Term-Structure Slope Risk: *Convexity Revisited*

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Conventional approaches to measuring the risk of fixed-income portfolios are usually based on one of two forms of analysis. In the historically older approach, duration is used to provide a measure of the sensitivity of the value of the portfolio to a shift in the level of the term structure. Convexity may also be computed to give a more accurate approximation for the effect of large parallel moves. In the second approach, the sensitivity is measured to a number of distinct factors. These could be tent-shaped shocks distributed along the term structure, or those revealed by principal components analysis of term-structure changes. Typically in the latter case, this involves three factors which correspond roughly to changes in the level, slope, and curvature of the term structure. These analyses are often augmented by considering cash flows within distinct "maturity buckets," "gap analysis," or other more *ad hoc* procedures.

This article provides a new and practical approach to measuring and hedging term-structure risk which integrates these forms of analysis. We show how to measure the risk of changes in the slope and curvature of the term structure, as well as in its level, either globally or in maturity buckets. This is motivated by the philosophy of calculating the exposures to principal components, but obviates the need to estimate them. We demonstrate that the new measures are related to the existing measures of duration and convexity, and to the (generalized) moments of the (present-value-weighted) distribution of cash-flow dates, thus linking with the older form of analysis.

Duration was first formulated by Macaulay [1938] and has been a standard method of managing interest-rate risk at least since Fisher and Weil [1971]. There are a number of precedents in the literature for the use of multifactor generalizations of duration related to ours. Chambers, Carleton, and McEnally [1988] suggested the use of a duration vector to immunize bond portfolio exposures. Willner [1996] also proposes something very similar, in the context of a parametric model. The idea has been further extended in the article of Nawalkha and Chambers [1997] and Crack and Nawalkha [2000]. However, all these articles describe global approximations to the term structure, often involving a high-order polynomial.[1] We propose here that more convenient and intuitive risk measures may be obtained by using only one or two terms in a localized approximation.

Nevertheless, it is natural to start with a global analysis. We will also echo the sentiment of Crack and Nawalkha [2000] that the empirical importance of convexity is not that it measures the second derivative of value under a parallel shift in rates, but that it measures the exposure to a change in the slope of the term structure. This important fact has been missed by many researchers and by almost all text books in the area.

The article is structured as follows. Following this introduction, the next section introduces our notation and summarizes the standard definitions of duration and convexity. We then introduce more novel material: we characterize the risk of slope (or rotational) changes in the term structure about an arbitrary point. This is particularly useful because once duration has been immunized this is the largest remaining source of market risk. It turns out that the sensitivity to a change in slope is a function of both duration and convexity. We also establish how it is related to the distribution of the dates of cash flows and their present values. This approach is then extended further to the risk of curvature changes in the term structure, and an example is provided to illustrate the procedure. Finally, we describe how to calculate the incremental-risk savings by laying on further hedges and to optimize hedges in this framework, and also how to incorporate the risk of options positions.

NOTATION AND CONVENTIONAL MEASURES

We assume that initially a curve of zero-coupon yields has been constructed. The notation P_T is used to denote the price of a zero-coupon bond with maturity T years from today.

P_T may, therefore, be expressed equivalently as either

$$P_T = (1 + Y_T)^{-T} \tag{1}$$

in terms of the annually compounded zero-coupon yields, Y_T, or

$$P_T = e^{-y_T T} \tag{2}$$

in terms of the continuously compounded zero-coupon yields, y_T.

We prefer to use continuous compounding for our derivations, but it is straightforward to translate these results into other compounding conventions.

We will consider the sensitivities of a portfolio of cash flows to the shape of the term structure, where the portfolio has certain cash flows of c_i to be received (or paid, if negative) at dates T_i from today where $i = 1,....,N$. It will also be convenient to use the notation $PV(c_i) = c_i \times P_{T_i}$ for the present value of the i^{th} cash flow, evaluated using the zero-coupon bond prices.

Conventional duration and convexity measure the effect of parallel shifts in the term structure. If the initial term structure changes to one with yields of $y_T + \alpha$, then since

$$V = \sum_{i=1}^{N} c_i \exp\{-T_i(y_{T_i} + \alpha)\} \tag{3}$$

it is well known that the first and second derivatives of V with respect to α give

$$\begin{aligned}\left(\frac{\partial V}{\partial \alpha}\right)_{\alpha=0} &= -\sum_{i=1}^{N} T_i c_i \exp\{-T_i y_{T_i}\} \\ &= -\sum_{i=1}^{N} T_i PV(c_i) = -V\,D(\mathbf{c})\end{aligned} \tag{4}$$

where $D(\mathbf{c}) = \frac{\Sigma_i T_i \times PV(c_i)}{\Sigma_i PV(c_i)} =$ Duration of cash flow vector $\mathbf{c}$, and

$$\begin{aligned}\left(\frac{\partial^2 V}{\partial \alpha^2}\right)_{\alpha=0} &= \sum_{i=1}^{N} T_i^2 c_i \exp\{-T_i y_{T_i}\} \\ &= \sum_{i=1}^{N} T_i^2 PV(c_i) = V\,C(\mathbf{c})\end{aligned} \tag{5}$$

where $C(\mathbf{c}) = \frac{\Sigma_i T_i^2 \times PV(c_i)}{\Sigma_i PV(c_i)} =$ Convexity of cash flow vector $\mathbf{c}$.

The second-order Taylor series expansion in α gives the return on V resulting from a shift α as

$$\frac{\Delta V}{V} = -\alpha D(\mathbf{c}) + \tfrac{1}{2}\alpha^2 C(\mathbf{c}) + O(\alpha^3) \tag{6}$$

If we wish to work with a shift in the annually compounded rate Y_T, instead of the continuously compounded one, we simply need to recall that

$$y_T = \ln(1 + Y_T)$$

so

$$\frac{dy_T}{dY_T} = \frac{1}{1 + Y_T} \tag{7}$$

and

$$\frac{dV}{dY_T} = -V.\frac{D(\mathbf{c})}{1 + Y_T}, \text{ as usual}$$

The expression $D(\mathbf{c})/(1+Y_T)$ is also called modified duration.

TERM-STRUCTURE LEVEL AND ROTATIONAL EXPOSURE

In practice, shifts in the term structure are seldom exactly parallel, especially if they are large enough for the convexity term to matter. Instead, we may be interested to know the effect of a rotation in the curve about some arbitrary point. Consider term structures where the yields near to maturity T_0 shift by amounts α in level and β in slope, so that the initial term structure changes to one with yields of

$$y_T + \alpha + \beta\,(T - T_0) \tag{8}$$

The portfolio value is given as

$$V = \sum_{i=1}^{N} c_i \exp\{-T_i(y_{T_i} + \alpha + \beta(T_i - T_0))\} \tag{9}$$

We have already seen that the derivative with respect to α gives a conventional duration measure (when evaluated at $\alpha = \beta = 0$). The derivative with respect to β gives a new measure of slope risk, which happens also to be related to convexity

$$\begin{aligned}\left(\frac{\partial V}{\partial \beta}\right)_{\alpha=\beta=0} &= -\sum_{i=1}^{N} T_i(T_i - T_0)c_i \exp\{-T_i y_{T_i}\} \\ &= -\sum_{i=1}^{N} T_i(T_i - T_0)PV(c_i) \\ &= -\left[C(\mathbf{c}) - T_0\, D(\mathbf{c})\right]V\end{aligned} \tag{10}$$

where $C(\mathbf{c})$ and $D(\mathbf{c})$ are convexity and duration as before. We call this quantity "the exposure to rotation of the term structure about maturity T_0." For a portfolio which is already immunized in a duration sense, so that $D(c) = 0$, this risk exposure equals, minus the convexity, whatever point we choose for T_0.

The higher the convexity the more the portfolio is exposed to the risk of an increase in the slope of the term structure. In conventional bond risk analysis, convexity is usually regarded as a "good thing," providing insurance against large moves in the same way as the gamma of an option. Our analysis points out that, just as option gamma brings with it a risk exposure to volatility, convexity creates an exposure to the slope of the term structure. The following example illustrates that this risk is typically far more significant than the risk of a large parallel move.

Barbell Portfolio Example

A barbell portfolio, such as the one in Exhibit1, shows a small increase for any instantaneous parallel shift in rates, either up or down, as illustrated in scenarios A and B of Panel 3.

However, it will also lose a lot of value if the term structure becomes more upward sloping, and gain a lot of value if it becomes less upward sloping. The convexity rotational risk measure of 3,200 predicts that the gains and losses will be plus or minus 8.00 in each case (3,200 times the change in slope of 0.0025). The risk of such gains or losses is clearly much more significant than the much smaller gains from the curvature (or gamma) effect.

PV-Weighted Maturity Date Distribution

Duration and convexity are weighted averages of times to maturity and squared times to maturity, respectively, such that

$$D(\mathbf{c}) = \sum_i w_i T_i, \quad C(\mathbf{c}) = \sum_i w_i T_i^2$$

where

$$w_i = \frac{PV(c_i)}{\sum_j PV(c_j)}, \text{ so } \sum w_i = 1 \tag{11}$$

In the case where all cash flows are non-negative, we can describe the weights as if they were probabilities, and interpret duration and convexity as expectations under this rather unusual probability measure:[2]

EXHIBIT 1

Performance of a Barbell Portfolio

Panel 1: Portfolio PVs, Duration, and Convexity				
Maturity	**PV**	**t PV**	**t^2 PV**	
1	$100	100	100	
5	-$200	-1000	-5000	
9	$100	900	8100	
Total	**$0**	**0**	**3200**	
Panel 2: Four Interest-Rate-Change Scenarios				
Maturity	**A**	**B**	**C**	**D**
1	1%	-1%	-1%	1%
5	1%	-1%	0%	0%
9	1%	-1%	1%	-1%
Panel 3: Change in Portfolio Value ($)				
Maturity	**A**	**B**	**C**	**D**
1	-1.00	1.00	1.01	-1.00
5	9.75	-10.25	0.00	0.00
9	-8.60	9.42	-8.61	9.42
Total	**0.15**	**0.17**	**-7.60**	**8.42**

$$D(\mathbf{c}) = E^{PV}[T_i] = \mu,$$
$$C(\mathbf{c}) = E^{PV}[T_i^2] = \mu_2' = \sigma^2 + \mu^2 \tag{12}$$

where μ is the mean, μ_2' is the second moment about zero, and σ^2 is the variance.

EXPOSURE TO CURVATURE CHANGES

Finally, we will also show how the sensitivity to the curvature of the term structure may be computed, and summarize the resulting three-parameter, but first-order expansion for the aggregate change in value. Suppose we have a local change of the term structure to

$$y_T + \alpha + \beta(T - T_0) + \gamma(T - T_0)^2 \tag{13}$$

so at maturity T_0 the level changes by α, the slope by β, and the second derivative by 2γ. The sensitivity to γ is given by

$$\left(\frac{\partial V}{\partial \gamma}\right)_{\alpha=\beta=\gamma=0} = -\sum_{i=1}^{N} T_i (T_i - T_0)^2 PV(c_i) \tag{14}$$

All three sensitivities, to α, β, and γ, are simple to calculate, either for all cash flows in a portfolio or disaggregated into buckets, and provide risk measures which are both accurate and intuitive to interpret. We will illustrate this with a representative example.

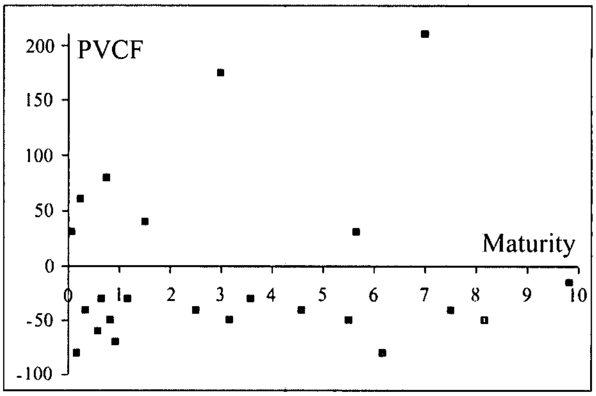
Cash-Flow Aggregation Example.

Exhibit 2 provides a short example of the technique applied to cash flows aggregated, in a conventional way, into three maturity ranges.

We suppose that these cash flows are aggregated into buckets for 0–1, 1–5 and 5–10 years, and we provide ex-ante and ex-post analysis of the exposures to the shape of the term structure. Exhibit 3 shows the analysis.

The top part of the exhibit shows the (unsigned) exposures to one-standard-deviation (1-SD) changes in level, slope, and curvature at each part of the curve. If we desired, we could also calculate the overall 1-SD exposures for each bucket, for each type of shock, and, in total, using the whole covariance matrix to do so. The bottom part of the exhibit shows the corresponding outcomes attributable to these risks (relative to their earlier forward values) and, also, the actual outcomes. The unexplained residual is very small and is attributable to the market shock not quite taking a quadratic shape between years 5 and 10.

EXHIBIT 2
Maturity Profile of the PV of Cash Flows



This example was constructed using monthly term-structure data from J. Huston McCulloch at the Department of Economics, Ohio State University.[3] We have used the monthly changes from January 2002 to June 2006 to estimate the 1-SD moves; the shock described is the change from January 31–February 28, 2006.

HEDGE CONSTRUCTION

The framework we have just developed is a particularly nice one in which to measure Value at Risk (VaR), calculate the effect of adding hedges, and also to do hedge optimization. We will work in a mean–variance framework.

The changes in the level, slope, and curvature of each segment of the term structure are the drivers for this analysis. It is quite easy to estimate these each period, calculate the historical covariances, and adjust these if desired to obtain a forward-looking covariance forecast. The exposures to each factor are also derived directly from the forward cash flows, as described earlier.

We will use the notation V to denote the $N \times N$ covariance matrix of factor changes and use a to denote the vector of raw exposures. In our previous numerical example we had nine factors ($N = 9$) with the $N \times 1$ vector a given (as the three columns) in the third panel of Exhibit 3. Obviously, the prospective variance (in \$-squared units) is given by the standard formula

$$\text{Variance} = a'Va \tag{15}$$

EXHIBIT 3
Risk Analysis by Maturity Bucket

	Bucket 0-1	Bucket 1-5	Bucket 5-10	Total
Hedge Point	0.5	3	7	-
1 SD Exposure				
Level	0.000	0.003	0.051	
Slope	0.020	0.143	0.092	
Curvature	0.001	0.039	0.091	
Outcomes				
Level	0.000	−0.001	−0.017	**−0.018**
Slope	−0.007	−0.052	−0.115	**−0.174**
Curvature	0.000	0.015	−0.134	**−0.119**
Total	**−0.007**	**−0.038**	**−0.266**	**−0.311**
Actual	−0.007	−0.039	−0.278	
Raw Exposures				
Level	0.000	−0.008	−0.158	
Slope	0.300	3.552	4.474	
Curvature	0.133	5.082	24.740	
Change				
Level	0.205	0.153	0.105	
Slope	−0.021	−0.015	−0.026	
Curvature	−0.002	0.003	−0.005	
1 SD Change				
Level	0.289	0.313	0.320	
Slope	0.068	0.040	0.021	
Curvature	0.006	0.008	0.004	

Extending the previous example, the historical covariance matrix provides a variance estimate of 0.0260 and a standard deviation of \$0.1612. Because of the correlation structure and the low exposure to parallel movements, this is quite a lot less than the total of the entries in the first panel in the exhibit of \$0.440.

Suppose we wish to consider the use of K ($<N$) instruments for further hedging. We will show how their marginal impact and optimal quantities may be calculated. Each instrument has an $N \times 1$ vector of exposures to the N factors, so these define an $N \times K$ matrix of exposures B. Following the Garman [1996] analysis, the $K \times 1$ vector $2\ B'Va$ gives the derivative of the variance with respect to the quantity used of each instrument. This is easily translated into the sensitivity of a VaR defined as a multiple of the standard deviation.

Further, to minimize the total variance, we should choose the quantities x of these instruments as

$$x = -(B'VB)^{-1}B'Va \tag{16}$$

As an illustration, suppose we can add 4- or 8-year bullet payments as a hedge in the previous example. The B' matrix is trivially

$$\begin{bmatrix} 0 & 0 & 0 & -4 & -4 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & -8 & -8 \end{bmatrix} \tag{17}$$

The sensitivity of the variance to adding unit PV amounts is

$$\begin{bmatrix} \frac{\partial \text{Variance}}{\partial x_1} \\ \frac{\partial \text{Variance}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0.002355 \\ 0.003139 \end{bmatrix} \tag{18}$$

Finally, the optimal amounts of the two instruments are −35.97 and 15.70, respectively, and these reduce the variance from 0.02600 to 0.00829, bringing the standard deviation down from \$0.1612 to \$0.0910.

OPTIONS EXPOSURES

The article has described a new approach to measuring and hedging term structure exposures, defined by future cash flows. We may also include the exposures arising from caps, floors, swaptions, and other kinds of interest-rate-sensitive options in this analysis. Any option valuation model driven by Brownian diffusions enables the option to be represented as a dynamically equivalent portfolio of zero-coupon bonds, so it is natural to include this representation in the risk analysis. Unfortunately, there is usually some ambiguity in the choice of this portfolio; in a model driven by K Brownian motions, in principle, an option position can be hedged using any $K + 1$ discount bonds. Nevertheless, there is often a natural way to define the portfolio which provides, in some sense, a robust choice. Equally though, there are pitfalls to be avoided in extracting such a portfolio.

For example, where a European swaption is valued using the standard practitioner model as

$$V_0 = A_{M,N,0}\left(C_X N(d_1) - Y_0 N(d_2)\right) \tag{19}$$

where $A_{M,N,0}$ is the value of a forward-starting annuity, C_X is the strike rate, and Y_0 is the forward swap rate, it might be tempting to regard this as equivalent to $(C_X N(d_1) - Y_0 N(d_2))$ units of the forward-starting annuity. This would be incorrect; it has the correct value, but the wrong dynamics. We should instead rewrite the formula as

$$V_0 = C_X A_{M,N,0} N(d_1) - (B_{M,0} - B_{M+N,0})N(d_2) \tag{20}$$

which provides a representation as $N(d_1)$ units of the forward fixed stream minus $N(d_2)$ units of the floating one (itself represented as the difference between the two discount bond prices).

It is known (for example, see Chance [2003]) that in the case of a derivative whose value is homogenous of degree 1 in the underlying and strike, so that

$$V(\lambda S, \lambda X) = \lambda V(S, X) \tag{21}$$

it follows from Euler's rule that there is a natural representation of the form

$$V(S,X) = S\frac{\partial V}{\partial S} + X\frac{\partial V}{\partial X} \tag{22}$$

This is the situation above, and a paper by Joshi [2005] describes the quite large class of "log-type" models which give rise to it. However, there can still be a natural dynamically equivalent portfolio without this property holding, and the topic deserves further research.

CONCLUSION

The article has developed a new and practical approach to measuring the risk of changes in the slope and curvature of the term structure, as well as its level. This is related to the principal components commonly found in term-structure movements, but obviates the need to estimate them. Once these sensitivities are known, together with the corresponding sensitivities of appropriate hedging instruments, it is straightforward to choose the hedge which minimizes them. The technique is also particularly suitable for reconciling the changes in market values of cash buckets with the term-structure movements that created them.

We have also seen how the new measures are linked to the existing measures of duration and convexity. Some new insights were obtained by interpreting these conventional risk measures as the (generalized) moments of the (present-value-weighted) distribution of cash-flow dates.

In conventional bond risk analysis, convexity is usually regarded as a "good thing", providing insurance against large moves in the same way as the gamma of an option. Our analysis points out that, just as option gamma brings with it a risk exposure to volatility, convexity creates an exposure to the slope of the term structure, even when there is no change in its level at the duration of the cash flows.

ENDNOTES

[1]Nalwalkha and Chambers have used a vector of 5 elements, for example.

[2]We may continue to use this notation even for the case where cash flows vary in sign, and write:

$$\left(\frac{\partial V}{\partial \alpha}\right)_{\alpha=\beta=0} = -\mu\ V$$
$$\left(\frac{\partial V}{\partial \beta}\right)_{\alpha=\beta=0} = -\left[\sigma^2 + (\mu - T_0)\mu\right]V$$

[3]The data was downloaded from http://economics.sbs.ohio-state.edu/jhm/ts/ts.html.

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