Modeling of CDO Squareds: *Capturing the Second Dimension*

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This article is an extract of a more complete and detailed work on a specific class of credit derivatives, namely Collateralized Debt Obligations made out of Collateralized Debt Obligations, widely known as CDO Squareds (CDO^2). This asset class has become quite popular in recent years as it deserves investors' appetite for higher credit spread via leverage when investing in bonds. This article focuses more on the modeling aspects of plain vanilla structures than on factors which would allow for tailoring a CDO^2 for specific investors' risk appetites.

First we give a brief introduction to CDO^2 and recall important specifics which have to be taken into account for modeling. The following section intends to put John Hull's ideas of pricing CDO^2, which he proposed during the ICBI Global Derivatives Conference of May 2005, in concrete and formal mathematical terms.

First, we model the total loss conditional on the market factor on the respective underlying portfolios via a double-t factor copula model. This allows us to calculate the correlation between these underlying entities and to put the values into a correlation matrix. We refer to this step as "capturing the second dimension, i.e., the cross dependence structure." With the help of this matrix, it is possible to calculate—conditional on the commonality factor—the total loss distribution of the CDO^2 via a one-factor Gaussian Copula Model and numerical methods. The obtained CDO^2 loss distribution is used for pricing a CDO^2 tranche inspired by Credit Default Swap (CDS) pricing methods.

THE MODELING OF CDO SQUAREDS

General Notations

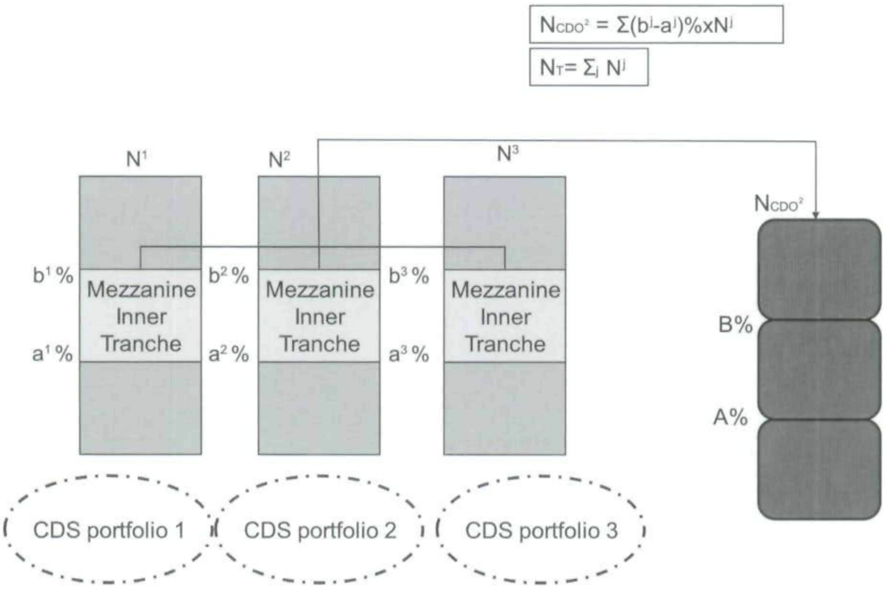
In the following, we focus on a plain-vanilla CDO^2 structure made of n inner mezzanine CDO tranches (see Exhibit 1). Each inner CDO j relies on a portfolio composed of m Credit Default Swaps (CDS). We refer to each name associated with a CDS as a Credit Entity (CE). Thus CE_i^j represents credit entity i within the underlying CDS portfolio associated to inner CDO j.

Each underlying inner CDO tranche is defined by its Attachment Point α^j and its Detachment Point b^j.

In the following table, we define frequently used notations:

Abbreviation	Definition
n	Total number of underlying inner CDOs
j	Counting index related to an inner CDO or the associated CDS portfolio

EXHIBIT 1
Plain Vanilla CDO^2 Set-Up



m	Total number of CDS (CE)in each inner portfolio (the same for all)
i	Counting index related to CE with one inner portfolio
τ_i^j	Default time of CE i in underlying portfolio j
$\alpha^j\%$	Attachment Point of underlying CDO tranche j
$b^j\%$	Detachment Point of underlying CDO tranche j
N_i^j	Notional of credit entity CE_i^j
N^j	Reference Notional of underlying CDS portfolio j (associated to inner CDO j)
$\bar{N}^j$ or $N_{[a^j;b^j]}$	Notional or thickness of underlying inner CDO tranche j
N^j_{CDO}	Reference notional of inner CDO j
$A\%$	Attachment Point of CDO^2 tranche
$B\%$	Detachment Point of CDO^2 tranche
$N_{[A;B]}$	Notional or thickness of outer CDO^2 tranche [A;B]
R	Constant Recovery Rate (in %)

We conclude that

$$N^j = \sum_{j=1}^{m} N_i^j$$

We assume that the underlying CDOs refer to the entire inherent CDS portfolio and thus,

$$N_j = N^j_{\text{CDO}}$$

$$N_{[a^j;b^j]} = (b^j - a^j)\% \times N^j, N_{CDO^2} = \sum_{j=1}^{n} N_{[a^j;b^j]}$$

$$N_{[A;B]} = (B - A)\% \times N_{CDO^2}$$

MODELING JOINT DEFAULTS WITHIN ONE UNDERLYING PORTFOLIO

A CDO^2 structure implies that one has to model losses on two levels: first, the loss distribution of each individual inner CDO has to be taken into account. Various models are already known in order to solve that task. However, what is trickier is to model the joint loss distribution of all underlying CDOs. But the phenomenon of Loss Cap as well as path dependence has already showed us that only the securitized inner CDO tranche is relevant for experienced losses on outer tranches. As a consequence, only the loss distributions of the securitized mezzanine tranches are of importance.

But let us have a look at these modeling constraints difficulty step-by-step.

At the origin of every loss distribution is always the default of one credit entity i in portfolio j. This default can be modeled via various ways: the so-called reduced form models which generally use the copula approach; the structural models inspired by the ideas of Merton; and the semi-analytic models which aim to simulate a loss distribution via approximation in order to price the tranches.

Here we opt for a reduced form approach, the one-factor-copula model.

The Copula Concept: A General Overview

The following are general assumptions:

Abbreviation	Definition
τ_i	Time of default of the ith entity
$Q_i(t)$	Cumulative risk-neutral probability that company i will default before time horizon t
M, Z_i	Independent random variables with zero-mean and unit variance distributions
x_i	Random Variable
F_i	Cumulative distribution of the x_i
H	Cumulative distribution of the Z_i

According to Merton's approach, we assume that default occurs when the asset return of obligor i crosses the threshold X. To generate a one-factor model for the τ_i, the asset value follows a process defined by random variables x_i:

$$x_i = a_i\, M + \sqrt{(1-a_i^2)} \times Z_i$$

with $-1 \leq \alpha_i < 1$. The defaults are measured by the Market Factor M.

The lower the M, the earlier the defaults occur!

The equation above defines a correlation structure between the x_i dependent on a single common factor M. The correlation between x_i and x_j is $\alpha_i \alpha_j$. By the way, calculations show that α_i is the correlation with the market factor M (which is important for calibration).

Marginal Default Probability. Under the copula model, the x_i are mapped to the τ_i using a percentile-to-percentile transformation. The five-percentile point in the probability distribution for x_i is transformed to the five-percentile point in the probability distribution of the τ_i; the 10-percentile point in the probability distribution for x_i is transformed to the 10-percentile point in the probability distribution of the τ_i; and so on.

In general, the point $x_i = x$ is transformed to $\tau_i = t$, where

$$t = Q_i^{-1}\,[F_i\,(x)]$$

It follows from the definition of x_i that

$$P(x_i < x \mid M) = H\left[\frac{x - a_i\, M}{\sqrt{1-a_i^2}}\right]$$

when

$$x = F_i^{-1}\,[Q_i(t)] \Rightarrow P(t_i < t) = P(x_i < x)$$

Hence,

$$P(\tau_i < t \mid M) = H\left[\frac{F_i^{-1}[Q_i(t)] - a_i M}{\sqrt{1-a_i^2}}\right]$$

Marginal Survival Probability. Thus, the conditional probability that the ith bond will survive beyond time t is, therefore,

$$S_i(t \mid M) = 1 - H\left[\frac{F_i^{-1}[Q_i(t)] - a_i M}{\sqrt{1 - a_i^2}}\right]$$

The cumulative individual risk-neutral default probability $Q_i(t)$ could be modeled via various structural models.

But contrary to theory, $Q_i(t)$ is often implied by the obligor's default probability p_i. p_i can be derived from the implied default probability of quoted CDS spreads for a better matching of market prices. In order to calibrate to that implied default probability, the threshold is chosen appropriately.

The cumulative probability that credit entity i defaults by time t is given by

$$p_i = Q_i(t) = P[x_i < X] = F_i(X) \Rightarrow X = F_i^{-1}(p_i)$$

I personally refer to the expression $X = F_i^{-1}(p_i)$ as "inverse fingerprinting" as F_i^{-1}, together with the correlation parameters, gives the multidimensional fingerprint to the individual risk-neutral cumulative distribution function $Q_i(t)$ (or p_i in the implied case).

For the modeling of the marginal joint default probability of an individual CE i within a CDS portfolio j, we opt for a one-factor *double-t* copula model. Calculations showed that this model best fits market prices of CDS (Hull [2005]). Besides, calculation effort is not too tremendous as we stay in the universe of one-factor copula models.

Modeling of the Default of One Credit Entity in an Underlying CDS Portfolio

Before starting modeling, let's have just a quick look on the *double-t* copula and the way of modeling joint default.

The Double-t Copula

The following are general assumptions.

Abbreviation	Definition
$Q_i^j(t)$	Cumulative risk-neutral probability that company i will default before time horizon t in CDS portfolio j
$V = V(V_1,\ldots,V_N)$	Vector which follows a non-specified cumulative distribution F
M, Z_i	Independent random variables $\rightarrow t_\nu$, both with respectively ν and $\bar{\nu}$ degrees of freedom
R	Recovery Rate

The asset value of the obligor related to CE_i^j follows the process:

$$V_i = \varphi\left(\frac{\nu-2}{\nu}\right)^{\frac{1}{2}} \times M + \sqrt{1-\varphi^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{\frac{1}{2}} Z_i$$

with $\varphi \geq 0$. Note that in consequence also L_i^j follows the *Marginal Joint Default Probability Conditional on M,* Conditional on M, we have:

$$P(Z_i \leq V|M) = P\left[\frac{V-\varphi\left(\frac{\nu-2}{\nu}\right)^{\frac{1}{2}} M}{\sqrt{1-\varphi^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{\frac{1}{2}}} \,|\, M\right]$$

$$= t_\nu\left[\frac{F^{-1}\left(Q_i^j(t)\right)-\varphi\left(\frac{\nu-2}{\nu}\right)^{\frac{1}{2}} M}{\sqrt{1-\varphi^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{\frac{1}{2}}}\right]$$

Here, we write F, not t_ν, as one might consider, as the *student-t* distribution is *not* stable under convolution. Even though the Z_i are *t-student* distributed, the copula associated with $(V_1, \ldots, V_n)$ is not necessarily a *student-t* copula. H^* has to be computed by statistic approximation.

Thus, with percentile-to-percentile transformation, we have:

$$P(\tau_i^j < t\,|\,M) = t_\nu\left[\frac{F^{-1}\left(Q_i^j(t)\right)-\varphi\left(\frac{\nu-2}{\nu}\right)^{\frac{1}{2}} M}{\sqrt{1-\varphi^2}\left(\frac{\bar{\nu}-2}{\bar{\nu}}\right)^{\frac{1}{2}}}\right]$$

and

$$S(\tau_i^j < t\,|\,M) = 1 - P(\tau_i^j < t\,|\,M)$$

The default time is given by:

$$\tau_i^j = F_i^{-1}\left(Q_i^j(t)\right)$$

If the degrees of freedom are large enough, the difference between Gaussian and *double-t* copulas vanishes if they rely on the same Correlation matrix Σ. This is due to the fact that for large ν, t_ν is approximately normal. If one decreases the degrees of freedom, the copula will show more tail dependency.

For CDO/CDO^2 modeling this has an important consequence. More tail dependency means a higher potential for joint defaults which, in turn, implies higher stress for senior tranches in CDO transactions.

Hull showed that for CDOs (and here we model the inner CDO j), $\nu = 4$ and $\bar{\nu} = 5$ fit market prices best.

We introduce a Random Loss Variable

$$L_i^j(t) = N_i^j \times (1 - R) \times 1_{\{\tau_i^j < t\}}$$

with

$$E[1_{\{\tau_i^j < t\}}] = P(\tau_i^j < t \mid M)$$

Starting from the random loss variable L_i^j, we define a loss variable for the entire inner CDS portfolio j:

$$L^j(t) = \sum_{i=1}^{m} (1 - R) \times N_i^j \times 1_{\{\tau_i^j < t\}} = \sum_{i=1}^{m} L_i^j(t)$$

Note that the defaultable entity associated to $L_i^j(T)$ is the underlying CDO portfolio j.

We know that the only relevant defaultable entities for the outer CDO^2 tranches are the securitized inner mezzanine CDO tranches. Assume an inner mezzanine tranche of portfolio j with Attachment Point α^j and Detachment Point b^j. The random loss variable on this tranche can be defined by:

$$Y^j(t) = [L^j(t) - a^j \times N^j]^+ - [L^j(t) - b^j \times N^j]^+$$

Approximating the Cross Dependency

With the help of $L^j(t)$, it is possible to approximate the correlation among the inner CDOs and, thus, among inner CDO tranches. This correlation is known as cross correlation or cross dependency.

Previously, we calculated via a *double-t* copula the conditional cumulative marginal joint default probability (conditional on market factor M) that credit name i defaults before time horizon t. We did so for every portfolio j of the total n underlying CDS portfolios.

Conditional on the market factor M, the defaults of credit entities within one portfolio j are independent. However, cross correlation is still present and is, thus, the only source of correlation at this stage.

We scale covariance in order to obtain correlation. We provide the general formula for covariance below.

Definition of Covariance. Let x_i and y_i be two random variables with respective mean μ_x and μ_y. Hence,

$$\mathrm{Cov}(x_i, y_i) = E[(x_i - \mu_x)(y_i - \mu_y)]$$

Recall that

$$L^j = \sum_{i=1}^{m} L_i^j = \sum_{i=1}^{m} (1 - R) N_i^j 1_{\{\tau_i^j < t\}}$$

Thus,

$$E[L^j] = \sum_{i=1}^{m} (1 - R) N_i^j P(\tau_i^j < t \mid M)$$

and

$$\mathrm{Cov}(L^j, L^j) = \mathrm{var}(L^j) = E[(L^j)^2] - E[L^j]^2$$
$$= (1-R)^2 \sum_{i=1}^{m} (N_i^j)^2 P(\tau_i^j < t \mid \mathrm{M})(1 - P(\tau_i^j < t \mid \mathrm{M}))$$

Due to the symmetry property of covariance, this necessarily gives us a symmetric covariance matrix:

$$\begin{aligned}
\tilde{\Xi}_{i,j} &= \mathrm{Cov}(L^j, L^{j'}) \\
&= (1 - R)^2 \mathrm{Cov}\left(\sum_{i=1}^{m} N_i^j 1_{\{\tau_i^j < t\}}, \sum_{i=1}^{m} N_i^{j'} 1_{\{\tau_i^{j'} < t\}} \right) \\
&= E[(L^j - E[L^j])(L^{j'} - E[L^{j'}])] \\
&= (1 - R)^2 E\left[\left(\sum_{i=1}^{m} N_i^j 1_{\{\tau_i^j < t\}} - \sum_{i=1}^{m} N_i^j P(\tau_i^j < t \mid M) \right) \right. \\
&\qquad \left. \times \left(\sum_{i=1}^{m} N_i^{j'} 1_{\{\tau_i^{j'} < t\}} - \sum_{i=1}^{m} N_i^{j'} P(\tau_i^{j'} < t \mid M) \right) \right] \\
&= (1 - R)^2 N_i^j N_i^{j'} E\left[\left(\sum_{i=1}^{m} 1_{\{\tau_i^j < t\}} - \sum_{i=1}^{m} P(\tau_i^j < t \mid M) \right) \right. \\
&\qquad \left. \times \left(\sum_{i=1}^{m} 1_{\{\tau_i^{j'} < t\}} - \sum_{i=1}^{m} P(\tau_i^{j'} < t \mid M) \right) \right]
\end{aligned}$$

with constant N_i^j for all i, i.e., the CDS notional are constant in each underlying CDS portfolio. This results in a $n \times n$ matrix as the CDO^2 is made of n inner CDO tranches.

Finally, the correlation matrix $\Xi_{i,j}$ is obtained when considering that the correlation is given by

$$\rho_{i,j} = \frac{\mathrm{Cov}(L^j, L^{j'})}{\sigma^j, \sigma^{j'}}$$

with σ^j, $\sigma^{j'}$ representing the respective standard deviations.

Loss Distribution on the Underlying CDS Portfolio

In order to elaborate the joint loss distribution on the underlying inner CDS portfolios (and hence for inner CDOs), we have to calculate the conditional cumulative default probability of the inner tranche by time horizon t. We have already calculated the marginal joint default probability $P(\tau_i^j < t \mid M)$ as well as the corresponding survival probability $S(\tau_i^j < t \mid M)$.

Numerical methods allow calculation of the probability of exactly k defaults by time horizon t.

Determining the Probability of Exactly k Tranche Defaults. The aim is now to calculate the distribution function $F_i^j(x)$ conditional on M of the underlying CE via a moment generating function, the so-called recursive iterative approach.

In order to elaborate that loss probability distribution on a portfolio j, we have to calculate the probability of exactly k defaults on the portfolio.

Theorem: Iterated Expectation Theorem. Let (Σ, F, P) be a probability space and X, M: $\Sigma \rightarrow \Re$ be two random variables.
Then

$$E[X]=E[E[X \mid M]]$$

Corollary

$$P(A)=E[1_A]=E[E[1_A \mid M]]=E[P(A \mid M)]$$

for any subset $A \subset \Sigma$.

By the Iterated Expectation Theorem, the joint distribution of the default times within an inner CDS portfolio j can be written as:

$$\begin{aligned} F^j(t_1,\ldots,t_m) &= P(t_1^j < t_1, \ldots, t_m^j < t_m) \\ &= E[P(t_1^j < t_1, \ldots, \tau_m^j < t_m \mid M)] \end{aligned}$$

As $(X_1, \ldots, X_m)$ and thus $(\tau_1, \ldots, \tau_m)$ are independent *conditional* on the realization of the common factor M,

$$\begin{aligned} F^j(t_1,\ldots,t_m) &= E\left[\prod_{i=1}^{m} P(\tau_i^j < t_i \mid M)\right] \\ &= E\left[\prod_{i=1}^{m} P\left(\frac{V - \varphi\left(\frac{\nu-2}{\nu}\right)^{\frac{1}{2}} M}{\sqrt{1-\varphi^2}\left(\frac{\overline{\nu}-2}{\overline{\nu}}\right)^{\frac{1}{2}}} \middle| M\right)\right] \\ &= \int_{-\infty}^{\infty} p(M) \left[\prod_{i=1}^{m} P\left(\frac{V - \varphi\left(\frac{\nu-2}{\nu}\right)^{\frac{1}{2}} M}{\sqrt{1-\varphi^2}\left(\frac{\overline{\nu}-2}{\overline{\nu}}\right)^{\frac{1}{2}}} \middle| M\right)\right] dM \end{aligned}$$

with $\rho(M)$ representing the density of the common factor M.

Remark: The joint distributions of the default times in a one-factor copula model can be expressed as a one-dimensional integral.

Now we extend the reasoning to calculate the probability of k names being default at time t.

Let $\Gamma_i^j(t)$ denote the indicator function of the event $\{\tau_i^j \leq t\}$ and let

$$\Gamma^j(t) = \sum_{i=1}^{m} \Gamma_i^j(t)$$

be the counting process associated to the number of defaults within CDS portfolio j.

Conditional on the realization of M, the $\Gamma_1^j(t), \ldots, \Gamma_m^j(t)$ are independent Bernoulli-distributed random variables. Hence, we can use one of the standard tools to compute their conditional convolution.

The Moment Generating Function: A Recursive Approach. Assume a moment generating function of $\Gamma^j(t)$, named Ψ, and a deterministic counting process for defaults, named $\Gamma^j(t)$. By definition,

$$\Psi_{\Gamma^j(t)} = E\left[u^{\Gamma^j(t)}\right] = \sum_{k=0}^{m} P(\Gamma^j(t) = k) u^k$$

With the Iterated Expectation Theorem, we have

$$\Psi_{\Gamma^j(t)} = E[u^{\Gamma^j(t)}] = E[E[u^{\Gamma^j(t)} \mid M]]$$
$$= E\left[E[u^{\sum_{i=1}^m \Gamma_i^j(t)} \mid M] \right] = E\left[\prod_{i=1}^m E[u^{\Gamma_i^j(t)} \mid M] \right]$$

Let us focus on the last expression, namely $E[\Pi_{i=1}^m E[u^{\Gamma_i^j(t)} \mid M]]$:

$$E[u^{\Gamma_i^j(t)} \mid M] = P(\Gamma_i^j(t) = 0 \mid M)u^0 + P(\Gamma_i^j(t) = 1 \mid M)u^1$$

$$=1-P$$
$$\times\left(V_i \leq \left(\frac{\bar{v}}{\bar{v}-2}\right)^{\frac{1}{2}} \times \frac{F_i^{-1}(Q_i^j(t)) - \varphi\left(\frac{v-2}{v}\right)^{\frac{1}{2}} \times M}{\sqrt{1-\varphi^2}} \mid M \right)$$
$$+P\left(V_i \leq \left(\frac{\bar{v}}{\bar{v}-2}\right)^{\frac{1}{2}} \times \frac{F_i^{-1}(Q_i^j(t)) - \varphi\left(\frac{v-2}{v}\right)^{\frac{1}{2}} \times M}{\sqrt{1-\varphi^2}} \mid M \right)$$
$$\times u = 1-t_v$$
$$\times\left(\left(\frac{\bar{v}}{\bar{v}-2}\right)^{\frac{1}{2}} \times \frac{F_i^{-1}(Q_i^j(t)) - \varphi\left(\frac{v-2}{v}\right)^{\frac{1}{2}} \times M}{\sqrt{1-\varphi^2}} \mid M \right)$$
$$+t_v\left(\left(\frac{\bar{v}}{\bar{v}-2}\right)^{\frac{1}{2}} \times \frac{F_i^{-1}(Q_i^j(t)) - \varphi\left(\frac{v-2}{v}\right)^{\frac{1}{2}} \times M}{\sqrt{1-\varphi^2}} \mid M \right) \times u$$

This can be written because $\Gamma_i^j(t)$ is the counting process of default of credit name i until t, so there is the possibility of *at least* one default, i.e., the process takes the value 0 or 1.

Let $\rho(M)$ be the density function of Market Factor M. Consequently:

$$\Psi_{\Gamma^j(t)} = \int_{-\infty}^{\infty} \rho(M) \left[\prod_{i=1}^m [(1-t_v(\approx \mid M)) + t_v(\approx \mid M) \times u] \right] dM$$
$$= \int_{-\infty}^{\infty} \rho(M) \left[\sum_{k=1}^m \xi_k(M) \times u^k \right] dM$$

with $\xi_k(M)$ being the coefficient of u^k when multiplying

$$\prod_{i=1}^m [(1-t_v(\approx \mid M)) + t_v(\approx \mid M) \times u]$$

By comparing the coefficient of u^k with

$$\Psi_{\Gamma^j(t)} = E[u^{\Gamma^j(t)}] = \sum_{k=0}^m P(\Gamma^j(t)=k)u^k$$

we conclude that

$$P(\Gamma^j(t)=k) = \int_{-\infty}^{\infty} \rho(M)\xi_k(M)dM$$

Here we are more interested in the conditional probabilities. Individual conditional independence of credit events will be necessary later when trying to model the total loss distribution for outer tranches. To do so, we will have to take into account cross dependency among the underlying portfolios.

Hence the probability of exactly k defaults conditional on M within portfolio j is:

$$P(\Gamma^j(t)=k \mid M) = \binom{m}{k} \times t_v(\approx \mid M)^k$$
$$\times (1-t_v(\approx \mid M))^{(m-k)} \cong \xi_k^j(M)$$

Consequently, we define

$$F^j(x) = \sum_{k=0}^x P(\Gamma^j(t)=k \mid M)$$

MODELING DEFAULTS ACROSS INNER PORTFOLIOS

Linking the Underlying Mezzanine Inner CDO Tranches. Now, we aim to link all underlying securitized inner CDO tranches together via a second copula and the cross-correlation matrix Ξ_i^j. To do so, we make use of a One-Factor Gaussian Copula Model.

The One-Factor Gaussian Copula

In the case of a one-Factor Gaussian Copula model—when still referring to our general overview on copula models—H, F follow a standard normal distribution $H\,F \rightarrow \Phi$ and analog, default times are given by:

$$\tau_i = F_i^{-1}\left[\Phi\left(a_i M + \sqrt{1-a_i^2} \times Z_i\right)\right],\ i \in \{1, \ldots, N\}$$

If the value of the common factor M is known, we can calculate the conditional probability that the ith obligor defaults before time t:

$$P(\tau_i < t \mid M) = \Phi\left[\frac{\Phi_i^{-1}[Q_i(t)] - a_i M}{\sqrt{1-a_i^2}}\right]$$

and, therefore,

$$S(\tau_i < t \mid M) = 1 - \Phi\left[\frac{\Phi_i^{-1}[Q_i(t)] - a_i M}{\sqrt{1-a_i^2}}\right]$$

represents the survival probability of the ith credit entity.

M is necessary for calibration and reflects the market tendency. Hence, we define:

$$y^j = f_j \times C + \sqrt{1-f_j^2} \times W^j$$

C as well as W_j have independent standard normal distributions.

We refer to C as the *Commonality Factor*.

Further, we assume that the aggregated random loss variable of the entire underlying portfolio, $L^j(t)$, follows y_j. This assumption implies that

$$G^j(L^j(T)) = N(y^j)$$

where G^j is the cumulative loss distribution for $L^j(t)$.

As we operate with a one-factor Gaussian Copula model, the f_j can easily be matched to the correlation matrix Ξ:

$$\mathrm{Corr}(y^i, y^j) = f_i f_j = \Xi_{i,j} \text{ and } \mathrm{Corr}(C, y^j) = f_j$$

In other words, we consider a vector $(y^1, \ldots, y^n)$, one y^j for each entity.

When we set up the first copula, the individual-name random loss variable L_i^j followed the x_i's. The defaultable entity related to the L_i^j was credit name CE_i^j.

Now, the L^j, which represent the sum of all losses from inner portfolio j, follow the y^j. The defaultable entity associated with the $L^j(t)$ is the sum of losses of all CE comprising the portfolio—hence the entire portfolio j itself.

However, we illustrated that the only relevant loss variable for outer tranches is $Y^j(t)$. By definition, it follows that when the L^j follow the y^j, so do the $Y^j(t)$. The associated defaultable entity to $Y^j(t)$ is the mezzanine inner tranche of inner CDO j. Note that by "default of the mezzanine inner tranche," we understand the total erosion of that tranche by time horizon t.

Hence for calculating the Gaussian Copula, we have to calculate the cumulative default probability conditional on M that the mezzanine tranche of portfolio j defaults before time horizon t. We refer to that probability as

$$\tilde{Q}^{j|M}(T)$$

This is analog to the risk neutral default probability $Q_i^j(t)$ of entity CE_i^j when it came to set up the first copula.

Simplification. In order to facilitate our reasoning, we assume infinitesimal underlying inner tranches, i.e., inner tranches that can be wiped out just by one default in their respective portfolio.

Hence, we assume a constant CE notional N_i^j all over the n inner portfolios. The Attachment Point α^j and the Detachment Point b^j of the inner tranche is chosen in a way that

$$k - 1 = \frac{a^j \% \times N_{CDO}^j}{(1-R)N_i^j}$$

$$k = \frac{b^j \% \times N_{CDO}^j}{(1-R)N_i^j}$$

So it takes exactly k defaults in order to default the securitized tranche of inner portfolio j. Consequently, the default probability conditional on M of inner tranche j is:

$$\tilde{Q}^{j|M}(t) = P(\Gamma^j(t) = k \mid M)$$

The survival probability can be written as:

$$S(\tilde{\tau}^{j|M})=1-\tilde{Q}^{j|M}(t)$$

with $\tilde{\tau}^j$ representing the default time of inner tranche j.

Generally, and by definition, the Merton approach with barrier threshold can be applied here to an entire tranche value.

The default of one inner tranche corresponds to the default of k credit names until time horizon t in the corresponding underlying portfolio (here credit name-related assets are concerned).

Hence one can write:

$$P(y^j < B^j \mid M,C)=\Phi\left[\frac{B^j - f_j}{\sqrt{1-f_j^2}}\right]$$

With percentile-to-percentile transformation, the marginal joint probability (conditional on M,C) that inner tranche j "defaults" before time horizon t is:

$$P(\tilde{\tau}^j < t \mid M,C)=\Phi\left[\frac{\Phi^{-1}(\tilde{Q}^{j|M}(t))-f_j}{\sqrt{1-f_j^2}}\right]$$

and thus the inner tranche's survival probability can be written as

$$S(\tilde{\tau}^j < t \mid M,C)=1-\Phi\left[\frac{\Phi^{-1}(\tilde{Q}^{j|M}(t))-f_j}{\sqrt{1-f_j^2}}\right]$$

In order to price the outer CDO^2 tranches, it remains to compute the Cash Flows related to the $\tilde{\tau}^j$. The CDO^2 outer tranches refer to a portfolio made of all inner CDO mezzanine tranches. This portfolio has the notional

$$N_{CDO^2}=\sum_{j=1}^{n} N_{[a^j;b^j]}$$

In our case, the mezzanine tranche thickness $N_{[a^j;b^j]}$ corresponds to the notional of a defaulted credit entity in portfolio j:

$$N_{CDO^2}=\sum_{j=1}^{n}(1-R)\times N_i^j$$

In order to evaluate the expected cash flows on this portfolio, and hence on the outer tranches, we have to calculate the total CDO^2 Loss Distribution.

The Total CDO² Loss Distribution

As before, we have to derive via numeric methods the probability of exactly d defaults on the CDO^2 portfolio from the marginal joint default probabilities obtained via the second copula. (Note that every default on the securitized inner CDO tranche is accounted as a default on the CDO^2.) We achieve this via our previously presented recursive method.

We define the moment generating function as follows:

$$\Psi_{\Gamma(t)}=E[u^{\Gamma(t)}]=\sum_{d=0}^{n}P(\Gamma(t)=d)u^d$$

With the Iterated Expectation Theorem, we have

$$\begin{aligned}\Psi_{\Gamma(t)} &= E\left[u^{\Gamma(t)}\right]=E\left[E[E[u^{\Sigma_{j=1}^{n}\Gamma^j(t)} \mid M,C]]\right]\\ &= E\left[E\left[\prod_{i=1}^{n}E[u^{\Gamma^j(t)} \mid M,C]\right]\right]\end{aligned}$$

Let us focus on the last expression, namely $E[u^{\Gamma^j(t)} \mid M,C]$:

$$\begin{aligned}E[u^{\Gamma^j(t)} \mid M,C] &= P(\Gamma^j(t)=0 \mid M,C)u^0\\ &\quad + P(\Gamma^j(t)=1 \mid M,C)u^1\\ &= 1-P\left(w^j \le \frac{\Phi^{-1}(\tilde{Q}^{j|M}(t))-f_j}{\sqrt{1-f_j^2}} \mid M,C\right)\\ &\quad + P\left(w^j \le \frac{\Phi^{-1}(\tilde{Q}^{j|M}(t))-f_j}{\sqrt{1-f_j^2}} \mid M,C\right)\times u\end{aligned}$$

Consequently:

$$\Psi_{\Gamma(t)}=\int_{-\infty}^{\infty}\rho(C)\int_{-\infty}^{\infty}\rho(M)\left[\prod_{j=1}^{n}\eta^j(M,C)\right]dMdC$$

with

$$\eta^j(M,C) = 1 - P\left(W^j \leq \frac{\Phi^{-1}(\tilde{Q}^{j|M}(t)) - f_j}{\sqrt{1-f_j^2}} \mid M,C\right) + P\left(W^j \leq \frac{\Phi^{-1}(\tilde{Q}^{j|M}(t)) - f_j}{\sqrt{1-f_j^2}} \mid M,C\right)$$

and $\rho(M)$, $\rho(C)$ being the respective density functions of M and C.

This can be written as

$$\Psi_{\Gamma(t)} = \int_{-\infty}^{\infty} \rho(C) \int_{-\infty}^{\infty} \rho(M)\left[\sum_{d=0}^{n} \Phi_d(M,C)u^d\right] dMdC$$
$$= \sum_{d=0}^{n}\left[\int_{-\infty}^{\infty} \rho(C)\int_{-\infty}^{\infty} \rho(M)\Phi_d(M,C)dMdC\right] \times u^d$$

with $\Phi_d(M,C)$ being the coefficient of u^d when multiplying

$$\prod_{j=1}^{n} \eta^j(M,C)$$

When we compare the coefficients of u^d with the expression

$$\Psi_{\Gamma(t)} = E[u^{\Gamma(t)}] = \sum_{d=0}^{n} P(\Gamma(t) = d)u^d$$

we have

$$P(\Gamma(t) = d) = \int_{-\infty}^{\infty} \rho(C)\int_{-\infty}^{\infty} \rho(M)\, \Phi_d(M,C)\, dMdC$$

Note that the expression above represents the *unconditional* probability distribution concerning defaults on the CDO^2 portfolio (the probability of d credit name defaults on the CDO^2 portfolio).

Analog, we have

$$F(X) = P(X \leq x) = \sum_{d=0}^{x} P(\Gamma(t) = d)$$

It is now possible to introduce a new unconditional and relevant random loss variable:

$$Z^j(t) = N_{[a^j;b^j]} \times 1_{\{\hat{\tau}^j < t\}}$$

with

$$E[1_{\{\tilde{\tau}^j < t\}}] = P(\tilde{\tau}^j < t) = \int_{-\infty}^{\infty} \rho(C)\int_{-\infty}^{\infty} \rho(M)\, \Phi\left[\frac{\Phi^{-1}(\tilde{Q}^{j|M}(t)) - f_j}{\sqrt{1-f_j^2}}\right] dMdC$$

Taking into account our simplification assumption, this can also be written as

$$Z^j(t) = (1-R)N_i^j \times 1_{\{\tilde{\tau}_k^j < t\}}$$

with $\tilde{\tau}_k^j$ being the default time of the *kth* credit name in CDS portfolio j.

Hence the overall random loss variable is defined as follows:

$$Z(t) = \sum_{j=1}^{n} Z^j(t)$$

THE EXPECTED CASH FLOW ON OUTER TRANCHES

In the following, we consider an outer CDO^2 tranche which is characterized by its Attachment Point $A\%$ and its Detachment Point $B\%$, both referring to the CDO^2 notional

$$N_{CDO^2} = \sum_{j=1}^{n} N_{[a^j;b^j]}$$

As for CDOs, pricing a CDO^2 tranche is very similar to pricing a basket of credit default swaps. In case of CDO^2, the default of an underlying inner CDO tranche is equivalent to a default of a credit entity in the CDS case.

First, we have to estimate the present value of tranche losses triggered by credit events during the CDO^2 lifetime. Second, we calculate the present value of the premium payments weighted by the outstanding capital (original tranche amount minus accumulated losses). The former is called the Protection Leg and we refer to the latter as Premium Leg.

The fair spread s is defined such that the expected value of both legs is equal (no free lunch). The expectation is taken under the risk-neutral measure:

$$s = \frac{E_Q[\text{Protection leg}]}{E_Q[\text{Premium leg}]}$$

The Premium Leg

The protection buyer pays a regular fixed spread (premium) on the outstanding notional of the tranche (premium leg). We refer to the outer tranche's outstanding notional with $X(t)$.

Assume that the payment dates are fixed in regular intervals until the end of the CDO^2 life cycle. Hence, these dates can be considered as $(t_1, \ldots, t_i, \ldots, t_p)$, for $i = 1, \ldots, p$

The expression for the Premium Leg, therefore, is:

$$\text{Premium leg} = s \times \sum_{i=1}^{p} E_Q[X(t_i)]e^{-rt_i} \times \delta(t_{i-1}, t_i) + \text{Accrual}$$

In most cases, however, the default time of a credit name is situated between two regular spread payment dates. In this case, the spread payment has to be accrued with respect to the elapsed time between the last regular payment date and the default time.

Accrual:

$$s \times \int \delta(t_{i-1}, t_i) e^{-r(t+\Delta t)} dE_Q[X(t)]$$
$$\approx \sum_{i=1}^{n} \ldots (E_Q[X(t_{i-1})] - E_Q[X(t_i)])$$

with Δt representing the accrued time.

Let us focus on the expression $E_Q[X(t_i)]$.

We associated, with each underlying CDO tranche, the random loss variable Z^j representing the overall loss implied by default time $\tilde{\tau}^j$ with a marginal default distribution:

$$\tilde{F}^j(t) = P(\tilde{\tau}^j \leq t), \quad t \geq 0$$

Therefore, the CDO^2 portfolio loss is given by

$$Z(t) = \sum_{j=1}^{n} Z^j(t)$$

We are interested in how the total losses affect an outer CDO^2 tranche $[A\%; B\%]$. From the payment subordination, we know that the tranche suffers a loss at time t if:

$$A\% \, N_{CDO^2} < Y(t) < B\% \, N_{CDO^2}$$

We choose the following notations:

$$A = A\% \times N_{CDO^2}$$
$$B = B\% \times N_{CDO^2}$$

This is relevant for calculations where leverage intervenes.

Consequently, we can express the tranche loss $Z_{[A;B]}(t)$ as:

$$Z_{[A;B]}(t) = [Z(t) - A]1_{\{Z(t)\in[A;B]\}} + (B - A)1_{\{Z(t)\in[B;N_{CDO^2}]\}}$$

Note that the overall *percentage* loss random variable can be written as:

$$\tilde{Z}(t) = \frac{Z(t)}{N_{CDO^2}} = \frac{\Sigma_{j=1}^{n} N_{[a^j;b^j]} \times 1_{\{\tilde{\tau}^j < t\}}}{N_{CDO^2}}$$

Here the parallels to *nth*-to-default CDS pricing become obvious as we calculate the expected loss on the tranche via the Protection Leg.

We assume that the risk-free interest rate r is constant.

Now, with $A < B$, the outstanding tranche notional for the outer tranche can be written as

$$\begin{aligned} X(t) &= [B - Z(t)]^+ - [A - Z(t)]^+ \\ &= [B - Z(t)]1_{\{Z(t)<B\}} - [A - Z(t)]1_{\{Z(t)<A\}} \\ &= [B - A]1_{\{Z(t)\leq A\}} + [B - Z(t)]1_{\{A<Z(t)\leq B\}} \\ &= [B - A]1_{\{Z(t)\in[0;A]\}} + [B - Z(t)]1_{\{Z(t)\in[A;B]\}} \end{aligned}$$

A Discrete Closed Form Expression. This case is quite important as discretization simplifies computational implementation.

$$
\begin{aligned}
E[X(t_i)] &= [B-A]\times Q(Z(t)\le A) \\
&\quad + [B-E_Q[Z(t)]]1_{\{A<Z(t)<B\}} \\
&= [B-A]P(\Gamma(t)\le x) \\
&\quad + [B-E_Q[Z(t)]]\times P(x<\Gamma(t)\le y) \\
&= [B-A]F(x)+[B-E_Q[Z(t)]] \\
&\quad \times P(x<\Gamma(t)\le y)
\end{aligned}
$$

$$
\begin{aligned}
&= [B-A]\times\sum_{d=0}^{x}P(\Gamma(t)=d)+B\times\sum_{d=x+1}^{y}P(\Gamma(t)=d) \\
&\quad -\sum_{d=0}^{\hat{n}}(1-R)N_d P(\Gamma(t)=d)\times\sum_{d=x+1}^{y}P(\Gamma(t)=d)
\end{aligned}
$$

with

$$
\begin{aligned}
x &= \mathrm{Int}\left[\frac{A\%\times N_{CDO^2}}{N_{[a^j;b^j]}}\right]=\mathrm{Int}\left[\frac{A\%\times N_{CDO^2}}{(1-R)N_i^j}\right] \\
y &= \mathrm{Int}\left[\frac{B\%\times N_{CDO^2}}{N_{[a^j;b^j]}}\right]=\mathrm{Int}\left[\frac{B\%\times N_{CDO^2}}{(1-R)N_i^j}\right] \\
\hat{n} &= \mathrm{Int}\left[\frac{N_{CDO^2}}{N_{[a^j;b^j]}}\right]=\mathrm{Int}\left[\frac{N_{CDO^2}}{(1-R)N_i^j}\right] \\
N_d &= N_{[a^j;b^j]}=(1-R)N_i^j
\end{aligned}
$$

Note:

The Expected Trancheloss is the following:

$$\sum_{d=x}^{y}(1-R)N_d P(\Gamma(t)=d)$$

An alternative formulation would be

$$
\begin{aligned}
E[X(t_i)] &= [B-A]F(x) \\
&\quad + [B-E[Z(t)]]\times P(x<\Gamma(t)\le y) \\
&= [B-A]F(x)+\max(B-E[Z(t)];0) \\
&= [B-A]F(x) \\
&\quad +\max\left(B-(1-R)\sum_{j=1}^{n}N_{[a^j;b^j]}P(\tilde{\tau}^j\le t);0\right)
\end{aligned}
$$

A Continuous Time Expression:

With

$$l_d=\frac{d\times N_{[a^j;b^j]}}{N_{CDO^2}}$$

being the [%] loss caused by exactly d defaults on the CDO^2 portfolio, we can write:

$$P(\Gamma(t)=d)=P(\Gamma(t)=l_d)$$

as the probability of having d defaults on the CDO^2 portfolio is the same as having a l_d% erosion of the CDO^2 portfolio caused by d defaults.

Further, let

$$f(l_d,t)=\frac{\partial F(l_d)}{\partial l_d}$$

be the density.

Then the expected outstanding CDO^2 tranche notional can be written as:

$$
\begin{aligned}
E[X(t_i)] &= [B-A]\times Q(Z(t)\le A) \\
&\quad + [B-E_Q][Z(t)]]\,Q(A<Z(t)\le B) \\
&= [B-A]\,P(\Gamma(t)\le x) \\
&\quad + [B-E_Q[Z(t)]]\times P(x<\Gamma(t)\le y) \\
&= [B-A]F(l_x\le A) \\
&\quad + [B-E_Q[Z(t)]]\times P(x<\Gamma(t)\le y) \\
&= [B-A]\int_0^A f(l_d,t)dl_d+B\int_{A+}^{B}f(l_d,t)dl_d \\
&\quad -\int_0^{N_{CDO^2}}(1-R)\,l_d f(l_d,t)dl_d\times\int_{A+}^{B}f(l_d,t)dl_d
\end{aligned}
$$

The Protection Leg

The protection seller pays a single payment on each default that affects and reduces the tranche. Hence, the Protection Leg or Default Leg is calculated as follows:

$$\int e^{-r(t+\Delta t)} dE_Q[X(t)] \approx \sum_{i=1}^{p} e^{-rt_i} \times (E_Q[X(t_{i-1})] - E_Q[X(t_i)])$$

The above can be written as any variation on the outstanding outer tranche notional between periods $i - 1$ and i would indicate a loss. By discounting the spread paid on each CDO^2 tranche with respect to the CDO^2 life cycle, one can now evaluate a CDO^2 and its tranches.

Revisiting the Concept in Detail

The concept of calculating the different legs with the outstanding tranche notional can be illustrated by the following concrete example.

The most current implementation for CDO pricing is the one-factor copula model approach. Remember that the probability that there will be at least d defaults by time T is:

$$\sum_{k=d}^{n} P(\Gamma(t) = k)$$

Do not forget either that we assume that the principles associated with the underlying reference entities are the same, or that recovery rates R are non-stochastic and are the same for all credit entities.

Example: Let's consider a CDO^2 made of 100 inner CDO tranches with constant recovery rate of 40%. Suppose that the junior tranche is defined by an Attachment Point of 5% and a Detachment Point of 15%, i.e., the tranche's thickness is 10%. Calculations have shown the following scenario:

The junior tranche bears the costs of, respectively,

- 66:67% of the 9th to default CDS
- 100% of the 10th to default CDS
- 100% of the 11th to default CDS

⋮

- 100% of the 25th to default CDS

The Mezzanine tranche bears the costs of, respectively,

- 100% of the 26th to default CDS
- 100% of the 27th to default CDS

⋮

- 100% of the 35th to default CDS

N = Principal of each entity

$s\% \times N$ = promised payment to the tranche holder (spread) at time t^*.

The Expected Payment at time t^* is:

$$\begin{aligned} EP(t^*) &= 10 \times N \times s \times \sum_{k=0}^{8} P(\Gamma(t^*) = k) \\ &+ 10 - 0.667 \times 0.6) \times N \times s \times P(\Gamma(t^*) = 9) \\ &+ 10 - 1.667 \times 0.6) \times N \times s \times P(\Gamma(t^*) = 10) \\ &+ 10 - 2.667 \times 0.6) \times N \times s \times P(\Gamma(t^*) = 11) + \cdots \\ &+ 10 - 15.667 \times 0.6) \times N \times s \times P(\Gamma(t^*) = 24) \end{aligned}$$

The first expression,

$$10 \times N \times s \times \sum_{k=0}^{8} P(\Gamma(t^*) = k)$$

might be surprising as the premium leg of an ordinary CDS is calculated by a survival function

$$1 - P(\Gamma(t^*) = k)$$

However, the context here is not quite the same. As the junior tranche is not affected by defaults one to eight, the tranche holder perceives the total amount of his contractual spread unless more than eight defaults have occurred. Thus, we have to price the tranche with the probability function which gives the probability of at least k defaults at $T = t^*$.

The following expressions of the pricing formula are explained by the CDO^2's structural environment: every default superior to eight until the tranche's Detachment Point of 15% (which corresponds to losses until the CDO^2 default) diminishes the tranche's thickness and, therefore, the notional on which the spread is paid. When the losses associated to the defaults exceed the thickness, the investor perceives a negative spread as a tranche holder in a CDO is considered as a seller of protection.

Note, however, that we still have to differentiate between losses on an underlying CDO and losses on a securitized inner tranche in order to take into account loss cap. Path dependence is already taken into consideration. This can be achieved by considering all the possible outcomes via simulation.

Otherwise, we have to identify defaults on inner tranches (for example, all defaults superior to five in portfolio *j*) but we still have to class *j* potential first to defaults on the CDO^2 (i.e., the *j* FtD on inner tranches) by degree of their probability. This is the only way to obtain the probability of one default.

EXTENSIONS OF OUR MODELING APPROACH—PRICING THICKER TRANCHES

Individual Name Approach—A Recursive Algorithm

Previously we assumed a specific thickness of inner tranches so that they were whipped out by one default on that tranche. This assumption had the advantage that the independent mezzanine tranche default probability conditional on $M;C$ was at the same time the conditional probability of an individual name default on the CDO^2 portfolio.

$$P(\tilde{\tau}^j < t \mid M,C) = P(\tau_k^j < t \mid M,C)$$

Thus it became possible to calculate the unconditional probability distribution of credit name defaults on the CDO^2 portfolio:

$$P(\Gamma(t) = d)$$

Let us now suppose a thickness of inner CDO tranches that allows more than one default on each securitized tranche.

But how to take every default on an inner tranche into consideration? We can not just link every credit name in portfolio *j* to a credit name in portfolio *i* and hence merge all underlying portfolios to one huge Master CDO. This approach would not take into account path dependence of defaults and loss cap. Loss cap describes the phenomenon that when aggregate loss on one inner CDS portfolio exceeds the securitized tranche's Detachment Point, the outstanding Excess Loss is not taken into account.

Besides it is not certain that the first default on tranche *j* will be the first default passed through outer tranches. Perhaps even the first three defaults relevant for the CDO^2 outer tranches are the first three defaults on tranche *i*, etc.

It becomes obvious that here we have to deal with a problem due to combination of different loss scenarios:

Take a standard CDO^2 made of three inner tranches. The scenario of three defaults on the CDO^2 portfolio is subject to different default patterns. The three overall defaults might be the result of two defaults on inner tranche 1 and one default on inner tranche 2 (and hence no default on the third tranche). Or perhaps the third inner tranche bears all the three defaults, etc...

Only simulation methods would help to solve that problem.

It is not an easy task to find a closed-form solution, however Baheti et al. applied a well-known recursive algorithm for CDO evaluation on CDO^2 in order to consider all individual name defaults on the securitized mezzanine inner CDO tranches:

Baheti et al.[2004] propose a recursive procedure which is a multivariate extension of a well-known recursive algorithm used for plain vanilla CDOs (cf, for example, Greenberg et al. [2004]) and hence avoid constructing a second copula to overcome cross dependency.

Losses are first discretized in the event of default by associating each credit with the number of loss units that its default would produce in each of the underlying portfolios: the representative element of the loss matrix $(\lambda_{k,j})$ indicates the integer number of loss units in underlying portfolio *j* due to the default of name *k*. (credit overlap!)

Next, they construct an *n*-dimensional hyper-cube whose *jth* side consists of all possible loss levels for the *jth* underlying portfolio, i.e.,

$$\left(0,1,\ldots,\sum_{k=1}^{m}\lambda_{k,j}\right)$$

For ease of explanation, and without loss of generality, we consider here a two-dimensional example ($k = 2$). In this case, our hyper-cube is simply a matrix (Z_{h_1,h_2}) where we can store the conditional joint distribution of the two underlying portfolios; for example, we store in $(Z_{3,5})$ the probability of jointly having three loss units in the first underlying portfolio and five in the second.

In order to obtain the correct set of joint probabilities, we first initiate each state (recursion step $k = 0$) by setting:

$$Z^0_{h_1,h_2} = 1, \text{ if } h_1 = 0 \text{ and } h_2 = 0$$
$$Z^0_{h_1,h_2} = 0, \text{ otherwise}$$

Then we feed the m credit names, one at a time, through the following recursion:

The *kth* recursion step when considering credit name k can be written as:

$$\begin{aligned} Z^k_{h_1,h_2} &= (1 - P(\tau_k < t \mid M)) \times Z^{k-1}_{h_1,h_2} \\ &\quad + P(\tau_k < t \mid M) \times Z^{k-1}_{(h_1-\lambda_{k,1}),(h_2-\lambda_{k,2})} \\ &\quad \text{if } h_1 > \lambda_{k,1} \wedge h_2 > \lambda_{k,2} \\ Z^k_{h_1,h_2} &= (1 - P(\tau_k < t \mid M)) \times Z^{k-1}_{h_1,h_2} \text{ otherwise} \end{aligned}$$

Explanation: Check out

$$\begin{aligned} Z^k_{h_1,h_2} &= (1 - P(\tau_k < t \mid M)) \times Z^{k-1}_{h_1,h_2} \\ &\quad + P(\tau_k < t \mid M) \times Z^{k-1}_{(h_1-\lambda_{k,1}),(h_2-\lambda_{k,2})} \\ &\quad \text{if } h_1 > \lambda_{k,1} \wedge h_2 > \lambda_{k,2} \end{aligned}$$

Suppose that the default of credit name k implies one default in portfolio 1 ($\lambda_{k,1} = 1$) and one default in portfolio 2 ($\lambda_{k,2} = 1$).

Hence the joint probability of having two defaults in portfolio 1 and three defaults in portfolio 2 is:

$$(Z^k_{2,3}) = S(\tau_k < t \mid M) \times Z^{k-1}_{2,3} + S(\tau_k < t \mid M) \times Z^{k-1}_{2-1,3-1}$$

with S being the survival probability.

This means that we have two different ways of achieving two defaults in portfolio 1 and three defaults in portfolio 2:

If CE_k does not default, it is up to credit name $k - 1$ (or its predecessors as recursion and the previous information included in the probability) to cause two defaults in portfolio 1 and three defaults in portfolio 2.

Or (and thus the "+") CE_k defaults, then credit name $k - 1$ (or its predecessors) only have to "assure" one default in portfolio 1 and three defaults in portfolio 2 (thus $Z_{1,2}$).

The default probabilities P and Z (and thus S) are independent conditional on M so that Z can be written as a recursive product.

$$(Z^k_{h_1,h_2}) = (1 - P(\tau_k < t \mid M)) \times Z^{k-1}_{h_1,h_2} \text{ otherwise}$$

is a logical consequence as if the requested default scenario is already "enabled" by the previous credit names, credit k must survive in order to allow such a scenario and not to explode the default rate.

Each credit can either survive, and every state then "keeps" its position, or default, in which case every state "moves" in the direction$[\lambda_{k,1}, \lambda_{k,2}]$. After including all the issuers, we set:

$$(Z_{h_1,h_2}) = (Z^m_{h_1,h_2})$$

The matrix (Z_{h_1,h_2}) now holds the joint loss distribution of the two mini portfolios conditional on the realization of the market factor. It is then straightforward to recover the conditional joint distribution of losses on the underlying tranches, the conditional loss distribution of the CDO^2 portfolio and the conditional loss distribution of the CDO^2 tranche. One can then proceed to integrate over the market factor, construct the tranche survival curve, and price the outer tranche as described in the previous section.

However. Here we have still to face combination problems concerning the different default scenarios in order to set up an overall loss distribution function:

First all defaults on inner CDO tranches have to be identified. This should not be too difficult as all defaults on each individual underlying CDS portfolio exceeding a minimum number of defaults have to be taken into account. (For example more than five defaults on the second inner portfolio and so on.)

But still there are different combinations possible for an overall default of exactly k names on the CDO^2. Here in our cause we would have to sum up all the Z's which sum of indexes is equal to k ($h_1 + h_2 + \cdots = k$).

Once the overall distribution is set up, pricing of CDO^2 tranches is analog to what we have already seen previously.

Tranche-Default Approach

Here we just consider the entire securitized CDO tranche as a defaultable entity:

Just assume thicker inner CDO tranches defined by their respective Attachment Point a^j and Detachment Point b^j so that

$$k = \frac{a^j \times N^j_{CDO}}{(1-R)N^j_i}$$

and

$$k + 5 = \frac{b^j \times N^j_{CDO}}{(1-R)N^j_i}$$

We still suppose N^j_i constant among all the n inner CDS portfolios.

Hence it would take $k + 5$ defaults on portfolio j that the entire inner tranche j defaults. We have already calculated

$$P(\Gamma^j(t) = k \mid M)$$

As a consequence the default probability conditional on M of inner tranche j until time horizon t is given by

$$\tilde{Q}^j(t) = P(\Gamma^j(t) = k + 5 \mid M)$$

Concerning the second copula, the Merton barrier concept is also applicable to entire tranches.

Explanation. Nowadays entire synthetic tranches are quoted on markets (index tranches). We assumed that the L^j follow the Y^j. Hence the associated defaultable entity to this problem is the entire portfolio as explained before.

Comparable to the whole CDS portfolio, index tranches with an Attachment Point equal to 0 are available on the market.

Thus, it is possible to duplicate the same CDS exposure as the inner portfolio via quoted tranches. When the portfolio gets heavily eroded, i.e., experiences huge losses, the corresponding pool of quoted index tranche loses value.

Therefore it is possible to imply a default barrier which equals total erosion of the portfolio.

Now as the defaults on mezzanine tranches are only relevant for outer tranches, we introduced the loss variable Y^j. Due to definition, the Y^j follow also the y^j.

Mezzanine tranches have an Attachment Point different from 0.

Fortunately one can "*construct*" a virtual quoted tranche by recurrence out of index tranches with different Detachment Points.

This method is similar to the one of the Base Correlation framework.

As a result we can observe the market price of a synthetic tranche which is equivalent in all aspects to the inner mezzanine CDO tranche. Now the same concept applies:

As the quoted tranche loses value when it experiences losses, one can derive a default barrier in order to simulate the default of the inner tranche.

The following steps are now equal to the ones already explained before, taken into consideration the modified default probability above when it comes to establishing the copula, as well as the new inner tranche notional $\widetilde{N}^j$.

However this approach can only be seen as a rough approximation. By defining the entire tranche as a defaultable entity across the inner portfolios, we neglect for example the probability of 1; 2; 3; 4 defaults on the tranche. These defaults are actually passed through outer CDO^2 tranches but are *NOT* taken into consideration by this approach.

This means that the thinner the inner tranche, the more precise the approximation.

In the following we quantify this approximation error:

Let l^j_k be the [%] loss caused by k defaults on CDS portfolio j.

Hence one can write

$$P^j(\Gamma^j(t) = k \mid M) = P^j(l^j = l^j_k \mid M)$$

and therefore

$$F^{j|M}(X) = P(X \le x \mid M) = \sum_{k=0}^{x} P^j(\Gamma^j = k \mid M)$$

The density function is

$$p^{|M}(l^j = l^j_k) = \frac{\partial F^{j|M}(l^j = l^j_k)}{\partial l^j_k}$$

Therefore

$$E[L^j \mid M] = \int_{a^j}^{b^j} l_k^j \times p^{|M}(l^j = l_k^j)\, dl_k^j$$

would be the Expected Loss on the inner tranche conditional on M.

In our previous case, the expected tranche loss conditional on M was:

$$E[L^j \mid M] = (1-R)\tilde{N}^j P(\Gamma^j(t) = k+5 \mid M)$$

Pricing is analog to what we have seen before.

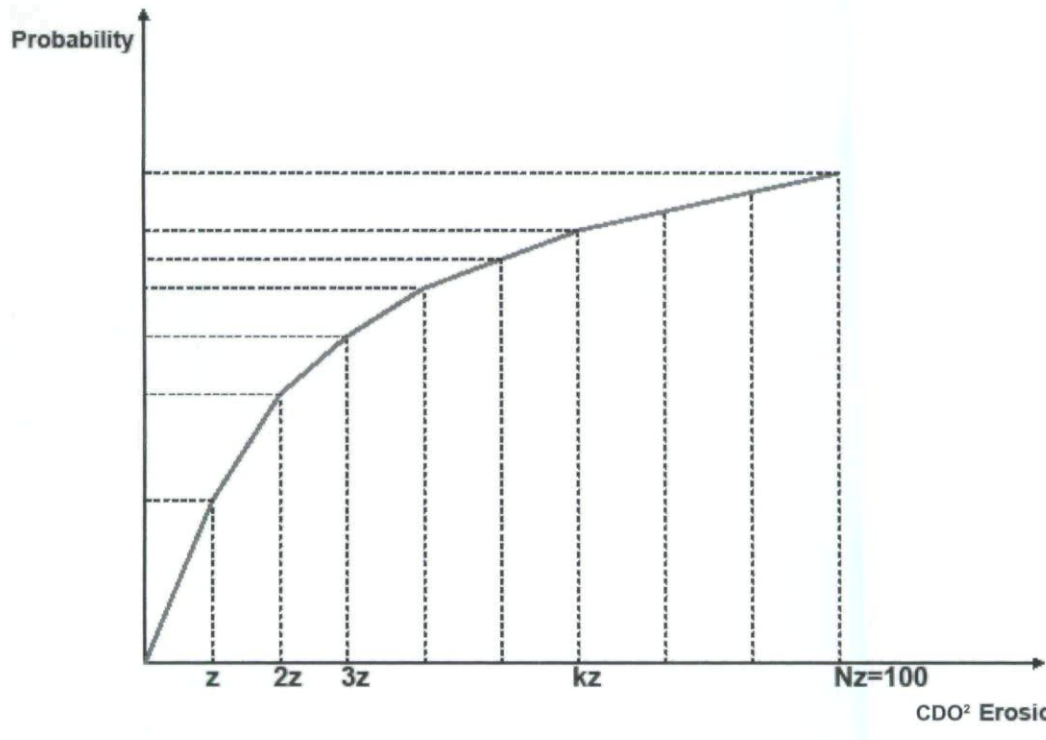
Calibrating the overall loss distribution

The problem of underestimation of idiosyncratic default risk on inner CDO tranches can be overcome.

Suppose as in our previous example a standard CDO^2 structure whose inner tranches are capable to support five defaults.

With our tranche default modeling approach we obtain a rough overall loss distribution on the CDO^2 portfolio as we have only calculation results concerning the multiples of the overall percentage loss a default of a "5-name tranche" would cause.

EXHIBIT 2
Rough Tranche Default Approach with k = 5



Explanation: Assume that a default of one inner tranche capable of absorbing five defaults would cause a z% loss on the CDO^2 portfolio. Due to our copula framework and numerical methods we can only calculate the effect of k tranche defaults on the CDO^2 portfolio, hence $k \times z$% losses. Therefore if we wanted to illustrate our overall loss distribution graphically (% overall loss in function of overall default probability), we could only calculate and draw the points (z%, $2z$%, …, nz%).

The rest would have to be obtained by linear interpolation.

In order to get a more precise overall loss distribution we need more information.

This is why one may calculate different constellations:

First we calculate (still with the tranche default approach) the overall loss distribution of the same standard CDO^2 structure, but this time with inner tranches capable of absorbing one default (like seen before).

EXHIBIT 2 (*Continued*)

We repeat the previous step with inner tranches capable of absorbing two defaults.

We repeat the previous step with inner tranches capable of absorbing three defaults.

We repeat the previous step with inner tranches capable of absorbing four defaults.

Now a default of a inner tranche capable of absorbing one default causes a $v\%$ erosion on the outer CDO^2 portfolio.

A default of an inner tranche capable of absorbing two defaults causes a $w\%$ erosion on the outer CDO^2 portfolio.

A default of an inner tranche capable of absorbing three defaults causes a $x\%$ erosion on the outer CDO^2 portfolio.

A default of an inner tranche capable of absorbing two defaults causes a $y\%$ erosion on the outer CDO^2 portfolio.

By definition

$$v\% < w\% < x\% < y\% < z\%$$

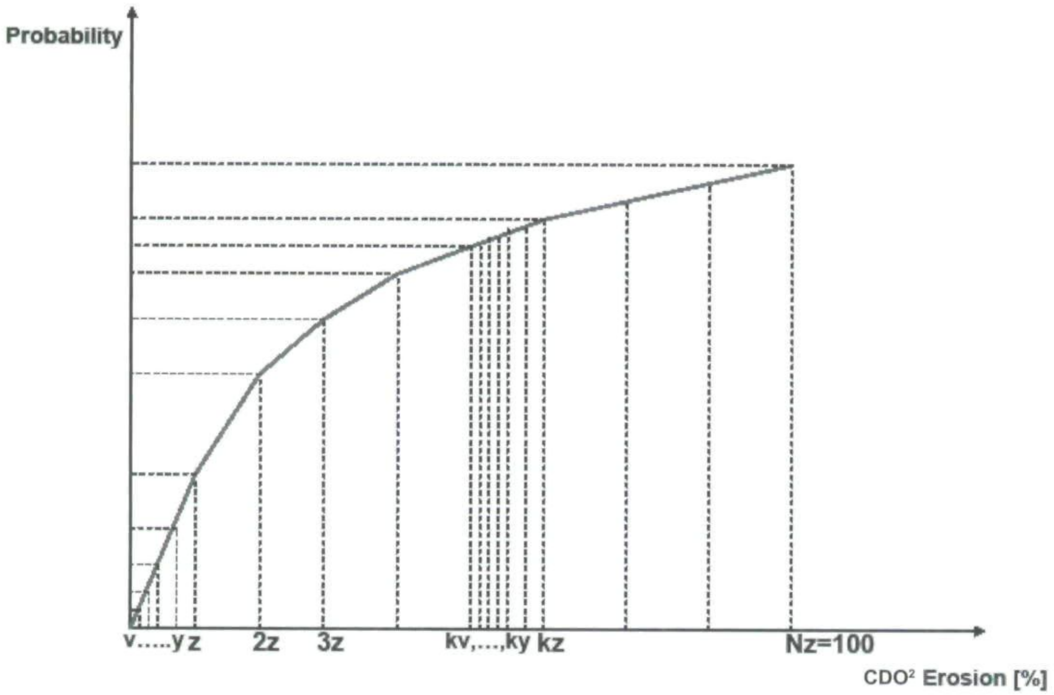
Numerical methods provide us with the corresponding default probability of the multiples of these variables on the CDO^2 portfolio.

Now if we wanted to draw a graph of the overall CDO^2 loss distribution we would dispose of more precise information:

The exact probability amounts can be related to the following erosion scenarios:

$$v\% < w\% < x\% < y\% < z\% < 2v\% < 2w\% < 2x\% < 2y\% < 2z\% < \cdots$$
$$< kv\% < kw\% < kx\% < ky\% < kz\% < \ldots < nv\% < nw\% < nx\% < ny\% < nz\%$$

EXHIBIT 3
Cumulative Tranche Approach



It becomes obvious that now linear interpolation becomes necessary only on a significantly smaller interval which reduces approximation error in evaluation.

In fact this procedure amounts to the following:

We focus on each individual name default absorbed by one inner tranche in a cumulative manner: one default before time horizon t, then two defaults, three defaults, etc. until five defaults (*loss cap*). For each of these assumptions we set up a second copula to capture cross dependence and calculate an overall unconditional loss distribution. Each individual loss distribution is part of a "puzzle" and they are merged in order to furnish more information about the possible outcomes.

Path dependence as well as loss cap is taken into consideration.

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