Integrating Market and Credit Risk Using a Simplified Frailty Default Correlation Structure

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From its inception, the Value at Risk (VaR) framework was constructed as a market risk assessment system. With a given probability and over a predetermined horizon, VaR estimates the maximum potential changes in the value of a portfolio due to market fluctuations. With the development of credit risk assessment, attempts have been made to incorporate credit risk into the system to better describe the overall portfolio risk, particularly for fixed income portfolios.

It has been well known that market risk correlates with credit risk (e.g., Fridson et al. [1997]). Separating the analysis of market and credit risk could misestimate security and portfolio risk levels. However, in computing the VaR for overall portfolio risk, the general practice is to assess market and credit risk separately. This is mostly caused by the general belief that in portfolio credit risk modeling, estimating the correlation structure is already such a complex task (e.g., Duffie and Garleanu [2001] and Zhou [2001]), not to mention the further complication of combining market risk. Given the number of significant variables and the complexity of their correlation structure, a closed form analytical solution for portfolio VaR is not available. As a result, burdensome recursive numerical procedures are often practiced.

For example, Barnhill and Maxwell [2002] conduct an extensive study. They suggest a simulation approach to assess the portfolio VaR of fixed income securities with correlated risk-free interest rate, interest rate spread, foreign exchange rate risk, and firm-specific credit risk. By simultaneously simulating both the future financial environment in which financial instruments will be valued, and the credit rating of specific firms, a portfolio VaR can be obtained with market volatility correlated with credit risk. However, the scale of simulation could incur tremendous costs. Therefore, simpler models, such as the CSFP CreditRisk$^+$ model [1997], or related modified versions that assume only default or non-default scenarios, are used perhaps due to their cost-effectiveness.

The other approach is to propose integrated models. Kijima and Muromachi [2000] integrate interest rate risk into an intensity-based credit portfolio model. The risk-free short rate and the intensity are modeled as correlated extended Vasicek process. Jarrow and Yildirim [2002] propose a model for integrating market and credit risk of a fixed-income security. The market risk comes from the exposure to dynamics of short rate, while the credit risk is identified in a Poisson process in which the default arrives at some intensity. The two sources of risks are then related by a linear approximation. This modeling, when applied to the computation of portfolio VaR, requires estimations of correlation parameters for individual bonds, as well as credit risk correlation parameters for all portfolio components. This implies the estimation task could become very technically intensive. Kiesel, Perraudin, and Taylor [2003] investigate the consequences of adding rating-specific credit spread risk to defaultable bond portfolios. The risk-free interest rate is assumed constant. Grundke [2005] extends the credit risk model underlying the Internal Ratings-Based (IRB) approach proposed by the Basel Committee on Banking Supervision to large portfolios of defaultable bonds. One drawback of this approach is that there are only two credit states, default and no default. Also, the credit spreads are assumed to be non-stochastic.

This article adopts a simplified approach, as suggested by Lee, Kuo, and Urrutia [2004], to assess the correlation structure of credit risk. The approach could significantly reduce the numbers of estimated parameters in credit risk measurement. Thus it provides a simple way to integrate market risk smoothly that leads to a unified framework for computing fixed-income portfolio VaR. The integrated model is based on the proportional hazard model that is often used for analyzing the effect of covariates on the hazard rate. Chava and Jarrow [2004] and Hillegeist et al. [2004] use it to predict bankruptcy.

Furthermore, based on recent research findings that frailty factors can induce a large estimated increase in default clustering, we also consider frailty variables in our integrated model. Frailty factors are unobservable risk factors, to which many firms could be jointly exposed. The consequence of ignoring frailty factors is that intensity estimation in credit risk model could be biased. In particular, when unobservable interest rates and macro risk factors affect default intensity, the portfolio credit risk could be underestimated.

Fama and Gibbons [1982] use an unobservable variable, representing the rate of inflation anticipated by the bond market, to study the behavior of the real interest rate. Stock and Watson [1996] postulate the existence of an unobservable variable that represents the state of business cycle. A set of observable macroeconomic variables is then assumed to be influenced by the business cycle. Jarrow and Turnbull [2000] point out that a firm's asset values are unobservable and model parameters are unknown. Elizalde [2005] uses frailty macro risk factors to quantify the effects of common factors and to analyze their impacts on risk management models.

Duffie et al. [2006] and Duffie et al. [2007] introduce a dynamic frailty-correlated default model. Common dependence by firms on unobservable time-varying default covariate is shown to cause large changes in conditional mean default rates above and beyond those predicted by observable factors. It is concluded that frailty factors could induce a large estimated increase in default cluster. We base our frailty factor estimation on Duffie et al. [2006].[1]

Using a fixed-income portfolio as illustration, we show that the traditional approach where the correlation of market and credit risk is not considered, or frailty is not accounted for, may underestimate VaR.[2]

The remainder of the article is organized as follows. The second section presents the model for integrating market and credit risk. The third section makes the comparison of VaR computed by the new approach with VaR by the traditional approach for a fixed-income portfolio. The final section concludes the article.

AN INTEGRATED MARKET AND CREDIT RISK MODEL WITH FRAILTY VARIABLES

This section illustrates how to apply proportional hazard model with frailty variables to compute credit and market risk in a unified VaR framework. The computation procedure is divided into three steps. First, we build a proportional hazard model with some common and firm-specific factors. Second, we illustrate how to use maximum likelihood estimator (MLE) method and grouping method to estimate coefficients. Third, a portfolio loss function is derived based on the new model specification. With the derived loss function, the appropriate portfolio value is defined. Finally, VaR is computed by using the exposed fixed-income portfolio value information.

Deriving Proportional Hazard Model with Frailty Factor

This sub-section illustrates how to apply a proportional hazard model with frailty variables to compute hazard rates. According to Das et al. [2007], using a traditional proportional hazard assumption, we take intensity λ_{it} for every firm the log-linear form:[3]

$$\begin{aligned}\lambda_{it} &= \lambda_{i0}\exp\left[\beta_{i0} + \beta_{i1}w_{1t} + \beta_{i2}w_{2t} + \sum_{j\neq i}^{n} q_j D_{jt}\right] \\ &= \lambda_{i0}\exp\left[\theta_i^T \mathbf{X}_{it}\right]\end{aligned} \tag{1}$$

where w_{1t} is the U.S. three-month Treasury bill rate, w_{2t} the trailing one-year return of the S&P 500 stock index,

λ_{i0} the baseline hazard function, and β_{i0}, q_j parameters to be estimated. Denote D_{it} the default indicator of firm i, $\theta_i^T = [\beta_{i0}\ \beta_{i1}\ \beta_{i2}\ q_{i1} \cdots q_{i,i-1}\ q_{i,i+1} \cdots q_{i,n}]$, $X_{it}^T = [1\ w_{i1}\ w_{i2}\ D_{1t} \cdots D_{i-1,t}\ D_{i+1,t} \cdots D_{nt}]$. Thus, $\ln(\lambda_{it}/\lambda_{i0}) = \theta_i^T X_{it}$. The prices of nine zero-coupon bonds with standard maturities are usually chosen as market variables: 1 month, 3 month, 6 month, 1 years, 2 years, 5 years, 7 years, 10 years, 30 years (see Hull [2006]). They are the driving common risk factors generating credit risk.[4] In this case, in addition to the U.S. three-month Treasury bill rate w_{1t}, w_{it}, i = 3 to 10 could denote other interest rate market variables.

By partial differentiation, we get $\partial\lambda_{it}/\partial\theta_i = \lambda_{it}X_{it}$. The estimation of the unknown coefficients θ_i is usually conducted by a full maximum likelihood procedure, as detailed in Kavvathas [2000]. Following Duffie et al. [2006], the primary likelihood equation of the model is

$$L_i = \exp\left(-\sum_{s=1}^{t}\lambda_{is}\Delta s\right)\prod_{s=1}^{t}\lambda_{is}^{D_{is}} \tag{2}$$

where Δs is the unit time, setting Δs = 1yr. Thus,

$$\ln L_i = -\sum_{s=1}^{t}\lambda_{is} + \sum_{s=1}^{t}D_{is}\left(\ln\lambda_{i0} + \theta_i^T X_{is}\right) \tag{3}$$

Assuming $L = \Pi_{i=1}^{n} L_i$, the log-likelihood can be written as

$$\ln L = \sum_{i=1}^{n}\ln L_i = \sum_{i=1}^{n}\left(-\sum_{s=1}^{t}\lambda_{is}\right) + \sum_{i=1}^{n}\left[\sum_{s=1}^{t}D_{is}\left(\ln\lambda_{i0} + \theta_i^T X_{is}\right)\right] \tag{4}$$

However, if the number of firms in the portfolio is large, such as with CDO products, the estimation would be a problem. Similar to Lee, Kuo, and Urrutia [2004], we group firms with the same credit ratings. Thus,

$$\ln L = \sum_{k=1}^{K}\left(-N_k\sum_{s=1}^{t}\lambda_{ks}\right) + \sum_{k=1}^{K}\left\{\sum_{s=1}^{t}\left[N_k\left(\ln\lambda_{k0} + \theta_k^T X_{ks}\right)\sum_{i\in G_k}D_{is}\right]\right\} \tag{5}$$

where k is the index number of groups, K the number of groups, N_k the number of firms in the k^{th} group, and G_k is a group k for firms with same ratings. Hence, the likelihood equations are

$$\partial\ln L/\partial\theta_k = -N_k\sum_{s=1}^{t}\lambda_{ks}X_{ks} + \sum_{s=1}^{t}\left(X_{ks}\sum_{i\in G_k}D_{is}\right) \tag{6}$$

The Hessian is $\frac{\partial^2 \ln L}{\partial\theta_k\partial\theta_k^T} = -N_k\Sigma_{s=1}^{t}\lambda_{ks}X_{ks}^T X_{ks}$, negative definite for all X_{ks} and θ_k. From Equation (6) for every t, we get $\hat{\lambda}_{kt} = \Sigma_{i\in G_k} D_{it}/N_k$, thus $\hat{\lambda}_{it} = \hat{\lambda}_{kt}$ for $i \in G_k$. We then use $\hat{\lambda}_{it} = \lambda_{i0}\exp[\hat{\theta}_i^T X_{it}]$ to calculate $\hat{\theta}_i$, denoting the estimator of θ_i. In the following, we consider frailty factors to extend this model.

When frailty factors are considered, the estimation would be more complicated. One simple way to deal with this matter is to extend the basic model, by introducing frailty factors $Y_t = [y_{1t}\ y_{2t}\ y_{3t} \cdots \cdots y_{Mt}]$, following Duffie et al. [2006]. To be specific, the log-linear model of the intensity λ_{it} for each firm i is:

$$\lambda_{it} = \lambda_{i0}\exp\left[\theta_i^T X_{it}\right]\exp\left[\eta_{i1}y_{1t} + \eta_{i2}y_{2t} + \ldots + \eta_{iM}y_{Mt}\right] \tag{7}$$

where η_{im} is the coefficient. To be more specific, the conditional likelihood is the product of n firm specific conditional likelihoods:

$$L^F = \prod_{i=1}^{n} L_i^F(\theta_i, \eta_i \mid X_{is}, Y_s, s = 1 \sim t) \tag{8}$$

where L_i^F is the conditional likelihood function for firm i. For simplicity, we assume only one frailty factor, i.e., M = 1. Thus, $\ln\lambda_{it} = \ln\lambda_{i0} + \theta_i^T X_{it} + \eta_i Y_t$.

By nature, frailty variables cannot be observed. Therefore, an expectation and maximization (EM) algorithm is used for the computation of the MLE parameters in models involving missing or incomplete data.[5]

Deriving the Loss Function

After obtaining MLE hazard rate estimation for each ratings group, we can further derive the loss function for each group portfolio, and for the whole portfolio by adding up all group loss functions.

Assigning each firm within a group k an equal notional amount M_k, the total notional amount in each group is $M_k N_k$. With the assumption of constant recovery rate, loss function variable S, or the dollar default loss for the total portfolio before debt maturity, can be written as:

$$S = \sum_{k=1}^{K} S_k \tag{9}$$

where $S_k \sim Poisson(\lambda_{kt})$ is the loss function for group k. Therefore, loss function of each group is correlated by common shock in the proportional hazard model to describe multi-asset default correlation. The common shock refers to the link of individual companies via industry-specific and/or general economic conditions. Without loss of generality, in our model it is assumed that all components of the portfolio are subjected to common shocks of interest rate level w_1, stock market volatility w_2, and frailty factor y_t. As interest rate fluctuation is generally considered the major source of market risk, market and credit risk are thus integrated.

It has been shown that this approach can significantly reduce the number of parameters to be estimated, because the intensity model is based on groups of firms with similar default probability rather than on individual firms. Barnhill and Maxwell [2002] undertake complex simulation of bond defaults in a contingent claims analysis framework. Stock returns and volatilities are used to simulate the dynamics for firm values, and default probabilities are calculated. Firm correlated defaults are related through the market index and beta parameters in CAPM. The number of parameters to be estimated is $2n$, where n is the number of all firms. Jarrow and Yildirim [2002] relate market and credit risk by a linear approximation, $\lambda_j = \max[\lambda_{ij} + k_j\, r, 0]$, where λ_j is the default intensity for firm j, λ_{ij} the firm-specific component of default intensity for firm j, r the single market risk factor, k_j the coefficient. Assuming a single common risk factor, the number of parameters to be estimated is also $2n$. In comparison, without grouping, the number of parameters to be estimated is $(n + 3)n$ in our model. Though the number of parameters in their models is less than ours without grouping, they assume independent default conditional on a common factor, the risk-free rate. This assumption is inappropriate as illustrated by Lee, Kuo, and Urrutia. More importantly, the number of parameters in our model after grouping will be reduced by $n^2 - K^2 + 3(n - K)$. Consequently, the number of parameters is $(K + 3)K$. Since n is usually a much larger number than K, the reduction is significant.

Defining the Portfolio Value

Equations (2) ~ (4) are used to estimate the hazard rates and coefficient parameters in the loss function. Following Gregory and Laurent [2003], the discounted loss value can be written as $\int_0^T B(t)dS(t)$, where $B(t)$ stands for the discount factor for maturity t, and $S(t)$ is value of loss function S at time t. The previous stochastic integral turns out to be a discrete sum $\Sigma_{k=1}^{K} B(\tau_k)[S(\tau_k^+) - S(\tau_k)]$, where τ_i is the default time of name i. It is in turn used to compute the fixed-income portfolio value P_t, as defined by:

$$P_t = \sum_{k=1}^{K} M_k N_k - E\left(\sum_{k=1}^{K} B(\tau_k)[S(\tau_k^+) - S(\tau_k)] \right) \tag{10}$$

The expectation is obtained by simulation. The risk-free short rate is modeled as a mean-reverting Ornstein-Uhlenbeck process introduced by Vasicek [1977]: $dr(t) = \kappa(\theta - r(t))dt + \sigma dW_r(t)$, where κ, θ, $\sigma \in R^+$ are constants and $W_r(t)$ is a standard Brownian motion under P measure. This term structure model is chosen here for the sake of simplicity, but the qualitative results of this article should not change using a different term structure model.

Measuring VaR

In our article, VaR is defined as losses from market and credit risk. Market risk, referring to interest rate risk, following the naming convention by the fixed-income industry, is calculated as treating the portfolio as default-free so that future payments of the nominal value are discounted with stochastic risk-free rate. Credit risk, similar to Grundke [2005] where only default risk is considered, is calculated by discounting the cash flows with risk-adjusted interest rates of respective rating grade. Combining stochastic models describing risk, we will have a simple and consistent VaR model that integrates market and credit risk. In the next section, we will compare VaRs calculating by both the traditional and integrated models. The main difference is that credit risk is independent of market risk in the traditional model, while credit risk is correlated with market risk in the integrated model.

In the following, we illustrate how to calculate VaR. Firstly, we calculate market risk VaR_M. In computing fixed-income portfolio VaR, the first derivative of the portfolio value, its duration, is usually used. The potential loss in value P_t is computed as $dP_t = \frac{\partial P_t}{\partial r} dr$, which involves the potential change due to change in interest rate *dr*. Following the practice of the delta-normal method, the portfolio VaR_M can be represented by:

$$\text{VaR}_M = \alpha \left| \frac{\partial P_t}{\partial r} \right| \sigma_r \tag{11}$$

where α is the standard normal deviate corresponding to the specified confidence level, e.g., 1.645 for a 95% confidence level, and σ_r is the volatility of interest rates.

Secondly, credit risk is calculated. If the risk-free rate dynamics is assumed as above, the hazard rate would follow an exponential stochastic process. Thus, the loss function would also be stochastic in nature, resulting in credit risk exposure. It is thus possible to estimate VaR using price changes of credit risk products that are driven by changes in hazard rates. Therefore, credit risk VaR_C in our traditional model is defined as the value change due to change in hazard rate, which is independent of interest rate:

$$VaR_C = \alpha \left| \frac{\partial P_t}{\partial \lambda} \right| \sigma_\lambda \tag{12}$$

where σ_λ is volatility of hazard rate. In the integrated model, hazard rate based on the proportional hazard model is affected by interest rate. Therefore, credit risk using the integrated model can be represented by:

$$\text{VaR}_c^* = \alpha \left| \frac{\partial P_t}{\partial \lambda} \right| \sigma_\lambda^* \tag{13}$$

where σ_λ^* is volatility of hazard rate, considering effect of interest rate volatility. That is, the traditional VaR calculating credit risk measure uses Equation (12), in contrast to an integrated approach where hazard rate volatility is assumed related to interest rate volatility.

By adding up market and credit risk, an integrated VaR is obtained. In an integrated approach, risk-free rate volatility is not only a market risk factor, but also a credit risk factor because hazard rate is related to interest rate.

In the following section, we will use a fixed income portfolio as an example to illustrate how the new approach compares with the traditional approach where credit risk is considered separately.

A COMPARISON STUDY

Assuming the usual mapping process has been completed, and ignoring coupon and maturity effects, the bond portfolio constructed comprises issuers categorized into five rating classes, i.e., $K = 5$. The five rating classes are Aaa, Aa, A, Baa, and Ba, tracked by Moody's over the period 1982–2005. Bonds with Baa or better ratings are generally known as investment grade bonds. Ba refers to the low credit quality. The portfolio with $500 notional amount is equally allocated to each rating class category. Exhibit 1 shows summary statistics of probability of default (PD) samples for each rating class over the period 1982–2005. For instance, in column Ba, "Mean" 13.172% denotes average default ratio, which is the default number of firms with rating Ba in five years divided by the number of all Ba rating firms in five years, from 20 observations.

Parameter Values

The first step to perform our portfolio risk analysis is to estimate parameters in the proportional hazard model, with and without frailty. We choose firm numbers for the five groups N_1, N_2, N_3, N_4, N_5 to be all 100. Then, using the MLE method, the estimates of model parameters are calculated as shown in Exhibit 2 and Exhibit 3. All bonds are assumed to have a five-year maturity. The portfolio is subject to the impact of common interest rate and stock market volatility risk factor, idiosyncratic default risk specific to other obligors, and frailty factor, where risk-free interest rate is the direct market risk factor for the portfolio.

Model Comparison

To check the validity of our approach, we compute market and credit risk using historical data. Group loss function variables S_k are obtained by simulation with the hazard rate λ_{kt}. Thus, we have the simulated distribution and expectation of S_k. We then examine the differences between our new approach and the traditional method separating market and credit risk. The VaR values are shown in Exhibit 4 and Exhibit 5.

EXHIBIT 1

Summary Statistics of Common Factor and PD Samples over the Period 1982–2005

3m T-bill, Trailing S&P, and Ratings Aaa denote variables three-month treasury bill rate, volatility of trailing 1-year stock return on S&P500 index, and ratings Aaa firms default rate, respectively.

Common factor	Risk-free rate	Stock return volatility
Mean	6.22450	0.98101
Median	5.81000	0.91414
Maximum	11.09000	2.12501
Minimum	3.07000	0.49127
Std. Dev.	2.17252	0.38153
Skewness	0.61438	1.26260
Kurtosis	2.65282	5.02967
Jarque-Bera	1.35867	8.74681
Probability	0.50695	0.01261
Observations	20	20

Ratings	Aaa	Aa	A	Baa	Ba
Mean	0.19120	0.32315	0.69055	2.34705	13.17235
Median	0.00000	0.00000	0.52300	2.76900	12.18000
Maximum	2.39500	1.79700	2.53000	5.36000	24.07500
Minimum	0.00000	0.00000	0.00000	0.00000	2.38200
Std. Dev.	0.60901	0.48717	0.74675	1.50929	7.12812
Skewness	2.98209	1.71130	0.95351	0.06229	0.13582
Kurtosis	10.45506	5.41034	2.92618	2.20760	1.68684
Jarque-Bera	75.95769	14.60330	3.03517	0.53619	1.49849
Probability	0.00000	0.00067	0.21924	0.76484	0.47272
Observations	20	20	20	20	20

Results

Comparing Exhibit 4 and Exhibit 5, it can be seen that frailty is not a factor to be ignored. In addition, for the portfolio constructed, Exhibit 5 shows that the traditional approach where credit risk is considered separately underestimates VaR. In particular, the 95% VaR using traditional approach where credit risk is not considered with market risk is \$194.94, versus \$208.92 using the integrated one. For the Ba-rated bonds, the 95% VaR using traditional approach is \$84.87, versus \$98.42 using the integrated one. The VaR difference percentages are 6.69% and 13.76%, for the portfolio and Ba-rated bonds, respectively. For the investment grade bonds, the biases are generally low.

The bias for the overall portfolio is lower than the Ba-rated bonds, implying there is portfolio effect. However, it should be noted that the portfolio effect can be complicated. For example, when interest rate rises, there may be potential losses due to market risk exposure. But the high interest rate may result from booming economic growth, thus implying lower firm default probability. Therefore, when the correlation of market and credit risk is not obvious, the underestimation may improve because the effect of interest rate volatility may diminish.

CONCLUSION

Without a reliable overall portfolio risk measure, it is difficult to determine adequate capital and hedging

EXHIBIT 2

MLE of the Intensity Parameters in the Model without Frailty

Maximum likelihood estimates of the intensity parameters in the model without frailty. Coefficients of variables in the model are shown in each ratings category, followed by standard errors, given in parentheses. These variables are three month T-bill rate, S&P stock index return, and "Aa d.p." denoting default probability of ratings Aa group. "Constant" is the constant term in the regression model.

Aaa Ratings Group			
Constant	**3m T-bill**	**Trailing S&P**	**R_square**
-8.258	0.708	116.341	0.713
(1.432)	(0.224)	(123.055)	
Aa d.p.	**A d.p.**	**Baa d.p.**	**Ba d.p.**
4.280	-1.985	0.541	-0.216
(1.061)	(0.771)	(0.365)	(0.089)
Aa Ratings Group			
Constant	**3m T-bill**	**Trailing S&P**	**R_square**
-0.999	-0.750	-169.307	0.675
(2.027)	(0.365)	(156.733)	
Aaa d.p.	**A d.p.**	**Baa d.p.**	**Ba d.p.**
2.607	2.078	-0.672	0.341
(0.960)	(0.946)	(0.540)	(0.123)
A Ratings Group			
Constant	**3m T-bill**	**Trailing S&P**	**R_square**
-7.220	0.424	145.809	0.810
(1.341)	(0.241)	(110.384)	
Aaa d.p.	**Aa d.p.**	**Baa d.p.**	**Ba d.p.**
-0.844	1.716	1.214	-0.118
(0.667)	(0.905)	(0.309)	(0.091)
Baa Ratings Group			
Constant	**3m T-bill**	**Trailing S&P**	**R_square**
-2.328	-0.008	49.741	0.560
(1.206)	(0.203)	(90.575)	
Aaa d.p.	**Aa d.p.**	**A d.p.**	**Baa d.p.**
0.790	-1.471	1.057	0.078
(0.605)	(0.988)	(0.591)	(0.070)
Ba Ratings Group			
Constant	**3m T-bill**	**Trailing S&P**	**R_square**
-2.065	0.195	27.324	0.777
(0.379)	(0.051)	(30.878)	
Aaa d.p.	**Aa d.p.**	**A d.p.**	**Baa d.p.**
-0.309	0.403	-0.148	0.185
(0.205)	(0.314)	(0.241)	(0.098)

Exhibit 3

MLE of the Intensity Parameters in the Model with Frailty

Maximum likelihood estimates of the intensity parameters in the model with frailty. Coefficients of variables in the model are shown in each ratings category, followed by standard errors, given in parentheses. These variables are three month T-bill rate, S&P stock index return, latent factor, and "Aa d.p." denoting default probability of ratings Aa group. "Constant" is the constant term in the regression model.

		Aaa Ratings Group		
Constant	**3m T-bill**	**Trailing S&P**	**Latent Factor**	**R_square**
-8.301	0.672	126.648	0.166	0.960
(0.553)	(0.087)	(47.542)	(0.019)	
Aa d.p.	**A d.p.**	**Baa d.p.**	**Ba d.p.**	
4.435	-2.142	0.479	(0.188)	
(0.410)	(0.298)	(0.141)	(0.035)	
		Aa Ratings Group		
Constant	**3m T-bill**	**Trailing S&P**	**Latent Factor**	**R_square**
0.301	-1.136	-102.479	-0.237	0.853
(1.462)	(0.275)	(111.268)	(0.062)	
Aaa d.p.	**A d.p.**	**Baa d.p.**	**Ba d.p.**	
4.740	2.745	-1.019	0.371	
(0.876)	(0.686)	(0.389)	(0.087)	
		A Ratings Group		
Constant	**3m T-bill**	**Trailing S&P**	**Latent Factor**	**R_square**
-8.091	0.666	156.317	0.139	0.888
(1.116)	(0.210)	(88.525)	(0.048)	
Aaa d.p.	**Aa d.p.**	**Baa d.p.**	**Ba d.p.**	
-2.235	2.517	1.284	-0.185	
(0.721)	(0.777)	(0.249)	(0.076)	
		Baa Ratings Group		
Constant	**3m T-bill**	**Trailing S&P**	**Latent Factor**	**R_square**
-2.572	0.039	62.420	0.023	0.565
(1.416)	(0.247)	(99.993)	(0.063)	
Aaa d.p.	**Aa d.p.**	**A d.p.**	**Baa d.p.**	
0.505	-1.182	0.912	0.069	
(1.002)	(1.293)	(0.730)	(0.077)	
		Ba Ratings Group		
Constant	**3m T-bill**	**Trailing S&P**	**Latent Factor**	**R_square**
-2.505	0.268	29.161	0.052	0.881
(0.319)	(0.045)	(23.481)	(0.016)	
Aaa d.p.	**Aa d.p.**	**A d.p.**	**Baa d.p.**	
-0.973	1.009	-0.626	0.283	
(0.258)	(0.303)	(0.236)	(0.080)	

EXHIBIT 4

Delta-Normal versus Integrated VaR Analysis for Bond Portfolios Using Model without Frailty

1. This Exhibit shows the VaR values for the bond portfolio with $100 in each ratings bonds at a one-year time step facing market and credit risk.
2. The column "Market Risk" denotes market risk VaR_M, "Credit Risk" the credit risk VaR_C not considering relation with market risk, "Traditional Model" the VaR calculated by the traditional model, "I-Model" the VaR calculated by the integrated model.

	Market Risk	Credit Risk	Traditional Model	I-Model
Rating Aaa	16.49	8.44	24.92	24.92
Bias			0.003%	0.003%
Rating Aa	16.48	6.56	23.04	23.12
Bias			-0.193%	0.153%
Rating A	16.46	10.05	26.50	26.63
Bias			-1.121%	-0.663%
Rating Baa	15.89	19.83	35.72	35.72
Bias			0.064%	0.075%
Rating Ba	11.85	76.11	87.96	95.94
Bias			-10.628%	-2.515%
Portfolio	77.15	120.98	198.14	206.33
Bias			-5.160%	-1.240%

EXHIBIT 5

Delta-Normal versus Integrated VaR Analysis for Bond Portfolios Using Model with Frailty

1. This Exhibit shows VaR values for the bond portfolio with $100 in each ratings at a one-year time step facing market and credit risk.
2. The column denotations are as in Exhibit 4.

	Market Risk	Credit Risk	Traditional Model	I-Model
Rating Aaa	16.485	8.44	24.92	24.92
Bias			-0.000%	
Rating Aa	16.481	6.56	23.04	23.08
Bias			-0.171%	
Rating A	16.442	10.04	26.48	26.80
Bias			-1.197%	
Rating Baa	15.839	19.78	35.62	35.69
Bias			-0.211%	
Rating Ba	11.304	73.57	84.87	98.42
Bias			-13.761%	
Portfolio	76.55	118.39	194.94	208.92
Bias			-6.691%	

strategies. This article shows that the complex correlation structure involved in modeling overall portfolio risk measure can be greatly simplified by grouping firms with the same quality and using a proportional hazard model with frailty accounting for common shocks. Both market and credit risk can be treated in a unified framework and the portfolio VaR can be computed in a more consistent manner.

An empirical example is provided to confirm the validity of the new approach by comparing with the VaR using traditional method. The strength of our new approach can be seen when market risk correlates more with credit risk.

ENDNOTES

[1]Correlation in our model is represented by coefficient of common factor. Due to the large number of correlation parameters, we group firms with the same credit ratings. This method is also used in Lee, Kuo, and Urrutia [2004].

[2]The traditional approach is thus defined because the Basel Committee on Bank Supervision requires capital for market and credit risk separately.

[3]Some proportional hazard models do not have a constant term because the baseline hazard can be regarded as an individual specific constant. Thus, baseline hazard is $\lambda_{i0} \exp(\beta_{i0})$ in their model. The associated time durations are assumed independent, conditional on an exogenous macro-economic process.

[4]The aspect of risk before the default happens is generally considered to be market risk. The actual default risk is considered credit risk (see Marrison [2002]).

[5]Dempster, Laird, and Rubin [1977] were the first to provide a comprehensive treatment of the EM method.

REFERENCES

Barnhill, T., and W. Maxwell. "Modeling Correlated Market and Credit Risk in Fixed Income Portfolios." *Journal of Banking and Finance*, 26 (2002), pp. 347–374.

Chava, S., and R. Jarrow. "Bankruptcy Prediction with Industry Effects." *Review of Finance*, 8 (2004), pp. 537–569.

CreditRisk+: A Credit Risk Management Framework. Credit Suisse Financial Products, 1997.

Das, S., D. Duffie, N. Kapadia, and L. Saita. "Common Failings: How Corporate Defaults are Correlated." *Journal of Finance* (2007), forthcoming.

Dempster, A.P., N.M. Laird, and D.B. Rubin. "Maximum Likelihood from Incomplete Data via the EM Algorithm." *Journal of the Royal Statistical Society* B, 39 (1977), pp. 1–38.

Duffie, D., A. Eckner, G. Horel, and L. Saita. "Frailty Correlated Default." Working paper, Stanford University, 2006.

Duffie, D., L. Saita, and K. Wang. "Multi-Period Corporate Default Prediction with Stochastic Covariates." *Journal of Financial Economics*, 83 (2007), pp. 635–665.

Duffie, D., and N. Garleanu. "Risk and Valuation of Collateralized Debt Obligations." *Financial Analysts Journal*, 57 (2001), pp. 41–59.

Elizalde, A. "Do We Need to Worry about Credit Risk Correlation?" *The Journal of Fixed Income*, 15 (2005), pp. 42–59.

Fama, E.F., and M.R. Gibbons. "Inflation, Real Returns, and Capital Investment." *Journal of Monetary Economics*, 9 (1982), pp. 297–324.

Fridson, M., C. Garman, and S. Wu. "Real Interest Rates and the Default Rates on High-Yield Bonds." *The Journal of Fixed Income*, 7 (1997), pp. 27–34.

Gregory, J., and J.-P. Laurent. "I Will Survive." *Risk* (June 2003), pp. 103–107.

Grundke, P. "Risk Measurement with Integrated Market and Credit Portfolio Models." *Journal of Risk*, 7 (2005), pp. 63–94.

Hillegeist, S.A., E.K. Keating, D.P. Cram, and K.G. Lundstedt. "Assessing the Probability of Bankruptcy." *Review of Accounting Studies*, 9 (2004), pp. 5–34.

Hull, J.C. *Options, Futures, and Other Derivatives*. Prentice Hall, 2006.

Jarrow, R.A. and S.M. Turnbull. "The Intersection of Market and Credit Risk." *Journal of Banking and Finance*, 24 (2000), pp. 271–299.

Jarrow, R. and Y. Yildirim. "Valuing Default Swaps under Market and Credit Risk Correlation." *The Journal of Fixed Income*, 11 (2002), pp. 7–19.

Kavvathas, D. "Estimating Credit Rating Transition Probabilities for Corporate Bonds." Working paper, University of Chicago, 2000.

Kiesel, R., W. Perraudin, and A. Taylor. "The Structure of Credit Risk: Spread Volatility and Ratings Transitions." *Journal of Risk*, 6 (2003), pp. 1–27.

Kijima, M., and Y. Muromachi. "Evaluation of Credit Risk of a Portfolio with Stochastic Interest Rate and Default Processes." *Journal of Risk*, 3 (2000), pp. 5–36.

Lee, C.-W., C.-K. Kuo, and J. Urrutia. "A Poisson Model with Common Shocks for CDO Valuation." *The Journal of Fixed Income*, 14 (2004), pp. 72–81.

Marrison, C. *The Fundamentals of Risk Measurement.* McGraw-Hill, 2002.

Stock, J.H., and M.W. Watson. "Evidence on Structural Instability in Macroeconomic Time Series Relations." *Journal of Business and Economic Statistics*, 14 (1996), pp. 11–30.

Vasicek, O. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, 5 (1977), pp. 177–188.

Zhou, C. "An Analysis of Default Correlations and Multiple Defaults." *Review of Financial Studies*, 14 (2001), pp. 555–576.