A Copula Approach to Value-at-Risk Estimation for Fixed-Income Portfolios

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Since its introduction in the 1990s, and despite recent criticism of its shortcomings from a theoretical standpoint (see Artzner and Alii [1999]), Value-at-Risk (VaR), defined as the conditional quantile of a portfolio distribution, has become a standard for risk measurement, and is widely used to determine the possible loss that might be sustained for a given period and a given confidence level. A variety of methods (including historical VaR, various forms of parametric VaR, and Monte Carlo VaR) are available for VaR estimation, and their respective merits and drawbacks have been well documented from a general standpoint.[1]

Relatively little is known, however, on the performance of competing VaR estimates in the context of fixed-income portfolios. VaR estimation for fixed-income securities actually displays a series of specific challenges. In particular, because bond prices can be regarded as (non-linear) functions of underlying interest rate risk factors, any analysis of extreme risks in fixed-income portfolios should be based on risk estimates for these underlying risk factors. The most commonly used methodology, initiated by Litterman and Scheinkman [1991], is based on principal component analysis to extract systematic risk factors from a series of bond returns. One key problem with applying this technique to the context of VaR estimation for bond portfolios is that correlation-based measures of co-movements embedded in principal component analysis are ill-suited for the analysis of the dependence structure for non-linear functions of the underlying risk factors.[2]

In this article, we suggest the use of a copula approach to the modelling of the non-linear dependence structure of these factors. Stated simply, copulas are useful because they allow one to model the dependence relationships among random variables independently of their marginal distributions. As a result, copulas provide a means of handling dependence that is valid for any multivariate distribution, whereas correlations are only valid with multivariate normal or near-normal distributions. Copula functions, the general theory of which was developed in the late 1950s by Sklar [1959], have recently been successfully applied to various domains of financial risk management (see Jouanin, Riboulet, and Roncalli [2003] or Cherubini, Luciano, and Vecchiato [2004] for an overview of the use of copula functions in finance). Among the successful applications to copulas in finance, one could refer to the pricing of derivative securities (Cherubini and Luciano [2002] [2003a]; Bennet and Kennedy [2004], etc.), the analysis of dependence across international financial markets (Hu [2006]), the analysis of credit risk (Li [2000]; Schonbucher and Schubert [2001]; Cherubini and Luciano [2003b], etc.), as well as the portfolio construction process (Hennessy and Lapan [2002]). More related to this article, several authors have used copula functions for risk measurement purposes in general, and VaR estimates in particular (see for example Ane and Kharoubi [2003]).

This article can be regarded as one of the very first attempts to apply a copula approach to the estimation of extreme risk measures in fixed-income portfolios. We first use a parsimonious model of the yield curve to extract the time-varying parameters that will be used as a proxy for factors (level, slope, and curvature) affecting the shape of the yield curve. We then estimate the yield curve model as well as the copula function parameters on the basis of a sample of daily prices borrowed from the French bond market, and subsequently forecast the VaR measure for a large set of Treasury coupon-bearing bonds using a flexible semi-parametric approach. We finally provide formal statistical testing of the accuracy of the VaR forecasts based on a methodology recently introduced by Christoffersen [1998].

Our article is perhaps most closely related to Junker, Szimayer, and Wagner [2006], who, to the best of our knowledge, were the first to use copula functions in the context of fixed-income portfolios. The focus in their article is different, however, and relates to an ex-post in-sample estimation of the dependence in risk factors affecting the shape of the yield curve, while we aim at providing out-of-sample forecasts of extreme risk measures.

The remainder of this article is organized as follows. In the *Value-at-Risk Estimation for Fixed-Income Securities* section, we present our approach for estimating Value-at-Risk for fixed-income portfolios. The empirical validation of our model is given in the *Empirical Analysis* section, and we present our conclusions in the *Conclusion* section.

VALUE-AT-RISK ESTIMATION FOR FIXED-INCOME SECURITIES

Our approach to VaR estimation for fixed-income portfolios is based on a two-step process. We first introduce a parsimonious factor model for changes in shape of the yield curve, regarded as the underlying infinite-dimensional risk factor. In a second step, we then use a copula function to model the dependence structure of these factors and make VaR forecasts.

Fitting a Factor Model to the Observed Yield Curve

Following Dolan [1999]; Diebold and Li [2002], or Fabozzi, Martellini, and Priaulet [2006], we use the Nelson and Siegel [1987] model as a parsimonious way to extract a time series of a few key parameters that will be used as a proxy for factors affecting the shape of the yield curve. This model can be written as:

$$R(0,\theta) = \beta_0 + \beta_1\left[\frac{1-\exp(-\theta/\tau)}{\theta/\tau}\right] + \beta_2\left[\frac{1-\exp(-\theta/\tau)}{\theta/\tau} - \exp(-\theta/\tau)\right]$$

where:

$R(0, \theta)$ = the rate at time 0 with maturity θ;
β_0 = the limit of $R(t, \theta)$ as θ goes to infinity. In practice, β_0 should be regarded as a long-term interest rate;
β_1 = the limit of $\beta_0 - R(t, \theta)$ as θ goes to 0. In practice, β_1 should be regarded as the short- to long-term spread;
β_2 = a curvature parameter;
τ = a scale parameter that measures the rate at which the short-term and medium-term components decay to zero.

The advantage of this model is that the three parameters β_0, β_1, and β_2 can directly be interpreted as level, slope and curvature changes in the yield curve. These parameters can easily be estimated using an OLS optimization program, which consists, for a basket of bonds, in minimizing the sum of the squared spread between the market price and the theoretical price as obtained with the model. After calibrating the model on market data, the next step consists of analyzing the potentially non-linear dependence between changes in the parameters, which is the reason we now turn to a (multi-variate) copula approach.

Introducing Copulas Functions

In what follows, we provide a simple non-technical introduction to copulas, and we refer interested readers to Joe [1997] or Nelsen [1999] for more details. As stated in the introduction, copulas are functions that make it possible to obtain multivariate distribution functions from the marginal distribution functions. Sklar's theorem actually allows every joint distribution function to be defined as a copula and two or more univariate cumulative distribution functions. Conversely, one can express any copula function with a joint distribution function and the "inverse" of two margins. Starting from two variables X and Y with marginal distribution functions F_1 and F_2, one can write (a result known as Sklar's theorem) the unique *copula* function C such that:

$$\begin{aligned} H(x, y) &\equiv P(X \leq x, Y \leq y) \\ &= P(F_1(X) \leq F_1(x), F_2(Y) \leq F_2(y)) \\ &= C(F_1(x), F_2(y)) \end{aligned}$$

for every$(x, y) \in \Re^{*2}$ is:

$$\begin{aligned} C(u,v) &= P\left(F_1(X) \leq u, F_2(y) \leq v\right) \\ &= P\left(X \leq F_1^{-1}(u), Y \leq F_2^{-1}(v)\right) \\ &= H\left(F_1^{-1}(u), F_2^{-1}(v)\right) \end{aligned}$$

In other words, copula functions allow for the construction of a bivariate (multivariate) distribution to be split into two components: i) the choice of the marginal distribution functions, and ii) the choice of the copula function that merges them.

There then remains the question of which copula function should be used. Perhaps the most commonly used copula is the Gaussian copula which density is given in the multivariate context by:

$$C^{ga}(u_1, \ldots, u_N) = \frac{1}{|R|^{1/2}} \exp\left\{-\frac{1}{2}\alpha'(R^{-1} - I_N)\alpha\right\}$$

where R is the correlation matrix, I is the identity matrix and $\alpha = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N))'$.

One problem with this choice for a copula function is that it does not allow extreme co-movements to be taken into account, since it does not exhibit tail dependence unless the variables are perfectly correlated. To exhibit tail dependence, one possible alternative is to consider another copula, the Student copula, which density is given by:

$$C^{student}(u_1, \ldots, u_N) = |R|^{-1/2} \frac{\Gamma\left(\frac{v+N}{2}\right)\left[\Gamma\left(\frac{v}{2}\right)\right]^{N-1}\left(1+\frac{1}{v}\varsigma^{-1}R^{-1}\varsigma\right)^{-\frac{v+N}{2}}}{\left[\Gamma\left(\frac{v+1}{2}\right)\right]^{N}\prod_{i=1}^{N}\left(1+\frac{\varsigma_i^2}{v}\right)^{-\frac{v+1}{2}}}$$

where, $\varsigma_i = t_v^{-1}(u_i)$, $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_N)'$, t_v is the univariate Student's t distribution function with v degrees of freedom, and $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$ is the gamma distribution.

Even though the Student copula converges to the Gaussian one as the number of degrees of freedom v increases, it generates fatter tails. Exhibit 1 shows the difference between a Gaussian copula and a Student-t copula. Both are based on a correlation coefficient of 0.1.[3] As we can see, t-copula exhibits tail dependence and gives fatter tails than those of the Gaussian copula. Multivariate distributions obtained from this copula exhibit fatter tails even with Gaussian marginals, as we can observe in Exhibit 2. Using a t-copula also makes it possible to take upper tail dependence into account, as can be seen from Exhibit 3, where we show simulated values for nine

EXHIBIT 1

Density of a Gaussian and a Student-t Copula with $\rho = 0.1$ and $v = 2$

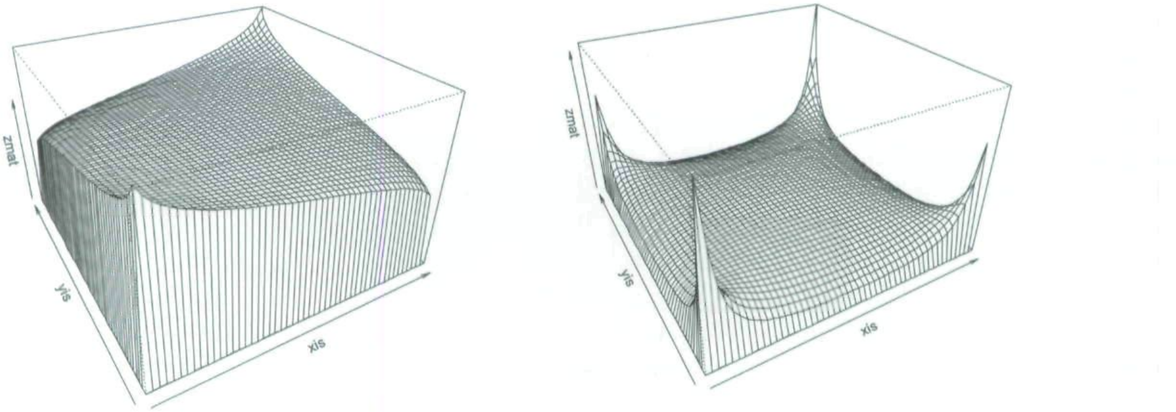
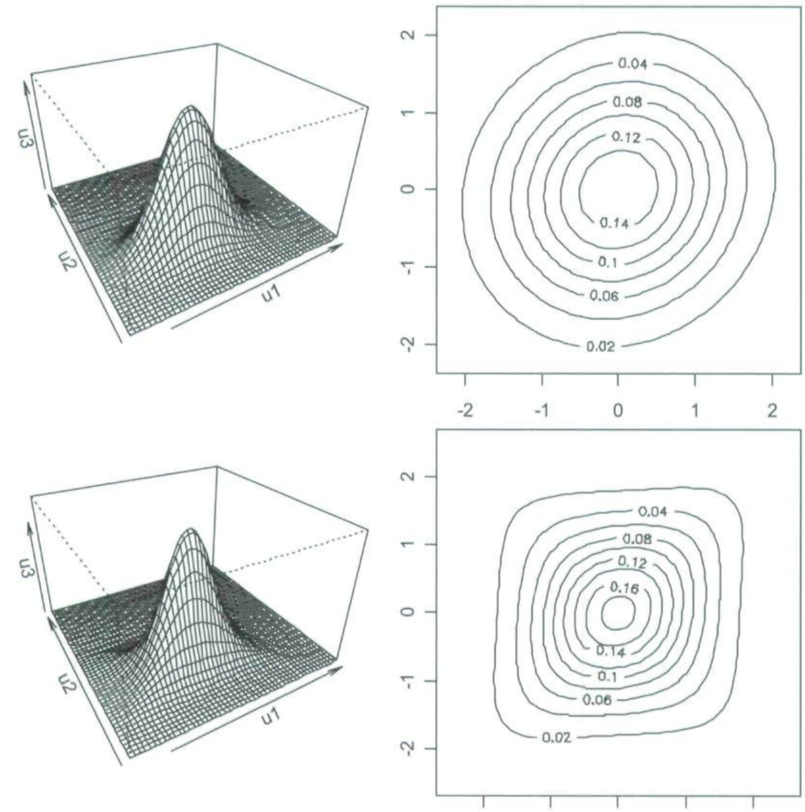


Exhibit 2

Comparison of Densities and Level Curves of Two Bivariate Distributions Obtained from a Gaussian (Up) and a Student-*t* Copula (Down) with Two Standard Gaussian Margins and $\rho = 0.1$, $\nu = 3$



bivariate t-copulas with uniform margins and different values for the correlation coefficient ρ.

We now turn to the VaR estimation methodology we have chosen, which is a mixture of historical simulation and Monte Carlo simulation approaches, where we use marginal distributions of the parameters for historical simulation and a Monte Carlo simulation for the dependence parameters only.[4] Instead of estimating the multivariate distribution function in a global approach, we separate the estimation of the copula and the margins using the *Canonical Maximum Likelihood* (CML) methodology of Romano [2002]. Unlike in the *Inference for Margins* (IFM) approach developed by Joe and Xu [1996], which requires the specification of parametric univariate marginal distributions, we avoid the need to make any parametric assumption by transforming the marginals into uniform variables through the use of the empirical cumulative distribution function. [5]

Hence the initial series (x_i, y_i) is transformed into $(u_i, v_i) = (\frac{rank(x_i)}{n+1}, \frac{rank(y_i)}{n+1})$. In our case we estimate the copula parameters via maximum likelihood estimation rather than least squares, as it gives equal importance to all data, and not only where the maximum amount of data is, i.e., near the middle of the distribution. Hence we consider the following program:

$$l(\theta) = \sum_{i=1}^{n} \ln c(u_1, \ldots, u_n)$$

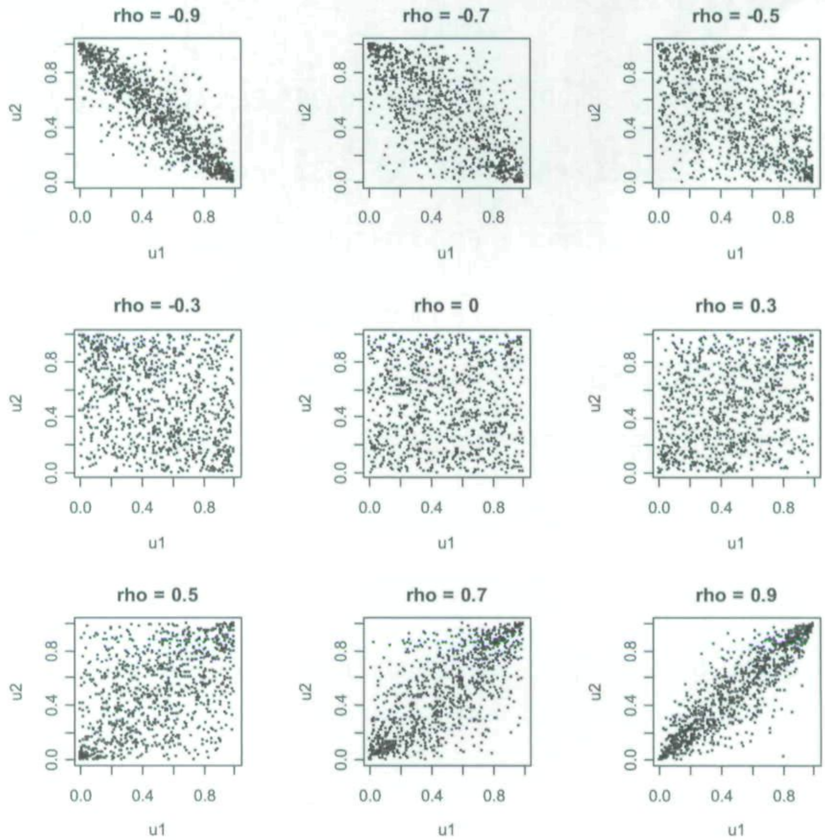
$$\hat{\theta} = \arg\max_{\theta} l(\theta)$$

with $c(\cdot)$ the density of the copula under consideration and θ the set of parameters to estimate.

One additional empirical challenge is related to the need to jointly estimate the correlation matrix of the four

EXHIBIT 3

Simulation of Nine Bivariate T-Copulas with $\nu = 10$ and Different Values for ρ



parameters in the Nelson-Siegel model and the number of degrees of freedom in the Student copula. Lindskog, McNeil, and Schmock [2003] have argued that it is possible to simplify the parameters estimation by using the following relationship that holds, for all non-degenerate elliptical distributions, between the correlation coefficient ρ and the Kendall's tau coefficient τ:

$$\tau = \frac{2}{\pi}\arcsin(\rho)$$

Inversing this relationship allows us to obtain a sample estimator of the correlation matrix, reducing to one the remaining number of parameters to estimate the number of degrees of freedom.

The second step in the VaR estimation process involves simulating a large number of draws (100,000 in our case) from the copula distribution and using the empirical marginal distributions of each parameter to generate the simulated uniform parameters. From this series, we generate a simulated daily term structure that allows us to obtain possible yield curves and thus possible bond prices. We then select the chosen percentile to obtain the Value-at-Risk estimate.

We now turn to the empirical validation of the copula model that we use to generate out-of-sample forecasts of the VaR for a bond portfolio.

EMPIRICAL ANALYSIS

Our dataset consists of the daily end of day midquotes of yields to maturity of all the strip bonds proposed by six *Spécialistes en Valeurs du Trésor* (the French equivalent to the U.S. Primary Dealers) on the French bond market between February 6, 2004, and October 12, 2005 (resulting in 434 trading days). The database also includes the price of the same bonds on the Euronext

EXHIBIT 4
Characteristics of the Bonds Used in the Validation Sample

Name	Isin Code	EuroMTS Index
OAT 5.5% 25/04/07	FR0000570574	EuroMTS 3-5 Y
OAT 5.25% 25/04/08	FR0000570632	EuroMTS 3-5 Y
OAT 4% 25/04/09	FR0000571432	EuroMTS 5-7 Y
OAT 4% 25/10/09	FR0000186199	EuroMTS 5-7 Y
OAT 5% 25/04/12	FR0000188328	EuroMTS 7-10 Y
OAT 4% 25/04/13	FR0000188989	EuroMTS 7-10 Y
OAT 5% 25/10/16	FR0000187361	EuroMTS 10-15 Y
OAT 8.5% 25/04/23	FR0000571085	EuroMTS > 15 Y
OAT 6% 25/10/25	FR0000571150	EuroMTS > 15 Y

market.[6] From this database of 101,617 prices/YTM, we only select the bid-ask spreads posted by IXIS Capital Markets, which was the most active Primary Dealer in our database, with more than a quarter of the quotes (25,498). Using a single market maker ensures homogeneity in the data. We also use a set of coupon-bearing bonds issued by the French Treasury Agency. We have only selected those that are considered as benchmarks by EuroMTS and that are components of EuroMTS indices. The main characteristics of the nine bonds selected are given in Exhibit 4. Again, for the benchmark series we use end of day mid-quotes posted by Ixis.

As we can see in Exhibit 4, our sample encompasses a large range of maturities extending from one year to 20 years. We first fit daily yield curves by estimating the various parameters of the Nelson-Siegel model. We have chosen to estimate these through a classic non-linear least squares estimation. Exhibit 5 presents various statistics for each parameter estimated.

The second step consists of checking the quality of the yield curve fitting for the model we analyze. We report the results obtained for three usual criteria (the adjusted R-Square ($\bar{R}^2$), the AIC, and the BIC criteria) in Exhibit 6.

We observe that the Nelson-Siegel model achieves a high quality of fit, with an average adjusted $\bar{R}^2$ higher than 99% and never below 99% in our estimation window.

EXHIBIT 5
Summar Statistics on the Nelson-Siegel Parameters

	β_0	β_1	β_2	τ_1
Mean	0.0498	-0.0295	-0.0268	2.0298
Standard error	0.0056	0.0062	0.0067	0.1746
Min	0.0403	-0.0389	-0.0458	1.6366
Max	0.0574	-0.0194	-0.0119	2.8561

EXHIBIT 6
Goodness-of-Fit Tests

	$\overline{R}^2$	AIC	BIC
Mean	0.9991	5.6302e-8	6.6459e-8
Standard error	0.0004	4.2517e-8	5.0652e-8
Min	0.9973	1.5100e-8	1.7900e-8
Max	0.9997	2.0700e-7	2.4507e-7

This quality of fit is visually confirmed by the two graphs shown in Exhibit 7. Exhibit 8 shows the mean pricing error ε and the absolute mean pricing error ($|\varepsilon|$) given by the use of the Nelson-Siegel model on our validation sample of nine bonds. The mean error varies from -0.09 cents to 0.53 cents, which is small compared to bid-ask spread values, and again signals a good quality of fit.

We use the methodology described in Section I to generate Value-at-Risk forecasts for the bonds in the sample. Several statistical tests have been developed to assess the quality out-of-sample performance of VaR estimates. One simple option is to measure the number of observations until a failure is observed (see Kupiec [1995]). Unfortunately, this test does not consider the total number of failures over the period under analysis; it also has a very poor accuracy on small samples, since the sample has to contain more than 438 observations to reject the null hypothesis at the 5% confidence level. An extension of this test consists of focusing on the proportion of failures. If we define the indicative function $I_t(\alpha)$

$$I_t(\alpha)=\begin{cases}1 \text{ if } x_{t,t+1} \geq -VaR_\alpha \\ 0 \text{ if } x_{t,t+1} < -VaR_\alpha\end{cases}$$

where $x_{t,t+1}$ denotes the profit and loss between time t and t+1, then it should be i.i.d. and follow a Bernoulli process of parameter α. The likelihood ratio test of non-rejection of the null hypothesis is given by:

$$LR_{UC}=-2\ln\left[\left(\frac{1-\alpha}{1-\frac{I_\alpha}{T}}\right)^{T-I_\alpha}\left(\frac{\alpha}{\frac{I_\alpha}{T}}\right)^{I_\alpha}\right]$$

with $I_\alpha = \Sigma_{t=1}^{T} I_t(\alpha)$. Under the null hypothesis that the proportion of failure corresponds to the theoretical level, the likelihood ratio is distributed as a $X^2(1)$. While this

EXHIBIT 7

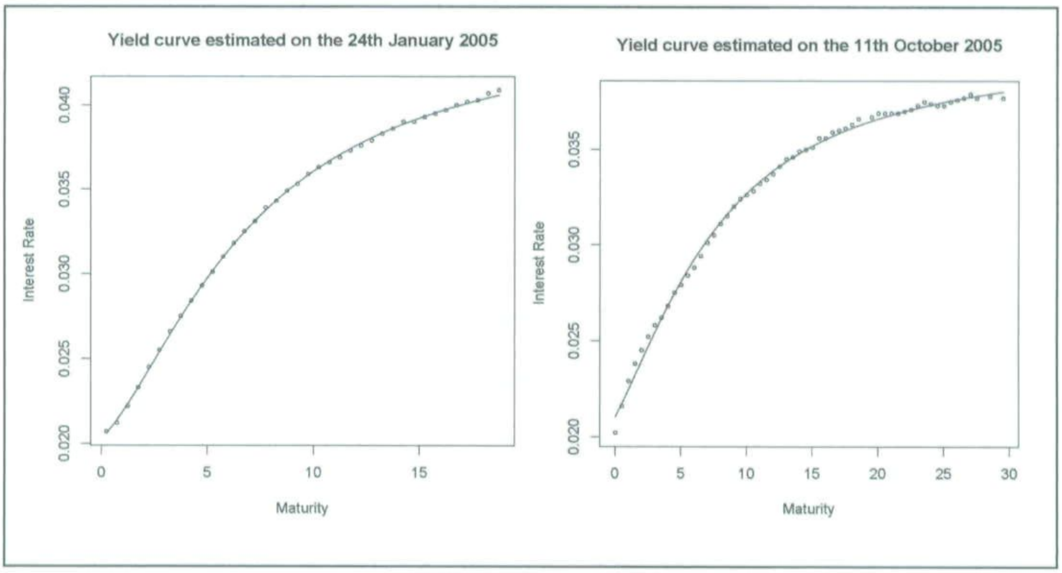


EXHIBIT 8

Summary Statistics for the Pricing Errors in the Calibration Sample (in %)

	ε				$\lvert\varepsilon\rvert$			
	Mean	St. Dev.	Min	Max	Mean	St. Dev.	Min	Max
OAT1	-0.0627	0.0430	-0.2372	0.1463	0.0636	0.0416	3.12e-4	0.2372
OAT2	-0.0933	0.0555	-0.3121	0.1917	0.0944	0.0537	6.05e-4	0.3122
OAT3	-0.0320	0.0484	-0.1761	0.3405	0.0442	0.0377	1.42e-4	0.3405
OAT4	-0.0160	0.1515	-0.5095	0.5521	0.0118	0.0966	1.08e-3	0.5521
OAT5	0.0128	0.0986	-0.5028	0.3615	0.0820	0.0560	5.03e-4	0.5028
OAT6	0.0204	0.0915	-0.5274	0.3821	0.0739	0.0739	1.92e-4	0.5274
OAT7	0.1427	0.1104	-0.4795	0.9779	0.1502	0.0996	5.34e-4	0.9779
OAT8	-0.4722	-0.4722	-1.5058	0.9499	0.4781	0.1938	2.65e-2	1.5058
OAT9	-0.5271	-0.5271	-1.5395	0.0552	0.5274	0.1796	5.52e-2	1.5395

EXHIBIT 9

VaR Back-Testing of the Calibration Sample for 95% (Solid Line) and 99% (Dashed Line)

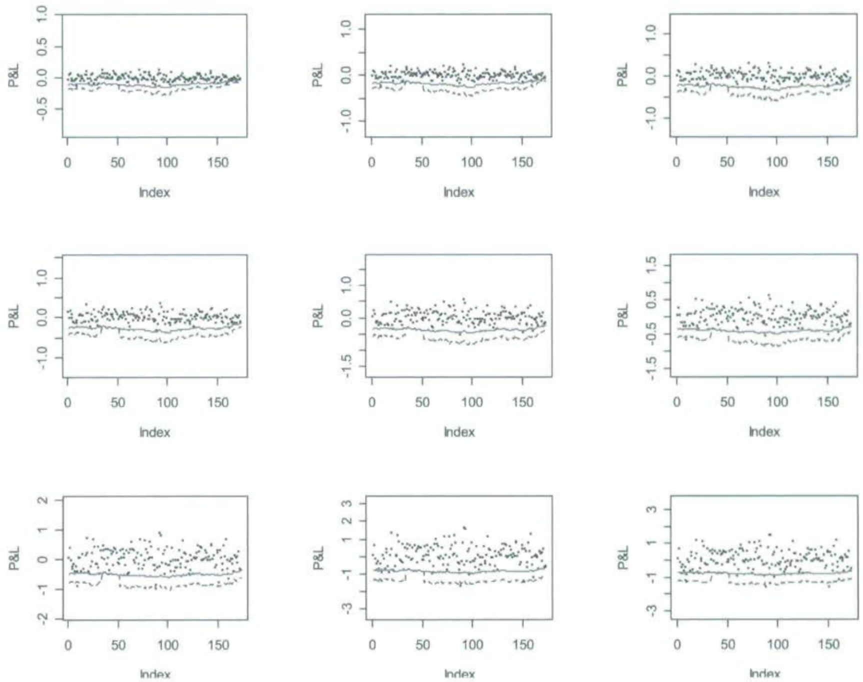


EXHIBIT 10

Out-of-Sample Number of Failures of the VaR Estimate with the Results of the Christoffersen Test Reported in Paranthesis (Probability of Rejecting the Null Hypothesis in the Global Conditional Coverage Test)

	NS	
	99%	95%
5.5% 25/04/2007	0 (3.48)	9 (3.81)
5.25% 25/04/2008	0 (3.48)	5 (2.14)
4% 25/04/2009	0 (3.48)	4 (3.45)
4% 25/10/2009	1 (0.38)	4 (3.45)
5% 25/04/2012	1 (0.38)	5 (2.2)
4% 25/04/2013	1 (0.38)	6 (1.38)
5% 25/10/2016	1 (0.38)	7 (0.94)
8.5% 25/04/2023	4 (2.38)	14 (2.98)
6% 25/10/2025	4 (2.38)	13 (2.01)

test has the advantage of being easy to implement and is frequently used to validate VaR models, it also suffers from a lack of power for small samples, which is a problem for all "Event Probability Forecast Evaluation" tests.[7]

To alleviate this concern, we have chosen to report the results of our methodology both for a 99% and a 95% confidence threshold. This allows us to ensure that our VaR measures are not overestimated by reducing the number of data needed to construct the non-rejection test. We also select the Christoffersen [1998] test, which extends the Kupiec test by explicitly accounting not only for the number, but also for the sequence, of VaR violations taken into account. In fact, the violations should be independent of each other if the VaR model is well-specified. The Christoffersen test consists in forming a join test regrouping the unconditional test of Kupiec and the following independence test:

$$LR_{ind} = -2\ln\left[\frac{(1-\hat{\pi}_2)^{T_{00}+T_{10}}\,\hat{\pi}_2^{\,T_{01}+T_{11}}}{(1-\hat{\pi}_{01})^{T_{00}}\,\hat{\pi}^{T_{01}}(1-\hat{\pi}_{11})^{T_{10}}\,\pi_{11}^{\,T_{11}}}\right]$$

with $\hat{\pi}_{ij} = P\{I_t = i,\ I_{t-1} = j\}$, $\hat{\pi}_{01} = T_{01}/(T_{00}+T_{01})$, $\hat{\pi}_{11} = T_{11}(T_{10} + T_{11})$ and $\hat{\pi}_2 = (T_{01} + T_{11})/T$.

Hence, by adding the null hypothesis to the unconditional coverage test of Kupiec, one can merge those two tests in a global conditional coverage test that is simply $LR_{CC} = LR_{UC} + LR_{ind}$ and $LR_{CC} \rightarrow \mathrm{X}^2(2)$.

The back-testing analysis we perform on our out-of-sample period represented in Exhibit 9 reveals that both 99% and 95% VaR estimates obtained with our methodology cannot be rejected at a 5% confidence threshold. In Exhibit 10 we have given the results of the back-test for the nine bonds in the calibration sample.[8] Our VaR measure passes the Christoffersen test successfully, for all bonds in our calibration sample. One can also note that the higher the maturity of the bond, the higher the number of failures, except for the two first maturities.

In Exhibit 11, we report for comparison purposes the performance of naïve alternative methods such as Gaussian VaR and historical VaR, both of which fail the Christoffersen test.

CONCLUSION

The approach we propose in this article offers a potentially useful alternative to the standard methods used for Value-at-Risk estimation in a fixed-income environment. We have proposed to use the time-varying parameters of a popular yield curve model to proxy for factors affecting changes in shape of the yield curve, and have introduced a copula modelling to characterize the dependence structure of those various risk factors.

While we have chosen to focus on a Student copula, the approach presented in this article could easily be extended to other copulas, such as the skew Student-t copula, which offers an additional flexibility to fit the observed dependence structure between risk factors. This flexibility comes, however, at the cost of additional parameters, and the benefits of using a less parsimonious model should be assessed on an out-of-sample basis.

EXHIBIT 11

Out-of-Sample Number of Failures for Alternative VaR Estimation Methods with the Results of the Christoffersen Test Reported in Paranthesis (Probability of Rejecting the Null Hypothesis in the Global Conditional Coverage Test)

	Gaussian VaR		Historical VaR	
	99%	95%	99%	95%
5.5% 25/04/2007	52 (239.03)	1 (0.38)	10 (64.44)	7 (30.11)
5.25% 25/04/2008	50 (192.8)	1 (0.38)	9 (58.54)	5 (3.75)
4% 25/04/2009	53 (202.19)	0 (3.48)	4 (7.31)	3 (6.5)
4% 25/10/2009	55 (206.34)	0 (3.48)	0 (17.75)	0 (3.48)
5% 25/04/2012	55 (202.33)	0 (3.48)	0 (17.75)	0 (3.48)
4% 25/04/2013	55 (195.92)	0 (3.48)	0 (17.75)	0 (3.48)
5% 25/10/2016	53 (184.67)	2 (6.81)	0 (17.75)	0 (3.48)
8.5% 25/04/2023	44 (135.32)	7 (55.58)	0 (17.75)	0 (3.48)
6% 25/10/2025	43 (135.98)	7 (55.58)	0 (17.75)	0 (3.48)

ENDNOTES

[1]See for example Jorion [2000] for a detailed introduction to VaR estimation.

[2]See Singh [1997] for VaR estimation with PCA analysis.

[3]In the bi-variate case, the useful information in the correlation matrix R can be summarized in the value of a unique correlation coefficient that we denote by ρ.

[4]The copula parameters are estimated on a daily basis from a rolling window of 250 trading days (see Section II).

[5]It is also possible to use a kernel estimator to obtain the density of the different random variables. Other non-parametric methods, such as the empirical method (Deheuvels [1978]) or the kernel copula (Scaillet [2000]) are presented with empirical applications in Cherubini [2004].

[6]The French market is organized upon a dual structure; see Grimonprez and Meyfredi [2005] for a full description.

[7]As Christoffersen [1998] pointed out, an accurate VaR needs to simultaneously satisfy two criteria. The first one is the unconditional coverage property corresponding to a number of failures equal to the chosen risk threshold. The second is the independence property, assuming that the failures are independently distributed.

[8]With the Christoffersen test two results could display different values for the same risk level and the same number of failures.

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