Valuation of Bond Illiquidity: *An Option-Theoretical Approach*

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Liquidity is an essential pricing factor. It measures the ease with which a security can be converted into money or other assets. Holding illiquid assets imposes additional risks to an investor, as her portfolio cannot be adjusted without cost or delay. Consequently, the modeling of liquidity is an important problem for e.g., the design of investment strategies and risk management systems. A sound risk management methodology has to consider that thinly traded securities might not be liquidated in an appropriate time span and that prices can be at a considerable discount relative to those of comparable liquid instruments. Hence, risk measures such as value at risk have to be corrected for the degree of liquidity. Likewise, investors must take liquidity effects into account for the successful implementation of trading strategies.

While these fundamental considerations are valid for any class of assets, liquidity is especially important in fixed-income markets where liquidity spreads are directly visible. In most cases, illiquid and liquid but otherwise similar bonds simultaneously exist such that liquidity spreads can be easily calculated from the price differences of these bonds. This is not true for many other instruments such as stocks, because cash flows from equities usually differ substantially. Bond spreads form the basis for a variety of trading strategies. However, the existence of these spreads per se does not necessarily offer valuable trading opportunities. An evaluation of these strategies rather requires the knowledge of the fair size of a spread. As liquidity is a major factor for bond spreads, it is not surprising that liquidity risk is crucial for investors who follow investment strategies based on relative bond price differences. A drastic demonstration of the possible dangers arising from liquidity risk are the huge losses that hedge fund Long-Term Capital Management (LTCM), which pursued spread convergence arbitrage strategies, suffered during the aftermath of the 1998 Russian crisis.

The effects of liquidity risk on bond prices are well-documented in several empirical studies. For example, Amihud and Mendelson [1991] and Kamara [1994] find that illiquid Treasury Notes trade at a discount compared to more liquid Treasury Bills of the same maturity. The studies of Warga [1992]; Redding [1999], and Krishnamurthy [2002] show that there is a significant yield difference between U.S. government bonds of the latest issuance and comparable older bonds. This 'on-the-run' effect is attributed to the superior liquidity of the newly-issued bonds. Similarly in the German market, Kempf and Uhrig-Homburg [2000] find a significant illiquidity discount for government bonds with lower liquidity.[1]

There is also a large volume of literature which examines the theoretical impact of liquidity on asset prices. The studies can be roughly classified by the description of liquidity. For example, Amihud and Mendelson [1986] examine in their seminal paper the price dimension of liquidity by incorporating transaction costs. Other studies such as Grossman and Miller [1988] focus on the time dimension of liquidity and identify liquidity risk as immediacy risk; i.e., an asset is considered to be more liquid than some other one if it can be traded more often. Many of these theoretical studies focus on stocks. The analysis of bond liquidity, however, allows one to investigate further issues. As opposed to stocks, which basically are securities with infinite maturity, bonds mature in a finite time. Therefore, liquidity spreads of bonds inevitably vary over time. In particular, bond spreads must vanish when time to maturity approaches zero, because at maturity the illiquid bond is redeemed in the same manner as the liquid bond.

The aim of our article is to analyze the pricing consequences of bond liquidity. We provide an easily applicable method for the calculation of the resulting illiquidity discounts. We derive a methodology for the valuation of illiquid bonds by extending a model by Longstaff [1995] originally designed to value the marketability of equities. Longstaff argues that liquidity offers a valuable option to trade. The option analogy enables Longstaff to determine the maximum discount for a stock that is restricted from trading for one period of time.

Our approach extends the work of Longstaff [1995] in two ways. First, we focus on fixed-income markets. We develop a model within a framework of stochastic interest rates of the Vasicek type. Second, we discard the assumption that the security is restricted from trading for one time span only and becomes perfectly liquid afterwards. In our model, the illiquid bond can be traded on any desired set of dates during its lifetime. This specification enables us to model time-varying liquidity of bonds such that we can investigate the effects of liquidity dynamically.

Our model of liquidity spreads shows several meaningful and plausible properties. Liquidity spreads are humped-shaped functions of maturity and increase with interest rate volatility. Furthermore, liquidity spreads are not only influenced by the number of possible trading dates, but also by their distribution over time. In particular, a bond which is traded more often at the beginning of its lifetime bears a lower liquidity spread compared to another otherwise identical bond with the same number of trading dates. Although illiquid zero bond values depend on the current short rate level, the liquidity spread is invariant of the short rate.

We conduct an empirical study using data of the German Jumbo Pfandbrief market. Applying an implicit estimation method to calibrate our model, we find that our liquidity model is able to explain to a large part the empirically observed spreads, while credit risk has nearly no explanatory power for the bond prices. The obtained estimates for the parameters of our model allow one to translate observable spreads into more intuitive parameters such as the effective trading frequency of a bond.

OPTION-THEORETICAL VALUATION OF ILLIQUIDITY

In this section, we describe our notion of liquidity. We show that the liquidity advantage of perfectly liquid bonds can be determined by fair prices of exotic interest rate options.

Notion of Liquidity

Although the majority of market participants consider the liquidity of assets in portfolio decisions, it is a challenge in itself to present a precise formal definition of liquidity that everyone can agree on. For example, Keynes [1930, pp. 67], considers an asset as more liquid if it is "more certainly realizable at short notice without a loss." The basic insight of this definition is the combination of at least two dimensions of liquidity: a time dimension ("at short notice") and a price dimension ("without a loss"). The at least two-dimensional nature of liquidity can also be found in other definitions, e.g., Hirshleifer [1972], who defines illiquid assets as "characterized by a relatively large discount for 'premature' realization." Since liquidity is a bundle of properties rather than a single property, it is hard to develop a generally accepted formal model of liquidity effects.

Many studies in the literature define liquidity by its price dimension. Typically, these models impose exogenous bid-ask spreads or transaction costs that cause portfolio revisions to be costly. Examples for models of this class include amongst others the work of Amihud and Mendelson [1986]; Vayanos and Vila [1999], and Huang [2003].[2] The degree of illiquidity is measured by the size of the transaction costs.

Illiquid fixed-income markets are often characterized by longer non-trading periods. Therefore, we define liquidity in our theoretical model by the possible frequency of trading, concentrating on the time dimension of liquidity. In this sense, a perfectly liquid asset can be traded continuously by investors throughout its lifetime. This is not true for bonds in thin markets. It is possible to trade illiquid assets only at certain points in time. Related models with similar specifications of liquidity include Rogers and Zane [2002]; Subramanian and Jarrow [2001], and Longstaff [1995]. In our approach, the total information about the liquidity of a bond is contained in the set of possible (not necessarily equidistantly distributed) trading dates $0 \leq t_1 < t_2 < \ldots < t_N = T$. In most cases, we consider trading dates that are uniformly allocated over time. In these cases, we measure the magnitude of liquidity by one parameter, namely, the distance $\Delta t = t_{i+1} - t_i$ between two adjacent trading dates. However, our valuation model is flexible enough to also cover arbitrarily distributed trading dates.

Another class of studies focuses on a third dimension of liquidity—quantity. If the market impact (price dimension) or the frequency of possible trades (time dimension) depend on the desired size of a trade, the quantity dimension is already implicitly contained in the other two dimensions of liquidity outlined above. Models like Longstaff [2001] consider the quantity dimension in a more explicit way by limiting the maximum trading volume per time for illiquid securities.

Illiquidity Discount under Stochastic Interest Rates

In this section, we present the basic ideas behind our option-theoretical approach to calculating illiquidity discounts for thinly traded bonds. Our goal is to establish a link between option payoffs and benefits of liquidity similar to the approach of Longstaff [1995] for valuation of the marketability of stocks. To achieve this aim, we describe the benefits of zero bonds with different degrees of liquidity by regarding the potential trading gains of an investor with perfect market timing ability.

However, our notion of illiquidity differs from the concept of non-marketability over a single period that Longstaff [1995] employs for equities. Since we assume that an illiquid bond is tradeable only on a certain set of dates during its lifetime, we have to take a modified approach that accounts for multiple periods in which trading of the illiquid bond is not possible. Our approach enables us to identify the benefits of liquidity as exotic interest rate option payoffs. The difference of the option prices yields the illiquidity discount.[3]

We consider a financial market with a stochastic short rate r_t. There are two zero bonds with the same maturity T. One bond is perfectly liquid, i.e., it can be traded at any time in $[0, T]$. Its market value at time t is denoted by $P(t,T)$. The other bond is illiquid. Transactions comprising this bond can take place only on a set of dates $0 \leq t_1 < t_2 < \ldots < t_N = T$. At time T, both bonds pay their face value (normalized to 1) with certainty, i.e., there is no difference in credit risk.

Next, we consider an investor with perfect market timing. For each bond, we characterize the investor's proceeds at time T if she sells the security optimally in $[0, T]$.

First, we investigate the investor's proceeds if she holds a perfectly liquid bond at time $t = 0$. At a time t^*, she chooses to sell the bond and invests the received sale price $P(t^*, T)$ in short term securities with the variable interest rate r_t from t^* to T. As a result of her perfect market timing ability, the investor will choose the optimal t^* for switching from long-term bonds to short-term money market instruments, such that her proceeds at time T will equal

$$\max_{0 \leq t \leq T} \left\{ P(t,T) \cdot e^{\int_t^T r_s ds} \right\} \tag{1}$$

Evidently the gains of this strategy can be identified as the payoff of a fixed strike (with strike price equal to zero) lookback call option[4] on the fictive underlying $P(t, T)\, e^{\int_t^T r_s ds}$ which equals the price of the liquid bond including short term interests accrued from t to T.

Second, we repeat the analysis for the case of an illiquid bond. There are two differences from the previous case. On the one hand, the investor can no longer sell the bond at any time throughout $[0, T]$ but is restricted to the set of possible trading dates $\{t_1, t_2, \ldots, T\}$. Consequently, the maximum is now formed over these discrete points in time. On the other hand, the investor cannot prematurely sell the illiquid bond at the same price as the liquid bond. In fact, the bond price is reduced by the future liquidity discount at each date t_i that still has to be determined.

To reduce the complexity of the necessary calculations for the liquidity spread $\gamma(T)$ for a bond with maturity T, we impose—for calculation purposes—the assumption that the liquidity spread of this bond is equal to $\gamma(T)$ at every trading date in its lifetime. Thus, the sale price of the illiquid bond at t is given by

$$e^{-\gamma(T)\cdot(T-t)}P(t,T)$$

As a consequence, the investor's proceeds at time T of an optimal sale strategy for the illiquid bond read

$$\max_{i=1,\ldots,N}\left\{e^{-\gamma(T)\cdot(T-t_i)}P(t_i,T)\cdot e^{\int_{t_i}^{T} r_s ds}\right\} \tag{2}$$

The gains from this strategy also equal the payoff of a fixed strike (with a strike price equal to zero) lookback call option. However, the underlying is now a different fictive asset with price $P(t,T)e^{\int_t^T (r_s-\gamma(T))ds}$. Additionally, the maximum is now monitored discretely rather than continuously as in the case of the perfectly liquid bond. The value of the proceeds in the liquid bond case exceeds the corresponding value in the illiquid bond case. The difference stems from the fact that the better liquidity of the first bond offers valuable trading opportunities for investors holding this bond. Therefore, we claim that the relative difference of the value in the illiquid case to the value in the liquid case yields the discount for illiquidity. Thus, we can derive an implicit equation for $\gamma(T)$.

Evidently, liquidity spreads $\gamma(T)$ arising from our model depend on the time to maturity of the illiquid bond. We simplify the determination of $\gamma(T)$ by setting future liquidity spreads equal to $\gamma(T)$. In principle, our model framework allows time-varying liquidity spreads $\gamma_{t_i}(T)$ at any trading date t_i at the stage of the computation of $\gamma_0(T) = \gamma(T)$. In this case, the liquidity spreads have to be determined recursively. The computation is always feasible because of the one-directional dependence of the spreads, i.e., future spreads affect current spreads but not vice versa. Thus, we have to apply a backward-solving algorithm. Given the corresponding liquidity spreads $\gamma_{t_i}(T)$ for all trading dates $t_i > t_j$, we can employ these liquidity spreads in Equation (2) to find an implicit solution for the liquidity spread $\gamma_{t_j}(T)$ for a longer time to maturity $T-t_j$. Obviously, the option analogy valuation leads to a solution for $\gamma_{t_j}(T)$ which is unique within the interval from zero (perfect liquidity) to $\gamma_{t_j}^{\max}(T)$. The maximum liquidity spread $\gamma_{t_j}^{\max}(T)$ corresponds to the case in which trading is not possible during the security's whole lifetime.

The more complex recursive algorithm requires a number of steps. The steps themselves again contain many iterations as we are forced to apply numerical methods such as Monte Carlo simulation or numerical integration in each step. The fast-growing complexity of our model works against our goal to develop an easily applicable method for the determination of illiquidity discounts. A comparative analysis of the two methods in the appendix shows that there is a large gain in computing speed while the resulting spread differences for reasonable parameter values are small in comparison to the empirical changes of liquidity spreads in our data sample. Furthermore, the qualitative properties of the liquidity spread are the same for both algorithms. Thus, we use the simplified algorithm for the numerical determination of liquidity spreads.

After ascertaining the equivalence between certain options' payoffs and the corresponding proceeds from liquid and illiquid assets, we can apply our basic valuation principle, the option analogy in the spirit of Longstaff [1995]. The principle circumvents a more complex valuation. Instead of computing a complex general equilibrium model where some markets are illiquid, we introduce a virtual complete fixed-income market. Within this virtual market, the above specified options are priced by using risk-neutral valuation techniques as presented in the next section. The relative difference of both option values is then interpreted as the liquidity discount in the illiquid market. Note that the option analogy does not claim the equivalence between bond prices and option prices. It claims rather that the relative difference of the option prices equals the one of the bond prices. Furthermore, our model considers strategies of informed investors that involve only long positions in the liquid and the illiquid bond such that possible arbitrage opportunities between both securities are ruled out. Hence, we have to solve the following fixed-point problem to calculate liquidity spreads:

$$\begin{aligned} e^{\gamma(T)\cdot T} &= \frac{\text{value liquid bond}}{\text{value illiquid bond}} \\ &= \frac{\text{option value liquid case}}{\text{option value illiquid case (depends on } \gamma(T))} \end{aligned} \tag{3}$$

Though we have to use numerical methods to solve Equation (3), it is an easier valuation formula than those resulting from an equilibrium model of the illiquid, incomplete market.

One might feel that our model allows for arbitrage strategies of the kind that an investor buys illiquid bonds and sells liquid bonds with the same maturity. This is a general problem for every model dealing with liquidity spreads of assets with identical cash flows. However, our model circumvents this problem because we never analyze liquid and illiquid bonds simultaneously. Since we rather regard both assets in two separate portfolio choice problems, our model is arbitrage-free. Similar models such as Longstaff [2001] use the same technique to avoid arbitrage opportunities.

A second objection states that perfect market timing is a too strong trading motive such that our model exaggerates the magnitude of liquidity spreads. Longstaff [1995] declares perfect foresight to be an extreme case which allows one to derive upper bounds for illiquidity discounts. On the other hand, the investor is restricted to a buy-and-hold strategy. Clearly, the investor's earnings could be greatly increased if she were in a position to apply truly dynamic strategies, as already mentioned by Kempf [1999, pp. 74]. This is the reason why we scale the liquidity discount by the introduction of a correction factor in order to vary the intensity of the trading motive. This factor can also able reflect the effects described above.

The information correction factor c measures the degree of the deviation of the theoretically assumed from the real trading motives, i.e., the liquidity spread is given by

$$\gamma^{\text{adj}}(T) = c \cdot \gamma(T) \tag{4}$$

Due to its origin, c is a systematic factor. It is not bond-specific, which allows its empirical identification. In our empirical study, we expect c to be positive and smaller than 1, which indicates a weaker trading motive than in the case of perfect information. In principle, a c higher than 1 is possible (albeit not particularly likely) as—unlike the investor in our model—real investors are not constrained in their strategy space.

Discussion of Modeling Approach

Although the clearness and simplicity of the valuation by the option analogy is appealing, this approach can be questioned. Typically, concerns rest on the assumption of perfect market timing. The investor's market timing ability creates her trading motives. Both trading motives and trading possibilities are the two key components of every model that determine the size of the equilibrium liquidity spread. At this point, we emphasize that our liquidity spreads come primarily from restricted trading possibilities rather than perfect foresight. This is due to the fact that we compare the value of a liquid bond (trading is always possible) with an illiquid bond (trading is restricted) from the perspective of a specific investor. To give the investor an incentive to trade, we provide her with additional information. However, it is important only that the investor has a trading incentive at all; the type of the trading motive is less crucial. It could be easily exchanged for a different motive. Moreover, it is irrelevant whether the investor actually has superior information or just believes so.

The chosen specification is only one way out of many analyze liquidity spreads. Every approach requires the formulation of a trading motive. While other approaches like those assuming, e.g., heterogeneous investors in an imperfect market setting, reveal economic insights about liquidity spreads, the practical implementation of these models is hardly achievable. Typically other problems in terms of the calibration arise.

Furthermore, the desire to trade induced by other reasons does not necessarily differ considerably from that created by perfect foresight. The trading wishes of a consumption-smoothing investor with a standard utility function show similarities to those of an investor with perfect information in our model. For example, a rising volatility leads in both cases to an increase of the trading needs, as the following comparison shows. An investor without superior information wants to adjust her portfolio more often if prices change more quickly, to smooth her consumption stream. As we will show when analyzing the properties of the liquidity spread, the described option analogy reveals that an investor with perfect foresight can also realize a greater value in more volatile markets.

As a result, we find that—though the assumed trading motive certainly does not represent all possible forms of trading needs—it captures to a large extent the economically relevant factors that drive actual trading wishes. For this reason, we understand our approach as a reduced-form model with meaningful properties which has fundamental computational advantages compared to other settings in this field. The simplicity of our approach only yield a good tractability of the model but also eases the development of an economic intuition about the price of liquidity. Ultimately, the relevance of the model can be verified only by empirical considerations.

DETERMINATION OF THE LIQUIDITY SPREAD

The liquidity spreads $\gamma^{adj}(T)$ and $\gamma(T)$ arise from the values of two interest rate lookback options. In particular, we have to evaluate both a continuously and a discretely monitored lookback call option with strike price zero written on a zero bond plus earned interest. The valuation of these exotic interest rate options itself is an interesting and challenging issue from a theoretical perspective. However, we cannot obtain the required option values from the literature. This is why we regard it as necessary to sketch the derivation of the required lookback options first. Then, we show how to determine $\gamma^{adj}(T)$ and $\gamma(T)$ from these option values.

Valuation of Interest Rate Lookback Derivatives

In this section, we determine the value of lookback derivatives having a payout as specified in Equations (1) and (2). We consider a complete and frictionless market for fixed-income securities. The short term interest rate r_t follows a Vasicek [1977] process and its well-known dynamic is given by[5]

$$dr_t = \kappa(\theta - r_t)\cdot dt + \sigma \cdot dz_t \tag{5}$$

where θ stands for the long-term mean of the short rate and z_t for the value of a standard Wiener process under the risk-neutral (short rate) measure. κ denotes the speed of mean-reversion. This specification uniquely determines the prices of zero bonds $P(t,T)$ with a face value of unity and expiration date T as

$$P(t,T) = e^{A(t,T) - r_t \cdot B(t,T)} \tag{6}$$

where

$$B(t,T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa},$$
$$A(t,T) = \left(\frac{\sigma^2}{2\kappa^2}\right)(T-t) - \theta\cdot(T-t) + \left(\theta - \frac{\sigma^2}{\kappa^2}\right)B(t,T) + \left(\frac{\sigma^2}{4\kappa^3}\right)(1 - e^{-2\kappa(T-t)})$$

In the first step, we consider the continuously monitored lookback call option with strike price zero written on the zero bond plus earned interest. We denote the value of this exotic option by $\overline{S}(t,T)$. The payoff from this contract at maturity T reads:

$$\overline{S}(T,T) = \max_{0\le t\le T}\left\{P(t,T)\cdot e^{\int_t^T r_s\,ds}\right\}$$

Recall that $\overline{S}(T, T)$ is the value obtained from selling a liquid zero bond maturing at time T at the optimal time. Therefore, $\overline{S}(t, T)$ stands for the value at time t obtained with one zero bond when its holder has perfect market timing ability. In order to find a representation for the value of $\overline{S}(0, T)$ at time 0, we apply risk-neutral valuation. This leads us to the following representation:

$$\overline{S}(0,T) = \mathbb{E}_0\left(e^{-\int_0^T r_s ds}\max_{0\le t\le T}\left\{P(t,T)\cdot e^{\int_t^T r_s ds}\right\}\right) = \mathbb{E}_0\left(\max_{0\le t\le T}\left\{P(t,T)\cdot e^{-\int_0^t r_s ds}\right\}\right)$$

For ease of notation, we define:

$$y_T := \ln\left(\max_{0\le t\le T}\left\{P(t,T)\cdot e^{-\int_0^t r_s ds}\right\}\right) - \ln(P(0,T))$$

The corresponding density function $f(y_T)$ is given by

$$f(y_T) = \frac{1}{\sqrt{\xi(T)}}\Phi'\left(\frac{y_T + \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + e^{-y_T}\cdot\Phi\left(-\frac{y_T - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + e^{-y_T}\cdot\Phi'\left(\frac{y_T - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right)$$

where $\Phi'(\cdot)$ stands for the density function of the standard normal distribution and $\Phi(\cdot)$ for the corresponding distribution function, respectively. $\xi(\tau)$ is defined as:

$$\xi(\tau) = \sigma^2\frac{4e^{-T\kappa}(1 - e^{\kappa\tau}) - e^{-2T\kappa}(1 - e^{2\kappa\tau}) + 2\tau\kappa}{2\kappa^3}$$

Hence, the value of $\overline{S}(0, T)$ amounts to:

$$\overline{S}(0,T) = \mathbb{E}_0\left(\max_{0 \le t \le T}\left\{P(t,T)\cdot e^{-\int_0^t r_s ds}\right\}\right)$$
$$= \int_0^\infty e^{y_T + \ln P(0,T)} \cdot f(y_T) dy_T$$

The evaluation of the integral leads us to a closed-form solution for $\overline{S}(0, T)$:

$$\begin{aligned}\overline{S}(0,T) &= \int_0^\infty e^{y_T + \ln P(0,T)} \cdot f(y_T) dy_T \\ &= P(0,T)\cdot\left(2\cdot\Phi\left(\frac{1}{2}\sqrt{\xi(T)}\right)\right. \\ &\left. + e^{\frac{-\xi(T)}{8}}\sqrt{\frac{\xi(T)}{2\pi}} + \frac{\xi(T)}{2}\Phi\left(\frac{1}{2}\sqrt{\xi(T)}\right)\right)\end{aligned} \tag{7}$$

For the complete derivation, we refer to the appendix.

It follows from representation (7) that $\overline{S}(0, T)$ is proportional to the zero bond price. In particular, the proportionality rate $\overline{S}(0, T)/P(0, T)$ is strictly increasing in σ and T and decreasing with κ. These properties can be verified by taking the corresponding derivative. Hence, in line with our intuition, the proportionality rate benefits from a higher interest rate volatility.[6] The reason for this finding is analogous to the typical argument used for plain vanilla options. This argument states that a higher volatility is associated with better chances of higher final payoffs, but the loss-potential is limited. Note that a higher volatility corresponds not only to a higher σ but also to a lower κ. When κ rises, r_t approaches θ more severely; this effect leads to a decline of the volatility.

The property that a longer time to maturity T implies a higher proportionality rate $\overline{S}(0, T)/P(0, T)$ is also consistent with our intuition. This is because the right embedded in $\overline{S}(t, T)$ to sell a zero bond optimally is related to a longer time period.

Since $\xi(T)$ depends neither on the short-rate r_t nor the mean-reversion factor θ, the proportionality rate is also independent of r_0 and θ and the relative change of $\overline{S}(0, T)$ with a change of r_0 or θ coincides with the corresponding relative change of the zero bond price $P(0, T)$.

In the next step, we determine the value of a similar lookback option $\underline{S}(t, \delta; t_1, \ldots, t_N)$ that monitors the underlying at discrete dates $0 \leqslant t_1, \ldots, t_N = T$. Additionally, this derivative takes into account that the monitored underlying $P(t, T) \cdot e^{\int_t^T r_s ds}$ experiences a deterministic discount before maturity by the factor $e^{-\delta\cdot(T-t)}$, $\delta \geqslant 0$. Hence, we can classify $\underline{S}(t, \delta; t_1, \ldots, t_N)$ as the value of a discrete lookback call option written on the fictive underlying $e^{-\delta\cdot(T-t)}\cdot P(t, T) \cdot e^{\int_t^T r_s ds}$ with strike price zero. The payoff from this contract at time T reads:

$$\underline{S}(T, \delta; t_1, \ldots, t_N) = \max_{i=1,\ldots,N}\left\{e^{-\delta\cdot(T-t_i)}\cdot P(t_i, T)\cdot e^{\int_{t_i}^T r_s ds}\right\}$$

Following the arguments in the previous section, we can interpret $\underline{S}(t, \delta; t_1, \ldots, t_N)$ as the wealth an investor with perfect market timing ability can obtain with one zero bond when trading takes place subject to two restrictions; i.e., trading occurs only at $t_1, \ldots, t_N$ and the trading price at time t equals $e^{-\delta\cdot(T-t)} \cdot P(t, T)$.

The value of $\underline{S}(t, \delta; t_1, \ldots, t_N)$ at time 0 is given by:

$$\begin{aligned}&\underline{S}(0, \delta; t_1, \ldots, t_N) \\ &\quad= \mathbb{E}_0\left(e^{-\int_0^T r_s ds}\cdot\max_{i=1,\ldots,N}\left\{e^{-\delta\cdot(T-t_i)}\cdot P(t_i, T)\cdot e^{\int_{t_i}^T r_s ds}\right\}\right) \\ &\quad= \mathbb{E}_0\left(\max_{i=1,\ldots,N}\left\{P(t_i, T)\cdot e^{-\delta(T-t_i)-\int_0^{t_i} r_s ds}\right\}\right)\end{aligned} \tag{8}$$

In the appendix, we show that

$$\begin{aligned}&\underline{S}(0, \delta; t_1, \ldots, t_N) = P(0,T)\cdot e^{-\delta\cdot T} \\ &\cdot\sum_{i=1}^{N} e^{\delta\cdot t_i}\cdot\Phi(d_{i,1}^1, d_{i,2}^1, \ldots, d_{i,i-1}^1; \Theta_i^1)\cdot\Phi(d_{i,i+1}^2, \ldots, d_{i,N-1}^2, d_{i,N}^2; \Theta_i^2)\end{aligned} \tag{9}$$

with

$$d_{i,j}^1 := \frac{\delta\cdot(t_i - t_j) + \frac{1}{2}\cdot(\xi(t_i) - \xi(t_j))}{\sqrt{\xi(t_i) - \xi(t_j)}} \qquad (1 \le j < i)$$

$$d_{i,j}^2 := -\frac{\delta\cdot(t_j - t_i) - \frac{1}{2}\cdot(\xi(t_j) - \xi(t_i))}{\sqrt{\xi(t_j) - \xi(t_i)}} \qquad (i < j \le N)$$

$$\Theta_i^1 := \begin{pmatrix} \phi_{1,1}^i & & \phi_{1,i-1}^i \\ & \phi_{j,k}^i & \\ \phi_{i-1,1}^i & & \phi_{i-1,i-1}^i \end{pmatrix} \qquad (1 \le j,k < i)$$

$$\Theta_i^2 := \begin{pmatrix} \psi^i_{i+1,i+1} & & \psi^i_{i+1,N} \\ & \psi^i_{j,k} & \\ \psi^i_{N,i+1} & & \psi^i_{N,N} \end{pmatrix} \quad (i < j,k \leq N)$$

$$\phi^i_{j,k} := \frac{\xi(t_i) - \max(\xi(t_j), \xi(t_k))}{\sqrt{\xi(t_i) - \xi(t_j)}\sqrt{\xi(t_i) - \xi(t_k)}} \quad (1 \leq j,k < i)$$

$$\psi^i_{j,k} := \frac{\min(\xi(t_j), \xi(t_k)) - \xi(t_i)}{\sqrt{\xi(t_j) - \xi(t_i)}\sqrt{\xi(t_k) - \xi(t_i)}} \quad (i < j,k \leq N)$$

where $\Phi(\cdot;\cdot)$ denotes the corresponding multivariate cumulative standard normal distribution function.

Regarding representation (9), we can easily see that $\underline{S}(0, \delta; t_1, \ldots, t_N)$ is proportional to the zero bond price $P(0, T)$. The proportionality rate $\underline{S}(0, \delta; t_1, \ldots, t_N)/P(0, T)$ increases with a higher volatility. The intuition for this property is—same as for $\overline{S}(0, T)/P(0, T)$—that a higher volatility is associated with a higher upside potential, but the loss potential is restricted. It is obvious that $\underline{S}(0, \delta; t_1, \ldots, t_N)/P(0, T)$ benefits from each additional trading date t_i. The higher δ, the lower the proceeds from selling the bond prematurely; this effect results in a lower value of $\underline{S}$ $(0, \delta; t_1, \ldots, t_N)$. Moreover, $\underline{S}(0, \delta; t_1, \ldots, t_N)/P(0, T)$ does not depend on r_0 and θ, because $P(0, T)$ is the only term of the right-hand side of Equation (9) that is impacted by r_0 or θ.

Computation of the Liquidity Spread

The values of $\overline{S}(t, T)$ and $\underline{S}(t, \delta; t_1, \ldots, t_N)$ allow us to quantify the liquidity spread described by Equation (3). For this purpose, we relate $\overline{S}(t, T)$ and $\underline{S}(t, \delta; t_1, \ldots, t_N)$ to the previously defined "option value liquid case" and the "option value illiquid case" respectively. This link enables us to rewrite the full liquidity spread $\gamma(T)$ for a zero bond which trades only at $t_1, \ldots, t_N$ as follows:

$$\begin{aligned} \gamma(T) &= \frac{1}{T}\ln\left(\frac{\text{option value liquid case}}{\text{option value illiquid case}}\right) \\ &= \frac{1}{T}\ln\left(\frac{\overline{S}(0,T)}{\underline{S}(0,\gamma(T);t_1, \quad ,t_N)}\right) \end{aligned} \tag{10}$$

The computation of $\gamma(T)$ poses two problems. First, $\underline{S}(0, \gamma(T); t_1, \ldots, t_N)$ depends on the value of a multidimensional cumulative standard normal distribution function. As there exists no general closed form solution for this function, we have to employ a numerical procedure like numerical integration or Monte Carlo simulation to get the value of the distribution function. Alternatively, we could apply a Monte Carlo simulation to the expectation in Equation (8). At this point, we opt for the second opportunity: i.e., all computations will be made by applying the Monte Carlo simulation to Equation (8) throughout the article.

Hereby, we have to draw one random number for each trading date $t_1, \ldots, t_N$ to get one value for $\max_{i=1,\ldots,N}\{P(t_i, T)\cdot e^{-\delta\cdot(T-t_i)-\int_0^{t_i} r_s ds}\}$. As we work with 100,000 repetitions, a total of $N\cdot 100{,}000$ random variables are required to get one value for $\underline{S}(0,\gamma(T);\ t_1, \ldots, t_N)$.[7] Second, recall that $\gamma(T)$ requires the value of $\underline{S}(0, \gamma(T); t_1, \ldots, t_N)$ which depends on $\gamma(T)$ itself. As mentioned in the previous section, we have to solve a fixed-point problem to obtain $\gamma(T)$. In particular, we use a binary search algorithm with a relative accuracy of one percent[8] to quantify the solution of the fixed point problem $\gamma(T)$. Obviously, $\gamma^{adj}(T)$ results directly from $\gamma(T)$ by a scale reduction.

PROPERTIES OF THE LIQUIDITY SPREAD

In this section we analyze the properties of the liquidity spread $\gamma^{adj}(T)$. Throughout the analysis, we set the information correction factor c equal to 1 such that $\gamma^{adj}(T) = \gamma(T)$. Obviously, the results for different correction factors are analogous.

As Exhibit 1 shows, the liquidity spread $\gamma^{adj}(T)$ lies between 15.1 and 29.5 basis points (bp) for a time to maturity from 0.5 to 15 years. For a lower or a higher time to maturity, the corresponding liquidity spread is closer to zero. The interest rate parameters used for this example are those that are obtained by empirical estimations presented in the following section. Moreover, the distance between each pair of consecutive trading dates $t_{i+1} - t_i$, for all $i = 1, \ldots, N-1$, is equal to $\Delta t = \frac{1}{10}$. This value of Δt corresponds to the case that trading occurs ten times a year. Without regarding the empirical liquidity spreads yet, we find that the easily applicable option-theoretical approach provides us with proper values for the liquidity spread; $\gamma^{adj}(T)$ is of a significant size and is not of an unexpected high magnitude. Next, we examine how $\gamma^{adj}(T)$ is affected by a variation of time to maturity and volatility. Furthermore, we regard the aging-effect and discuss the impact of further parameters on $\gamma^{adj}(T)$.

Impact of Time to Maturity

When a variation of the time to maturity T is regarded, the time Δt between each two consecutive trading dates is held constant. This condition implies that $T/\Delta t$ is an integer.

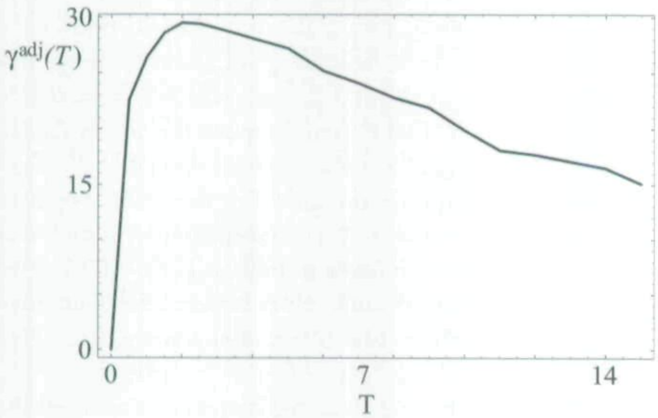
With an increase of time to maturity T for a fixed Δt, both option values divided by the liquid zero bond price, $\overline{S}(t, T)/P(t, T)$ and $\underline{S}(t, \delta; t_1, \ldots, t_N)/P(t, T)$, rise. However, the first ratio benefits more from an increase of T than the second one. This is due to the fact that an increase of T is associated with more opportunities when holding a lookback option with continuous monitoring of the underlying rather than a lookback option with discrete monitoring. Therefore, the ratio between the values of a liquid and an illiquid zero bond, given by $e^{\gamma^{adj}(T)\cdot T} = e^{c\cdot\gamma(T)\cdot T} = (\overline{S}(t, T)/\underline{S}(t, \gamma(T); t_1, \ldots, t_N))^c$, increases with T.

In the next step, we regard the liquidity spread $\gamma^{adj}(T) = c\cdot\gamma(T)$ rather than the expression $e^{c\cdot\gamma(T)\cdot T}$. At first glance, it is not clear whether the increase of $e^{\gamma^{adj}(T)\cdot T}$ with T arises from a potential increase of $\gamma^{adj}(T)$ or from the increase of T. It turns out that the liquidity spread $\gamma^{adj}(T)$ first rises with T and then declines, even though $\gamma^{adj}(T)\cdot T$ always increases with T. This behavior is illustrated by Exhibit 1. Thus, for high values of T, the decrease of $\gamma^{adj}(T)$ is less severe than the increase of T. In particular, $\gamma^{adj}(T)$ approaches zero when T tends to zero or infinity. This property can be shown as follows. According to Equation (10), $\gamma^{adj}(T)$ equals $\frac{1}{c\cdot T}\ln(\overline{S}(0, T)/\underline{S}(0, \gamma(T); t_1, \ldots, t_N))$. Since $\underline{S}(t, \delta; t_1, \ldots, t_N)$ always exceeds $P(t, T)$, the inequality $\gamma(T) \leq \frac{1}{c\cdot T}\ln(\overline{S}(0, T)/P(0, T))$ holds. Since $\frac{1}{c\cdot T}\ln(\overline{S}(0, T)/P(0, T))$ is given in closed form, we can show that this upper bound for the liquidity spread converges to zero when T tends to zero or infinity. As the spread is always positive, it must approach zero for $T \to 0$ or $T \to \infty$.

Economically, it is intuitive that $\gamma^{adj}(T)$ first increases with T, since $\gamma^{adj}(0)$ is equal to zero. The subsequent decrease stems from the fact that we regard an annualized yield spread rather than the total price difference. While the value of a lookback option is increasing with its time to maturity, the marginal increase in value declines with rising T, because it is more and more probable that the maximum of the underlying is attained before T. Thus, $\gamma^{adj}(T)$ will decrease with T from the point where the marginal increase of the option value is less than proportional to the change in T.

EXHIBIT 1
Liquidity Spread as a Function of Time to Maturity

The exhibit shows the liquidity spread measured in bp as a function of the time to maturity T. The distance between two consecutive trading dates is a constant equal to Δt. The used parameter values are: $c = 1$, $\kappa = 0.16643$, $\sigma = 0.01097$, $\Delta t = 0.1$.



Impact of Volatility

There are two ways to look at the volatility of zero bonds. First, the higher σ, the higher the uncertainty of the short rate r_t, and consequently, the higher the uncertainty of the liquid zero bond price $P(t, T)$. It follows from the option pricing formulae that both option values, $\overline{S}(t, T)$ and $\underline{S}(t, \delta; t_1, \ldots, t_N)$, benefit from an increase of σ. But in the case of $\overline{S}(t, T)$, the option value benefits from the increase of σ over the total time period $[t, T]$, while in the case of $\underline{S}(t, \delta; t_1, \ldots, t_N)$ the increase is in effect only at the discrete trading dates $t_1, \ldots, t_N$. Thus, $\underline{S}(t, T)$ rises more sharply with σ than $\underline{S}(t, \delta; t_1, \ldots, t_N)$, which leads to an increase of the liquidity spread $\gamma^{adj}(T)$.

Exhibit 2 illustrates that the liquidity spread increases with σ. In this example for $c = 1$, an increase of σ by 0.01 is related to an increase of $\gamma^{adj}(5)$ of about 23 bp.

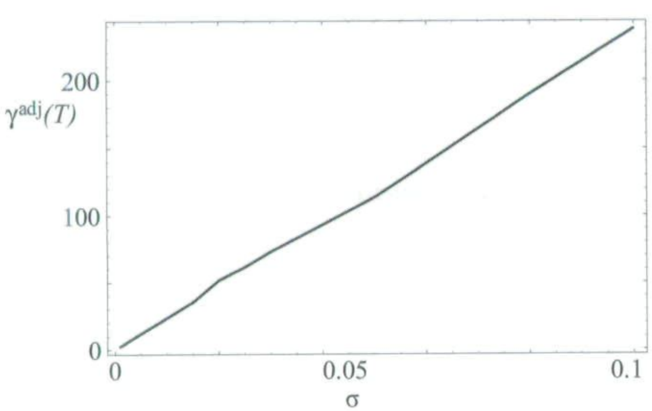
The second way to impact the volatility is to vary κ, the speed of reversion. The higher κ, the stronger is the mean-reversion effect that drives the short rate r_t towards θ; this effect decreases the volatility of r_t as well as the volatility of the zero bond price. Following the same argument as for σ, we also find that the liquidity spread increases when the volatility is increased by a lower κ.

One can argue that volatility stems from the arrival of new information. According to this view, the higher the volatility, the higher the density of new information and/or the greater the importance of new information. In the case of interest rate certainty, i.e., $\sigma = 0$, the complete information about the succeeding short rates r_t is fully reflected by the current zero bond prices $P(t, T)$. Therefore, the ability to find the optimal date to sell the zero bond is worthless and $\gamma^{adj}(T) = 0$.

Since the arrival of new information can be seen as an incentive to trade, the willingness of investors to trade becomes more severe with the density and the importance of new information. This effect results in a higher liquidity spread $\gamma^{adj}(T)$. In other words, when the volatility rises due to a higher density of the arrival of new and important information, the incentive to trade increases and, therefore, the liquidity spread is higher. This intuitive view concerning the effect of the volatility on the liquidity premium is reflected by the liquidity spread $\gamma^{adj}(T)$ arising from our model.

EXHIBIT 2
Liquidity Spread as a Function of σ

The exhibit shows the liquidity spread measured in bp as a function of σ. The distance between each pair of adjacent trading dates is a constant equal to Δt. The used parameter values are: $c = 1$, $\kappa = 0.16643$, $T = 5$, $\Delta t = 0.1$.



Impact of the Distribution of Trading Dates

Since our model is not restricted to the case of equidistant trading dates as considered in the examples of Exhibits 1 and 2, we can also deal with arbitrarily distributed trading dates $t_1, \ldots, t_N$. It has been found empirically that many bonds are traded more often at the beginning of their lifetime than at the end. Typically, bond trading volume decreases with age since more and more bonds are absorbed by buy-and-hold investors. In the following, we examine how this bond aging effect impacts the liquidity spread. We compare the liquidity spread of equidistantly distributed trading dates with a different distribution of the same number of trading dates exhibiting the aging-effect property. To keep things simple, we first consider the case of three trading dates, at time $t_1 = 0$, at maturity $t_3 = T = 5$, and at one arbitrary date $t_2 \in [0, 5]$ in between. Next, we analyze how the location of t_2 effects the liquidity spread. It is especially interesting whether there exists a particular date t_2 which minimizes the liquidity spread. An optimal value of t_2 below $T/2 = 2.5$ would indicate that it is favorable for the illiquid bond value if the bond is traded more often at the beginning of its lifetime. This case corresponds to the observed aging effect. On the contrary, an optimal intermediate trading date t_2 above $T/2$ suggests that the illiquid bond especially benefits from trading dates close to maturity and that the aging-effect distribution of trading dates is rather harmful for the illiquid bond value.

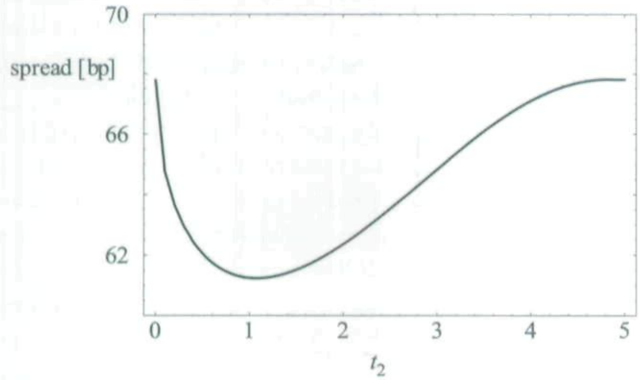
Exhibit 3 shows the liquidity spread as a function of the second trading date. For $t_2 = 0$ and $t_2 = 5$, we effectively have only two trading dates and therefore the spread is maximal for these values for t_2. As a consequence, the liquidity spread declines first with t_2 until it reaches a local minimum and increases afterwards. Of course, the possibility to trade the bond at the second date is especially valuable if the second trading date is not too close to one of the other two trading dates. According to Exhibit 3, the optimal trading date lies at about 1.2 years which is remarkably below $T/2 = 2.5$. This finding supports the fact that a distribution of trading dates satisfying the aging effect is advantageous for the illiquid bond value. The fact that the optimal trading date lies at this level is not surprising. It is such that $\xi(t_2) - \xi(0)$ is approximately as high as $\xi(T) - \xi(t_2)$. This condition means that the differences between the discounted zero bond values from two consecutive trading dates, $P(t_2, T) \cdot e^{-\int_0^{t_2} r_s ds} - P(0, T)$ and $P(T, T) \cdot e^{-\int_0^{T} r_s ds} - P(t_2, T) \cdot e^{-\int_0^{t_2} r_s ds}$, have the same variance. Since the liquidity discount of the illiquid bond at the intermediate trading date also affects the optimal trading date, the optimal second trading date t_2 slightly deviates from the time t that solves the condition $\xi(t) - \xi(0) = \xi(T) - \xi(t)$.

Now, we extend this idea and regard the case with an arbitrary number of trading dates. Since $\xi(t)$ is a strictly concave function of t, a distribution of the trading dates t_i, such that the differences $\xi(t_{i+1}) - \xi(t_i)$ are constant for all $i = 1, \ldots, N-1$, obviously exhibits this aging-effect property. Following the idea of our three-trading-date example, we see that this particular distribution ensures that every difference between the discounted zero bond values for two consecutive trading dates $P(t_{i+1}, T) \cdot e^{-\int_0^{t_{i+1}} r_s ds} - P(t_i, T) \cdot e^{-\int_0^{t_i} r_s ds}$ has the same variance. According to Exhibit 1, we get a liquidity spread equal to 27.2 bp for $T = 5$ and a constant $\Delta t = 0.1$. Using the same parameter values except for the fact that the trading dates t_i are distributed—as described above—according to the conditions $t_1 = 0$, $\xi(t_{i+1}) - \xi(t_i) = const$ for all $i = 1, \ldots, N-1 = 50$, and $t_N = T = 5$, we get a much lower liquidity spread of 15.4 bp.

Hence, this particular distribution exhibiting the aging effect is more favorable to the value of the illiquid zero bond than the equidistant distribution of the same number of trading dates. This example suggests that the observable aging-effect distribution might result in lower liquidity spreads and higher bond values respectively.

EXHIBIT 3
Liquidity Spread as a Function of Trading Date t_2

The exhibit shows the liquidity spread measured in bp as a function of t_2. In this case the illiquid bond is tradeable at the following three dates $t_1 = 0$, $t_2 \varepsilon [0, T]$, and $t_3 = T$. The used parameter values are: $c = 1$, $\kappa = 0.16643$, $\sigma = 0.01097$, $T = 5$.



The intuition for this lower liquidity spread in the case of the aging-effect-distribution is that the volatility of the zero bond $P(t, T)$ decreases when time to maturity shortens. According to Ito's lemma, the instantaneous variance of the zero bond return is given by $B(t, T)^2 \cdot \sigma^2 = (1 - e^{-\kappa \cdot (T-t)})^2 \cdot \kappa^{-2} \cdot \sigma^2$. As seen in the previous subsection, the illiquid zero bond price benefits from the opportunity to trade as a response to new information. Taking the view that a higher volatility causes stronger trading incentives, we can see that the illiquid zero bond price might be higher, when most of the trading dates are at the beginning of the lifetime during a period with a high volatility and a high information density. Conversely, our term structure model with the property of a higher zero bond volatility at the beginning of a bond's lifetime already contains an idea of a sort of aging-effect, since high volatility is usually accompanied with high trading activity.

Impact of Other Parameters

Recall that the current interest rate level r_t and the mean-reversion θ neither impact $\overline{S}(t, T)/P(t, T)$ nor $\underline{S}(t, \delta; t_1, \ldots, t_N)/P(t, T)$. Therefore, a change of r_t and θ influences the price of the liquid and the illiquid zero bond but the liquidity spread $\gamma^{adj}(T)$ is independent of both variables.

EMPIRICAL BEHAVIOR OF LIQUIDITY SPREADS

After presenting our theoretical model, we regard empirical liquidity spreads. For this purpose, we first describe the data set used for our empirical analysis. We explore the statistical properties of the empirical spreads and verify whether some qualitative properties of our theoretical model can be reconciled with the empirical data. These considerations lay the foundations for the calibration of our model in the next section.

Analyzed Segments of the German Fixed-Income Market

An empirical study of a model of liquidity spreads requires a data set containing bonds that are homogenous in all respects but liquidity. To achieve this goal, we analyze our model with data from two segments of the German fixed-income market.

The first segment is formed by German government securities. We take the price of government securities as the reference value for perfectly liquid securities.[9] For the illiquid sector, we have to take bonds with lower liquidity that are otherwise (nearly) equal. Therefore, we use the prices of €-denominated German Jumbo Pfandbriefe. The institutional design of the Pfandbrief bonds makes them very comparable to German Government bonds regarding their interest rate and credit risk. Furthermore, both security classes fall under the same taxation rules.

Pfandbriefe are one of the oldest types of asset backed securities.[10] They are covered by first-ranking mortgages or loans to the public sector. Safety for Pfandbrief investors is guaranteed through strict legislative regulation by the German Mortgage Bank Act and the Public Pfandbrief Act. Investor security is enforced by restricting of the business activities of Pfandbrief issuers to low-risk loans. The security of the Pfandbriefe is further increased by mandatory conservative rules for the determination of the quality and size of the cover assets and issuing limits. The effectiveness of the legal framework can be seen by the fact that there has been neither a default of a German Pfandbrief nor a default of a German mortgage bank since the German Mortgage Bank Act in 1900.

Jumbo Pfandbriefe were first issued in 1995. They are based on the same legal framework as Pfandbriefe. However, there are additional standards. The primary difference from traditional Pfandbriefe is their minimum issue size of 500 million €. A reduction of the outstanding volume by way of premature redemption is not allowed. Jumbos must have straight bond format (i.e., fixed coupon payable annually in arrears, bullet redemption). Thus, Jumbos have no embedded options (and in particular no prepayment risk as opposed to U.S. mortgage backed securities). Furthermore, each Jumbo Pfandbrief must obtain an official listing on a German stock exchange. At least three banks must act as market makers for a specific issue. A recommendation of the Association of German Mortgage Banks limits the maximum spread from four cents up to 20 cents depending on the maturity. As a consequence, the prices on the stock exchange are likely to reflect fair equilibrium prices though most of the trading volume is on the OTC market.

The aim of the high degree of standardization is to increase the liquidity of the Jumbo Pfandbrief. However, market participants regard their liquidity to be lower than that of government bonds. Unfortunately, we cannot quantify the liquidity disadvantage on a bond-by-bond level since the majority of both government bond and Jumbo Pfandbrief transactions are on OTC markets. As opposed to exchange transactions, there are no official statistics for OTC transactions. Perhaps the rise of electronic trading platforms like EuroCredit MTS will allow a more direct analysis of liquidity in the future.

The Data

The prices of the government bonds that serve as the liquid benchmark are calculated from the parameters of the Svensson [1994] term structure model. The used parameters are estimated and publicly provided by the Deutsche Bundesbank on a daily basis using all government securities.

The sample of the illiquid bonds is chosen in order to minimize the influence of a potential residual credit risk. Therefore, we include only Jumbo Pfandbriefe that are Aaa-rated by Moody's.[11] Because of the special safety requirements and our additional sample selection, the analyzed bonds bear only a marginal portion of credit risk that is not supposed to be a relevant determinant of their spread over government securities. Nevertheless, we control in our empirical analysis for effects of residual credit risk and the different capital requirements for the two security classes.[12] To get a complete time series for every bond with proper liquidity spreads, we consider those bonds that were traded within the whole sample period and that had a time to maturity above one year at any point in this period. There are two reasons why we exclude the short-term bonds. The first reason is that marginal deviations in the prices of short-term bonds result in relatively high deviations in the bond's yield. The shorter the time to maturity, the more severe this effect is. Therefore, the inaccuracy of the observed liquidity spreads of these bonds might be especially high. Second, for similar reasons, the estimates for the government term structure arising from the Svensson model might be especially biased for short maturities. This effect results in additional inaccuracies for the liquidity spreads of short-term bonds.

Our sample period ranges from January 2000 to December 2001. We use weekly bond prices of a total of 32 Pfandbrief bonds issued by nine different institutions. We use the prices of the last offcial trade at the Frankfurt Stock Exchange on each Wednesday. The data is provided by Datastream. All bonds have been issued between June 1995 and October 1999. Their initial maturity is, on average, 8.31 years and varies between four and 15 years.

The coupon ranges from 3.25% to 6.50% (4.76% on average). Both the range of maturities and the range of coupon sizes are comparable to the corresponding features of government bonds.

At this point, we face the problem that data about illiquid bonds is available for coupon only bonds, but our theoretical model is designed for zero bonds. In principle, our valuation method for zero bonds could be applied directly to coupon bonds. However, the intermediate coupon payments would increase the complexity of the necessary calculations.[13] Thus, we use a simple transformation of coupon bond prices into liquidity spreads to sustain the applicability of our model in a reasonable way. We treat coupon bonds as zero bonds with a time to maturity that is determined such that the elasticity with respect to the short rate r_t is equal for both securities according to the Vasicek term structure model. The resulting time to maturity of bond j, which we call Vasicek duration D_j, is implicitly (and uniquely) given by the following equation[14]

$$\frac{1}{P(t,t+D_j)} \cdot \frac{\partial P(t,t+D_j)}{\partial r_t} = -\frac{1-e^{-\kappa \cdot D_j}}{\kappa}$$

$$\stackrel{!}{=} \frac{\frac{\partial}{\partial r_t}(\Sigma_{t_i} P(t,t_i)\cdot c + P(t,T)\cdot 100)}{\Sigma_{t_i} P(t,t_i)\cdot c + P(t,T)\cdot 100}$$

where $P(t, \tau)$ denotes the Vasicek value of a $(\tau - t)$-year zero bond and t_i and T stand for the coupon dates and the maturity of the considered coupon bond.

Our procedure ensures that the coupon bond and the zero bond with adjusted maturity have (locally) the same exposure to interest rate risk. This fact suggests that the observed liquidity spread of a coupon bond is a reasonable proxy for that of the corresponding zero bond. Moreover, since the coupon payments in our sample are rather low relative to the face value, the Vasicek duration is not far below the time to maturity and the coupon payments are not supposed to play an essential role. According to this view, the empirically observed liquidity spread $\Gamma_j(t)$ obtained from Jumbo Pfandbrief j amounts to:

$$\Gamma_j(t) = -\frac{1}{D_j}\ln\left(\frac{P_j(t) + \text{accrued interest}}{PV(j)}\right)$$

where $P_j(t)$ stands for the observed price of Jumbo j, $j = 1, \ldots, J = 32$ and the term $PV(j)$ is equal to the present value of the outstanding payments of bond j discounted with the current term structure for government bonds. Thus, $PV(j)$ is the value of a bond which is liquid and otherwise identical to j. Then, the theoretically predicted liquidity spread that corresponds to Jumbo Pfandbrief j is equal to $\gamma^{\text{adj}}(D_j)$.

Statistical Properties of Liquidity Spreads

The analyzed data sample contains 3,328 weekly observations of liquidity spreads in total. Most of the liquidity spreads range from 0 up to 30 basis points. The average spread is 20.4 bp and the standard deviation amounts to 11.6 bp. It turns out that 1.3% of the spreads are negative. However, there is no theoretical justification for this finding. Therefore we think that these effects are not driven by liquidity. We mainly consider errors in the estimation of the government bond term structure, effects from nonsynchronous prices, and unobservable demand shocks to be the reasons for negative spreads. Since all these factors are not present in our model, we ignore these observations in the empirical implementation of our model.

The size of the liquidity spread is higher than in a recent study of the German fixed-income market by Kempf and Uhrig-Homburg [2000], who found a liquidity spread below 10 bp. The lower spread can be probably attributed to the fact that Kempf and Uhrig-Homburg analyzed only bonds from the government or state-operated funds (like the then state-owned railway service and federal mail). The size of the empirical spreads coincides approximately with the one observed by Perraudin and Taylor [2003] for illiquid AAA-rated U.S.-dollar denominated bonds (after adjusting for tax effects). It is also comparable to the results of a recent study of €-denominated bonds, for which Houweling et al. [2003] determine a liquidity premium (over all rating classes) of between 9 and 24 bp.

Not surprisingly, the liquidity spreads of different bonds show a similar evolution through time. This is indicated by the average inter-spread correlation of 0.52. In addition, this effect is illustrated in Exhibit 4. Exhibit 4 graphs the median, the 20%-quantile and the 80%-quantile of the spreads over time.

As Exhibit 4 shows, the cross-sectional median of the liquidity spreads evolves between 0.3 and 32.6 bp. In fact, most of the spreads are close to the (time-varying) average spread. Furthermore, the evolution of liquidity spreads of particular bonds, which we have not included in this diagram for the sake of clarity, is closely related to that of the median or that of the plotted quantiles.

Qualitative Behavior of Liquidity Spreads

As shown in the analysis of the properties of liquidity spreads, the theoretical liquidity spread is affected by both term-structure and bond-specific parameters. In this section, we check whether the comparatively static results of our model can be reconciled with the properties of empirical data. Basically, there are two implications of our model which are testable with our data set. The first hypothesis is that the liquidity spread is a hump-shaped function of the bond's maturity. The second one states that the liquidity spread does not depend on the current short rate r_t. To test the first hypothesis, we run cross-sectional regressions over all 32 bonds with different maturities for some fixed point in time. Moreover, we include some standard proxies for liquidity in these regressions to analyze the explanatory power of these factors. In order to verify the second hypothesis that the liquidity spread does not depend on r_t, we perform time-series regressions. In addition, we can use the time series analysis to check for the influence of other factors, in particular credit risk.

First, we focus on the impact of time to maturity T on liquidity spreads. For each observation date, we perform a cross-sectional regression of liquidity spreads (measured in bp). The time to maturity is represented by the Vasicek duration D_j of the coupon bonds. In order to test for a potential humped-shaped dependence of liquidity spreads on time to maturity, we include not only the linear term D_j, but also a truncated function $\max\{0, D_j - 7.0\}$ of D_j in the regression. Together with the constant, these terms form a piecewise linear function with a kink at $D_j = 7.0$ that can capture a humped-shaped relation.[15] Besides the time to maturity, we incorporate two other variables which are often used in the literature as proxies to capture liquidity effects: the time since issuance age_j measured in days and the amount issued $issue\_size_j$ (measured in million €). Finally, we use the size of the coupon $coupon_j$ (in percent) of each Jumbo bond j as an exogenous variable to consider a possible coupon effect due to taxes. Formally, the estimation equation is given by:

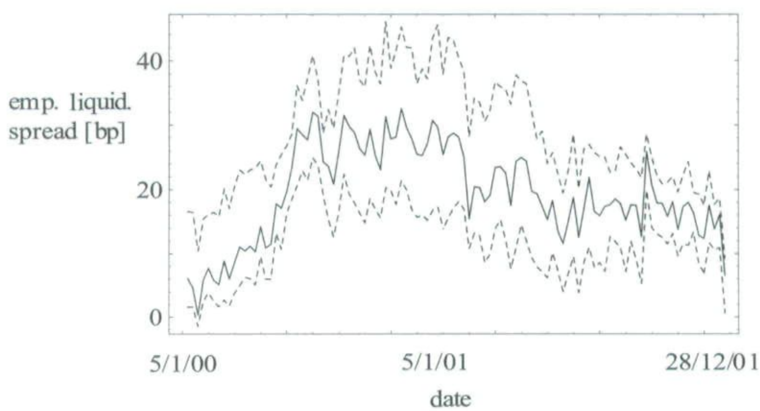
$$\Gamma_j = \beta_0 + \beta_1 \cdot D_j + \beta_2 \cdot \max\{0, D_j - 7.0\} + \beta_3 \cdot \text{age}_j + \beta_4 \cdot \text{issue\_size}_j + \beta_5 \cdot \text{coupon}_j + \epsilon_j \quad (11)$$

Since the sample period comprises two years of weekly observations, we can run a total of 104 cross-sectional regressions. The results of the regressions are reported in Exhibit 5.

The regression results in Exhibit 5 support the existence of a significantly positive linear dependence of the liquidity spread on the time to maturity in almost the whole sample period. The value of parameter β_2 indicates a negative relation between liquidity spreads and

EXHIBIT 4
Evolution of the Empirical Liquidity Spreads

The exhibit graphs the empirical liquidity spreads measured in bp over the observation period from January 5, 2000 until December 12, 2001. The solid line indicates the median, while the dashed lines stand for the 20% and the 80% quantiles respectively.



time to maturity for high values of T which would speak in favor of a humped-shaped dependence. However, the parameter is only rarely significant, such that there is no fully convincing support for a humped-shaped dependence of liquidity spreads on time to maturity T as predicted by our model. Of course, results are affected by the actual choice of location of the kink. Further inspections show that the inclusion of time to maturity truncated at 6.0 years and above into the regressions leads to a negative relation between duration and liquidity spreads for higher durations. However, the downward-sloping part of the function—regardless of the precise position of the kink—is insignificant for at least 90% of the regressions. Kempf and Uhrig-Homburg [2000] observe a similar pattern (a noticeable rise followed by a not very pronounced descent) for liquidity spreads of German government bonds. Perraudin and Taylor [2003] even find a very distinct humped-shaped dependence of liquidity spreads on time to maturity for U.S.-dollar denominated AAA-rated bonds. The reason for the absence of a significant downward-sloping part of the function might be that our data sample contains too few bonds with longer maturities. The insignificance might also be driven by the fact that our piecewise linear function is only a first and rough approximation to the (nonlinear) theoretical humped-shaped relation.

A further important finding is the mostly insignificant impact of the liquidity proxies, age and issue size, on the spread. Furthermore, the results indicate that there is no notable effect of coupon size.

Second, we explore the influence of the level of the short rate r_t on the liquidity premium. Our theoretical model predicts no impact of the short rate. In addition, we analyze the impact of credit risk on the observed bond spreads. Therefore, we need a proxy for the default risk of the Jumbo Pfandbriefe. A straightforward proxy would be the bond spread of AAA-rated non-government bonds. However, this spread is caused not only by credit risk but also by other factors like liquidity disadvantages. As the credit risk of these bonds is small, the liquidity component is likely to represent a large part of the total bond spread.[16] Thus, we would test a tautology and explain AAA-spreads by AAA-spreads. As a consequence, we approximate the pure credit risk of bonds with an excellent rating by bond spreads of worse-rated bonds. Since these bonds carry more credit risk, the default premium probably represents a big part of the spread. Thus, we include credit spreads cs_t determined by DZ-Bank for A-rated bonds with medium maturity in the regression to control for credit risk. The data is regularly published in the Frankfurter Allgemeine Zeitung.

EXHIBIT 5

Results of the Cross-Sectional Regressions

The exhibit provides a summary of 104 cross-sectional regressions computed for each observation date. The estimation equation is given by:

$$\Gamma j = \beta_0 + \beta_1 \cdot D_j + \beta_2 \cdot \max\{0, D_j - 7.0\} + \beta_3 \cdot \text{age}_j + \beta_4 \cdot \text{issue\_size}_j + \beta_5 \cdot \text{coupon}_j + \epsilon_j$$

Parameter	β_0	β_1	β_2	β_3	β_4	β_5
Average parameter value (all 104 regressions)	-12.1722	5.3529	-90.8158	-0.0031	-0.0007	2.6082
No. of regressions in which parameter is significant**	62	99	4	3	6	33
Average parameter value when significant**	-17.7217	5.6089	-36.8704	-0.0067	-0.0014	3.6615

*****significant at the 1% level.*

For each of the 32 Jumbo Pfandbriefe, we run a regression of the time series of liquidity spread changes. We proxy the unobserved short rate by the interest rate for $t = \frac{1}{10}$ calculated with the current Svensson [1994] parameters. As described above, we check for credit risk effects by including the exogenous variable cs_t, the spread of A-rated bonds. In order to control for the influence of the time to maturity T as observed in the cross-sectional analysis, we also include the first differences of the duration and the truncated duration as exogenous variables. The regression equation for each bond reads

$$\Delta\Gamma_t = \beta_0 + \beta_1 \cdot \Delta r_t + \beta_2 \cdot \Delta cs_t + \beta_3 \cdot \Delta D_t + \beta_4 \cdot \Delta(\max\{0, D_t - 7.0\}) + \epsilon_t \tag{12}$$

where Δx_t denotes the change of variable x from week t to week $t-1$. The variances of the estimates are adjusted for first-order autocorrelation of the residuals. To improve the readability of the results, we measure the short rate, the credit spread and the liquidity spread in basis points. Exhibit 6 summarizes our results.

Since the aim of our time series regression is mainly to exclude possible sources that influence the liquidity spread, we now use a higher level of significance, namely 5%. Exhibit 6 shows that the regression coefficient β_1 for the short rate is significant in only four out of 32 cases. Hence, there is almost no support for a dependence of the liquidity spread on the short term rate. This finding is fully consistent with our theoretical model. The credit spread is in several cases (eight out of 32) relevant for the liquidity spread. This result suggests that credit risk is not a key explanatory variable for the observed spreads. However, these findings indicate that we should be aware of a possible dependency on credit risk and control for it in the quantitative study carried out in the following section.

As opposed to the cross-sectional regressions, hardly any significant relation between liquidity spreads and duration is found in the time series data. Part of this result can be explained by the fact that the piecewise linear function is only a rough approximation for a nonlinear humped-shaped dependence. Another possible explanation is the existence of systematic time-varying influence. A changing market-wide factor that shifts all spreads simultaneously up or down interferes the linear dependence when time series data rather than cross-sectional data is analyzed. We pursue this issue further in the next section.

MODEL IMPLEMENTATION

To implement our liquidity spread model, we need three types of parameters, namely bond-specific, market-specific, and term structure parameters. In the first step, we show how to determine the required parameters. Then we present a calibration of the model and interpret our findings.

Calibration of Model Parameters

The specific parameters of an illiquid bond comprise the time to maturity and the set of feasible trading dates. The maturity of the observed liquidity spreads is represented by their duration, obtained from the Vasicek model. The duration can be easily calculated by the terms of the particular bond and the Vasicek term structure parameters.

The other bond-specific parameter is the set of feasible trading dates. To keep things simple, we assume equidistant trading dates, i.e., the distance Δt between two consecutive trading dates suffices for a complete characterization. Even with this simplification, this illiquidity parameter is difficult to observe empirically. In principle, Δt can be derived by analyzing trading volume data, turnover frequency, or other related measures. Unfortunately this is not possible because of the market structure. Detailed transaction data is available only for trades on the stock exchange. Since most Pfandbrief bonds are traded OTC by banks or institutional investors, the stock exchange volume data is fundamentally biased. Therefore, we determine the bond specific illiquidity parameters implicitly by inverting the pricing relation. Although implicit parameters are formally obtained by inverting the pricing relation, they convey more information than the spreads themselves. Implicit parameters make spreads obtained for different bonds at different dates comparable since the effects of other parameters relevant for pricing are ruled out. Thus, implicit parameters allow one to get an intuition about the 'pure liquidity' of a bond.

The market-specific parameters concern the factor c that measures the relation between the liquidity spread $\gamma(T)$ under perfect timing ability and the adjusted theoretical spread $\gamma^{adj}(T)$. This factor can be interpreted as the degree of timing ability reflected by market prices or, from an abstract point of view, as the intensity of the investor's trading motives.

EXHIBIT 6
Results of the Time Series Regressions

The exhibit provides a summary of 32 time series regressions computed for each Pfandbrief bond. The estimation equation is given by:

$$\Delta\Gamma t = \beta_0 + \beta_1 \cdot \Delta r_t + \beta_2 \cdot \Delta cs_t + \beta_3 \cdot \Delta D_t + \beta_4 \cdot \Delta(\max\{0, D_t - 7.0\}) + \epsilon_t$$

Parameter	β_0	β_1	β_2	β_3	β_4
Average parameter value (all 32 regressions)	-0.021	59.335	0.036	-1.383	-1.512
No. of regressions in which parameter is significant*	1	4	8	4	1
Average parameter value when significant*	-0.839	877.501	0.150	-21.881	-183.904

**significant at the 5% level.*

To control for a possible credit component in the observed liquidity spreads, which we could not rule out in the time-series regressions, we include the credit spread of A-rated bonds scaled by the factor c_{credit} in our pricing equation. The scaling factor is assumed to be constant for all bonds.

We regard a further additional term a_t which is constant for all bonds. The reason for inclusion of parameter a_t is to control for other systematic effects on the spreads that are not driven by liquidity or credit risk. In particular, these effects include different capital requirements, demand shocks, and errors in the estimation of the term structure. We allow for a certain time-dependence, where a_t differs among quarters in the sample period but stays constant within each quarter. Hence, a_t is given by the following step function:

$$a_t = \begin{cases} \alpha_1, & \text{for } t \text{ in 1st quarter of 2000} \\ \alpha_2, & \text{for } t \text{ in 2nd quarter of 2000} \\ \alpha_3, & \text{for } t \text{ in 3rd quarter of 2000} \\ \alpha_4, & \text{for } t \text{ in 4th quarter of 2000} \\ \alpha_5, & \text{for } t \text{ in 1st quarter of 2001} \\ \alpha_6, & \text{for } t \text{ in 2nd quarter of 2001} \\ \alpha_7, & \text{for } t \text{ in 3rd quarter of 2001} \\ \alpha_8, & \text{for } t \text{ in 4th quarter of 2001} \end{cases}$$

where each quarter consists of 13 weeks. Thus, we explain the liquidity spread $\Gamma_{j,t}$ of bond j by:

$$\begin{aligned} \Gamma_{j,t} &= a_t + c \cdot \gamma_{j,t}(D_j) + c_{credit} \cdot cs_t + \epsilon_{j,t} \\ &= a_t + \gamma_{j,t}^{adj}(D_j) + c_{credit} \cdot cs_t + \epsilon_{j,t} \end{aligned} \tag{13}$$

To determine liquidity spreads $\gamma(T)$, we need the parameters of the dynamics of the short rate which is assumed to follow a Vasicek process. In particular, we require only the interest rate volatility parameters σ and κ to compute $\gamma_{j,t}(D_j)$ for a given D_j, because the short rate r_t and the long term mean θ do not affect the theoretical spread $\gamma(T)$. The other parameters are needed to compute the Vasicek duration D_j of bond j. Since all short rate parameters, σ, κ, θ, and r_t, affect the observable bond prices, we must estimate all these parameters simultaneously. We use a standard estimation methodology that simultaneously uses information about the time series and the cross-section of bond prices. The approach consists of a maximum likelihood estimation applied to a state space model with a latent factor. The methodology, which is quite reminiscent about the Kalman filter, has been introduced by Chen and Scott [1993] and Pearson and Sun [1994].

The data sample for our econometric procedure consists of daily zero bond prices for maturities between three and ten years. The prices are derived from time series of Svensson [1994] parameters for German government bonds. In order to reduce the standard error of our estimates, we use a higher data frequency (daily) for the term structure parameter estimation than in the calibration of the illiquidity parameters.

The basis for our econometric model is formed by the dynamics equation of the short rate under the empirical measure. We assume that the market price of risk behaves in a way such that the long-term mean drift θ^{real} under the empirical measure is constant. Thus, the short rate process is given by the following standard Vasicek process

$$dr_t = \kappa(\theta^{real} - r_t)dt + \sigma dz'_t \tag{14}$$

with z'_t denoting a standard Wiener process under the empirical measure. A discretization of Equation (14) shows that the short rate r_t is conditionally normally distributed given the initial value at $t - \Delta t$ with the following mean and variance:

$$r_t | r_{t-\Delta_t}; \kappa, \sigma, \theta^{real} \sim N\Big(e^{-\kappa\Delta t} r_{t-\Delta t} + \theta^{real}(1 - e^{-\kappa\Delta t}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t})\Big)$$

The corresponding transition density is denoted by $\varphi(r_t | r_{t-\Delta t}; \kappa, \sigma, \theta^{real})$.

Our econometric model for each bond price with maturity $T = 3, 4, 5, \ldots, 10$ takes the following form (observation equation):

$$\ln P^{obs}(t, T) = A(t, T) - B(t, T) \cdot r_t + \epsilon_t^T \tag{15}$$

The formulation in terms of the logarithm of the price yields a linear model. The observation errors ϵ_t^T are assumed to be normally distributed and mutually independent. The state variable r_t itself is unobservable. We estimate its trajectory together with the parameters κ, σ, θ, θ^{real} and the error variances from the data. Therefore, we assume that the price of the bond with a remaining time to maturity of $\overline{T} = 5$ years is measured without error. This assumption yields an analytical expression for the latent r_t by rearranging Equation (15). Assuming further a constant standard deviation σ_B of the measurement errors for the other bonds, we can derive the density $f(t)$ of the observed log-prices in t conditional on the value of the state variable $r_{t-\Delta t}$ at the previous date:

$$f(t) = \varphi(r_t | r_{t-\Delta t}; \kappa, \sigma, \theta^{real}) \frac{1}{B(t, t+\overline{T})} \prod_{\substack{T=3 \\ T\neq 5}}^{10} \frac{1}{\sqrt{2\pi\sigma_B^2}} \exp\left(-\frac{\left(\epsilon_t^T\right)^2}{2\sigma_B^2}\right)$$

The parameters are estimated by maximizing the log-likelihood function for the complete sample period from 2000 to 2001:

$$\max_{\kappa,\sigma,\theta,\theta^{real},\ \sigma_B} \sum_{t=2000/01/04}^{2001/12/28} \log f(t)$$

The estimation of the parameters is repeated several times using different starting values. Further estimations using data only from the year 2000 or only from the year 2001 yield nearly the same results, which speaks in favor of the assumed Vasicek term structure model. The results are reported in Exhibit 7.

We take the values of the interest rate volatility parameters, σ and κ, to estimate the remaining parameters. Specifically, we calibrate the illiquidity parameters a_t, c, and Δt and the credit risk correction factor c_{credit} for every bond using Equation (13). The precise parameter values are obtained by a nonlinear least squares regression. Since some liquidity spreads Γj (t) are negative, which cannot be explained by any liquidity model, we exclude these observations from our regression. Formally, we calculate:

$$\min_{\alpha_1,\ \ldots,\alpha_8,c,c_{credit},\Delta t_1,\ \ldots,\Delta t_{32}} \sum_{j=1}^{J} \sum_{t=2000/01/05}^{2001/12/28} (\Gamma_{j,t} - (a_t + c\cdot\gamma_{j,t}) + c_{credit}\cdot cs_t)^2 \cdot \mathbf{1}_{\{\Gamma_{j,t}\geq 0\}}$$

The minimum has to be determined numerically, since there is no analytical formula for $\gamma_{j,t}(T)$. The estimates of the market-wide parameters a_t, c_{credit} and c for the full sample of bonds $J = 32$ over the observation period 2000 to 2001 are reported in Exhibit 8. A summary of the results for the bond-specific illiquidity parameters Δt_j is given in Exhibit 9.

Interpretation of Illiquidity Parameters

Exhibit 8 reports the estimates of the information correction factor c, the credit-risk adjustment factor c_{credit}, and the parameters $\alpha_1, \ldots, \alpha_8$ of the additional market-specific term a_t.

At first, we regard the impact of credit risk on the observed bond spreads. The estimate for the credit-risk adjustment factor c_{credit} is 0.016. A Wald test for this parameter indicates that c_{credit} is not significant at the 5% level. However, c_{credit} is significantly lower than 0.05. Since the average credit spread of A-rated bonds during the sample period is 100 bp, while the liquidity spreads are 20 bp on average, the statistical analysis can significantly rule out that credit risk causes more than a quarter of the total bond spread. Thus, these findings show—even more strongly than the results from the previous regressions—that credit risk is not an essential factor for the empirical liquidity spreads.

The relevance of our theoretical model for the observed spreads is documented by the information correction factor c. The estimate of this factor equals 0.58 and is significant at the 5% level. The significance of c speaks in favor of the applicability of our model to describe empirical liquidity spreads. The value of $c = 0.58$ indicates that empirical spreads can be explained by a component that is slightly more than half as high as the theoretical liquidity spread $\gamma(T)$ with a perfect timing ability. Since c is significantly positive, we can argue that empirical liquidity spreads are such that investors have a certain degree of timing ability but the ability is not perfect. Interpreted in a broader sense, the result shows that for the purpose of calculating liquidity spreads the assumed trading motive information is capable of describing empirical liquidity spreads, even though they might also stem partly from other reasons for trading.

There is an additional term, a_t, explaining the empirical liquidity spreads. According to Exhibit 8, the estimates of the parameters for a_t, i. e., $\alpha_1, \ldots, \alpha_8$, are significant for four out of eight quarters, but they are not significant for the other four quarters. Hence, half of the time, our model is able to explain the liquidity spreads better than a bond-independent function of time as well as a credit risk factor. In the other four periods, there might be effects from sources other than liquidity and credit risk, which have a significant influence on the empirical spreads.

A summary of the estimation results for the implicit lengths of non-trading intervals for our bond sample is given in Exhibit 9. The trading frequency ranges from every few minutes to less than three trades per year. At first glance, some of the estimated values for Δt seem to be too high, given the existence of market makers for all Jumbo Pfandbriefe. However, this seeming caveat can be explained by the notion of liquidity in our theoretical model. In our model, we describe liquidity exclusively by the time dimension as we assume that the investor can liquidate an arbitrarily sized Pfandbrief bond portfolio at each trading date without affecting the price. However, as far as empirical issues are concerned, liquidity cannot be fully assessed by a measure that relies solely on its time dimension. Investors in real markets cannot buy or sell arbitrarily high amounts without a price reaction. Therefore, we must not literally interpret Δt, but interpret Δt as the time span required to build up or to liquidate a bond position of reasonable size without considerable price effects. Thus, our interpretation of the results accounts for the fact that in real markets, all three liquidity dimensions—time, price, and quantity—are present simultaneously. The average of Δt equal to 12.35 days indicates that it takes about two and a half weeks to conduct considerable adjustments for German Pfandbrief bond portfolios without triggering a market impact. The

EXHIBIT 7

Estimates of the Term Structure Parameters for 2000–2001

The exhibit shows the estimates of the reversion factor κ, the volatility σ, the long term mean under the risk-neutral measure θ, the long term mean under the real measure θ^{real}, and the standard deviation σ_B of the measurement errors for the bonds. The values in brackets indicate the corresponding standard errors.

Parameter	κ	σ	θ	θ^{real}	σ_B
Value	0.16643	0.01097	0.06390	0.01748	0.00907
(Standard Error)	(0.00793)	(0.00030)	(0.00019)	(0.04889)	(0.00008)

EXHIBIT 8

Values for Market Parameters from Implicit Estimation

The exhibit shows the estimates for the liquidity correction factor c, the credit risk correction factor c_{credit}, and the time-dependent factor a_t. The values in the last column indicate the corresponding standard errors.

Parameter	Value	(Standard Error)
c	0.58*	(0.29)
c_{credit}	0.016	(0.0108)
α_1 [bp]	-1.89	(7.75)
α_2 [bp]	14.53*	(7.83)
α_3 [bp]	19.87**	(7.92)
α_4 [bp]	21.73**	(7.99)
α_5 [bp]	16.13*	(8.02)
α_6 [bp]	9.34	(8.06)
α_7 [bp]	9.49	(8.11)
α_8 [bp]	7.16	(8.20)

**significant at the 5% level.*

***significant at the 1% level.*

majority of Pfandbrief bonds in our sample are more liquid than the average non-trading interval of approximately two and a half weeks suggests. Half of the bonds trade every two days or more often, as the median Δt shows.

Even though the values of Δt for all bonds cover a relatively huge range, we find a certain pattern for Δt. We can see from Exhibit 10 that Δt increases with the Vasicek duration of the specific bond. Especially for bonds with an initial duration above six years, the estimation of Δt rises rapidly with the duration. Since Δt measures the degree of illiquidity of every individual bond, the empirical liquidity spreads suggest that the longer the bond's maturity the more illiquid (in terms of our model) the bond is supposed to be.

CONCLUSION

Liquidity is a major determinant of bond prices. In this article, we derive an easily applicable option-theoretical approach to calculating price discounts for illiquid bonds. We take the view that liquid assets offer a valuable option to trade. This interpretation of liquidity together

EXHIBIT 9

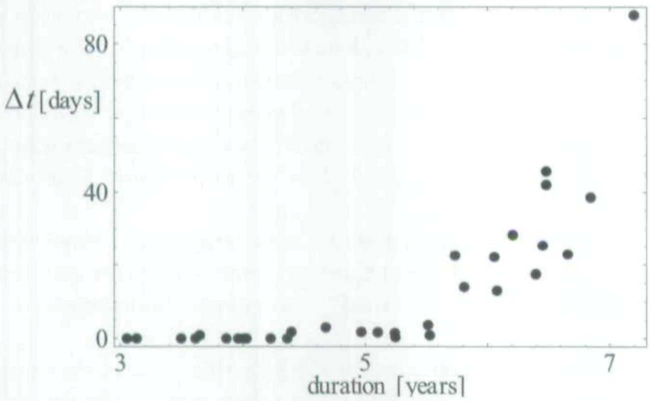
Summary Statistics for Illiquidity Parameters Δt_j

The exhibit shows a summary of the estimation results of the trading frequencies Δt of the 32 bonds. In particular, it provides the average of the 32 estimates of Δt, the average of the corresponding standard errors, and the median of Δt.

Average Δt [days]	Average Standard Error [days]	Median Δt [days]
12.35	24.64	1.75

EXHIBIT 10
Length of Trading Intervals

The graph plots the combinations of the length of trading intervals Δt measured in days and the Vasicek duration at the beginning of the observation period measured in years for every individual bond.



with the fiction of an investor with perfect foresight allows us to represent the benefits of liquidity, in the spirit of Longstaff [1995], as the payoff of certain financial options, namely lookback options.

We extend the work of Longstaff in two directions. First, since we want to focus on illiquid bonds rather than on equity, we introduce interest rate uncertainty. Second, we allow for different degrees of liquidity. In contrast to Longstaff, who considers one single non-trading period only, our model is flexible enough to deal with multiple non-trading periods which can be arbitrarily distributed over time. It turns out that the liquidity spread in our case is determined by the ratio of the price of a lookback option written on a zero bond with continuous monitoring of the underlying and the price of a corresponding discrete lookback option.

A theoretical analysis shows that the behavior of the liquidity spreads towards the variation of other economic variables has plausible properties. Liquidity spreads are humped-shaped functions of maturity. As one expects, they increase with bond volatility. Furthermore, the liquidity spreads are influenced not only by the number of trading dates but also by their distribution over time. Another essential feature of the model is the invariance of the liquidity spreads to the level of the short rate.

To assess the empirical quality of our model, we consider data of German Jumbo Pfandbrief bonds. We are able to reveal several parallels between the theoretical predictions of our model and the behavior of the regarded bond data. Consistent with our model, the empirical liquidity spreads do not significantly depend on the current short-rate. Though the observations speak for a humped-shaped dependence of the liquidity spreads on the time to maturity, this effect is not significant. Since reliable volume data is not available for these bonds, the main obstacle to the application of our model is to determine the precise dates when it is possible to trade an illiquid bond. We solve this problem through an implicit procedure. Interestingly, our analysis shows that the actual liquidity spreads are slightly more than half as high as they would be if investors had perfect market timing ability. This result shows that our model is able to explain empirical liquidity spreads even though information might not be the only reason for trading in real markets. While our model is able to explain the observed bond spreads to a considerable part, credit risk has nearly no explanatory power.

This article provides several opportunities for ongoing research. First, an obvious extension of our model is the consideration of trading dates that emerge stochastically rather than deterministically. Of course, this extension allows for a better calibration of our model. In principle, stochastic trading dates can be used to model the spread dynamics over time. However, the basic insights and main conclusions of our model are still valid if trading dates are stochastic.

Second, it is interesting to perform more direct tests of our model. If data about the trading volume of illiquid bonds is available and therefore enables conclusions about the trading dates to be drawn, it is possible to directly compare theoretical and observed spreads.

Third, our model takes a specific view towards liquidity to derive an easily applicable method for the determination of liquidity spreads. Therefore, a comparison of the outcomes of our simple approach with those of a fully-fledged equilibrium model will lead to further insights into the nature of liquidity spreads.

Fourth, we demonstrate how to use a mapping $\xi : t \rightarrow t$ to transform the motion of zero bond prices plus earned interest into a geometric Brownian motion with constant coeffcients. This approach allows for the easy determination of prices for various other exotic options on zero bonds within the Vasicek framework.

Appendix A

Analysis of the Numerical Procedure with Reduced Complexity

For simplicity, the spreads $\gamma(T)$ at a particular date t are computed as if they were constant throughout the lifetime of the bond.[17] Nevertheless, our model framework is flexible enough to consider time-varying spreads at the computation stage. In this case, we have to apply a recursive algorithm to determine $\gamma(T)$, since the values of the liquidity spreads at every future trading date have to be computed first. Hence, while the assumption of constant spreads enables us to use an algorithm with complexity $O(N)$, i.e., computing time grows linearly with the size of the problem (the number of trading dates), a recursive algorithm has the complexity $O(N^2)$, i.e., computing time grows quadratically. The additional complexity of the recursive algorithm has substantial consequences for the computing time. Each simulation step to determine the liquidity spread of a typical bond in our data sample (maturity: eight years, traded every two weeks, i.e., 24 times a year) requires the computation of 192 random numbers, ξ's and further manipulations of them. For the same bond, the recursive algorithm requires 18,528 (= 1 + 2 + … + 192) of these operations. Thus, the computing time rises approximately by a factor of 100.

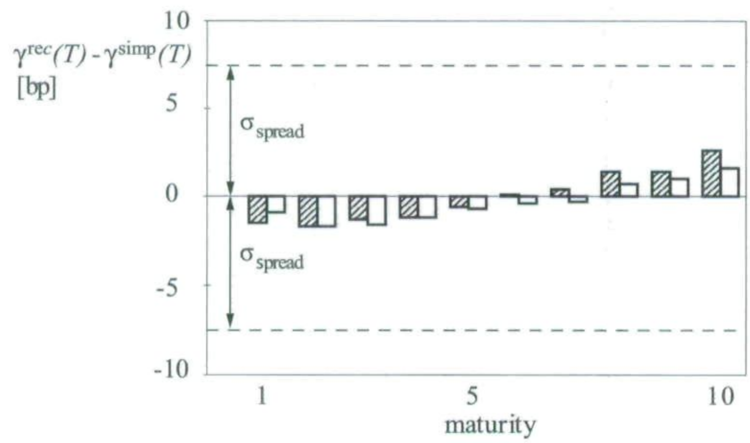
On the one hand, the used numerical algorithm has considerable advantages concerning the computing speed. Moreover, we find that the qualitative properties of the liquidity spreads such as the behavior for different times to maturity, different volatility levels, etc. when using the recursive algorithm are the same as if the simpler approach were used. Thus, these considerations provide some justification for the application of the simpler algorithm.

On the other hand, the resulting spreads might quantitatively differ from those obtained by the recursive method. As a test, we calculate the liquidity spreads with the recursive algorithm that can be traded two and four times a year, respectively, for several maturities. The resulting differences (measured in bp) to the liquidity spreads obtained by the less complex method are plotted for different maturities in Exhibit 11. The recursively computed liquidity spreads are lower for short maturities and higher for long maturities. However, the liquidity spread still attains its maximum in the same region. The average magnitudes of the differences are 1.2 bp for $\Delta t = 0.25$ and 1.0 bp for $\Delta t = 0.5$, respectively. Hence, an effect of the computing method cannot be neglected. However, its magnitude is modest compared to inevitable errors from other sources. First, the precision of the spread determination is bounded by the precision of the Monte Carlo simulation and the numerical fixed-point computation (set to 1%). Second and even more important, empirical liquidity spreads cannot be measured without error. The mean standard deviation of the weekly-observed spreads in our data sample amounts to 7.5 bp. As Exhibit 11 shows, its size considerably exceeds the differences between both methods for reasonable parameter values.

Exhibit 11

Differences of Liquidity Spreads

The exhibit shows the differences between the liquidity spreads $\gamma(T)$ for maturities T = 1, 2, …, 10 years computed with the recursive (rec) and the simplified (simp) algorithm. The distance between two consecutive trading dates is constant and equal to $\Delta t = 0.5$ (white bars) and $\Delta t = 0.25$ (hatched bars), respectively. The used parameter values are: $\kappa = 0.16643$, $\sigma = 0.01097$. The dashed lines indicate the positive and negative mean standard deviation of the weekly-observed spreads in our data sample.



As a result of the considerations that the computing speed increases rapidly when using the simplified algorithm, the qualitative properties are the same for both algorithms, and the numerical differences between both algorithms are small compared to empirical changes of spreads, we think it is appropriate to use the less complex method. With more computing power at hand, the recursive computation method could be applied within our model framework.

APPENDIX B

Derivation of Equation (7)

Applying Ito's lemma, we can write for the process of $M_t := P(t,T) \cdot e^{-\int_0^t r_s ds}$:

$$
\begin{aligned}
dM_t &= \frac{\partial M_t}{\partial r_t} dr_t + \frac{\partial M_t}{\partial t} dt + \frac{1}{2}\frac{\partial^2 M_t}{\partial r_t^2} dr_t^2 \\
&= -B(t,T)\, M_t \cdot dr_t + \left(\frac{\partial A(t,T)}{\partial t} - r_t \cdot \frac{\partial B(t,T)}{\partial t} - r_t \right) \cdot M_t \cdot dt \\
&\quad + \frac{1}{2} B(t,T)^2 M_t \sigma^2 \cdot dt \\
&= 0 \cdot dt - B(t,T) \cdot \sigma \cdot M_t \cdot dz_t
\end{aligned}
$$

Accordingly, the process of $d\ln M_t$ reads:

$$
d \ln M_t = -\frac{1}{2} B(t,T)^2 \sigma^2 \cdot dt - B(t,T) \cdot \sigma \cdot dz_t \tag{16}
$$

This representation leads us to the following solution of $\ln M_\tau$:

$$
\ln M_\tau = \ln M_0 - \int_0^\tau \frac{1}{2} B(s,T)^2 \sigma^2 \cdot ds - \int_0^\tau B(s,T) \sigma \cdot dz_s
$$

At this point, it is convenient to make the following definition:

$$
\xi(\tau) := \int_0^\tau B(s,T)^2 \sigma^2 \cdot ds = \sigma^2 \frac{4e^{-T\kappa}(1-e^{\kappa\tau}) - e^{-2T\kappa}(1-e^{2\kappa\tau}) + 2\tau\kappa}{2\kappa^3}
$$

This definition allows us to represent $\ln M_\tau$ as follows:

$$
\ln M_\tau = \ln M_0 - \frac{1}{2}\xi(\tau) - \int_0^\tau B(s,T)\, \sigma \cdot dz_s
$$

As the variance of $\ln M_\tau$ is equal to

$$
\int_0^\tau (B(s,T)\, \sigma)^2 \cdot ds = \xi(\tau),
$$

we can rewrite $\ln M_\tau$ by using a second standard Wiener-process z'_t at time $t = \xi(\tau)$:

$$
\ln M_\tau = \ln M_0 - \frac{1}{2}\xi(\tau) + z'_{\xi(\tau)} \tag{17}
$$

As $\xi(t)$ strictly increases from a value of 0 at $t = 0$ to $\xi(\tau)$ at $t = \tau$, each value of $\ln M_0 - \frac{1}{2}\xi(t) + z_{\xi(t)}$ for $t \in [0, \tau]$ is uniquely related to a value of $\ln M_0 - \frac{1}{2}t + z_t$, where $t \in [0, \xi(\tau)]$. Hence, we can interpret $\ln M_\tau$ as the result from an arithmetic Brownian motion at time $\xi(\tau)$ with drift $-\frac{1}{2}$ and unit variance. Therefore, we can deal with the less complex process $\ln M_0 - \frac{1}{2}t + z_t$ from $t = 0$ to $t = \xi(\tau)$ in order to examine the path of $\ln M_t$ from 0 to τ. Following Harrison [1985], we can determine the density function of $y_T := \ln(\max_{0 \le s \le \xi(T)}\{M_s\}) - \ln(M_0)$ as follows:

$$f(y_T) = \frac{1}{\sqrt{\xi(T)}} \cdot \Phi'\left(\frac{y_T + \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + e^{-y_T} \cdot \Phi\left(-\frac{y_T - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + e^{-y_T} \cdot \frac{1}{\sqrt{\xi(T)}} \cdot \Phi'\left(\frac{y_T - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right)$$

Applying this density function, we can evaluate the integral $\int_0^\infty e^{y_T + \ln P(0,T)} \cdot f(y_T)\, dy_T$ and we obtain:

$$\int_0^\infty e^{y_T + \ln P(0,T)} \cdot f(y_T) dy_T = P(0,T) \cdot \left(2 \cdot \Phi\left(\frac{1}{2}\sqrt{\xi(T)}\right) + e^{-\frac{\xi(T)}{8}}\sqrt{\frac{\xi(T)}{2\pi}} + \frac{\xi(T)}{2}\Phi\left(\frac{1}{2}\sqrt{\xi(T)}\right)\right)$$

APPENDIX C

Derivation of Equation (9)

In accordance with Equation (17), we can represent $\ln M'_\tau$, defined by the increment $d \ln M'_t := d \ln\left(P(t,T) \cdot e^{-\delta(T-t) - \int_0^t r_s ds}\right)$ as follows:

$$\ln M'_\tau = \ln M'_0 + \delta \cdot \tau - \frac{1}{2}\xi(\tau) + z'_{\xi(\tau)}$$

This definition enables us to write for the expectation in Equation (8):

$$\mathbb{E}_0\left(\max_{i=1,\ldots,N}\left\{P(t_i,T) \cdot e^{-\delta(T-t_i) - \int_0^{t_i} r_s ds}\right\}\right) = \mathbb{E}_0\left(\max_{i=1,\ldots,N}\left\{M'_{t_i}\right\}\right)$$

The expectation of the maximum value which is monitored discretely results in

$$\mathbb{E}_0\left(\max_{i=1,\ldots,N}\left\{M'_{t_i}\right\}\right) = \sum_{i=1}^{N} \mathbb{E}_0\left(M'_{t_i} \cdot 1_{\left\{M'_{t_i} > M'_{t_j}, i \neq j, j=1,\ldots,N\right\}}\right)$$

where $1_{\{\cdot\}}$ denotes the indicator function. We can rewrite the expectation as

$$\mathbb{E}_0\left(M'_{t_i} \cdot 1_{\left\{M'_{t_i} > M'_{t_j}, i \neq j, j=1,\ldots,N\right\}}\right) = M'_0 \cdot e^{\delta t_i} . \widetilde{\mathbb{E}}_0\left(1_{\left\{M'_{t_i} > M'_{t_j}, 1 \le j < i = 1,\ldots,N\right\}}\right) \cdot \widetilde{\mathbb{E}}_{t_i}\left(1_{\left\{M'_{t_i} > M'_{t_j}, i < j \le N, j=1,\ldots,N\right\}}\right)$$

where $\widetilde{\mathbb{E}}_0(\cdot)$ stands for the expectation under an equivalent probability measure $\widetilde{\mathbb{P}}$. According to this measure $\widetilde{\mathbb{P}}$, $\ln M'_\tau$ can be represented by $\ln M'_0 + \delta \cdot \tau + \frac{1}{2}\xi(\tau) + \tilde{z}_{\xi(\tau)}$ if $\tilde{z}_t$ is a standard Wiener-process under $\widetilde{\mathbb{P}}$. Heynen and Kat [1995] also make use of this equation in order to price discrete lookback options. The intuition for the idea behind this change of measure can be found in Johnson [1987]. The two expectations $\widetilde{\mathbb{E}}_0(1_{\{M'_{t_i} > M'_{t_j}, 1 \le j < i, j=1,\ldots,N\}})$ and $\widetilde{\mathbb{E}}_{t_i}(1_{\{M'_{t_i} > M'_{t_j}, i < j \le N, j=1,\ldots,N\}})$ can be represented by the values of multivariate cumulative standard normal distribution functions. Plugging them in Equation (18) yields formula (9).

ENDNOTES

This research was carried out while the second author was a research and teaching assistant at the Chair of Finance, University of Mannheim. The views expressed in this article represent the views of the authors and do not necessarily reflect those of DekaBank.

[1]Further empirical studies of liquidity effects include Boudhouk and Whitelaw [1991]; Elton and Green [1998]; Shen and Starr [1998]; Charkravarty and Sarkar [1999], and Longstaff [2002], among others.

[2]A vast range of literature on portfolio choice and equilibrium under transaction costs exists. Thus, we are forced not to mention a lot of relevant and insightful work in this area. Some references can be found in the survey of Sundaresan [2000] and the textbook of LeRoy and Werner [2001].

[3]Note that this approach does not lead to a value for the market timing ability. Since we compare the proceeds of two strategies, which both require perfect market timing ability, we essentially derive the excess benefit created by liquid markets.

[4]A (European) fixed strike lookback call pays the difference between the maximum price of the underlying during the monitoring period and the strike price at maturity given that the difference is positive.

[5]The following theoretical derivations also hold under the more general short rate process with a time-dependent θ as introduced by Hull and White (1990). Since we will estimate these interest rate parameters for the Vasicek case with a constant θ, we already use this notation at this point even though it is not necessary.

[6]We define interest rate volatility as the conditional standard deviation of r_τ given r_t (with $\tau > t$).

[7]Our simulation results for 100,000 repetitions show a sufficient precision. For typical bonds, the standard error of the simulated liquidity spread $\gamma(T)$ is below 0.5 bp.

[8]This means $\left|x - \frac{1}{T}\ln\left(\frac{\overline{S}(0,T)}{\underline{S}(0,x;t_1,\ldots,t_N)}\right)\right| / x < 0.01$.

[9]As e.g., Kempf and Uhrig-Homburg [2000], Krishnamurthy [2002], and Goldreich et al. [2002] show, the liquidity of individual government securities can vary from security to security and over time. However, we use the whole market for German government securities as a benchmark such that we can expect the liquidity of our benchmark to be sufficiently stable.

[10]Details of the institutional properties of Pfandbriefe and Jumbo Pfandbriefe are taken from Mastroeni [2001] and the Association of German Mortgage Banks, http://www.pfandbrief.org.

[11]The rating of Standard & Poors for all Jumbo Pfandbriefe is AAA given that they are rated at all.

[12]Rules by the Basel Committee for Banking Supervision oblige banks to maintain a capital reserve of 0,8 % of the book value for Pfandbriefe while German government securities are not subject to any capital requirements.

[13]In principle, it is straightforward to apply numerical methods to determine the required values of continuously and discretely monitored lookback options for the case of coupon bonds. However, it is no longer possible to obtain a closed-form solution for the continuously monitored lookback option. This is because the price of a coupon bond as a sum of zero bond prices with each being lognormally distributed within the Vasicek framework is in general not lognormally distributed. A numerical method for valuing a continuous lookback option is rather costly, since there is a large number of discrete time steps required to approximate the distribution of the relevant maximum with suffcient precision.

[14]The estimation of the required parameters of the Vasicek model is described in a separate section.

[15]This approximation of a humped-shaped relation is admittedly rough. However, we want to avoid overfitting.

[16]This view is also expressed in a number of recent articles like Perraudin and Taylor [2003] or Huang and Huang [2003].

[17]Note that the liquidity spreads resulting from the method change with the time to maturity of the bond and, as a consequence, they also change over time.

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