Separability and Pension Optimization

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Finance theorists since Markowitz have been deriving separation theorems, also known as mutual fund theorems, for efficient frontiers. While elegant and powerful, these results have generated little in the way of empirical applications. It is one thing to assert that all efficient portfolios can be characterized as combinations of two "base" portfolios. It is another to relate these results to investors' day-to-day operations.

We show here that separability results are empirically meaningful for managers of defined benefit (DB) pension plans and for others managing relative to a liability benchmark. That is, for surplus optimization, all efficient portfolios can be expressed as combinations of a full hedge of liabilities and exposure in that overlay with maximum Sharpe ratio—maximum return relative to risk.

Thus, portfolio optimization for DBs can be broken down into two parts: establishing a full hedge of liabilities, and seeking additional returns and risks in suitable asset sectors. The full liability hedge for a pension fund is obviously a "base" that is empirically relevant for pension managers. Moreover, this construction sets up pension optimization as parallel to the construction of portable alpha strategies.

Besides separability, we also discuss two other features of optimal pension allocations. First, just as surplus-optimal allocations can be decomposed into a liability hedge and an exposure in the "optimal" overlay, so too asset-optimal allocations[1] can be decomposed into the minimum-variance portfolio and exposure in that same optimal overlay. It follows that the difference between surplus-optimal and asset-optimal portfolios is independent of the target level of expected returns. Therefore, the effect on portfolio allocation of moving from asset-optimization to surplus-optimization frameworks can be fully described in terms of a single overlay portfolio and so is independent of return targets.

Second, our formulation lends itself to a discussion of the characteristics of full hedges of liability variation. Pension managers are familiar with hedging the interest rate risks of liabilities, via dedicated bond portfolios, duration-matched swaps, and so forth. However, liability valuations fluctuate for reasons other than interest rate swings, and a *full* hedge of liabilities should address all the systematic risks of liabilities. Our analysis identifies the full hedge of liabilities as the linear projection of liabilities on assets, thus the full extent to which systematic variation in liabilities can be addressed via asset allocations.

Once non-interest-rate risks are considered, there is no presumption that a dollar's worth of plan liabilities can be fully hedged with exactly a dollar of assets. In that case, even if a plan is fully funded on an accounting basis, available assets may be insufficient (or more than sufficient) to fully hedge existing liabilities. We introduce the concept of the *effective* funding ratio for a pension fund that measures the ability of existing plan assets to match the short-term volatility of existing liabilities. The effective funding ratio is the official funding ratio divided by the number of dollars of assets required to fully hedge a dollar of liabilities.

Regardless of whether there is a difference between "actual" and "effective" funding levels, we also show that there is no presumption that the expected rate of return on the full hedge of liabilities will exactly match the expected rate of change in liabilities. Therefore, there is potential conflict between pensions' desire to minimize surplus risk and to sustain a given level of funding. It need not be the case that any one strategy will effectively achieve both goals.

NOTATION

We work here with a universe of k assets, and we define the random variables x_1 through x_k as the holding period returns on a dollar invested in assets 1 through k. We consider both *full portfolios*, for which the sum of portfolio weights is one, and *overlays*, for which the sum of portfolio weights is zero. The vector w will typically refer to a full portfolio and z to an overlay. Thus, with *e* a k-dimension vector of ones, and a′b the inner product of vectors a and b, $w'e = 1$ and $z'e = 0$ for typical full and overlay portfolios, respectively. We use lower-case letters to indicate (column) vectors, upper-case, plain-face letters to indicate matrices, and upper-case, boldface letters to indicate scalars.

Define the random variable **L** as the holding-period rate of change on a dollar of plan liabilities. Defining the variables in this way allows us to work with return and risk components that are expressed in percent terms, simplifying the analysis. It also allows us to analyze funding-ratio issues by introducing the funding-ratio variable **F**, which shows pension assets as a percentage of liabilities at the beginning of the holding period. So the holding-period change in pension plan surplus per dollar of assets is: $\mathbf{A} - \mathbf{L}/\mathbf{F} \equiv w_1x_1 + w_2x_2 + \ldots + w_kx_k - \mathbf{L}/\mathbf{F} \equiv w'x - \mathbf{L}/\mathbf{F}$.

The distribution of the random vector x is described by its vector of expected values μ and its variance-covariance matrix Σ. The distribution of **L** is described by its expected value ρ and variance η^2 (both scalars but designated as lower-case), and by χ, the vector of covariances between *x* and **L**.

While covariance properly concerns two scalars, we use **cov**(x, x′) to designate covariance sets involving vectors and matrices, in this case, the covariances of *x* with itself, while covariance(**A**, **B**) designates that between two scalars. The order of the vectors/matrices and transposition designation within the **cov**() notation reflects the order and type of matrix multiplication for the arguments. Thus, **cov**(x, x′) returns the covariance matrix Σ, derived from the expectation of $x \cdot x'$, while **cov**(x′, x) returns a scalar: the variance of the scalar x′x.

Using this notation, it is convenient to express the covariance between two portfolios w′x and z′x solely in terms of the weight vectors w and z. Thus, define $\mathbf{cov}(w', z) \equiv \text{covariance}([x'w]', x'z) = w' \cdot \mathbf{cov}(x, x') \cdot z = w'\Sigma z$. Similarly, $\text{variance}(w'x) \equiv \mathbf{var}(w) \equiv \mathbf{cov}(w', w) \equiv w'\Sigma w$, and

$$\mathbf{cov}(w, \mathbf{L}/\mathbf{F}) \equiv \text{covariance}(w'x, \mathbf{L}/\mathbf{F}) \equiv w'\mathbf{cov}\ (x, \mathbf{L})/\mathbf{F} \equiv w'\chi/\mathbf{F}$$

So, surplus variance is $\text{variance}(w'x - \mathbf{L}/\mathbf{F})$

$$= \mathbf{var}(w) + \text{variance}(\mathbf{L}/\mathbf{F}) - 2\mathbf{cov}(w, \mathbf{L}/\mathbf{F}) = w'\Sigma w + \eta^2/\mathbf{F}^2 - 2w'\chi/\mathbf{F}$$

DERIVING THE EFFICIENT FRONTIER

The frontier of surplus efficient portfolios is the set of w which minimize surplus variance for given levels of expected return on surplus.[2] From the formula for surplus variance, it is apparent that if liabilities are uncorrelated with the assets—if χ is a vector of zeroes, then minimizing surplus variance amounts merely to minimizing portfolio variance $w\Sigma w$—the variance of liabilities η^2 is unaffected by asset allocation. If $\chi = 0$, then surplus optimization collapses down to standard asset optimization.

The expected return on surplus for portfolio w is $w'\mu - \rho/\mathbf{F}$. Since ρ and **F** are out of the control of the plan for a given holding period, minimizing surplus variance for a given level of expected surplus return is equivalent to minimizing surplus variance for a given level of expected total return, or $w'\mu$. Without loss of generality, we can then refer to the surplus optimization problem as minimizing surplus variance for a given level of expected total return on assets. This perfects the parallel between asset- and surplus-optimization cases, asset optimization being a special case of surplus optimization when fluctuations in liabilities are uncorrelated with all available assets.

So, for both asset- and surplus-optimization, the efficient frontier of optimal portfolios is the set of portfolios w that minimize $w'\Sigma w + \eta^2/F^2 - 2 \cdot w'\chi/\mathbf{F}$, subject to

$w'\mu = \mathbf{M}$ and $w'e = 1$. Solutions to this minimization problem are characterized by Lagrangian multipliers, 2λ and 2φ, and the Hamiltonian,

$$\mathbf{H} \equiv w'\Sigma w + \eta/\mathbf{F}^2 - 2 \cdot w'\chi/\mathbf{F} + 2\lambda(\mathbf{M} - w'\mu) + 2\varphi(1 - w'e)$$

such that the partial derivatives of H with respect to w, λ, and φ are zero:

$$\partial\mathbf{H}/\partial w = 2 \cdot \Sigma w - 2 \cdot \chi/\mathbf{F} - 2\lambda\mu - 2\varphi e = 0 \quad (1)$$

$$\partial\mathbf{H}/\partial\lambda = \mathbf{M} - w'\mu = 0 \quad (2)$$

$$\partial\mathbf{H}/\partial\varphi = 1 - w'e = 0 \quad (3)$$

Solving (1) for w, $w = \Sigma^{-1}(\lambda\mu + \varphi e + \chi/\mathbf{F})$, so (2) and (3) become

$$(\lambda\mu' + \varphi e')\Sigma^{-1}\mu = \mathbf{M} - \mathbf{xSm}/\mathbf{F} \quad (4)$$

$$(\lambda\mu' + \varphi e')\Sigma^{-1}e = 1 - \mathbf{xSe}/\mathbf{F} \quad (5)$$

where $\mathbf{xSm} \equiv \chi\Sigma^{-1}\mu$ and $\mathbf{xSe} \equiv \chi\Sigma^{-1}e$. So (4) and (5) become the system

$$\begin{bmatrix} mSm & mSe \\ mSe & eSe \end{bmatrix}\begin{bmatrix} \lambda \\ \varphi \end{bmatrix} = \begin{bmatrix} M - \mathbf{xSm}/\mathbf{F} \\ 1 - \mathbf{xSe}/\mathbf{F} \end{bmatrix} \quad (6)$$

where $\mathbf{mSm} \equiv \mu'\Sigma^{-1}\mu$, $\mathbf{mSe} \equiv \mu'\Sigma^{-1}e \equiv e'\Sigma^{-1}\mu$, and $\mathbf{eSe} \equiv e'\Sigma^{-1}e$.

The left side of (6) is unaffected by whether or not χ is nonzero; the optimization process is only indirectly affected by the presence of the liabilities. A unique solution to (6) will exist if the determinant $\mathbf{D} \equiv \mathbf{mSm} \cdot \mathbf{eSe} - (\mathbf{mSe})^2 > 0$. Since no mention of the liabilities occurs here, the conditions for existence and uniqueness of a solution are identical to those for the standard, asset-optimization case. Specifically, so long as no asset is perfectly correlated with any linear combination of other assets, then the matrix Σ is positive definite, so $\mathbf{mSm} > 0$ and $\mathbf{eSe} > 0$, being quadratic forms in a positive-definite matrix, and $\mathbf{D} > 0$.

Then the system (6) yields the solution

$$\lambda = [(\mathbf{M} - \mathbf{xSm}/\mathbf{F})\mathbf{eSe} - (1 - \mathbf{xSe}/F)\mathbf{mSe}]/\mathbf{D}$$

and

$$\varphi = [(1 - \mathbf{xSe}/\mathbf{F})\mathbf{mSm} - (\mathbf{M} - \mathbf{xSm}/\mathbf{F})\mathbf{mSe}]/\mathbf{D} \quad (7)$$

Substituting these into (1) and simplifying indicates that surplus-optimal w's, the w_+^*'s (the "$_+^*$" suffix indicates surplus-optimal portfolios), must satisfy:

$$w_+^* = \Sigma^{-1}[\chi/\mathbf{F} + (1 - \mathbf{xSe}/\mathbf{F})e/\mathbf{eSe}] + \{\mathbf{ese} \cdot [\mathbf{M} - \mathbf{xSm}/\mathbf{F} - (\mathbf{mSe}/\mathbf{eSe})(1 - \mathbf{xSe}/\mathbf{F})]/\mathbf{D}\}\Sigma^{-1}(\mu - \mathbf{mSe} \cdot e/\mathbf{eSe})\} \quad (8)$$

SEPARATING SURPLUS-OPTIMAL PORTFOLIOS INTO HEDGE AND OVERLAY PORTFOLIOS

Now, (8) indicates that the optimal portfolio weights are linear in the target return $\mathbf{M}$. As the target $\mathbf{M}$ rises, as the pension plan seeks more aggressive allocations, the optimal portfolio changes by the same fixed increments for each unit change in $\mathbf{M}$, which increments are proportional to vector $\Sigma^{-1}(\mu - \mathbf{mSm} \cdot e/\mathbf{eSe})$. We can express the terms in (8) in a more intuitively meaningful way by analyzing minimum-risk portfolios, those furthest to the left on efficient frontiers.

Consider, first, the case of asset optimization. That is, assume for the moment that $\chi = 0$. The minimum-variance portfolio (MVP) can be described by dropping the constraint $w'\mu = \mathbf{M}$ from the problem and minimizing $w'\Sigma w$ over the set of w's for which $w'e = 1$. Alternatively, asset variance is minimized in the present problem when $\lambda = 0$ (and $\chi = 0$). From (7), $\mathbf{M}_0 = \mathbf{mSe}/\mathbf{eSe}$, where $\mathbf{M}_0$ is the expected return on the MVP. It follows that $\mathbf{V}_0 = 1/\mathbf{eSe}$, where $\mathbf{V}_0$ is the variance of the MVP, and, finally, w_0, the MVP itself, is

$$w_0 = \Sigma^{-1}e/\mathbf{eSe} \quad (9)$$

Inserting this formula and $\chi = 0$ into (8), asset-optimal portfolios, w^*, satisfy

$$w^* = w_0 + [(\mathbf{M} - \mathbf{M}_0)/(\mathbf{V}_0\mathbf{D})]\Sigma^{-1}(\mu - \mathbf{M}_0 \cdot e) \equiv w_0 + (\mathbf{M} - \mathbf{M}_0)z^* \quad (10)$$

where $z^* \equiv \Sigma^{-1}(\mu - \mathbf{M}_0 \cdot e)/\mathbf{V}_0\mathbf{D}$. z^* is an overlay, since $e'z^* \approx (\mathbf{mSe} - \mathbf{M}_0 \cdot \mathbf{eSe}) = 0$.

So, asset-optimal portfolios are combinations of the MVP and the overlay portfolio z^*. (We represent z^* as the amount of this overlay necessary to raise target return by 1%. This is the case since $z^* = \partial w^*/\partial \mathbf{M}$ under our definition.)

Consider, now, the case of surplus optimization, $\chi \neq 0$. Then if $\lambda = 0$, from (7) and (8), the minimum surplus-variance portfolio (MSVP) w_{+0} is

$$w_{+0} = w_0 + \Sigma^{-1}(\chi - \mathbf{xSe}\cdot e/\mathbf{eSe})/\mathbf{F} \qquad (11)$$

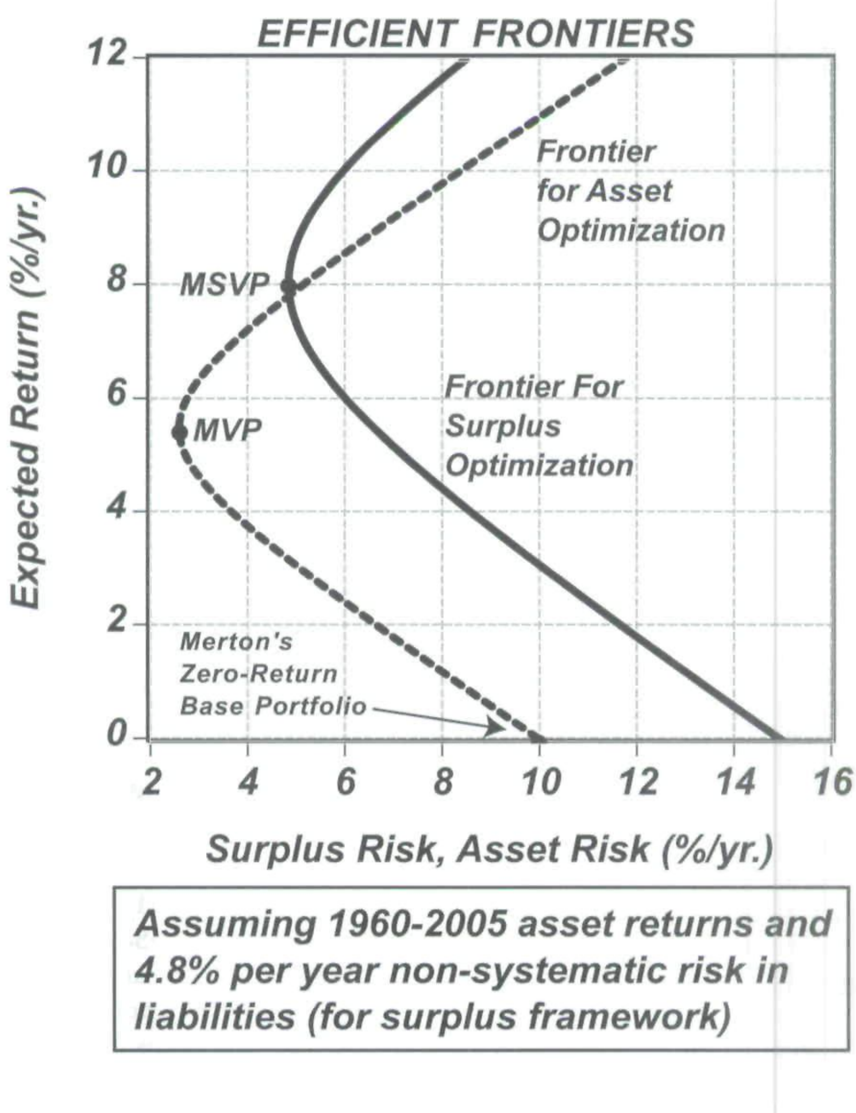
which is just a re-expression of the first term on the right side of (8). The return on the MSVP, $\mu' w_{+0}$, is $M_{+0} = M_0 + (\mathbf{xSm} - \mathbf{M}_0\mathbf{xSe})/\mathbf{F}$. (We interpret the meaning of w_{+0} in the second section following.) Equation (8) can be simplified to[3]

$$\begin{aligned} w_+^* &= w_{+0} + \{(\mathbf{M} - \mathbf{M}_{+0})(1/\mathbf{V}_0)/\mathbf{D}\}\Sigma^{-1}(\mu - \mathbf{M}_0\cdot e) \\ &\equiv w_{+0} + (\mathbf{M} - \mathbf{M}_{+0})\cdot z^* \end{aligned} \qquad (12)$$

Equation (12) states that surplus-optimal portfolios are combinations of the liabilities' hedge (MSVP) w_{0+} and an overlay z^* that is independent of the liabilities: the same overlay as in the asset optimization solution above.

Equation (12) is a restatement of the Separation Theorem for surplus-optimization. The exact form of this separation is somewhat counterintuitive. One would expect the plan's liabilities to be reflected in its optimal allocation, but one might also expect the correlation of various assets' returns with liabilities to affect the way

EXHIBIT 1



incremental risks are taken. Instead, incremental risks are optimally taken completely independently of the liabilities (via exposure in z^*).

It is easy to show that the overlay z^* has maximum Sharpe ratio across the set of overlay portfolios. That is, z^* is the solution to the problem: Maximize $z'\mu/(z'\Sigma z)^{1/2}$ subject to $z'e = 0$. It makes sense that as one moves along the efficient frontier away from the minimum-risk portfolio (MVP or MSVP, depending upon the framework), one adds overlays with maximum incremental return relative to incremental risk. After all, optimal portfolios themselves are maximum-Sharpe-ratio portfolios, or, in the case of surplus optimization, optimal portfolios have what we call "maximum surplus-Sharpe ratio": maximum values of expected return relative to surplus risk. So starting from the MVP or MSVP, which are surely on their respective frontiers and so have maximum Sharpe (or surplus-Sharpe) ratio for their levels of risk, it is logical that one would maximize the relevant Sharpe ratio of the resulting portfolio by adding on that overlay with maximum Sharpe ratio.[4] A potential problem is that there could be correlation between the initial position (MVP or MSVP) and the z^* overlay and that could affect the relevant Sharpe ratio for the composite position.

However, the MVP must have zero covariance with any overlay, since it is a minimum, so that the derivative of its variance is zero with respect to movements in any direction, with respect to the addition of any overlay. (The gradient of portfolio variance with respect to its weights is proportional to the portfolio's covariances with its components: $\mathbf{cov}(x, x'w) = \Sigma w \approx \partial\{w'\Sigma w\}/\partial w$.) Similarly, the "surplus covariance" between the MSVP and any overlay $[\mathbf{cov}(w'x - \mathbf{L}/\mathbf{F}, z)]$ is zero, since the MSVP is a minimum.

Because the minimum-variance portfolio has zero covariance with any overlay, the Sharpe ratio for the composite of the minimum-variance portfolio and any overlay is maximized by adding on that overlay which itself has maximum Sharpe ratio, z^*; this is true for both asset- and surplus-optimization. This would not be true for just any "base" portfolio, but it is the case for the appropriate minimum-variance base, and so it is also the case for any other portfolio on the appropriate efficient frontier.

Again, (12) is just an extension of the Separation Theorem of Markowitz, Sharpe, Merton, and others to the surplus optimization case. These writers demonstrated that portfolios on the asset-optimal efficient frontier can be shown to be combinations of two base portfolios or "mutual funds." In Merton [1972], the "base" portfolios chosen were our "optimal" overlay z^* and that portfolio on the efficient frontier with zero return. Since the overlay z^* is constant across the frontier, one can characterize the frontier as combinations of z^* and any "starting point; one starting point is as good as another. In the case of surplus optimization, the MSVP, w_{0+}, is clearly a relevant starting point, and so we choose it and its asset optimization counterpart, the MVP, as "bases" for our construction of the respective frontiers.

These results indicate that pension optimization involves identifying a full hedge of liabilities as the benchmark for asset management, achieving that hedge, and then overlaying on "alpha" sources. Those should have maximum return (alpha) relative to the amount of risk (tracking error) they introduce. This construction is identical to that of a portable alpha strategy for which the "beta" is set to the plan's liabilities, and it justifies the practice currently emerging in Europe of managing pension allocation by porting alpha onto a liability hedge.

THE IMPACT OF SURPLUS OPTIMIZATION

The diagram illustrates that there is nothing in surplus optimization requiring a pension fund to "abandon" total return maximization.[5] Both asset- and surplus-optimal frontiers here are portrayed as achieving the same levels of total return. The presence of liabilities and their correlation with assets merely change the relevant measure of risk. What is the impact of this?

While total returns are comparable across the two frontiers, risk measures clearly are not. However, a relevant measure of the impact of shifting to a liability benchmark (to surplus optimization) can be seen in the effects on portfolio weights. Since both asset- and surplus-optimal weights, w^* and w_+^*, respectively, can be expressed as combinations of the appropriate minimum-variance "bases" and z^*, their weights shift linearly by the same fixed amounts as target returns vary. Thus, the difference in portfolio weights between asset- and surplus-optimal portfolios is independent of the returns target. From (10) and (12) the following expression can be derived:

$$\begin{aligned} w_+^* - w^* &= \Sigma^{-1}(\chi - \mathbf{xSe} \cdot e/\mathbf{eSe})/\mathbf{F} \\ &\quad - [(\mathbf{xSm} - \mathbf{M}_0\mathbf{xSe})/\mathbf{F}]\Sigma^{-1}(\mu - \mathbf{M}_0 \cdot e) \end{aligned}$$

which is independent of the target return M. *Shifting from an asset framework to a surplus framework, (while holding target* *returns constant), results in the same shift in the optimal portfolio, regardless of the plan's risk tolerance* (i.e., regardless of the level of target returns).

This portfolio shift is dependent on the exact nature of the liabilities—on the magnitudes of χ, x**Se**, and **xSm**. However, insight into the magnitude of this shift for different financial environments can be estimated by comparing asset-optimal and surplus-optimal portfolio weights at the level of **M** equal to long-bond yields. That is, whatever the economic environment, we can approximate the hedge portfolio, w_{0+}, with a 100%-long-bond allocation, and we can replicate the return on long bonds with an asset-optimal portfolio a la (10) with **M** set appropriately, and μ and Σ determined by our expectations for the economic environment. The difference between a 100%-bond allocation and the asset-optimal portfolio with equal expected return is, thus, a good estimate of the difference between asset- and surplus-optimal portfolios everywhere across the range of target returns. Thus, it is a good estimate of the impact of surplus optimization on portfolio allocation in a particular economic environment (particular values of μ, Σ, and χ).

THE FULLY HEDGED PENSION ALLOCATION

A full pension hedge is the MSVP, so that $w_{+0} = w_0 + \Sigma^{-1}(\chi\text{-}\mathbf{xSe} \cdot e/\mathbf{eSe})/\mathbf{F}$. What do all these terms mean? Suppose liabilities were essentially equal to long bonds. That is, suppose that long bonds are asset x_B, $B \in \{1, ..., k\}$, and $\mathbf{L} = x_B + \varepsilon$, where $\mathbf{cov}(x, \varepsilon) = 0$. (Subtract long bonds from liabilities, and what is left is non-systematic variation.) By construction, $\chi = \mathbf{cov}(x, x_B)$. The χ is the Bth column of Σ. Since $\Sigma^{-1}\Sigma = I$, $\Sigma^{-1}\chi = i_B$, the Bth unit vector. Then, $\mathbf{xSe} \equiv e'\Sigma^{-1}\chi = e'i_B = 1$, and $\mathbf{xSm} = \mu'\Sigma^{-1}\chi = \mu' i_B = \mu_B$, the expected return on long bonds.

When liabilities are "bond-like," the MSVP is merely $(1 - 1/\mathbf{F})w_0 + i_B/\mathbf{F}$. To minimize surplus risk, a plan with bond-like liabilities should buy long bonds in the amount $1/\mathbf{F}$, which is equal to the amount of their liabilities, and any over-funding $(1 - 1/\mathbf{F} > 0)$ should then be invested in the MVP, w_0, while any under-funding $(1 - 1/\mathbf{F} < 0)$ should be financed by borrowing/shorting the MVP. Again, this is the allocation that minimizes surplus risk at all costs.

Now, what if liabilities are bond-like, but it takes a dedicated portfolio of bonds to hedge them? Then there will be some bond portfolio, h, such that, as was the case with a single bond above,

$$0 = \mathbf{cov}(x, x'h - \mathbf{L}) = \Sigma h - \chi$$

or

$$h = \Sigma^{-1}\chi$$

Then, again, $\mathbf{xSe} = h'e = 1$, and $\mathbf{xSm} = h'\mu$, the expected return on the dedicated bond portfolio. So in this case, the MSVP is $w_{0+} = (1 - 1/\mathbf{F})w_0 + h/\mathbf{F}$. The plan can minimize surplus risk by investing in a dedicated bond portfolio in amount equal to its liabilities, and then investing/financing any over-funding/underfunding in the MVP.

Even if there is no dedicated bond portfolio that can hedge *all* systematic risks of the liabilities, the "orthogonality principle" states that as long as the distributions of x and **L** are stationary, there will exist an optimal linear "projection" of liabilities on the assets, h, such that $\mathbf{cov}(x, x'h - \mathbf{L}) = 0$.[6] This h certainly resembles that in the preceding paragraph, but there is an important difference. In this case, there is no presumption that the hedge portfolio h is made up only of bonds; h is merely the least-squares estimator of y using x, potentially employing all available assets. While it will still be the case that $\mathbf{xSe} = h'e$, there is no presumption that these weights sum to one. There is no presumption that a dollar of liabilities can be fully hedged with exactly one dollar of assets. We can give two real-world examples where this sum will not be one—ABO valuations and long-dated obligations.

ABO Valuations

Suppose a pension plan attempts to hedge its accumulated benefit obligation (ABO) valuation which is calculated by evaluating accrued pension benefits at *current* wage levels and discounting them to a present value.[7] Since the ABO is based on current wages, it will rise over time as wages increase, and it will also rise at the rate of interest. One dollar of bonds could be best able to hedge the interest rate risk of each dollar of ABO, but it cannot also hedge the wage inflation risk. If wage gains are systematically correlated with the assets x, then a full hedge of each dollar of ABO liability will require both a dollar of interest rate hedge and some *additional* amount of inflation hedge.[8]

Long-Dated Obligations

Suppose a plan's obligations extend out past the farthest maturity in the fixed-income universe. Even though we must assign discount rates to those out-year obligations in order to evaluate liabilities, there will be no dedicated bond portfolio extant whereby one dollar of liabilities could be fully hedged by one dollar of dedicated bonds. A dollar present value of obligations fifty years out might be best hedged by two dollars present value of 25-year maturity zero-coupon bonds (dollar-duration matching). So the best hedge of one dollar of long-dated liabilities requires more than a dollar of assets.

One might argue that stocks, commodities, or real estate would be able to hedge those out-year obligations. However, there is no reason to expect that exactly one dollar's worth of those assets would fully hedge a dollar of liabilities.

A variation on this case would be where the only assets suitable to hedge long-dated liabilities are Treasury bonds or strips. Since corporate yields are used to discount liabilities—and these are presumed to be higher than Treasury yields—it will take more than one dollar's worth (market value) of Treasuries to hedge one dollar's worth (present value) of long-dated pension obligations.

PBO valuations, in general, move with interest rates, unexpected changes in wages, and actuarial "events."[9] The last two concern movements relative to the current projections of the plan's actuaries. While such variation would be *expected* to be zero, the *realized* variation could still be systematically correlated with (realized) variation in various assets (inflation hedges). Therefore, a full hedge of liabilities' variation—as required by the MSVP portfolio—would require a dedicated bond portfolio to address interest risks and investment in other assets to hedge inflation/actuarial risks. Therefore, even a PBO valuation with no extremely long-dated obligations could feature a *full* hedge (MSVP) that requires more than a dollar of assets for each dollar of liabilities.

In general, xSe measures the amount of assets required to fully hedge one dollar of liabilities.[10] Therefore, while $1/\mathbf{F}$ measures the *apparent*, or official, amount of over-investment required to hedge liabilities, $\mathbf{xSe}/\mathbf{F}$ measures the *effective* amount of over-investment required to hedge the liability position. That is, if $\mathbf{F} = 80\%$, if the plan is 80% funded, then 125% (1/0.8) of existing assets apparently must be invested in the hedge. In addition, however, if $\mathbf{xSe} = 1.1$, then each dollar of liabilities requires \$1.10 of assets to achieve a full hedge, and so 138% (1.1 times 125%) of existing assets must be invested to fully hedge liabilities (financed by borrowing 38% of asset-worth of the MVP). ***F/xSe*** *can be interpreted as the effective funding ratio for the plan.* For the example here, the effective funding ratio is 0.8/1.1, or 73% (= 1/1.38).

Finally, $\mathbf{xSm}$ can be interpreted as the expected return on the amount of MSVP required to hedge a dollar of liabilities. Even when all of the systematic variation of the liabilities is interest rate-related and when $\mathbf{xSe} = 1$, there is still no presumption that $\mathbf{xSm} = \rho$, since the nonsystematic variation in liabilities may still have nonzero expected rate of change. (This is certainly the case for ABO valuations.) As a result, fully hedged allocations may not be sufficient to sustain current funding levels. This is an unpleasant fact, but a fact nonetheless, and it suggests a trade-off to a pension plan between those allocations which minimize risk and those which can be expected to maintain or improve funding ratios.

CONCLUSION

Our Separation Theorem for surplus optimization decomposes surplus-optimal portfolios into combinations of a full hedge of liabilities and suitable exposure in that overlay with maximum Sharpe ratio. These results imply that the impact on portfolio allocation of shifting from an asset- to surplus-optimization framework depends on the financial environment and on the nature of liabilities, but not on the degree of risk tolerance. The full hedge of a pension plan's liabilities is the linear projection of liabilities on assets. The required level of investment in this hedge depends on the magnitude and variability of the liabilities, not on the magnitude of the assets. Generally, there is no presumption that a dollar of assets will be sufficient to hedge a dollar of liabilities, nor is there any presumption that a fully hedged allocation will sustain present funding levels.

ENDNOTES

[1]We use *asset optimization* to describe the standard, Markowitz process of minimizing portfolio variance for a given level of expected return and *surplus optimization* to describe the process facing DB pension funds, minimizing surplus variance for a given level of expected returns.

[2]As with standard optimization analyses, the efficient frontier is only the positively sloped portion of the locus of solutions to this problem, but this fact does not change our analysis any more than it changes the standard depiction.

[3]This is a restatement of the results presented in Keel and Muller [1995].

[4]Moving from any one portfolio to another merely involves adding an overlay, since if w1 and w2 are any two full portfolios, the difference, w2 – w1, must be an overlay: $[w2 - w1]'e = w2'e - w1'e = 0$.

[5]Contrary to the assertion of Sharpe and Tint [1990].

[6]See Sargent [1987].

[7]For FASB accounting protocols, liabilities are evaluated according to a projected benefit obligation (PBO) and an accumulated benefit obligation (ABO), with the PBO using projected wage levels at retirement and the ABO using current wage levels. While primary FASB pension funding reporting is in terms of the PBO, special reporting requirements are made for plans that are underfunded on an ABO basis. Meanwhile, the ABO is the valuation that plan sponsors are currently legally committed to and is essentially equal to the funding valuation recognized by ERISA regulations, so it could be a relevant target for some corporate plan managers.

[8]If wage gains and/or actuarial changes are not systematically correlated with the assets, then one dollar of a dedicated bond portfolio would be sufficient to minimize the *surplus variance* of one dollar of ABO liabilities. However, that hedge will not be sufficient to fully replicate all *expected change* in the ABO. That is, the ABO would still be expected to rise over time with interest rate changes and with the expected change in workers' wages. In this case, *there would be an inevitable tension between minimizing surplus variance and maintaining a given funding level.* No allocation will do both.

[9]See note 7 for a description of PBO valuations.

[10]Under our terminology, a perfect hedge leaves no residual variation, while a "full" hedge merely leaves no systematic, residual variation.

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