A Density-Dependent Model for Credit Ratings Migration Dynamics

DROR PARNES

DROR PARNES
is an assistant professor at Finance Department, College of Business Administration, University of South Florida in Tampa, FL.
dparnes@coba.usf.edu

Credit ratings are often assigned by rating agencies to public debt at the time of issuance, and then are periodically revised, typically once every quarter. A change in the credit rating reflects improvement or deterioration in the credit quality, from the rating agency's perspective. Although credit ratings are fundamentally meant to shed light on the current credit quality of a firm, several theories use the rating drift phenomenon to predict default events, and to price defaultable bonds. This article proposes a different approach for simulating ratings migration, by correlating the transition rates of credit ratings to a significant economic parameter: the market density.

The intuition behind this association is the fact that credit events often cluster, as recognized by CreditMetrics™. One of the important issues raised in this document is the evidence of joint credit-rating correlations that can be examined from data of default histories. The underlying hypothesis suggests that if defaults were uncorrelated, then default rates would have been uniformly distributed along the years, but this is not the case in reality. This outcome can be interpreted in other ways as well – for example, default correlations not to one another, but to a time-varied macroeconomic condition such as the market density.

There are times when defaults are scarce, perhaps due to a low market density, where surviving firms can supply a fixed customer's demand, and there are periods of rising density in the market, where a "competitive effect" leads to insufficient market share to some firms, and thus forces countless defaults.

Lang and Stulz [2002] discover some evidence for this competitive effect, and Jorion and Zhang [2006] further detect a strong competition effect for Chapter 7 bankruptcies. The latter demonstrate this hypothesis of negative default correlations through a prominent example, where Bethlehem Steel gained from the late-2000 bankruptcy filing of its major opponent, LTV Corporation.

This study recommends utilizing the density-dependent model as a natural alternative to existing ratings migration schemes. It is an explicit model of conditional transition probabilities. Its main advantages are ease of use and an improved ability to track observed default patterns. The proposed model statistically and economically dominates the homogeneous Markov chain process, the stochastic business-cycles notion, and the momentum technique in describing empirically examined ratings transitions.

The credit ratings migration methodology generally assumes three things: that rating agencies know how to tag the right rating label; that all firms tagged within a given rating label share the same default risk; and that historical sample data on credit rating migration has predictive strength regarding the future.

These assumptions are presumed to hold throughout this study.

The remainder of this article is structured as follows. We first review existing credit rating models and related empirical investigations. Next we outline the theoretical framework of the density-dependent model. Then we present the data sample, followed by an explanation of the research methodologies, and contrast the proposed model with several existing schemes. Finally we conclude and point to possible future lines of research.

LITERATURE REVIEW ON CREDIT RATINGS MIGRATION ANALYSIS

Numerous studies explore credit ratings migration using the Markov chain process. Jarrow, Lando, and Turnbull [1997] propose a discrete-space finite-state time-homogeneous Markov chain of credit ratings to describe the term structure of credit risk spreads. The authors price risky bonds by using an absorbing time-homogeneity transition matrix, and by assuming that the risk-free spot interest rate, and the firm's credit ratings migration, follow uncorrelated stochastic processes.

Kijima and Komoribayashi [1998] point to possible numerical problems that arise due to low default probabilities, for highly rated bonds, within relatively short periods, and offer different risk premium adjustments than those suggested by Jarrow, Lando, and Turnbull [1997]. The authors propose to consider transition probabilities as volatile functions of time rather than assuming homogeneity for all transition matrix entries.

Kijima [1998] discusses how prior ratings changes may help to predict future rating changes over time, conditional on a firm's survival. Using the first-order stochastic dominance in the Markov chain dynamic, the author explains how firms with low credit ratings are more likely to be upgraded, while firms with high credit ratings are more likely to be downgraded, as the time horizon lengthens, given that they do not default.

Arvanitis, Gregory, and Laurent [1999] contribute to the pricing of risky bonds and bond options, allowing memory in credit rating changes, stochastic jumps, and mean reversion in credit spreads. The authors consider a two-state world of default and non-default, a deterministic credit ratings migration matrix, along with the restriction for credit spread independent of the interest rates, but permit sudden jumps, memory, and mean-reverting diffusion processes in credit rating changes.

Thomas, Allen, and Morkel-Kingsbury [1999] propose a Markov chain model for the term structure and credit spreads of risky bonds, allowing dependency between two stochastic processes, interest rate movements, and credit ratings migrations. This establishes a link between credit rating movements and the state of the economy.

Bielecki and Rutkowski [2000] derive stochastic processes for pricing defaultable bonds and other credit derivatives from rating transition matrices. The authors model the intensities of credit migrations among various credit ratings classes by combining available data for credit spreads and recovery rates, and conditioning the martingale probabilities from the Markov chain process on interest rate risk.

Lucas, Klaassen, Spreij, and Straetmans [2001] derive an analytical approximation to the credit loss distribution of large portfolios by incorporating the Markov chain approach for credit ratings migration, along with several characteristics, such as credit quality, the degree of systematic risk, and the maturity profile. Assuming that rating transitions are normally distributed, the authors find that the rate of convergence of the actual credit loss quantiles to their analytic counterparts is influenced by the degree of portfolio heterogeneity.

Nevertheless, among others, Altman and Kao [1992a]; Altman [1998]; Lando and Skødeberg [2002], and Frydman and Schuermann [2006] collect overwhelming evidence opposing the homogeneity assumption underlying the standard Markov chain in describing credit rating dynamics.

The literature discusses a different approach of correlating credit ratings migration to stochastic business-cycles. Nickell, Perraudin, and Varotto [2000] quantify the dependency of rating transition probabilities both on the industry and domicile of the obligor, and on the stage of the business cycle. By defining three categories of business cycles—peak, normal times, and trough, depending on whether real GDP growth is at the top, middle, or bottom third from 1970 to 1997—the authors discover that the business-cycles variability is the most influential in explaining the volatility of transition probabilities. The authors conclude that applying the homogeneous transition matrix over high or low quality loans is likely to under- or overestimate credit risk respectively.

Bangia, Diebold, Kronimus, Schagen, and Schuermann [2002] empirically examine the homogeneous Markov chain model and condition transition matrices on stochastic business cycles. Using the National Bureau of Economic Research (NBER) database to identify certain months as expansionary or contractionary stages, the authors provide strong evidence for the momentum hypothesis, as most downgrade probabilities for the downward-momentum matrix are found to be larger than the corresponding values in the homogeneous matrix. The authors further conclude that distinguishing between expansionary and contractionary stages leads to a dramatic reduction in the level of uncertainty in transition estimations. However, the differences between the stochastic economic model and the unconditional simulation are relatively minor over short-term time horizons, though they gradually increase over time.

Also considering business cycles, Wei [2003] relaxes the assumption of a homogeneous Markov chain, and allows for rating inter-variations within the Markov chain states to price various credit derivatives.

Further attempts have been made to step away from the homogeneity assumption of the Markov chain dynamic. Gupton, Finger, and Bhatia [1997] illustrate a direct estimation of joint credit changes across companies, which are synchronized with each other. In theory, this method avoids explicitly specifying correlation estimation between defaults. Yet, the technique is difficult to implement over a large sample of firms, and becomes almost non-feasible when also considering ratings modifiers over a long time period.[1]

Israel, Rosenthal, and Wei [2001] identify the conditions for a true generator of Markov transition matrices to exist, present several techniques for estimating an approximate generator when the true one is not feasible, and apply those methods to ratings migration matrices. This methodology enjoys an ease of formulation and application. It helps to create different transition matrices through time, but it contains a major drawback: the generator lacks intuition, since it does not correlate the transition matrices to any meaningful economic variable.

Several scholars aim to accommodate for observed momentum. Bahar and Nagpal [2001] propose a non-homogeneous theory for that purpose. Their model enhances the standard Markov model with a partial, short-term memory. Each state describes not only the current rating category, but also the latest transitions. The authors divide the data sample into three sets according to the rating over the previous year: upgraded, stable, or downgraded. This methodology leads to three different transition matrices, one for each set of upgraded, stable, and downgraded companies. In the upgrade transition matrix, ratings migration probabilities are more prominent above the main diagonal, describing a common tendency of improvement in creditworthiness; in the downgrade transition matrix, ratings migration probabilities are more pronounced below the main diagonal, suggesting a general propensity of deterioration in creditworthiness.

Christensen, Hansen, and Lando [2004] present a continuous-time hidden Markov Model that accounts for the tendency of recent downgrades to continue deteriorating, with special attention toward rare events. The authors rely on the intuition that a typical path of default goes through a sequence of downgrades, and use the bootstrap methodology to obtain an interval of probabilities of defaults from credit ratings.

Giampieri, Davis, and Crowder [2005] further use a general Hidden Markov Model (HMM) along with an algorithm developed by Baum and Welch for signal processing applications to mimic observed momentum in a sequence of default events.

THE DENSITY-DEPENDENT MODEL

This section proposes an alternative approach to simulate ratings migration through a correlation of a firm's survivability and rating transitions to an insightful economic parameter: the market density.

In a competitive market, rating migrations may be correlated through another mechanism. When resources are limited by a finite market share, or a restricted number of potential customers, the credit rating migration of one company affects the others. A default of one company leaves more space for the others to thrive. An economic improvement for a different company narrows the market for the others, forcing them to reduce in size. When some firms have already defaulted, more market share is allocated to the survivors, allowing them to thrive. With fewer surviving companies supplying the fixed customers' demand, the survivability of those remaining companies is likely to improve. Market density may affect companies' survivability as well as their ratings transition probabilities.

The density-dependent model assumes a finite number of competitive companies and limited resources, customers, and market share.[2] It also assumes that a large cumulative default rate triggers an increase in the survivability of the remaining firms. In addition, it is assumed that a larger cumulative default rate triggers slower transition rates toward the absorption state of default. The density-dependent model considers the following definitions:

σ_i = survival probability within the next quarter for a company in state i at zero cumulative default rate (the starting point of the simulation)
γ_{ji} = unconditional probability to migrate from credit rating i to credit rating j at zero cumulative default rate (the starting point of the simulation)
s_{ji} = conditional transition probability
CDR = Cumulative Default Rate, the percentage of firms already defaulted as a measure of market density
α_i = coefficients of density-dependent survivability for companies in state i
β_i = coefficients of density-dependent transitivity for companies from state i

Without correlating the survivability and transitivity of credit ratings to the cumulative default rate, the model would have taken the following form:

$$s_{ii} = \sigma_i\left(1-\sum_{j\neq i}\gamma_{ji}\right)\ \forall i \tag{1}$$

$$s_{ji} = \sigma_i\gamma_{ji}\ \ \forall j \neq i \tag{2}$$

These transition probabilities would depend only on the survivability of the firms, and the odds to migrate between credit categories regardless of the competition effect. However, after correlating the survivability and transitivity of credit ratings to the cumulative default rate, the model considers the following dependencies:

$$\gamma_{ji}(CDR) = \begin{cases} \gamma_{ji}\exp(\beta_1 CDR) & \text{if } j < i \\ \gamma_{ji}\exp(\beta_2 CDR) & \text{if } j > i \end{cases} \tag{3}$$

$$\sigma_i(CDR) = \sigma_i \exp(\alpha_i CDR) \tag{4}$$

These conditional probabilities also account for the competition effect in the form of CDR. The cumulative default rate always lags one quarter behind the conditional transition probability. The conditions $\beta_1 > 0$, $\beta_2 < 0$, $\alpha_i \geq 0$ dictate that for a zero cumulative default rate, there is no dependency on the market density, but when the cumulative default rate increases over time, conditional downgrade (upgrade) migration probabilities decelerate (accelerate), while survival probabilities grow. A different set of conditions can also be:

$$\gamma_{ji}(CDR) = \gamma_{ji}\frac{1}{1+\beta_i CDR} \tag{5}$$

$$\sigma_i(CDR) = \sigma_i\frac{1}{1-\alpha_i CDR} \tag{6}$$

Each set of conditions dictates different coefficients that can be calibrated. In density-dependent models, solutions do not grow exponentially. Instead, they tend to converge to limited subsets of the space, which are often called attractors. These can be fixed points or equilibria, various cycles or periodics, or sometimes even more complicated structures. In general, density-dependent models are more sensitive to small perturbations of initial conditions, depending on the stability of the equilibria, as illustrated below.

The market starts from an arbitrary point having a zero cumulative default rate. After a while, some companies have already defaulted, and the market has reached an equilibrium point of $\hat{n} = 1 - CDR$. This point represents the percentage of companies that survived out of the initial pull. If the equilibrium is unstable, there exists a deviation vector x such that

$$x(t) = n(t) - \hat{n} \quad \text{or} \quad n(t) = x(t) + \hat{n} \tag{7}$$

Within this setting, the equilibrium in the subsequent time period will take the form

$$x(t+1) + \hat{n} = S_{x(t)+\hat{n}}(x(t)+\hat{n}) \tag{8}$$

where S denotes the survival matrix, representing transitions among livelihood states, without the row vector D that contains transition probabilities into the absorbing state of default. Applying the Taylor series expansion on the transition matrix around the equilibrium point, $\hat{n}$, gives

$$x(t+1) + \hat{n} \cong \left(S_{\hat{n}} + \sum_i x_i \left.\frac{\partial S}{\partial n_i}\right|_{\hat{n}} \right)(x(t)+\hat{n}) \tag{9}$$

Decomposing the right-hand side of Equation (9) yields

$$x(t+1)+\hat{n}=S_{\hat{n}}\hat{n}+S_{\hat{n}}x(t)+\left(\sum_i x_i \left.\frac{\partial S}{\partial n_i}\right|_{\hat{n}}\right)x(t)+\left(\sum_i x_i \left.\frac{\partial S}{\partial n_i}\right|_{\hat{n}}\right)\hat{n} \tag{10}$$

The first term on the right-hand side equals $\hat{n}$ and cancels out. The third term on the right-hand side is of order x_i^2 and can be ignored for small deviations. Thus, a linear approximation is obtained as

$$x(t+1)\cong S_{\hat{n}}x(t)+\left(\sum_i x_i \left.\frac{\partial S}{\partial n_i}\right|_{\hat{n}}\right)\hat{n} \tag{11}$$

We may define matrices H_i with $\hat{n}$ in column i and with zeroes elsewhere. Then Equation (11) can also be expressed as

$$x(t+1)\cong\left(S_{\hat{n}}+\sum_i \left.\frac{\partial S}{\partial n_i}\right|_{\hat{n}} H_i\right)x(t)\equiv Jx(t) \tag{12}$$

The Jacobian matrix J depends on the equilibrium point and can be decomposed into[3]:

$$J=S_{\hat{n}}+\left(\left.\frac{\partial S}{\partial n_1}\right|_{\hat{n}}\hat{n} \quad \left.\frac{\partial S}{\partial n_2}\right|_{\hat{n}}\hat{n} \quad \left.\frac{\partial S}{\partial n_3}\right|_{\hat{n}}\hat{n}\ldots\right) \tag{13}$$

Equation (12) is linear so its solution depends on the dominant eigenvalue of the Jacobian matrix J. Departures from the equilibrium point grow over time (i.e., an unstable equilibrium) when the dominant eigenvalue λ_1 of J falls outside the unit circle, but decay to zero (for an asymptotically stable equilibrium) when $|\lambda_1^{(J)}|<1$.

Considering constant survivability σ_i, and recalling that $n_t = 1 - CDR_t$, the Jacobian matrix J can be obtained by taking the partial derivatives with respect to:

$$s_{ii}=\sigma_i\left(1-\sum_{j\neq i}\gamma_{ji}\exp(\beta_i(1-n_t))\right)\ \forall i \tag{14}$$

$$s_{ji}=\sigma_i\gamma_{ji}\exp(\beta_i(1-n_t))\ \forall j\neq i \tag{15}$$

In practice however, constantly changing market demand and supply, along with a dynamic market size and an irregular number of competitive firms, may dictate only a periodic equilibrium. These circumstances often dictate pro-cyclical patterns. Allen and Saunders [2004] survey pro-cyclicality effects on operational risk, credit risk, and market risk measures.

DATA

The Compustat database reports on 130,559 quarterly S&P long-term credit ratings for 4,510 industrial companies from 1985 to 2004. Each credit rating is tagged by a number between 2 for 'AAA' and 27 for default. Within this period, there are 123,849 valid credit rating transitions in consecutive quarters.[4] These transitions are collected and assigned to a frequency matrix F. Since there are $s = 21$ states in the life cycle of a company and an additional absorbing state of default, matrix F is of dimensions $(s + 1) \times s = 22 \times 21$, where an entry f_{ij} represents the number of credit ratings migrations observed from state j to state i.

The homogeneous transition matrix T in Exhibit 1 is constructed by dividing each entry in the frequency matrix F by the sum of its corresponding column. Matrix T represents the independent probabilities for ratings migration among the various categories, under the homogeneity assumption.

Several observations are reported as "Not Rated" (NR). These observations are classified as 1, 3, 22, 25, 28, 29, and 90 in the database. This study adopts the industry standard and treats transitions to and from NR status as missing information, by excluding them from the data sample.[5] Conceptually, transition matrices can be estimated over a variety of time horizons. Nevertheless, those transition matrices estimated over relatively short time intervals might best represent credit migration dynamics. For short-time rating migrations, more observations are available in the data sample, and rating migrations tend to follow a smoother shape due to fewer extreme jumps. For these reasons, the transitions time interval is chosen to be a quarter of a year, the shortest possible time frame in a discrete space, since agencies report their credit ratings once every quarter.

For the simulation of the stochastic business-cycles model, further information is collected from the National Bureau of Economic Research (NBER) website. NBER divides the examined time frame into two categories,

EXHIBIT 1

The Homogeneous Transition Matrix

Below is the non-arbitrary part of the transition matrix T as calculated directly from the frequency matrix F. The upper submatrix is the survival matrix S and the lower row vector is the default state D. Entries inside the matrix stated as percentages. Each column sums to 100%. Like the frequency matrix F (not reported here), this exhibit is of dimensions $(s + 1) \times s = 22 \times 21$.

Transition Matrix (%)	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C
AAA	98.04	0.83	0.16	0.02	0.01	0.00	0.04	0.01	0.00	0.02	0.03	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA+	1.05	94.36	0.29	0.04	0.01	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	0.32	3.71	95.27	0.79	0.06	0.04	0.04	0.00	0.02	0.01	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.00	0.00
AA-	0.32	0.76	3.00	94.49	0.82	0.09	0.06	0.01	0.03	0.06	0.03	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A+	0.11	0.14	0.65	3.34	94.98	1.19	0.15	0.15	0.05	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.07	0.00	0.00	0.00	0.00
A	0.00	0.14	0.35	0.81	2.77	95.04	1.81	0.33	0.12	0.07	0.07	0.00	0.02	0.02	0.00	0.00	0.00	0.12	0.00	0.00	0.00
A-	0.04	0.00	0.08	0.25	0.85	2.32	93.71	1.72	0.43	0.17	0.10	0.01	0.04	0.02	0.08	0.00	0.00	0.12	0.00	0.00	0.00
BBB+	0.00	0.00	0.12	0.13	0.24	0.78	2.80	93.45	1.83	0.47	0.12	0.08	0.04	0.07	0.02	0.04	0.00	0.12	0.00	0.00	0.00
BBB	0.07	0.07	0.06	0.04	0.08	0.27	1.12	3.06	94.04	2.52	0.89	0.32	0.12	0.02	0.07	0.00	0.07	0.00	0.00	0.00	0.00
BBB-	0.00	0.00	0.02	0.04	0.01	0.05	0.12	0.77	2.32	92.90	3.40	0.86	0.16	0.10	0.02	0.07	0.15	0.00	0.00	0.00	0.00
BB+	0.00	0.00	0.00	0.00	0.02	0.07	0.01	0.14	0.57	2.03	90.86	2.39	0.66	0.13	0.07	0.07	0.00	0.23	0.24	0.00	0.00
BB	0.04	0.00	0.00	0.00	0.02	0.04	0.03	0.22	0.22	1.20	2.44	91.43	2.20	0.54	0.15	0.14	0.22	0.12	0.24	0.00	0.00
BB-	0.04	0.00	0.00	0.00	0.01	0.04	0.01	0.03	0.10	0.33	1.26	2.99	92.21	1.94	0.42	0.29	0.29	0.23	0.72	0.93	0.00
B+	0.00	0.00	0.00	0.02	0.04	0.02	0.04	0.04	0.16	0.09	0.37	1.17	2.98	92.85	2.69	0.98	1.11	0.58	0.96	1.86	0.00
B	0.00	0.00	0.00	0.05	0.04	0.00	0.01	0.03	0.03	0.07	0.16	0.32	0.76	2.57	89.82	2.17	1.47	1.05	0.72	0.62	10.00
B-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.07	0.16	0.29	0.86	2.54	86.64	2.36	1.40	0.96	2.79	0.00
CCC+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.01	0.00	0.05	0.16	0.31	1.64	3.03	82.96	1.63	1.20	1.24	10.00
CCC	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.03	0.07	0.10	0.22	0.66	2.46	2.51	82.09	2.16	2.17	10.00
CCC-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.03	0.04	0.07	0.05	0.40	1.05	2.58	1.63	81.29	1.55	0.00
CC	0.00	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.03	0.05	0.04	0.08	0.43	1.12	1.99	2.09	3.36	73.68	0.00
C	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.00	0.07	0.47	0.24	0.00	50.00
D	0.00	0.00	0.00	0.00	0.02	0.01	0.01	0.01	0.05	0.01	0.09	0.03	0.13	0.16	0.96	1.95	4.13	8.14	7.91	15.17	20.00

expansionary and contractionary quarters, constructed by examining a larger number of business indicators in addition to the GDP.

METHODOLOGIES AND RESULTS

This section reports the empirical findings and summarizes their economic meaning. Exhibit 1 points to a greater stability among highly rated debt issuers, relative to low credit categories. The main diagonal in Exhibit 1 shows that the probabilities of remaining at the same credit rating within a quarter declines from 98.04% for AAA-rated bonds to 50.00% for C-rated bonds. After remaining above 90% for the credit ratings of AAA to B+, stability declines at a faster rate for the low credit ratings, from 89.82% to 50.00% for the credit categories B to C. These findings corroborate previous results.

Several articles report that higher-rated bonds tend to be more stable than lower-rated bonds, with respect to the time of remaining at the same credit rating. Altman and Kao [1992b] examine major credit ratings of more than 7,000 bonds issued from 1970 to 1988 and find that AAA-rated issues had the greatest stability in terms of retaining their initial ratings, up to five years after issuance. BB-rated bonds are found to be the least stable. Carty and Fons [1994] test approximately 4,700 long-term public debt issuers over a 70-year period and 2,400 short-term public debt issuers over a 22-year period from the Moody's proprietary database. The authors discover that higher ratings are relatively more stable, as reflected by their longer average length of time holding the same rating, given that it subsequently changes. Fons [1994] investigates broad rating categories out of the Moody's long-term default studies from 1970 to 1993 and discovers that default rates are not particularly stable, especially at low rating categories.

A second conclusion derived from the first exhibit is that when the probabilities of default are included, almost all the firms are more likely to be downgraded over time. For each column in Exhibit 1 the sum of the probabilities below the main diagonal is larger than the sum of the entries above it, except for a slight opposite trend for rating category BB+, and a more pronounced opposite trend for C-rated companies. These results assess the hypothesis of a finite life cycle for all companies.

We turn now to compare the proposed density-dependent model with several known schemes: the homogeneous Markov chain of Jarrow, Lando, and Turnbull

[1997] as constructed in Exhibit 1, the momentum approach by Bahar and Nagpal [2001], and the stochastic business-cycles notion offered by Bangia et al. [2002].

The momentum model captures the cumulative default rate over time by considering a starting point of equally weighted portfolios of upgraded, stable, and downgraded companies, and then iterating each transition matrix independently. The total cumulative default rate measure eventually accumulates the defaults in all three dynamics within an interval of 10 consecutive years.

The stochastic business-cycles model utilizes the NBER definition for the probabilities of being in an expansionary or a contractionary economic stage in the next quarter, conditional on the current business cycle, from 1985 to 2004. We run a Monte Carlo simulation that describes the incremental percentage of holdings that reach the absorption state and default within a time frame of 40 quarters.

An illustration of the density-dependent model under the first set of conditions in Equations (3) and (4) is next performed, while considering the coefficients of density dependent transitivity as $\beta_1 = 10$ and $\beta_2 = -15$, and allowing different coefficients of density dependent survivability for different groups of credit categories as follows: $\alpha_i = 0$ for the group of the four highest credit ratings (AAA to AA–); $\alpha_i = 0.0001$ for the group of the next eight credit ratings (A+ to BB); $\alpha_i = 0.001$ for the group of the next four credit ratings (BB– to B–); and $\alpha_i = 0.01$ for the group of the five lowest non-default credit ratings (CCC+ to C). These coefficients are chosen to reflect the growing impact of *CDR* on survivability for lower groups of credit ratings under the constraint $0 \leq \sigma_i \leq 1$. For the comparative analysis, the starting point for this simulation, the unconditional migration probabilities, γ_{ji}, and the survival probabilities within the next quarter, $\sigma_i = 1 - D_i$, are collected from the homogeneous transition matrix in Exhibit 1.

To identify which of the models explored in this study best represent the actual ratings migration dynamic, we first conduct a comparative analysis. Back-testing is a relatively naïve yet robust method for that purpose. We calibrate all theoretical models over the first 10 years, from the beginning of 1985 to the end of 1994, and then simulate them over the next 10 years, from the beginning of 1995 to the end of 2004, while comparing the predictive power of all models to the actual observed cumulative default rate over the same investment horizon.

Exhibit 2 illustrates this comparative analysis within a 10-year period, for the various theoretical models, and the actual cumulative default rate as observed from 1995 to 2004. The homogeneous Markov chain model overstates expected CDR, and predicts a 28.9% CDR within these 10 years. The stochastic business-cycles model based on the NBER definition for business cycles slightly improves this prediction, and points to 26.8% expected CDR in the same time frame. The momentum effect model projects the highest CDR, a 39.6% within a 10-year period, and deviates the most from the actual CDR. The density-dependent model yields a 12.15% expected CDR, and remains the closest to the observed 8.3% CDR.

The results show that all three existing models overestimate credit ratings migration probabilities, and project much higher cumulative default rates than are observed in practice. The proposed density-dependent non-homogeneous dynamic portrays a more realistic scenario, and only slightly overshoots the actual CDR.

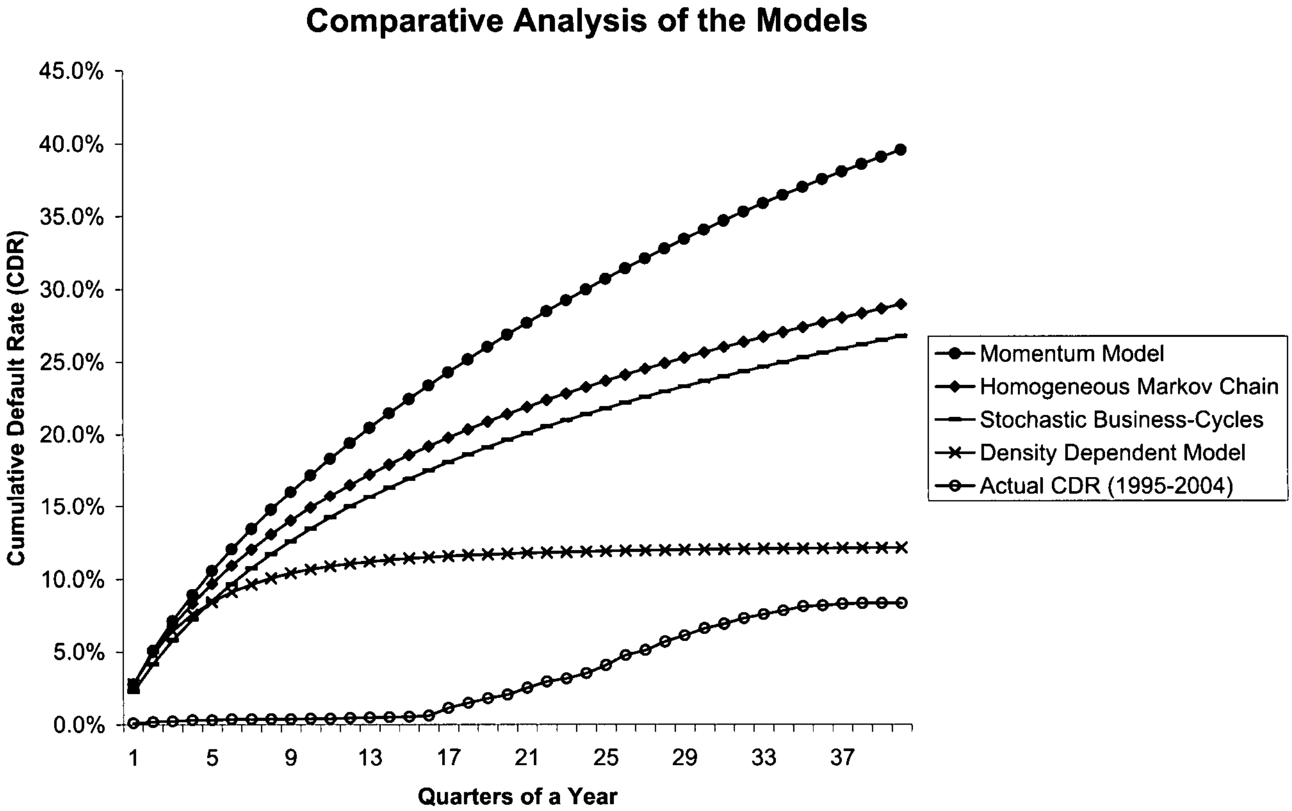
However all models thus far generate smooth CDR curves, and miss changes in patterns and slopes in the actual cumulative default rate, as can be spotted at the fourth year in the actual CDR curve. Correlating the models to further exogenous variables that capture fundamental changes in macroeconomic conditions may capture these inflection points in the cumulative default rate.

Another effective way to compare the models is by directly examining their assumptions. A suitable test to validate the homogeneity of the transition matrix is the chi-square test. The null hypothesis of homogeneity of credit rating migrations asserts that the multinomial probability distribution dictates independent transitions among different states. The chi-square statistic yields a value of 349.485, suggesting a strong rejection of the null hypothesis of the homogeneous Markov chain model, with a significance level of 0.999. By that, it refutes the assertion of independent credit rating transitions.

In essence, the momentum model, proposed by Bahar and Nagpal [2001], represents a high-order Markov process. Therefore, it can be validated through a stratified version of the chi-square test for homogeneity, with each possible rating category separating any two credit ratings, together forming a stratum. Sequences of three, four, and even a higher number of credit ratings can be examined through this method. However, a simpler way to examine this scheme is through a separate chi-square test for each of the three Markov chain processes, the upgraded, the stable, and the downgraded transition matrices. This

EXHIBIT 2
A Comparative Analysis of the Models

Below is a comparative analysis of the homogeneous Markov chain model of Jarrow, Lando, and Turnbull [1997], the momentum scheme of Bahar and Nagpal [2001], the stochastic business-cycles notion based on the NBER definition for business cycles of Bangia et al. [2002], the proposed density-dependent model, and the actual cumulative default rate. The theoretical models are first calibrated from the beginning of 1985 to the end of 1994, further simulated over a 40-quarter period, from the beginning of 1995 to the end of 2004, and then compared to the actual cumulative default rate as collected from the Compustat data sample.



method leads to a strong rejection of the null hypothesis of the momentum model as well.

We next validate the density-dependent model. Exhibit 3 presents sample results for survivability and transitivity measures from 1999 to 2004. Most of the defaults in the data sample take place within this time frame, and hence it can be used as an intensive test-field for validating the assumptions. A clear picture arises from the first panel, where survivability increases with the CDR. Among the lower credit ratings survivability tends to increase at a higher pace when the CDR rises relative to the higher credit categories. For the highest credit ratings, survivability remains constant, as suggested by the different coefficients α_i in the empirical simulation. The transitivity measures describe a somewhat less clear picture, but it is still possible to identify in the second and third panels several credit ratings with increasing transition probabilities above the main diagonal, as well as decreasing transition probabilities below the main diagonal when CDR arises, as predicted by the coefficients $\beta_1 > 0$ and $\beta_2 < 0$ in the theoretical model.

A word of caution is required here. These findings do not prove any causality between the cumulative default rate and the survivability or transitivity of the remaining companies. The test purely aims to validate the assumptions of the density-dependent theoretical model, and merely points to a plausible correlation between the mechanisms, and thus validates the assumptions.

The question of whether the density of companies in the market affects their default rate can be directly examined through the randomization test offered by Pollard, Lakhani, and Rothery [1987].[6] This test measures a population size, $N(t)$, at time t, computes $X(t) = ln(N(t))$, and compares three time series processes:

$$X_{t+1} = X_t + \varepsilon_t \tag{16}$$

$$X_{t+1} = \mu + X_t + \varepsilon_t \tag{17}$$

$$X_{t+1} = \mu + \beta X_t + e_t \tag{18}$$

Equation (16) describes a random walk in time and thus is density-independent. Equation (17) presents also a random walk, but with a drift. Equation (18) portrays a density-dependent dynamic, where the true slope β is estimated by the regression coefficient b as follows:

$$b = \sum_{t=1}^{n-1}(X_t - m_1)(X_{t+1} - m_2)/\sum_{t=1}^{n-1}(X_t - m_1)^2$$

where (19)

$$m_1 = \sum_{t=1}^{n-1} X_t/(n-1)\ m_2 = \sum_{t=1}^{n-1} X_{t+1}/(n-1)$$

For the null hypothesis of density independence, since Equation (16) is a special case of equation (17), it is always preferable to examine the second and third time series processes. In this setting, the likelihood ratio test-statistic is:

Exhibit 3

Survivability and Transitivity Measures for the Density-Dependent Model

A sample of the survivability and transitivity measures, above and below the main diagonal, from 1999 to 2004, with the corresponding cumulative default rates within this period is presented below for validating the density-dependent assumptions.

Year	CDR	Sample of Survivability Measures (in %)					
		AAA	AA	BBB	BB+	B+	CCC
1999	1.44%	100.00	100.00	99.843505	99.685535	99.607843	72.000000
2000	2.90%	100.00	100.00	99.864682	99.737877	99.663866	77.631579
2001	5.07%	100.00	100.00	99.872881	99.829497	99.672012	77.697842
2002	6.67%	100.00	100.00	99.848439	99.811202	99.722145	80.660377
2003	7.66%	100.00	100.00	99.884793	99.851998	99.749602	85.937500
2004	7.72%	100.00	100.00	99.896737	99.867374	99.770355	87.958115

Year	CDR	Sample of the Transitivity Measures (in %) Above the Main Diagonal (j < i)			
		B+	B-	CCC+	CC
1999	1.44%	1.568627451	0	1.428571429	5.555555556
2000	2.90%	1.568627451	1.089918256	1.621621622	9.756097561
2001	5.07%	1.494169096	1.208459215	2.265372168	7.142857143
2002	6.67%	1.555987774	1.857923497	2.678571429	11.20689655
2003	7.66%	2.048713863	2.629272568	3.442340792	11.94968553
2004	7.72%	2.171189979	2.762430939	4.423380727	11.69590643

Year	CDR	Sample of the Transitivity Measures (in %) Below the Main Diagonal (j > i)		
		AAA	B-	CCC-
1999	1.44%	0	10.85271318	14.28571429
2000	2.90%	4.184100418	7.629427793	4.761904762
2001	5.07%	3.768115942	9.516616314	4.255319149
2002	6.67%	3.628117914	10.49180328	3.529411765
2003	7.66%	3.047619048	9.991235758	4.838709677
2004	7.72%	3.169014085	9.629044988	4.964539007

$$T = \frac{\Sigma_{t=1}^{n-1}(X_{t+1} - m_2)^2 - b\,\Sigma_{t=1}^{n-1}(X_t - m_1)(X_{t+1} - n}{\Sigma_{t=1}^{n-1}(X_{t+1} - X_t)^2 - (X_n - X_1)^2/(n-1)} \quad (20)$$

To test the null hypothesis that observations arise from a density-independent market, the following steps are taken. First, the number of companies in the database at the beginning of the period and the number of defaults throughout the period are used to estimate the observed values X_t, and the test-statistic T. Second, changes in values, $D_t = X_{t+1} - X_t$ are permutated, and the corresponding X_t values are computed. Next, for each possible permutation, the corresponding T value is calculated. Finally, the percentages of T values that fall below (or exactly at) the test-statistic T are calculated. If, for example, less than 5% of the permutations' T values are smaller than or equal to the observed test-statistic T, then the null hypothesis of density-independence is rejected, at a significance level of 5% or better.

Exhibit 4 presents the results of four randomization tests. Each test starts at a different point in time and lasts for seven consecutive half-year intervals.[7] The first two tests suggest that some density-dependency exists, yet the results lack significance. However, the last two tests imply a strong rejection of the null hypothesis of density-independency. Both of the last two tests are statistically significant at a 1% level or better. These robust results clearly point to various levels of density-dependency among the market's cumulative default rate.

CONCLUSION

This study explores a new approach for simulating credit ratings migration analysis, by correlating credit ratings transition rates to a valuable economic variable: the market density. As demonstrated by Lang and Stulz [2002], and Jorion and Zhang [2006], negative default correlations could be the product of a "competitive effect," where a low market density allows more firms to prosper, while an increasing market density triggers a high rate of defaults.

The proposed density-dependent model is found to be highly pragmatic, and may improve the predictive strength of the credit ratings migration theory. Our empirical investigation endorses the assumptions of the density-dependent scheme while challenging the underlying assumptions of several other existing theories. A randomization test further approves the hypothesis of negative default correlations to various degrees, from 2000 to 2004. Thus, we conclude that the density-dependent model statistically dominates several known schemes for ratings migration.

A back-testing methodology authenticates that the density-dependent model also economically dominates the homogeneous Markov chain of Jarrow, Lando, and

EXHIBIT 4

Randomization Test for the Density-Dependent Model

The results of the randomization tests for seven consecutive half-year periods are given below. The first column describes the time frame of the test, including the first and last half-years in each interval. The second column presents the number of companies in the database at the beginning of the test. The third column shows the sequence of defaults (from left to right) as reported by S&P. The next four columns are the statistics m_1, m_2, b, and the test-statistic T as discussed in Lakhani and Rothery [1987]. The last column describes the significance level of each test. These values are the percentages of *T-values* computed for the permutations, and are smaller or equal to the test-statistic T in the previous column. Values less than 5% suggest a rejection of the null hypothesis of density independence, at 5% level of significance or better. Values less than 1% indicate a rejection of the null hypothesis of density independence, at 1% level of significance or better, etc.

Time Period	Number of Companies at the Starting Point	Number of Companies Defaulted in Consecutive Half-Year Periods	M1	M2	b	T	Significance Level
2000 H1 - 2003 H1	2,925	33,21,47,34,34,25,20	7.946691	7.924889	1.625152	0.663133	28.5714%
2000 H2 - 2003 H2	2,809	21,47,34,34,25,20,13	7.90569	7.895467	0.909386	0.679899	19.3254%
2001 H1 - 2004 H1	2,701	47,34,34,25,20,13,6	7.862521	7.852725	0.791643	0.056129	0.0198%
2001 H2 - 2004 H2	2,588	34,34,25,20,13,6,0	7.826733	7.819254	0.75175	0.110571	0.0397%

Turnbull [1997], the momentum approach by Bahar and Nagpal [2001], and the stochastic business-cycles notion offered by Bangia et al. [2002], by better converging to the actual cumulative default rate curve from 1995 to 2004.

Future lines of investigation may aim to correlate the density-dependent model to more exogenous variables that capture fundamental changes in macroeconomic conditions. This may help to identify inflection points in the cumulative default rate curve and better fit observed default patterns.

ENDNOTES

[1]In the CreditMetrics™ document, 1,234 firms are examined over 40 quarters with only seven non-default categories. This leads to 1.13 million pair combinations with 28 unique joint likelihood tables.

[2]This approach might be more applicable for specific industries rather than for the whole market. However, data on default rates within specific industries are scarce and relatively noisy.

[3] The Jacobian is the matrix of all first-order partial derivatives of a vector-valued function. Its importance lies in the fact that it represents the best linear approximation of a differentiable function near a given point. The Jacobian matrix J in this context describes the deviations dynamic and neither transition nor survival probabilities; therefore, it may contain negative values.

[4]The "SPDRC" data field assigns a unique credit rating based on the issuer overall credit worthiness, apart from its ability to service individual debt. Thus intra-company effects among different obligations are eliminated, yet cross-ratings correlations are allowed to exist.

[5]Bangia et al. [2002] discuss other alternatives, but eventually choose to exclude NR observations as well.

[6]The ecology literature suggests several more statistical tests for examining density-dependent population models, including a key factor analysis and an autocorrelation test, yet the randomization test is considered the most vigorous method.

[7]There are (n–1)! possible permutations, and thus sets of X_t values. Taking a large number of short consecutive periods creates computation difficulties, with either the lexicographic order or the recursive algorithms for permutations. On the other hand, considering a small number of longer time intervals may miss the statistical significance of the test. For these reasons, each of the four tests described here uses seven consecutive half-year intervals.

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