative, unlike shorter maturities above where the total relation is always positive. The results for Model 1 and Model 2 are very different as shown in this exhibit.

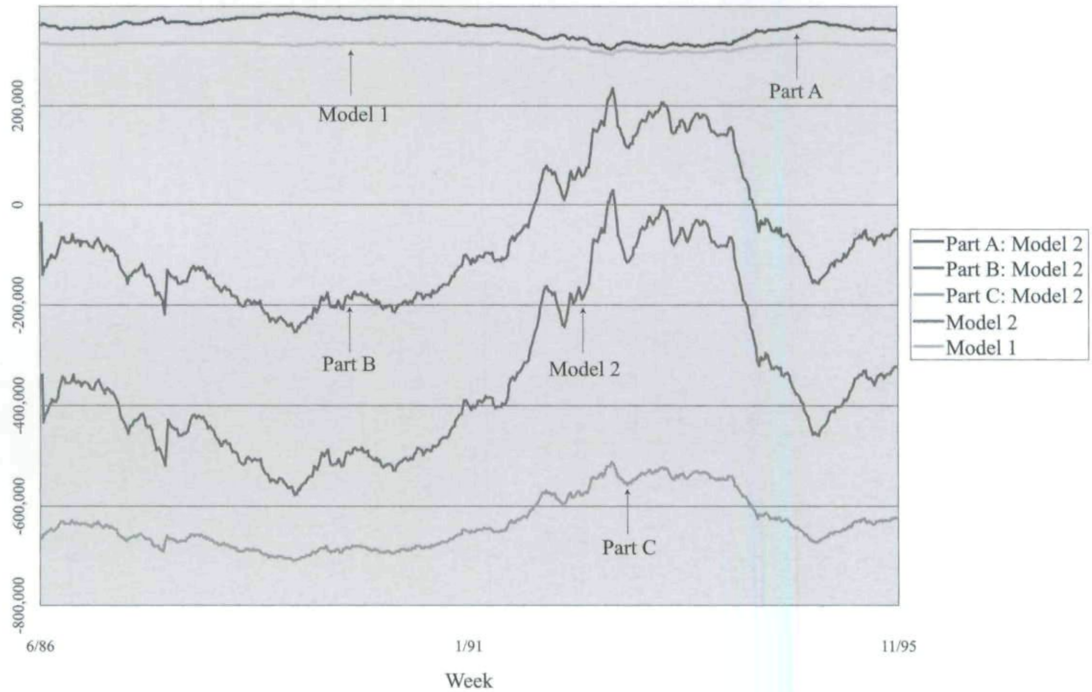


EXHIBIT 5A

Parts of Model 1 Derivative: Derivative of Variance with Respect to Yield

(Three-year maturity)

$$\frac{d[V_{P1}]_t}{dy_{t-1}} = P_{t-1}^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right] d_{t-1}^2 + 2P_{t-1}^2 \sigma_t^2 d_{t-1} \left[\frac{\partial d_{t-1}}{\partial y_{t-1}}\right]$$

Model 1 Relation = Part A + Part B

This exhibit depicts the different parts of the sensitivity of price variance to level of yield for Model 1. In this case, the total relation is always positive.

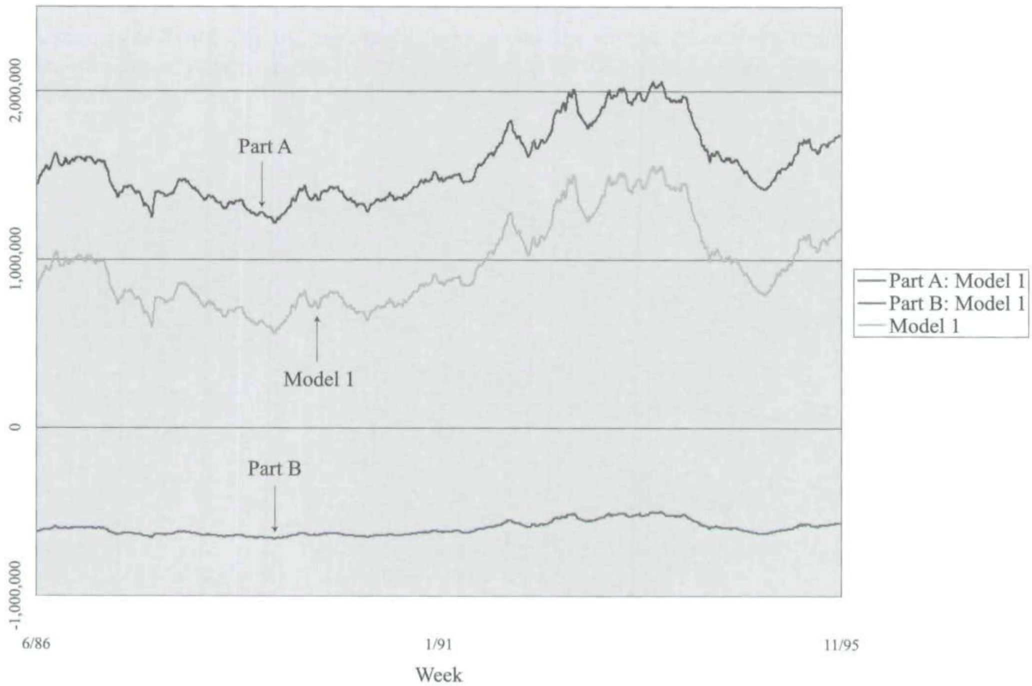


EXHIBIT 5B
Parts of Model 2 Derivative: Derivative of Variance with Respect to Yield

(Three-year maturity)

$$\frac{d[V_{P2}]_t}{dy_{t-1}} = \underbrace{P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]g_{t-1}^2}_{\text{Part A}} + \underbrace{2P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{\text{Part B}} + \underbrace{\frac{P_{t-1}^2}{2}\left\{\underbrace{\left[2C_{t-1}\frac{\partial C_{t-1}}{\partial y_{t-1}}\right](\sigma_t^2)^2}_{\text{C1}} + \underbrace{2C_{t-1}^2\sigma_t^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]}_{\text{C2}}\right\}}_{\text{Part C}}$$

Model 2 Relation = Part A + Part B + Part C

This exhibit depicts the different parts of the sensitivity of price variance to level of yield for Model 2. Part C is very sizeable compared to Parts A and B. Also, Part C is unique to Model 2 (absent from Model 1). In this case, the Model 2 total relation is sometimes positive and but usually negative, unlike shorter maturities above where the total relation is always positive. The results for Model 1 and Model 2 are very different as shown in this exhibit.

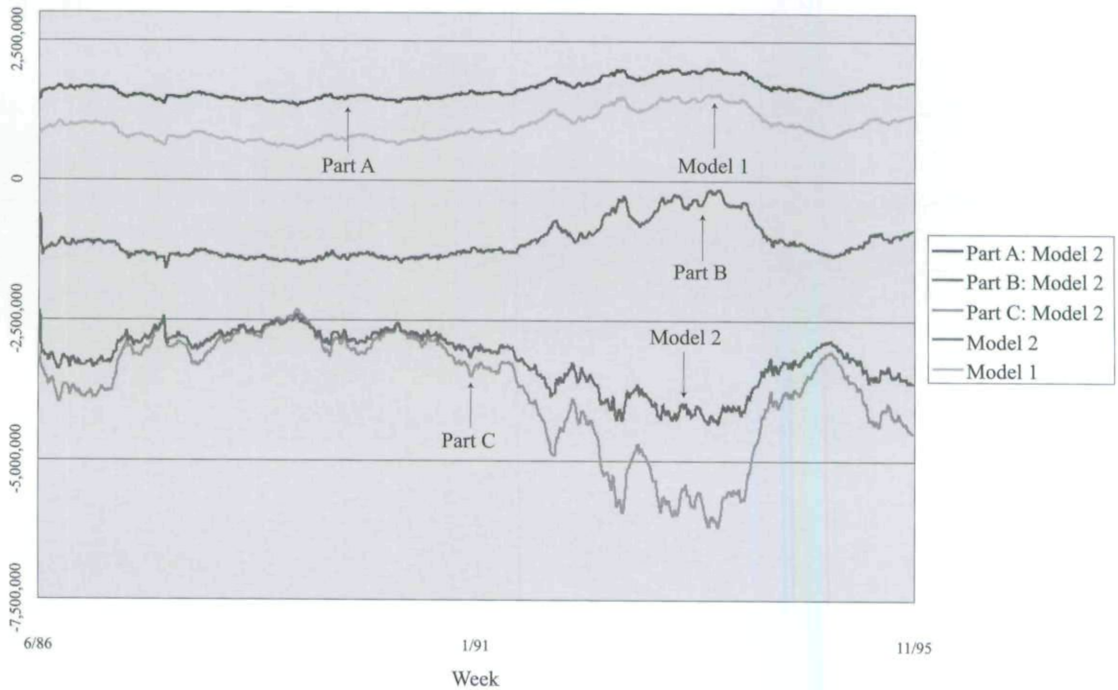


EXHIBIT 6

Signs for Parts of Model 2 Relation

$$\frac{d[V_{P2}]_t}{dy_{t-1}} = \underbrace{P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]g_{t-1}^2}_{\text{Part A}} + \underbrace{2P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{\text{Part B}} + \underbrace{\frac{P_{t-1}^2}{2}\left\{\underbrace{\left[2C_{t-1}\frac{\partial C_{t-1}}{\partial y_{t-1}}\right](\sigma_t^2)^2}_{\text{C1}} + \underbrace{2C_{t-1}^2\sigma_t^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]}_{\text{C2}}\right\}}_{\text{Part C}}$$

Maturity	Part A $P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]g_{t-1}^2$	Part B $2P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]$	Part C (please see above)	Value of Relation for V_{P2} $\frac{d[V_{P2}]_t}{dy_{t-1}}$
Three-month	Positive for all times	Negative for all times	Positive for all times	Positive for all times
Six-month	Positive for all times	Negative for all times	Positive for all times	Positive for all times
One-year	Positive for all times	Negative most of the time, sometimes positive	Negative for all times	Variable
Three-year	Positive for all times	Negative for all times	Negative for all times	Negative for all times

Appendix A

The Derivatives of Yield Volatility and Duration with Respect to Level of Yield

The derivative of yield volatility with respect to level of yields is

$$\frac{\partial \sigma_t^2}{\partial y_{t-1}} = \frac{2\gamma\sigma_t^2}{y_{t-1}} > 0$$

The derivative of duration with respect to yield for a zero coupon and a par coupon bond are, respectively,

$$\frac{\partial d_{t-1}}{\partial y_{t-1}} = -\frac{n}{(1+y_{t-1})^2} < 0$$

$$\frac{\partial d_{t-1}}{\partial y_{t-1}} = \frac{\left(n-d_{t-1}\right)}{y_{t-1}\left(1+y_{t-1}\right)} - \frac{\left(n+1\right)d_{t-1}}{\left(1+y_{t-1}\right)} < 0$$

Appendix B

Appendix B1: Critical γ for Zero Coupon Bond

To make the derivation briefer, the variance of $\Delta P_t/P_{t-1}$ is used. The same result obtains for variance of ΔP_t.

$$Var\left[\frac{(\Delta P)_t}{P_{t-1}}\right] = d_{t-1}^2\ \sigma_t^2$$

$$\frac{\partial Var\left[\frac{(\Delta P)_t}{P_{t-1}}\right]}{\partial y_{t-1}} = d_{t-1}^2\left(\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right) + 2d_{t-1}\left(\frac{\partial d_{t-1}}{\partial y_{t-1}}\right)\sigma_t^2$$

$$d_{t-1} = \frac{n}{(1+y_{t-1})}$$

$$\frac{\partial d_{t-1}}{\partial y_{t-1}} = -\frac{n}{(1+y_{t-1})^2}$$

$$\frac{\partial Var\left[\frac{(\Delta P)_t}{P_{t-1}}\right]}{\partial y_{t-1}} = \frac{2\mathrm{nd}_{t-1}\sigma_t^2}{(1+y_{t-1})^2} + d_{t-1}^2\left(\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right)$$

$$= \frac{-2n^2}{(1+y_{t-1})^3}\sigma_t^2 + \frac{n^2}{(1+y_{t-1})^2}\left(\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right)$$

$$= \frac{n^2}{(1+y_{t-1})^2}\left[\frac{-2\sigma_t^2}{1+y_{t-1}} + \frac{\partial \sigma_t^2}{\partial y_{t-1}}\right]$$

Here,

$$\sigma_t^2 = \psi^2\ y_{t-1}^{2\gamma}$$

$$\frac{\partial \sigma_t^2}{\partial y_{t-1}} = 2\gamma\ \frac{\psi^2\ y_{t-1}^{2\gamma}}{y_{t-1}} = \frac{2\gamma\sigma_t^2}{y_{t-1}}$$

$$\therefore \frac{\partial[\cdot]}{\partial y_{t-1}} = \frac{n^2}{(1+y_{t-1})^2}\left[\frac{-2\sigma_t^2}{1+y_{t-1}} + \frac{2\gamma\sigma_t^2}{y_{t-1}}\right]$$

$$= \frac{2\sigma_t^2 n^2}{(1+y_{t-1})^2}\left[\frac{\gamma}{y_{t-1}} - \frac{1}{y_{t-1}+1}\right]$$

$$= \frac{2\sigma_t^2 n^2}{(1+y_{t-1})^2}\left[\frac{\gamma(1+y_{t-1}) - y_{t-1}}{y_{t-1}\ (1+y_{t-1})}\right]$$

Now,

$$\gamma(1+y_{t-1}) - y_{t-1} = \gamma + y_{t-1}(\gamma - 1)$$

$$\therefore\ \gamma(1+y_{t-1}) - y_{t-1} \gtreqless 0$$

$$\gamma \gtreqless \frac{y_{t-1}}{1+y_{t-=1}}$$

$$\therefore\ \frac{\partial[\cdot]}{\partial y_{t-1}} \gtreqless {}^{0}_{0}$$

when $\gamma \gtrless \frac{y_{t-1}}{1+y_{t-1}}$. The same result obtains for ΔP_t. The zero coupon bond derivative is positive when $\gamma > y_{t-1}/(1 + y_{t-1})$. If γ is less than $y_{t-1}/(1 + y_{t-1})$, then part B (duration effect) dominates and the impact of yields on V_{P1} is negative.

Appendix B2

Derivation of Critical γ for Par Coupon Bond

To make the derivation briefer, the variance of $\Delta P_t/P_{t-1}$ is used. The same result obtains for variance of ΔP_t.

$$\text{Var}\left[\frac{\Delta P_t}{P_t}\right] = d_{t-1}^2\, \sigma_t^2 \tag{B1}$$

$$\frac{\partial[\cdot]}{\partial y_{t-1}} = 2\sigma_t^2 d_{t-1}\left(\frac{\partial d_{t-1}}{\partial y_{t-1}}\right) + d_{t-1}^2\,\frac{\partial \sigma_t^2}{\partial y_{t-1}} \tag{B2}$$

For a par coupon bond,

$$d_{t-1} = \frac{1}{y_{t-1}}\left[1 - \frac{1}{(1+y)^n}\right] \tag{B3}$$

$$\therefore \frac{\partial d_{t-1}}{\partial y_{t-1}} = \frac{(n-d_{t-1})}{y_{t-1}(1+y_{t-1})} - \frac{(n+1)d_{t-1}}{(1+y_{t-1})} \tag{B4}$$

$$\frac{\partial \sigma_t^2}{\partial y_{t-1}} = \frac{2\gamma\,\sigma_t^2}{y_{t-1}} \tag{B5}$$

$$\therefore \frac{\partial \text{Var}\left[\frac{\Delta P_t}{P_t}\right]}{\partial y_{t-1}} = 2\sigma_t^2 d_{t-1}\left[\frac{\partial d_{t-1}}{\partial y_{t-1}} + \frac{\gamma d_{t-1}}{y_{t-1}}\right] \tag{B6}$$

Substitute (B4) in (B6), and we see that the sign of the sensitivity depends on

$$\frac{\partial d_{t-1}}{\partial y_{t-1}} + \frac{\gamma d_{t-1}}{y_{t-1}}$$

Using this,

$$\frac{n-d_{t-1}}{y_{t-1}(1+y_{t-1})} - \frac{(n+1)d_{t-1}}{(1+y_{t-1})} + \frac{\gamma d_{t-1}}{y_{t-1}} > 0 \tag{B7}$$

when

$$\frac{n-d_{t-1}}{y_{t-1}(1+y_{t-1})} + \frac{\gamma d_{t-1}}{y_{t-1}} > \frac{(n+1)d_{t-1}}{(1+y_{t-1})} \tag{B8}$$

Multiplying both sides by $y_{t-1}(1 + y_{t-1})$,

$$(n - d_{t-1}) + \gamma d_{t-1}\,(1 + y_{t-1}) > (n + 1)y_{t-1}d_{t-1}$$

Dividing both sides by $(n+1)d_{t-1}$,

$$\frac{(n-d_{t-1})}{(n+1)d_{t-1}} + \frac{\gamma(1+y_{t-1})}{(n+1)} > y_{t-1} \tag{B9}$$

That is, $\frac{\partial[\cdot]}{\partial y_{t-1}}$ is positive when

$$\frac{n-d_{t-1}}{(n+1)d_{t-1}} + \frac{\gamma(1+y_{t-1})}{(n+1)} > y_{t-1} \tag{B10}$$

The critical γ is given by

$$\gamma > \frac{(n+1)}{(1+y_{t-1})}\left[y_{t-1} - \frac{(n-d_{t-1})}{\left[(n+1)d_{t-1}\right]}\right] \tag{B11}$$

Note that $\frac{n-d_{t-1}}{(n+1)d_{t-1}}$ is always positive and indirectly depends on y_{t-1}. The same result occurs for ΔP_t.

Appendix C

Details of Model 2 Sensitivity of Variance to Yield

Convexity for a zero coupon bond is

$$C_{t-1} = \frac{n(n+1)}{(1+y_{t-1})^2}$$

while convexity for a par coupon bond is[9]

$$C_{t-1} = \frac{1}{y_{t-1}^2}\left(1 - \frac{1}{(1+y_{t-1})^n}\right) - \left(\frac{n}{y_{t-1}(1+y_{t-1})^{n+1}}\right)$$

The derivations of components not involved in the first order model are given below.

$$\frac{\partial g_{t-1}}{\partial y_{t-1}} = \frac{\partial C_{t-1}}{\partial y_{t-1}} (z_{t-1}) + C_{t-1} \frac{\partial z_{t-1}}{\partial y_{t-1}} - \frac{\partial d_{t-1}}{\partial y_{t-1}}$$

where

$$\frac{\partial C_{t-1}}{\partial y_{t-1}} = 2\left\{\frac{2}{y_{t-1}^3}\left[-1+\frac{1}{\left(1+y_{t-1}\right)^{2+n}}\right]\right\}$$

$$+2\frac{\left[2\left(n+1\right)+n\left(n+2\right)y_{t-1}\right]}{y_{t-1}^2\left(1+y_{t-1}\right)^{2+n}}$$

for a par coupon and

$$\frac{\partial C_{t-1}}{\partial y_{t-1}} = \frac{-2n\left(n+1\right)}{(1+y_{t-1})^3} < 0$$

for a zero coupon bond. Computations show the derivative of par coupon convexity is also negative for realistic parameters. Finally,

$$\frac{\partial z_{t-1}}{\partial y_{t-1}} = \beta$$

ENDNOTES

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[1]See, for example, Fabozzi [2004], Chapter 4.

[2]Many classic models of interest rate processes include volatility (positively) dependent upon level of yields. For example, see Cox, Ingersoll, and Ross [1985].

[3]The simplified example does not address coupon income where, of course, coupon income increases any positive return and makes any (capital) loss smaller.

[4]Assuming similar declines in yields for longer maturities, if lower levels of yield reduce risk, one could also potentially increase maturity to get even greater bond returns and keep total portfolio variance lower or the same as before the change in yield.

[5]There is strong empirical evidence that many financial times series, including stock returns and changes in bond yields, are predictable. Bekaert [2001] comments on the general predictability of asset returns. Mills [1999] shows predictability in 20-year U.K. gilts and short-term interbank interest rates. Related to these, Campbell, Lo, and MacKinlay [1997] note that there is overwhelming evidence that autocorrelations in financial markets are statistically significant. Cochrane [1999] notes related predictability of returns in both equity and debt markets as one of the new facts of finance. Others such as Diebold and Li [2006] and Reisman and Zohar [2004] have analyzed the predictability of spot rates.

[6]Also, see an earlier footnote concerning Cox, Ingersoll, and Ross [1985] models of short-term interest rates.

[7]This is due to z_{t-1} being strongly positive for a significant time span, thus making the product $\partial C_{t-1}/\partial y_{t-1}$ times z_{t-1} strongly negative. See Appendix C.

[8]Of course, this assumes yields for short and long maturities are of same sign and similar magnitude. We note that greater expected capital gains for longer maturities may make longer maturities more attractive than shorter maturities, even though the risk of longer maturities increases with a lower level of yields.

[9]See Brooks and Livingston [2001] for the equations.

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