Explaining the Correlation Smile Using Variance Gamma Distributions

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In the 1980s, collateralized debt obligations (CDOs) were introduced for balance sheet risk management. The emergence of credit derivatives in the 1990s offered the possibility of synthetic risk transfer of a portfolio of bonds or loans. Since 2003, credit risk of standardized portfolios is traded in a liquid market in the CDS indices iBoxx and Trac-X. These indices merged into iTraxx in 2004. Standardized tranches that are linked to these indices started to be actively quoted. Thus, the entire distribution of portfolio losses (as seen by market participants) became an observable variable.

This development poses new challanges to credit risk models. One should expect the most common credit risk models to match the market implied loss distribution. However, this has not proven to be the case. Using the common one factor Gaussian copula (which underlies the most important industry models like CreditMetrics and KMV) different correlation parameters are needed to price different tranches.

The correlation smile has its counterpart in stock option valuation; namely the volatility smile, which arises when the Black Scholes model is applied. The underlying assumptions of the Black Scholes and the Gaussian copula model are basically the same: They assume stock or asset returns to be normally distributed. Furthermore, the Gaussian copula model assumes that the common distribution of asset returns is a multivariate normal distribution. But it is well known that stock options are not normally distributed. Instead, their distributions are skewed and have fat tails. It was mainly this idea that explained the volatility smile. The purpose of this article is to show that this idea may also explain the correlation smile in liquid CDS index tranches.

After a short introduction to CDS indices and a literature review in the next section (Valuation of CDOs), we extend a structural model based Variance Gamma (VG) processes that was proposed by Luciano and Schoutens [2005] in the subsequent section (The Structural Variance Gamma Model). A VG process is a Lévy process whose increments are VG distributed (just like Brownian motion is a Lévy process whose increments are normally distributed). VG distributions have fat tails and may be skewed; they are therefore suited for the modelling of stock return distributions. VG processes were introduced by Madan et al. [1998] as Brownian motions on a stochastic business time. In the same article, the authors also show that VG processes can explain the volatility smile. This representation was also chosen by Luciano and Schoutens [2005] and is therefore used in our extensions. We show that these extensions can explain the correlation smile in liquid index tranches of DJ iTraxx 5 year.

In the third section (VG Copula Model) we focus directly on the corresponding distributions of these processes. We show that they can be implemented into a factor copula approach in precisely the same way as the normal distribution is implemented for the Gaussian copula. We can therefore retain the ideas of the structural approach to generate a model that is analytically tractable. Moreover, this model has the advantage that all extensions of the Gaussian copula approach can be used for this VG copula as well. For example, nonhomogeneous spread curves, stochastic recovery rates, and random factor loadings are applicable.

We conclude in the final section. Appendix A gives the results concerning VG processes and distributions. We postpone proofs to Appendix B.

VALUATION OF CDOs

A CDO is a securitization of a portfolio of bonds or loans. The underlying portfolio is transfered to a special purpose vehicle that issues securities on the portfolio in several tranches. Each tranche is defined by an attachment point L_a and a detachment point L_d. For a percentual loss of $L_{\text{portfolio}}$ of the underlying portfolio, the tranche suffers a percentual loss of

$$L_{\text{tranche}} = \max\{\min(L_{\text{portfolio}}, L_d) - L_a, 0\}.$$

The lowest tranche has $L_a = 0$ and is called the equity tranche. Since it already suffers from the first loss in the portfolio, it is the riskiest tranche and has to pay the highest spread to investors. For attachment points between 3 and 7%, tranches are called mezzanine, whereas the highest tranches are called senior or super-senior.

In synthetic CDOs, portfolio credit risk is transfered via credit default swaps. With these instruments, not all tranches need to be sold to investors. On the basis of standardized tranches, two parties agree to act as protection buyer and protection seller for this particular single tranche. Standard portfolios exist in the indices DJ CDX NA for entities in Northern America and DJ iTraxx for European entities.[1] The main indices consist of 125 equally weighted entities. The attachment and detachment points are 0, 3, 7, 10, 15 and 30% for CDX NA and 0, 3, 6, 9, 12, and 22% for DJ iTraxx. There is also the possibility of trading the whole index. This corresponds to a tranche with $L_a = 0\%$ and $L_d = 100\%$. Spreads are quoted in base points per year for all tranches. The only exception is the equity tranche, where spread is quoted as a percentage upfront payment plus 500 bp running premium. Exhibit 1 shows market quotes of DJ iTraxx on June 24, 2005.

ONE FACTOR GAUSSIAN COPULA

The Gaussian copula model has become the standard market model for valuing synthetic CDOs. In its basic form, for every entity i in the portfolio a standard normal random variable X_i is defined by

$$X_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i. \tag{1}$$

M and Z_i are standard normally distributed and $|\rho| \leq 1$. The factor M represents a systematic and Z_i an idiosyncratic risk factor of entity i. For $i \neq j$, the correlation of X_i and X_j is given by ρ. Entity i defaults, if X_i is smaller than some default threshold C_i. The default threshold is determined so that the risk neutral default probability[2] $Q_i(\tau)$ of entity i for every time τ is given by $Q_i(\tau) = \Phi^{-1}(C_i)$, where Φ is the cumulative distribution function of a standard normal random variable. Then the distribution of the number of defaults can be obtained.[3] If one assumes constant recovery rates, this distribution implies a distribution of portfolio loss. By assuming a zero mark-to-market value for each tranche, spreads on the tranches can be calculated.

The main advantage of the model is the independence of defaults when conditioned to the common risk

EXHIBIT 1
Market Quotes of DJ iTraxx 5 Year on June 24, 2005

Tranche	0–3%	3–6%	6–9%	9–12%	12–22%	Index
Spread	30.0%	98 bp	34 bp	20 bp	14 bp	40.0 bp

Spread of the equity tranche is quoted as a percentage upfront plus 500-bp running premium. The other tranches are quoted as bp per year.
Source: Nomura Fixed Income Research.

factor. This allows a simple implementation and fast computations. However, when we apply the model to liquid tranches of the credit indices DJ CDX or iTraxx, different correlation parameters are needed to fit the prices of different tranches. For equity and senior tranches this implied correlation is higher than for mezzanine tranches. This phenomenon is known as the correlation smile of implied correlation. Exhibit 2 shows the implied correlations corresponding to the quotes of Exhibit 1.[4]

Market prices of liquid index tranches are used to calibrate models for the valuation of bespoke CDOs. If a model prices all liquid tranches correctly using the same parameter set, then a bespoke CDO tranche with nonstandard attachment and detachment points can be priced in a consistent manner. If this is not the case—as in the Gaussian framework—development of further techniques is required.

Within the Gaussian framework, one of these techniques consists of calculating base correlations. These are implied correlations of hypothetical equity tranches (i.e. tranches with attachment point $L_a = 0$) with varying detachment points. The advantage of base correlations over simplied correlations is that they monotonically increase along with the detachment point. One can therefore interpolate between base correlations so as to price nonstandard tranches.

However, this approach is more of an ad hoc method rather than a consistent means of CDO pricing. It does not resolve the fundamental inconsistency in using the Gaussian copula approach. The search for models that can price all tranches using a single parameter set has therefore been an active field of research in recent time. In the next subsection, we present a short survey of the methods proposed so far.

FURTHER METHODS

A natural idea is to try copulas other than the Gaussian one. The choices proposed so far include the student t, double t, Clayton, and Marshall-Olkin copula. Burtschell, and Gregory, and Laurent [2005] provide an overview of the results of calibrating these copula approaches to market spreads of DJ iTraxx. These authors find that the double t copula (first proposed by Hull and White [2004]) fits the observed spreads best. This double t copula arises when the factors M and Z_i in Equation (1) are t-distributed. The problem in this approach is that the latent variable X_i is no longer t-distributed, since the t distribution is not closed under convolution. The distribution of X_i depends on ρ and must be calculated numerically. This feature complicates the interpretation of the variable X_i as asset returns and may be the reason why this copula is not as famous for credit models as the student t copula, in which X_i is t-distributed.

Recently, Kalemanova, Schmid, and Werner [2005] proposed a factor copula approach based on normal inverse Gaussian distributions. They show that calibration to liquid index tranches is as good as through the use of the double t copula. It was this idea that inspired us to the use of the VG copula.

Empirical studies of de Servigny and Renault [2004] and Das et al. [2004] show that default correlations increase in times of a recession. Recently, several authors have proposed models that incorporate this fact. Andersen and Sidenius [2004] extend the Gaussian copula model as they correlate ρ and M in Equation (1) negatively.

Hull, Predescu, and White [2005] have developed a structural model where firm value processes are correlated Brownian motions. The degree of correlation may depend on the systematic part of the process and therefore on the state of the economy. When they assume a negative dependence of default correlation and the systematic part of the firm value processes, the authors show that spreads of CDX NA and iTraxx are fitted significantly better than by a constant correlation.

Another attempt to generate a dependence structure as observed in the market consists of the introduction of a stochastic business time. This business time is usually associated with the information flow in the market. In times where there is much information in the market, business times go fast and return volatilities are high.[5] The idea of a stochastic business time has been used for a long time for the valuation of equity derivatives, and it has

EXHIBIT 2
Implied Correlations of DJ iTraxx 5 Year on June 24, 2005

Tranche	0–3%	3–6%	6–9%	9–12%	12–22%
Implied Correlation	0.196	0.054	0.119	0.173	0.315

explained the volatility smile in equity options. In a structural approach where default occurs if the firm value hits some default barrier, a fast business time corresponds to a high probability to hit the barrier and thus to a high default probability.

Giesecke and Tomecek [2005] model default times in a portfolio as times of jumps of a Poisson process. The time scale of this process is varied depending on incoming information like economic environment and defaults. In this way, contagion effects can be considered.

Joshi and Stacey [2005] use Gamma processes to calibrate an intensity model to market prices of liquid CDO tranches. When business time is the sum of two Gamma processes, they show that their model can fit to the correlation smile of DJ iTraxx.

Cariboni and Schoutens [2004] model firm value processes and use Brownian motions subordinated by Gamma processes. The resulting VG processes are calibrated to single name credit curves. Luciano and Schoutens [2005] extend the approach of Cariboni and Schoutens [2004] for the valuation of default baskets.

The model we propose in this paper extends the approach of Luciano and Schoutens [2005] by allowing a more general dependence structure. In this way, we also shed some light on seemingly different economic findings that may all be caused by the same reason, namely the idea of a stochastic business time. These are leptokurtic asset (stock) returns, the volatility smile in equity options, and the correlation smile in liquid credit index tranches.

THE STRUCTURAL VARIANCE GAMMA MODEL

In this section, we briefly describe the structural model of Luciano and Schoutens [2005]. This serves as a base case for three extensions in the following subsections. In the last subsection, we examine the ability of the model and its extensions to explain the observed correlation smile in liquid index tranches. We provide the properties of VG processes needed for this article in Appendix A.

Base Case: Identical Gamma Processes, Independent Brownian Motions

Let N be the number of entities in the portfolio and for every $i \in \{1, \ldots, N\}$ let

$$X_t^{(i)} = \theta_i G_t + \sigma_i W_{G_t}^{(i)} \tag{2}$$

be a VG process with parameters $(\theta_i, \nu, \sigma_i)$. This means that $(G_t)_{t\geq 0}$ is a Gamma process with parameters (ν^{-1}, ν)[6] and for every i the process $(W_t^{(i)})_{t\geq 0}$ is a standard Brownian motion. For $i \neq j$, these Brownian motions are independent. The firm value process $(S_t^{(i)})_{t\geq 0}$ of entity i is given by

$$S_t^{(i)} = S_0^{(i)} \exp\left(r_t t + X_t^{(i)} + \omega_i t\right). \tag{3}$$

In Equation (3), r_t denotes the risk free interest rate at t and

$$\omega_i = \frac{1}{\nu}\log\left(1 - \frac{1}{2}\sigma_i^2\nu - \theta_i\nu\right)$$

a parameter to ensure the martingale property of the discounted firm value $\frac{S_t^{(i)}}{\exp(r_t t)}$ (see Appendix B for details).

Entity i defaults at time $\tau > 0$ if

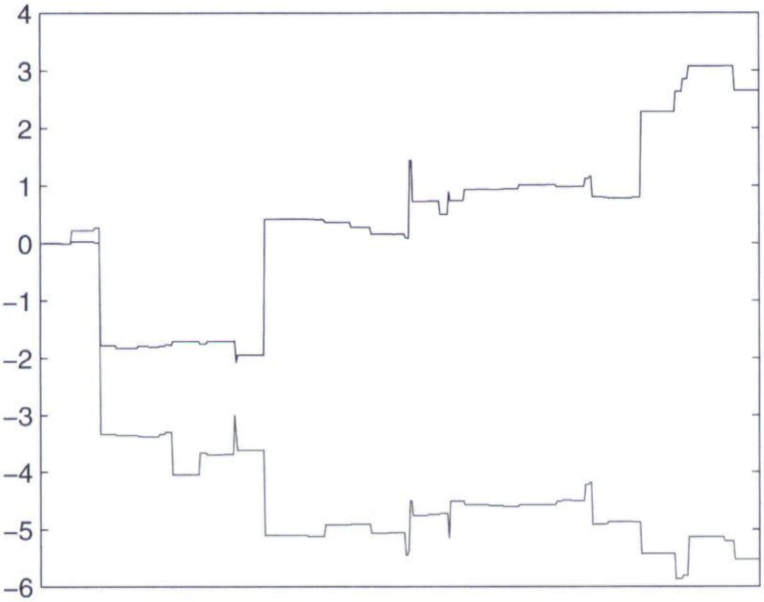
$$\tau = \min_{t\in T}\left\{S_t^{(i)} < L_t^{(i)}\right\}.$$

In this base case, all firm value processes [Equation (3)] follow the same Gamma subordinator $(G_t)_{t\geq 0}$. Luciano and Schoutens argue that all firms are subject to the same economic environment and thus information arrival should affect business time of all entities. The Brownian motions are independent, however. This means that all correlation involved is caused by the common business time.

Exhibit 3 shows two paths of VG processes with identical Gamma processes and independent Brownian motions. Jumps occur at identical times, but their directions are conditionally independent.

We extend this correlation structure in the following subsections. First, we allow business time to be correlated and not identical for all entities. We therefore allow changes in business time to be caused by systematic or idiosyncratic information. Second, we allow the Brownian motions to be dependent. This means that the directions of jumps are correlated. Finally, we integrate these two ideas into a third extension. In the next section we show that the last extension may be solved analytically under some simplifying assumptions. This analytical solution is a factor approach similar to the Gaussian copula in Equation (1).

EXHIBIT 3
VG Processes with Identical Gamma Processes



Extension A: Correlated Gamma Processes, Independent Brownian Motions

In this extension, changes in business time may be caused by information concerning the entire economy or by information about the individual firm. If we decompose $(G_t)_{t\geq 0}$ into a systematic and an idiosyncratic part, we can incorporate this idea into the model. For every i, we seperate $(G_t^{(i)})_{t\geq 0}$ into a sum of two Gamma processes

$$\mathrm{d}G_t^{(i)} = \mathrm{d}F_t + \mathrm{d}U_t^{(i)}.$$

In this decomposition, $(F_t)_{t\geq 0}$ and $(U_t^{(i)})_{t\geq 0}$ are independent Gamma processes with parameters $(a_F, b_F) = (a\nu^{-1}, \nu)$ and $(a_{U(i)}, b_{U(i)}) = ((1-a)\nu^{-1}, \nu)$ with $0 \leq a \leq 1$. It follows that $(G_t^{(i)})_{t\geq 0}$ is a Gamma process with parameters $((1-a)\nu^{-1} + a\nu^{-1}, \nu) = (\nu^{-1}, \nu)$.[7]

For $a \to 1$, this extension coincides with the base case. The parameter a controls for the pairwise correlation Corr $(X_1^{(i)}, X_1^{(j)})$. In Appendix B, we show that for independent Brownian motions $W^{(i)}$ and $W^{(j)}$ and for $i \neq j$ we have

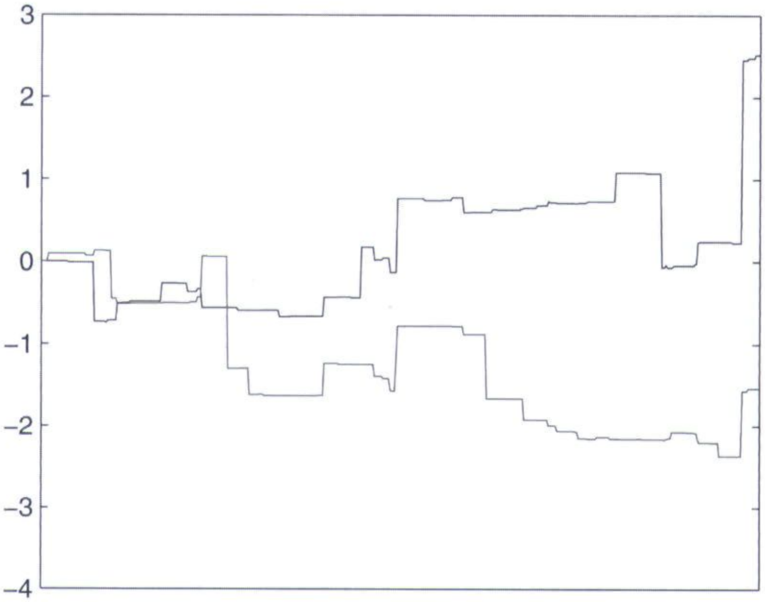
$$\mathrm{Corr}\left(X_1^{(i)}, X_1^{(j)}\right) = a\frac{\theta_i\theta_j\nu}{\sqrt{\theta_i^2\nu+\sigma_i^2}\sqrt{\theta_j^2\nu+\sigma_j^2}}. \qquad (4)$$

For a homogeneous portfolio (i.e. identical credit curves) all entities have identical parameters $\theta_i = \theta$, $\sigma_i = \sigma$, and all correlations in the interval $\left[0; \frac{\theta^2\nu}{\theta^2\nu+\sigma^2}\right]$ can be reached. The upper bound is obtained for $a \to 1$, which is the base case. The lower bound is reached for $a \to 0$, i.e. for independent Gamma processes.

One should bear in mind, however, that this correlation has to be treated with care. For example, if $\theta_i = \theta_j = 0$, the processes $X^{(i)}$ and $X^{(j)}$ are uncorrelated but not necessarily independent. For a better understanding of the dependence structure, copulas have to be considered.

Exhibit 4 shows sample paths of correlated VG processes for $a = 0.5$. There are times where only one process has a large jump. These jumps are caused by the idiosyncratic factor $U^{(i)}$. At other times, both realizations jump. Those jumps are caused by the systematic factor F.

EXHIBIT 4
VG Processes with Correlated Gamma Processes



Extension B: Identical Gamma Processes, Dependent Brownian Motions

The extension proposed in the last subsection correlates times of high activity of the VG processes. The directions of these jumps are conditionally independent since the Brownian motions $W^{(i)}$ and $W^{(j)}$ are independent. If jumps are caused by information concerning the state of the economy, this information should be (more or less) good or bad for all companies. The model reflects this fact if the Brownian motions are correlated.

The correlation of the Brownian motions is modelled via

$$dW_t^{(i)} = \sqrt{b}dM_t + \sqrt{1-b}dZ_t^{(i)}.$$

In this extension, we choose the same Gamma subordinator for each entity. Then, for the correlation of the VG processes we find for $i \neq j$ (see Appendix B)

$$\text{Corr}\left(X_1^{(i)}, X_1^{(j)}\right) = \frac{\theta_i\theta_j\nu + \sigma_i\sigma_j b}{\sqrt{\theta_i^2\nu + \sigma_i^2}\sqrt{\theta_j^2\nu + \sigma_j^2}}. \quad (4)$$

For a homogeneous portfolio with identical parameters $\theta_i = \theta$, $\sigma_i = \sigma$ all correlations in the interval $\left[\frac{\theta^2\nu}{\theta^2\nu+\sigma^2}, 1\right]$ can be reached, if positive values for b are considered. The lower bound corresponds to $b = 0$. If $b = 1$, the VG processes are identical and therefore have a correlation of 1.

Exhibit 5 shows two paths for $b = 0.5$. Large jumps occur at identical times and their directions are correlated.

Extension C: Sum of Two Independent VG Processes

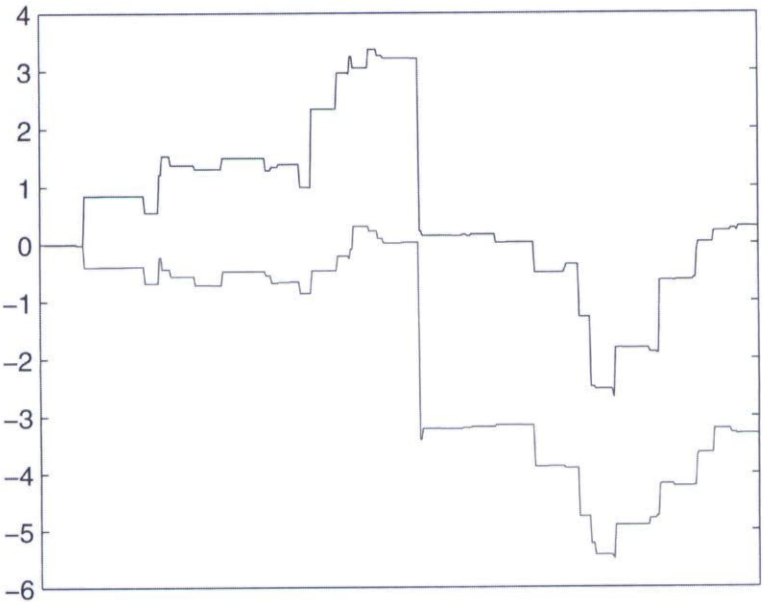
While in the above extensions we decompose either the Gamma or the Brownian part of the VG process, we now decompose the whole VG process into two VG processes. We restrict ourselves to the case where all VG processes $(X_t^{(i)})_{t\geq 0}$ have the same parameters (θ, ν, σ).

We choose $0 < c < 1$ and decompose $(X_t^{(i)})_{t\geq 0}$ into

$$X_t^{(i)} = cM_t + \sqrt{1-c^2}Z_t^{(i)} \quad (6)$$

EXHIBIT 5

VG Processes with Identical Gamma Subordinators and Correlated Brownian Motions



with

$$(\theta_M, \nu_M, \sigma_M) = \left(c\theta, \frac{\nu}{c^2}, \sigma \right), \tag{7}$$

$$(\theta_{Z^{(i)}}, \nu_{Z^{(i)}}, \sigma_{Z^{(i)}}) = \left(\sqrt{1-c^2}\theta, \frac{\nu}{(1-c^2)}, \sigma \right). \tag{8}$$

From the results of Appendix A it is clear that the sum $(X_t^{(i)})_{t\geq 0}$ is again a VG process and its parameters are given by

$$(\theta_{X^{(i)}}, \nu_{X^{(i)}}, \sigma_{X^{(i)}}) = (\theta, \nu, \sigma).$$

We show in Appendix B that

$$\mathrm{Corr}\left(X_1^{(i)}, X_1^{(j)} \right) = c^2 \tag{9}$$

for $i \neq j$.

Note that this extension is not the junction of extensions A and B for $a = b = c^2$. Nevertheless, the correlation of $(X_t^{(i)})_{t\geq 0}$ and $(X_t^{(j)})_{t\geq 0}$ stems from both Gamma and Brownian correlation.

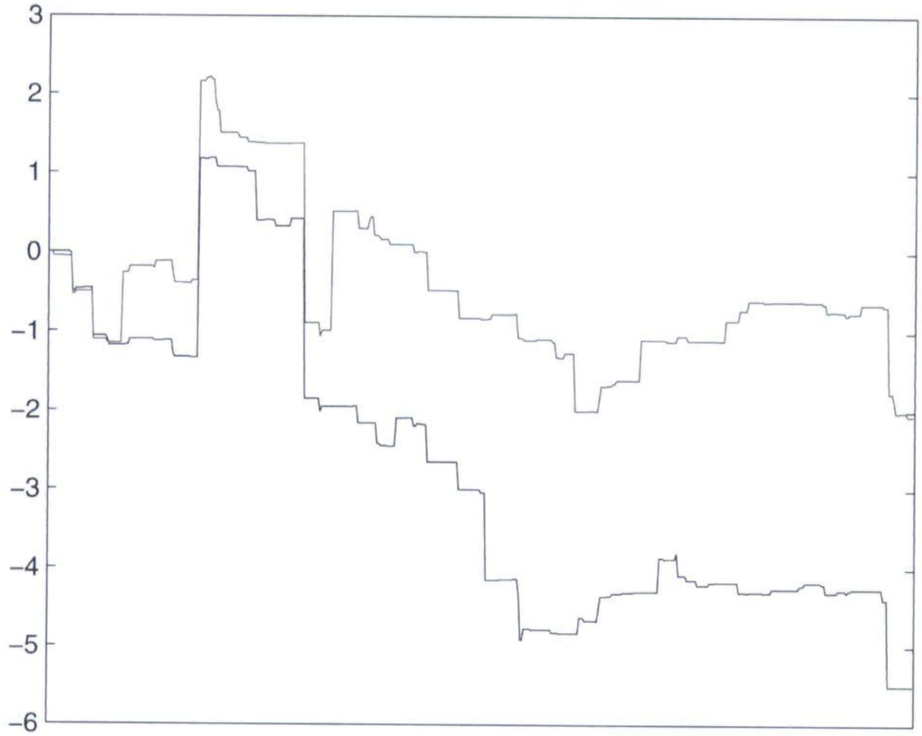
Exhibit 6 shows two paths of correlated VG processes for $c^2 = 0.5$. Jumps occur for systematic and idiosyncratic reasons. The directions of systematic jumps are correlated.

Since all firm value processes possess identical parameters, the entities in the portfolio are assumed to have identical credit curves. This simplification and the fact that all processes are of VG type allow for an analytical solution that is given in the next section. For the approach proposed there, the assumption of identical credit curves can be relaxed.

Application to Index Tranches

We now apply the model to data on tranches of DJ iTraxx 5 year from June 24, 2005, to December 9, 2005. For each of these dates, we also regard index spreads of DJ iTraxx for the maturities of 3, 5, 7, and 10 years. Our data set consists of 25 weekly quotes. The index spread curves are used to determine the parameters $(\theta_t, \nu, \sigma_t)$. Since we calibrate on index spread curves and not on single name spread curves, we assume that the portfolio

EXHIBIT 6
Correlated VG Processes



is homogeneous with respect to default probabilities. Thus all processes have identical parameters.

For each date, the calibration is accomplished in two steps: First, we determine the parameters (θ, ν, σ) to fit the index spread curve given by quotes of the indices with different maturities. In the base case, the entire correlation structure is determined and we calculate tranche spreads. In our extensions, we conduct a second step: We determine the correlation parameters a, b, and c such that the spread of the equity tranche is matched.

In all the computations of this subsection, we have set the initial firm value to $S_0 = 100$, the default barrier to $L = 50$, and recovery rates to $R = 40\%$. We simulated the firm value process on a discrete time grid with $dt = 0.02$.[8] The risk-free zero curve is the EUR zero curve.

We calculate par spreads in the usual way. We assume that all payments (fee and contingent legs) occur on quarterly payment dates T_i, $i = 1, \ldots, n$, The expected default loss $\mathrm{EL}(T_i)$ on the tranche up to payment date T_i is given by

$$\mathrm{EL}(T_i) = \int_0^1 \max(\min(x(1-R), L_\mathrm{d}) - L_\mathrm{a}, 0) f_{\mathrm{Loss}(T_i)}(x)\mathrm{d}x,$$

where R is the (constant) recovery rate, L_a and L_d denote the tranche's attachment and detachment points and $f_{\mathrm{Loss}(T_i)}$ is the probability density function of losses until T_i.

The tranche's mark-to-market (from the investors view) for a spread per annum of s is then given by[9]

$$\begin{aligned}\mathrm{MTM}(s) &= \mathrm{Fee} - \mathrm{Contingent}\\ &= s\sum_{i=1}^{n} \Delta_i e^{-rT_i}((L_\mathrm{d} - L_\mathrm{a}) - \mathrm{EL}(T_i))\\ &\quad - \sum_{i=1}^{n} e^{-rT_i}(\mathrm{EL}(T_i) - \mathrm{EL}(T_{i-1})),\end{aligned}$$

where Δ_i is the accrual factor for payment date i. The par spread is obtained by $\mathrm{MTM}(s_{\mathrm{par}}) = 0$.

EXHIBIT 7

Calibration to Spread Curves of DJ iTraxx from June 24, 2005, to December 9, 2005

	Maturity				
	3 years	5 years	7 years	10 years	Average APE
Market	21.87	37.00	46.68	57.31	
Restr. to $\theta = 0$	22.60 (0.92)	36.27 (0.77)	49.54 (2.87)	66.23 (8.93)	3.37
Restr. to $\nu = 1$	22.84 (0.99)	36.27 (0.70)	46.63 (0.28)	61.15 (3.88)	1.55
Unrestr.	25.18 (3.31)	37.12 (0.53)	46.39 (0.82)	57.08 (0.88)	1.39

Numbers in parentheses denote average pricing errors. Market quotes are mid quotes obtained by International Index Company.

Exhibit 7 shows the results of the calibration to the index spread curve. Calibration was done to minimize the absolute pricing error (APE).[10] In this calibration, the APE denotes

$$\mathrm{APE} = \frac{1}{\text{no. of maturities}} \times \sum_{\text{maturities}} \left|\text{spread}^{\text{Market}}_{\text{Maturity}} - \text{spread}^{\text{Model}}_{\text{Maturity}}\right|. \quad (10)$$

We have calibrated the full model to spread curves, as well as two cases where we restricted either to $\theta = 0$ or $\nu = 1$. If $\theta = 0$, we get unskewed distributions. If $\nu = 1$, the Gamma distributions are exponential distributions. We find that all cases lead to a good fit to observed spreads. This result confirms the findings of Cariboni and Schoutens [2004].

Given the parameters from this calibration, we can calculate model implied tranche spreads in the base case and the extensions. In the base case, there is no additional correlation parameter, and so tranche spreads are already determined by the values of (θ, ν, σ). We find that in all cases, the spread of the equity tranche is significantly underestimated. This shows that implied correlation of this approach is too high. Thus, we can use extension A and C to calibrate the correlation parameters a and c to match the equity tranche. We have determined a and c such that the spread of the equity tranche is matched. For comparison of the over all fit of the models, we have included the average APE into the exhibit, where the APE is given by

$$\mathrm{APE} = \frac{1}{\text{no. of tranches}} \times \sum_{\text{tranches}} \left|\text{spread}^{\text{Market}}_{\text{Tranche}} - \text{spread}^{\text{Model}}_{\text{Tranche}}\right|. \quad (11)$$

Since extension B further increases correlation compared with the base case, we cannot calibrate this extension to match the equity tranche.

We show the calibration results of the base case and extensions A and C to tranche spreads in Exhibit 8. For comparison, we have also included the values for the double t copula with 4 and 5 degrees of freedom. Burtschell et al. [2005] find the double t(4)copula to match tranches best when compared with several other copulas.

We find that in the base case, the model spread of the equity tranche is much lower than that is observed in the market. The negative spread of -12.8 for the case restricted to $\nu = 1$ indicates that the model implied spread is even smaller than the 500-bp running premium. In the extensions, the upfront premium of the equity tranche can be matched. In extension A, pricing errors are smaller than those for the Gaussian Copula, which indicates a model implied correlation smile. APEs are still quite large compared with the double t(4) copula for $\theta = 0$ and in the unrestricted case. The best fit is achieved in the case restricted to $\nu = 1$. Extension C leads to a fit that is comparable to the double t(4) copula in all three cases. This shows that extension C can simultaneously explain the index spreads for different maturities and the quotes of liquid index tranches of DJ iTraxx 5 year.

EXHIBIT 8

Average Market Spreads and Average Model Implied Spreads for DJ iTraxx 5 Year between June 24, 2005, and December 9, 2005

	0–3%	3–6%	6–9%	9–12%	12–22%	Average APE
Market	27.1%	86.7	26.8	14.1	8.4	–
Gauss	27.1%	193.3 (106.7)	45.3 (18.6)	12.6 (4.4)	1.7 (6.8)	34.1
Double t(4)	27.1%	97.1 (10.4)	36.4 (9.6)	21.1 (7.1)	10.8 (2.4)	8.3
Double t(5)	27.1%	112.9 (26.2)	38.5 (11.7)	20.3 (6.0)	9.1 (0.6)	11.4
VG Base Case, $\theta = 0$	17.0% (10.0)	281.8 (195.1)	88.6 (61.8)	23.5 (9.4)	1.7 (6.7)	–
VG Base Case, $\nu = 1$	–12.8% (39.8)	143.3 (56.6)	115.1 (88.3)	97.6 (83.5)	73.7 (65.3)	–
VG Base Case, Unrest.	6.3% (20.9)	278.5 (192.0)	153.4 (127.3)	89.8 (76.5)	32.3 (24.3)	–
VG Ext. A, $\theta = 0$	27.1%	182.4 (95.8)	42.1 (15.3)	8.5 (5.6)	0.5 (7.9)	31.2
VG Ext. A, $\nu = 1$	27.1%	106.1 (24.5)	32.7 (8.4)	15.9 (3.0)	5.9 (2.6)	9.6
VG Ext. A, Unrest.	27.1%	150.9 (64.2)	54.0 (27.2)	24.5 (10.4)	6.5 (2.0)	26.1
VG Ext. C, $\theta = 0$	27.1%	91.1 (13.5)	31.1 (6.2)	17.5 (3.8)	8.4 (1.4)	6.2
VG Ext. C, $\nu = 1$	27.1%	102.5 (20.4)	32.3 (7.5)	16.3 (3.3)	6.8 (1.7)	8.3
VG Ext. C, Unrest.	27.1%	98.5 (17.7)	36.7 (10.9)	21.3 (7.2)	10.4 (2.1)	9.5

Numbers in parentheses denote average pricing errors.

VG COPULA MODEL

We showed in the last section that the correlation smile in liquid index tranches may be explained by the structural variance gamma model. For practical purposes however, there are some disadvantages of the structural model. First, we have to use Monte Carlo simulations to calibrate the model to market quotes, which leads to slow implementations. Furthermore, we have used the parameters (θ, ν, σ) solely for the calibration to the index spread curve, and not for the calibration to tranche prices.

We therefore implement the idea of extension C above into a factor copula approach. We assume a homogeneous portfolio and constant default intensities for the entities. This default intensity is determined by the 5-year index spread. These assumptions are common for the Gaussian copula approach as well, although they can be relaxed in a straightforward way. We can use the parameters of the VG distribution and a correlation parameter to calibrate the approach to tranche quotes. Since VG distributions possess a fourth parameter μ in general (see Appendix A) and we restrict our distributions to have zero mean and unit variance, this gives us 2 degrees of freedom besides correlation.

Another assumption used in our examples is that the portfolio consists of an infinite number of entities. This allows us to use the large homogeneous portfolio (LHP) approximation,[11] which is a common procedure for other copula approaches. As with the Gaussian copula, this assumption may be relaxed. In this case, a semi-analytical approach similar to the ones for the Gaussian copula has to be conducted. However, the LHP method allows us to calculate an analytical solution for the portfolio loss distribution.

VG Copula

In analogy to the Gaussian Copula we define the one factor VG copula by

$$X_i = cM + \sqrt{1-c^2}Z_i, \tag{12}$$

where M and Z_i are independently VG distributed random variables. The distribution parameters are given by[12]

$$M \sim \mathrm{VG}\left(c\theta, \frac{\nu}{c^2}, \sqrt{1-\nu\theta^2}, -c\theta\right),$$

$$Z_i \sim \mathrm{VG}\left(\sqrt{1-c^2}\theta, \frac{\nu}{1-c^2}, \sqrt{1-\nu\theta^2}, -\sqrt{1-c^2}\theta\right)$$

Note that the distributions of M and Z_i correspond to the distributions of M_1 and $Z_1^{(i)}$ in Equations (7) and (8) with $\sqrt{1-\nu\theta^2}$ and $\mu = -\theta$ to obtain zero mean and unit variance. Using the results about scaling and convoluting VG variables at the end of Appendix A, we find

$$cM \sim \mathrm{VG}\left(c^2\theta, \frac{\nu}{c^2}, c\sqrt{1-\nu\theta^2}, -c^2\theta\right),$$

$$\sqrt{1-c^2}Z_i \sim \mathrm{VG}\left((1-c^2)\theta, \frac{\nu}{1-c^2}, \sqrt{1-c^2}\sqrt{1-\nu\theta^2}, -(1-c^2)\theta\right).$$

The distribution of X_i is

$$X_i = cM + \sqrt{1-c^2}Z_i \sim \mathrm{VG}\left(\theta, \nu, \sqrt{1-\nu\theta^2}, -\theta\right). \tag{13}$$

The third and fourth parameters were chosen such that all variables have zero mean and unit variance. We show in Appendix B that the correlation of X_i and X_j for $i \neq j$ is given by

$$\mathrm{Corr}(X_i, X_j) = c^2. \tag{14}$$

Using Equation (12) and the LHP approximation, we can deduce the loss distribution of the portfolio. If we denote the (common) default threshold by C and condition on the systematic factor M, then entity i defaults if

$$Z_i < \frac{C-cM}{\sqrt{1-c^2}}. \tag{15}$$

The conditional default probability is given by

$$P(X_i < C \mid M = m) = F_{Z_i}\left(\frac{C-cm}{\sqrt{1-c^2}}\right), \tag{16}$$

where F_{Z_i} denotes the cumulative distribution function of Z_i, which is identical for all i. We find that the loss distribution function of the portfolio is

$$F_{\text{portfolio loss}}(x) = 1 - F_M\left(\frac{C-\sqrt{1-c^2}F_{Z_i}^{-1}(x)}{c}\right), \tag{17}$$

if the recovery rates are zero. The proof from Vasicek [1987] applies here, too. We provide it in Appendix B.

Application to CDS Index Tranches

In this subsection, we calibrate the VG Copula model to the market quotes we have used in Application to Index Tranches section. The default threshold C is determined to match the index spread. We can thus use the parameters θ, ν, and c to calibrate the model to tranche spreads. The calibration is done by a minimization of the absolute princing error

$$\mathrm{APE} = \frac{1}{\text{no. of tranches}} \times \sum_{\text{tranches}} |\,\text{spread}_{\text{Tranche}}^{\text{Market}} - \text{spread}_{\text{Tranche}}^{\text{Model}}\,|, \tag{18}$$

while keeping the equity spread fixed to match the market spread. As in the last section, we assume a constant recovery rate of 40% for all entities. The risk-free zero curve is the EUR zero curve. First, we fix $\theta = 0$, which leads to unskewed distributions. In a second stage, we calibrate overall parameters. The results are given in Exhibit 9.

Both the restricted ($\theta = 0$) and the unrestricted case lead to a better overall fit than the double t(4) copula. In the unrestricted case, the fit is significantly better at a level

EXHIBIT 9

Calibration to DJ iTraxx 5 year Series 5 for Weekly Prices between June 24, 2005, and December 9, 2005

	0–3%	3–6%	6–9%	9–12%	12–22%	Average APE	$\bar{\rho}$
Market	27.1%	86.7	26.8	14.1	8.4	–	
Gauss	27.1%	193.3 (106.7)	45.3 (18.6)	12.6 (4.4)	1.7 (6.8)	34.1	0.160
double t(4)	27.1%	97.1 (10.4)	36.4 (9.6)	21.1 (7.1)	10.8 (2.4)	8.3	0.197
double t(5)	27.1%	112.9 (26.2)	38.5 (11.7)	20.3 (6.0)	9.1 (0.6)	11.4	0.193
VG(ν)	27.1%	86.6 (0.1)***	38.5 (11.7)	23.8 (9.7)	12.9 (4.5)	6.5	0.175
VG(ν,θ)	27.1%	75.4 (11.2)	26.3 (3.6)***	14.3 (1.6)***	6.5 (2.3)	5.2***	0.208

*VG(ν) denotes the VG Copula model restricted to $\theta = 0$ and VG(ν,θ) the unrestricted model. Average pricing errors for the tranches are given in parentheses. The values of $\bar{\rho}$ denote correlation averages. (***,**,*) denote a superior fit of the model compared with the double t(4) coupla at the 1, 5, and 10% level.*

of 1%. In this case, the fit is also significantly better at the 1% niveau for two tranches. In the unrestricted case, the average APEs of all tranches except the [3–6%]-tranche are below 4 bp, which is about the bid ask spread on a typical day.

Comparing the absolute pricing errors of Exhibit 8 (structural model) to those of Exhibit 9 (factor copula), we find that we could reduce these pricing errors by this analytical model. This is possible because we assumed a constant default intensity and ignored the CDS indices with maturities different than 5 years. One should not compare parameter values of ρ, θ, and ν, since we have restricted ourselves to unit variances for all distributions in this section. Further differences occur since defaults are not triggered by a constant default barrier L as in the last section.

Stability of Parameters

An important concern about the VG copula model is the stability of parameters. While the only parameter in the Gaussian copula model is correlation, the double t copula has one further parameter, namely the degrees of freedem, which is held fixed at a value of 4 or 5. In the VG copula model, there are one or two additional parameters in the restricted or unrestricted case besides correlation. These parameters θ and ν determine the implied skewness and implied kurtosis of the distributions (see Appendix A).

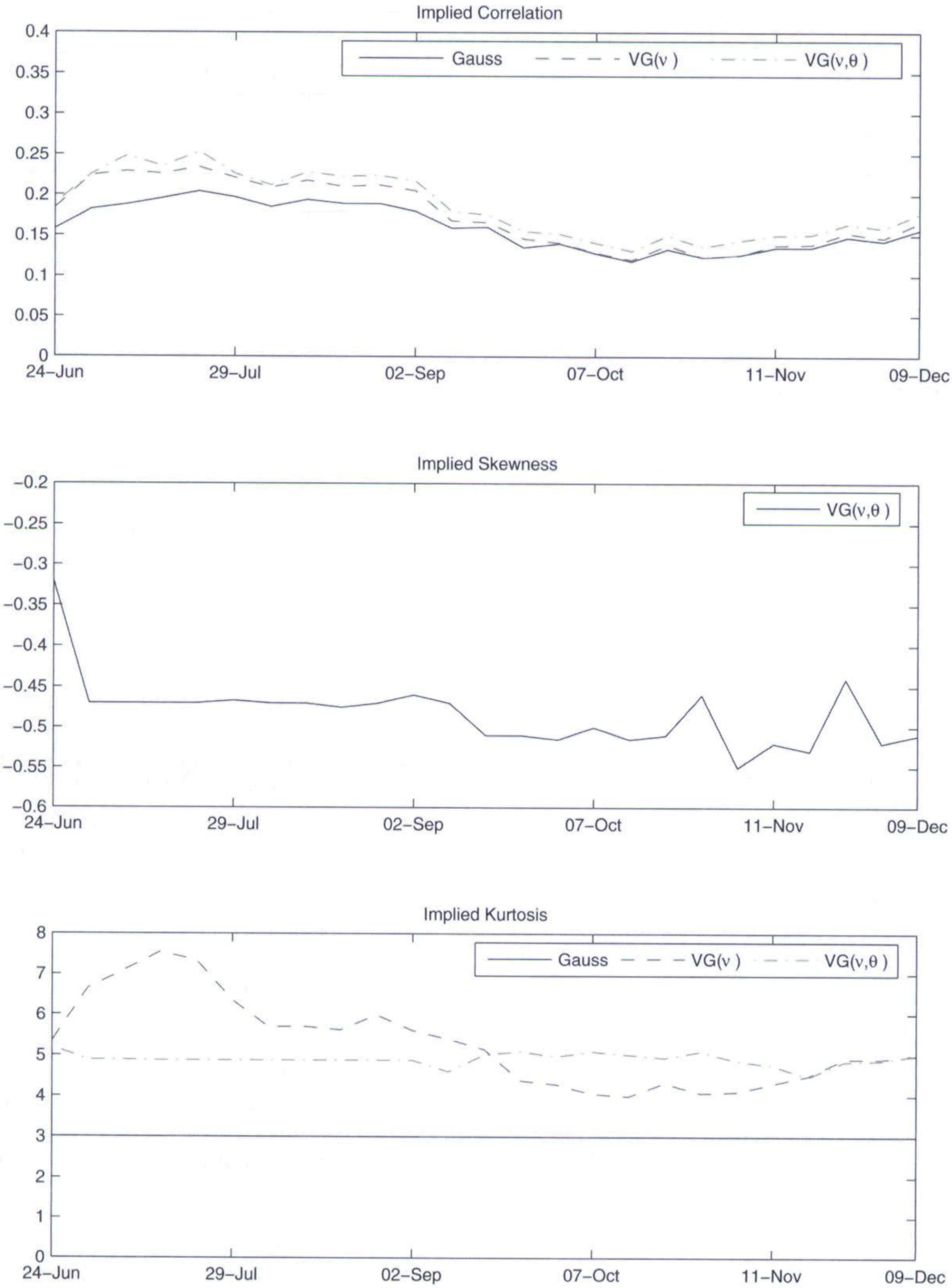
Exhibit 10 shows the implied moments for the Gaussian and the VG copula. The above figure shows that the implied correlations of all the models move together. In the unrestricted VG case, the implied correlation is a little higher than that for the other two models. The middle figure of Exhibit 10 shows the implied skewness of the unrestricted VG model. This implied skewness is 0 for in the Gaussian and the restricted case. We can see that this implied skewness is quite constant in the period considered and that it changes between -0.32 and -0.57. Finally, we consider the implied kurtosis in all the three models at the bottom of Exhibit 10. In the Gaussian case, this moment is fixed at a value of 3. We find a significant excess kurtosis in both VG cases. For the unrestricted model, this implied moment is very constant between 4.7 and 5.4. The restricted model shows a larger variation in implied kurtosis. Interestingly, this implied kurtosis seems to be highly correlated to the implied correlation of the model.

CONCLUSION

In this article, we proposed a valuation method for CDS index tranches by means of VG processes and distributions. We showed that extensions of a structural model developed by Luciano and Schoutens [2005] can simultaneously fit to index spread curves and to liquid tranche quotes of DJ iTraxx 5 year. We extracted the resulting dependence structure into a VG copula. This approach shares the advantages of the one factor Gaussian copula, and all extensions and implementation methods that have been developed for the Gaussian copula can be used for the VG copula as well. Yet, the VG Copula turns

Exhibit 10

Implied Correlation, Implied Skewness, and Implied Kurtosis for the Gaussian Copula, the Restricted and Unrestricted VG Copula Model



out to be more flexible and leads to a dependence structure that fits to observed tranche spreads. The additional parameters for skewness and kurtosis are stable for the data considered in this article.

Our model can be used in several ways. First, it leads to a consistent way of pricing bespoke tranches on CDS indices like DJ iTraxx or DJ CDX NA. Second, it gives an alternative to the normality assumptions in the CreditMetrics model. The approach is therefore not limited to application in credit portfolio derivatives but may also be used for credit portfolio management.

We envision several possibilities for further research based on the ideas proposed in this article. First, other distributions based on Lévy processes can be used in a copula approach if parameters are chosen suitably. For example, one could consider jump diffusions as proposed by Merton [1976] or Kou [2002]. Second, the impacts of our distribution and copula assumptions on risk measures like credit value at risk could be examined, thus carrying over the ideas of Frey, McNeil, and Nyfeler [2001] and Hamerle and Rösch [2005].

APPENDIX A: VG PROCESSES AND DISTRIBUTIONS

A random variable X is said to be VG distributed with parameters θ, ν, σ, and μ ($X \sim \mathrm{VG}(\theta, \nu, \sigma, \mu)$) if its density is given by

$$f_X(x) = \frac{2\exp\left(\frac{\theta x}{\sigma^2}\right)}{\nu^{\frac{1}{\nu}}\sqrt{2\pi}\sigma\,\Gamma(\frac{1}{\nu})}\left(\frac{(x-\mu)^2}{\frac{2\sigma^2}{\nu}+\theta^2}\right)^{\frac{1}{2\nu}-\frac{1}{4}} K_{\frac{1}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(x-\mu)^2\left(\frac{2\sigma^2}{\nu}+\theta^2\right)}\right).$$

K is the modified Bessel function of the third kind,

$$K_\nu(x) = \frac{1}{2}\int_0^\infty y^{\nu-1}\exp\left(-\frac{1}{2}x\left(y+y^{-1}\right)\right)\mathrm{d}y.$$

The parameter domain is restricted to $\mu, \theta \in \mathbb{R}$, and $\nu, \sigma > 0$.[13]

Since this distribution is infinitely divisible, there exists a Lévy process $(X_t)_{t\geq 0}$ such that X_1 has the above distribution. $(X_t)_{t\geq 0}$ is called VG process and can be represented by

$$X_t = \mu t + \theta G_t + \sigma W_{G_t},$$

where $(G_t)_{t\geq 0}$ is Gamma process parameters (ν^{-1}, ν) and $(W_t)_{t\geq 0}$ is a standard Brownian motion. In this article, we have set $\mu = 0$ when we consider VG processes.

If ,($X \sim \mathrm{VG}(\theta, \nu, \sigma, \mu)$,

the Laplace transform L_X of X is given by

$$L_X(z) = \mathrm{e}^{\mu z}\left(1-\theta\nu z-\frac{1}{2}\nu\sigma^2 z^2\right)^{-\frac{1}{\nu}}, \quad z \in \mathbb{R}, \tag{19}$$

We therefore obtain the moments:

$$\begin{aligned}
\mathrm{E}[X] &= \mu+\theta,\\
\mathrm{Var}[X] &= \nu\theta^2+\sigma^2,\\
\mathrm{S}[X] &= \theta\nu\frac{3\sigma^2+2\nu\theta^2}{(\sigma^2+\nu\theta^2)^{3/2}},\\
\mathrm{K}[X] &= 3(1+2\nu-\nu\sigma^4(\nu\theta^2+\sigma^2)^{-2}).
\end{aligned}$$

Besides infinite divisibility, the class of VG distributions is closed under scaling and convolution if parameters are chosen suitably:

1. If $X \sim \mathrm{VG}(\theta, \nu, \sigma, \mu)$ and $c > 0$, then $cX \sim \mathrm{VG}(c\theta, \nu, c\sigma, c\mu)$.
2. If $X_1 \sim \mathrm{VG}(\theta_1, \nu_1, \sigma_1, \mu_1)$ and $X_2 \sim \mathrm{VG}(\theta_2, \nu_2, \sigma_2, \mu_2)$ such that $\sigma_1^2\nu_1 = \sigma_2^2\nu_2$ and $\frac{\theta_1}{\sigma_1^2} = \frac{\theta_2}{\sigma_2^2}$ then

$$X_1 + X_2 \sim \mathrm{VG}\left(\theta_1+\theta_2, \frac{\nu_1+\nu_2}{\nu_1\nu_2}, \sqrt{\sigma_1^2+\sigma_2^2}, \mu_1+\mu_2\right).$$

APPENDIX B

Martingale Property of $S_t^{(i)}$ in Equation (3)

$$\begin{aligned}
E\left[S_t^{(i)}\right] &= S_0^{(i)}\exp(rt+\omega_i t)\cdot E\left[\exp(X_t^{(i)})\right]\\
&= S_0^{(i)}\exp\left(rt+\tfrac{1}{\nu}\log\left(1-\tfrac{1}{2}\sigma_i^2\nu-\theta_i\nu\right)t\right)\\
&= \exp\left(-\left(\tfrac{1}{\nu}\log\left(1-\tfrac{1}{2}\sigma_i^2\nu-\theta_i\nu\right)t\right)\right)\\
&= S_0^{(i)}\exp(rt).
\end{aligned}$$

The second equation holds as the characteristic exponent of $(X_t^{(i)})_{t\geq 0}$ is given by (see Appendix A)

$$\Psi_{X^{(i)}}(u) = -\frac{1}{\nu}\log\left(1+\frac{1}{2}u^2\sigma_i^2\nu - iu\theta_i\nu\right)$$

and since

$$E\left[\exp(X_t^{(i)})\right]=\exp\left(\Psi(-i)t\right).$$

Proof of Equation (4)

If

$$X_t^{(i)}=\theta_i G_t^{(i)}+\sigma_i W_{G_t^{(i)}}^{(i)}, \qquad X_t^{(j)}=\theta_j G_t^{(j)}+\sigma_j W_{G_t^{(j)}}^{(j)}$$

then

$$\begin{aligned}
\mathrm{Cov}(X_t^i, X_t^j) &= \mathrm{E}\left(\prod\nolimits_{k=i,j}\left(\theta_k\left(G_t^{(k)}-t\right)+\sigma_i\sqrt{G_t^{(k)}}W_t^{(k)}\right)\right)\\
&=\theta_i\theta_j\mathrm{Cov}\left(G_t^{(i)},G_t^{(j)}\right)+\theta_i\sigma_j\mathrm{E}\left(G_t^{(i)}\sqrt{G_t^{(j)}}\right)\underbrace{\mathrm{E}(W_t^{(i)})}_{=0}\\
&\quad+\theta_j\sigma_i\mathrm{E}\left(G_t^{(j)}\sqrt{G_t^{(i)}}\right)\underbrace{\mathrm{E}(W_t^{(j)})}_{=0}\\
&\quad+\sigma_i\sigma_j\mathrm{E}\left(\sqrt{G_t^{(i)}G_t^{(j)}}\right)\underbrace{\mathrm{E}(W_t^{(i)})\mathrm{E}(W_t^{(j)})}_{=0}\\
&=\theta_i\theta_j\mathrm{Cov}\left(G_t^{(i)},G_t^{(j)}\right)\\
&=\theta_i\theta_j\mathrm{Cov}\left(F_t U_t^{(i)},F_t+U_t^{(j)}\right)\\
&=\theta_i\theta_j\mathrm{Var}(F_t)\\
&=\theta_i\theta_j a\nu t.
\end{aligned}$$

In the same way (or using the results of Appendix A) it can be shown that

$$\mathrm{Var}(X_t^{(i)})=\left(\theta_i^2\nu+\sigma_i^2\right)t.$$

We therefore see that the correlation for all t is

$$\mathrm{Corr}(X_t^{(i)},X_t^{(i)})=a\frac{\theta_i\theta_j\nu}{\sqrt{\theta_i^2\nu+\sigma_i^2}\sqrt{\theta_j^2\nu+\sigma_j^2}}.$$

Proof of Equation (5)

If

$$X_t^{(i)}=\theta_i G_t+\sigma_i W_{G_t}^{(i)}, \qquad X_t^{(j)}=\theta_j G_t+\sigma_j W_{G_t}^{(j)}$$

then

$$\begin{aligned}
\mathrm{Cov}\left(X_t^{(i)},X_t^{(j)}\right) &= \mathrm{E}\left(\prod\nolimits_{k=i,j}\left(\theta_k\left(G_t-\mathrm{E}(G_t)\right)+\sigma_k\sqrt{G_t}W_t^{(k)}\right)\right)\\
&=\theta_i\theta_j\mathrm{Var}(G_t)+\theta_i\sigma_j\mathrm{E}\left(G_t\sqrt{G_t}\right)\underbrace{\mathrm{E}(W_t^{(i)})}_{=0}\\
&\quad+\theta_j\sigma_i\mathrm{E}\left(G_t\sqrt{G_t}\right)\underbrace{\mathrm{E}(W_t^{(j)})}_{=0}+\sigma_i\sigma_j\mathrm{E}(G_t)E\left(W_t^{(i)}W_t^{(j)}\right)\\
&=\theta_i\theta_j\mathrm{Var}(G_t)\\
&\quad+\sigma_i\sigma_j t\mathrm{E}\left(\left(\sqrt{b}F_t+\sqrt{1-b}U_t^{(i)}\right)\left(\sqrt{b}F_t+\sqrt{1-b}U_t^{(j)}\right)\right)\\
&=\theta_i\theta_j\nu t+\sigma_i\sigma_j tbE(F^2)\\
&=\left(\theta_i\theta_j\nu+\sigma_i\sigma_j b\right)t.
\end{aligned}$$

Proof of Equations (9) and (14)

If

$$X_t^{(i)}=cM_t+\sqrt{1-c^2}Z_t^{(i)}, \quad X_t^{(j)}=cM_t+\sqrt{1-c^2}Z_t^{(j)}$$

then

$$\begin{aligned}
\mathrm{Cov}\left(X_1^{(i)},X_1^{(j)}\right)&=\mathrm{Cov}\left(cM_1+\sqrt{1-c^2}Z_1^{(i)},cM_1+\sqrt{1-c^2}Z_1^{(j)}\right)\\
&=c^2\mathrm{Var}\left(M_1\right)\\
&=c^2\left(\tfrac{\nu}{c^2}\left(c\theta\right)^2+\left(\sigma\right)^2\right)\\
&=c^2\left(\nu\theta^2+\sigma^2\right).
\end{aligned}$$

Since

$$\mathrm{Var}(X_1^{(i)})=\left(\nu\theta^2+\sigma^2\right)1,$$

we get

$$\mathrm{Cov}(X_1^{(i)},X_1^{(j)})=c^2.$$

The proof of Equation (14) is identical.

Proof of Equation (17)

This proof is identical to the one proposed by Vasicek [1987] for the Gaussian copula. Let N be the number of entities in the portfolio. Conditional on $M = m$, the probability of $0 \leq k \leq N$ defaults is then

$$\begin{aligned}
&\mathbb{P}_k(M=m)\\
&\quad=\binom{N}{k}(\mathbb{P}(X_i<C\mid M=m))^k(1-\mathbb{P}(X_i<C\mid M=m))^{N-k}\\
&\quad=\binom{N}{k}\left(F_{Z_i}\left(\frac{C-mc}{\sqrt{1-c^2}}\right)\right)^k\left(1-F_{Z_i}\left(\frac{C-mc}{\sqrt{1-c^2}}\right)\right)^{N-k}.
\end{aligned}$$

We suppress the index i of Z_i in the following, since all Z_i are identically distributed. The unconditional probability is therefore

$$\mathbb{P}_k = \binom{N}{k} \int_{-\infty}^{\infty} \left(F_Z\left(\frac{C - mc}{\sqrt{1-c^2}} \right) \right)^k \left(1 - F_Z\left(\frac{C - mc}{\sqrt{1-c^2}} \right) \right)^{N-k} f_M(m)\mathrm{d}m.$$

If we substitute

$$s = F_Z\left(\frac{C - mc}{\sqrt{1-c^2}} \right),$$

we find for the percentage portfolio loss not exceeding x:

$$\begin{aligned} F_N(x) &= \sum_{k=0}^{[Nx]} \mathbb{P}_k \\ &= \sum_{k=0}^{[Nx]} \binom{N}{k} \int_0^1 s^k (1-s)^{N-k} \mathrm{d}W(s) \end{aligned}$$

with

$$W(s) = 1 - F_M\left(\frac{1}{c}\left(C - \sqrt{1-c^2} F_Z^{-1}(s) \right) \right).$$

Since

$$\begin{aligned} \lim_{N\to\infty} \sum_{k=0}^{[Nx]} \binom{N}{k} s^k (1-s)^{N-k} &= 0 \quad \text{if } x < s \\ &= 1 \quad \text{if } x > s, \end{aligned}$$

the cumulative distribution function in the limit $N \to \infty$ is

$$\begin{aligned} F_{\text{portfolio loss}}(x) &= W(x) \\ &= 1 - F_M\left(\frac{1}{c}\left(C - \sqrt{1-c^2} F_Z^{-1}(x) \right) \right). \end{aligned}$$

ENDNOTES

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[1]See Amato and Gyntelberg [2005] for detailed descriptions of these indices.

[2]The risk neutral default probablity can be determined from single name credit default swaps.

[3]See e.g. Gibson [2004] for details.

[4]These implied correlations slightly depend on certain assumptions about the portfolio structure. We have assumed an infinitely large portfolio of identical entities.

[5]Madan, carr, and chang [1998] is the reference for Gamma distributed business times.

[6]The reason why the parameter ν does not depend on i will be explained in the next subsection.

[7]For two independent Gamma variables $X \sim \Gamma(a_X, \lambda)$ and $Y \sim \Gamma(a_Y, \lambda)$ their sum is Gamma distributed with $X + Y \sim \Gamma(a_X + a_Y, \lambda)$. This is also the reason we chose the parameter ν to be identical for all i.

[8]We have tested the model to robustness with respect to the assumptions on L, R, and dt. Qualitatively, the results were the same, although computation speed obviously depends on dt.

[9]See for example Gibson [2004] for details.

[10]In this and the following calibrations, we have used other measures like the root mean squared error as well. The results were qualitatively the same.

[11]For the Gaussian copula model, this approximation was introduced by Vasicek [1987].

[12]See Appendix A for details on VG distributions and their properties as far as we need them for this article.

[13]This definition is taken from Bibby and Sørensen [2003] and we used the parameter transformation

$$\mu \to \mu, \alpha \to \sqrt{\frac{2}{\nu\sigma^2} + \frac{\theta^2}{\sigma^4}}, \beta \to \frac{\theta}{\sigma^2}, \lambda \to \frac{1}{\nu}$$

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