Models of Conditional Variance for Bond Prices

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The potential distribution of price changes from any investment is extremely important to any intelligent investor. Many studies have examined the historical distributions of equity price changes. Researchers have investigated such issues as the third moment and fourth moment of price changes (returns) and whether Gaussian, Student *t*, or stable Paretian distributions best represent equity price distributions. In contrast, there has been relatively little research on the distribution of bond price changes. The relative lack of research may be due to the fact that a crucial characteristic, maturity, is constantly changing so that the instrument is constantly changing.

Recent advances in modeling conditional distributions of economic time series have greatly improved models of conditional expected price changes and conditional (expected) variance of price changes. Conditional distributions are partially very popular because there is increasing evidence that price changes are predictable. Kirby [1998] notes that regression analysis to document return predictability has become commonplace. Xu [2004] has found that utilizing small levels of predictability can result in higher returns that are less risky. A special issue of the *Journal of Empirical Finance* [2001] was recently devoted to the predictability of asset returns. Dhillon and Lasser [1998] found that they can predict future interest rates to a degree. Cochrane [1999] lists return (price) predictability as one of the new facts of finance and maintains that bond returns are somewhat predictable. Papageorgiou and Skinner [2002] find that they can predict the direction of interest rate changes. Brenner, Harjes, and Kroner [1996] find predictability in monthly Treasury-bill returns.[1] Also, the second moment, conditional variance, has usually proven to be predictable.

The purpose of this research is to develop improved models for conditional variance of bond price change. We often refer to models for conditional volatility. A common example is the large literature that predicts variance using GARCH processes of conditional variance.[2]

Improved models allow us to address many important questions. First, what are the time varying functional forms of how duration and convexity explain conditional variance of price changes? Although elementary treatments of bond price volatility discuss the simple deterministic role of duration and sometimes convexity, the stochastic time series functional form of the relation has not been derived as done here.[3] For example, we will show that the time varying fourth moment interacts with duration and convexity.[4] Questions related to the time varying functional form of price changes include what are the unique and important components of conditional price variance? In this context, we will give expressions of conditional price variance that allow the analyst to examine how different statistical and financial concepts determine variance of price change.

Second, how does a second-order model of conditional price variance differ from a first-order model? How does the functional form differ from a first-order model? How much error will be introduced if a simple first-order model is used?

Third, does convexity tend to increase or decrease conditional price variance? The results should make bond portfolio managers reconsider the desirability of convexity in their portfolios. That is, if convexity tends to increase conditional price variance, then convexity may not be as desirable as elementary deterministic models would suggest. Models of required yield dependent upon convexity should be reconsidered.

Besides being useful to a portfolio manager, our work is also useful to regulators concerned about financial institution exposure to interest rate risk. Regulators frequently require analysis of the impact of, for example, parallel shifts in the yield curve upon institutional net worth.

The next section will derive the alternative mathematical models for conditional variance of price change. Alternative expressions for conditional variance will be described. We find an appealingly brief expression for conditional variance of price change which depends on duration, convexity, expected change in yield, conditional variance of yield changes, and the conditional fourth moment of yield changes. Also, we will illustrate how much first-order estimation of conditional variance differs from second-order. Then we discuss applications of our models. For example, the portfolio manager can see how convexity may not be as desirable as prior literature suggests. We briefly describe how our work can be useful to financial institution regulators. Finally, the last section concludes and summarizes our research.

MODELS FOR CONDITIONAL VARIANCE OF PRICE CHANGES

In this section, we derive two sets of naturally nested models for specifying the conditional variance of bond price changes. We suggest these models depend on two types of factors. The first type of factor is properties of the change in the yield such as the first, second and fourth conditional moments. For statistical analysis, we utilize Bollerslev [1986]. The second type of factor is standard properties of the bond such as duration and convexity.

Model I: The First-Order Model

It is well known that the first-order approximation to change in bond price, $\Delta P_t = P_t - P_{t-1}$, is given by

$$\begin{aligned}\Delta P_t &= -\left(\frac{D_{t-1}}{1+y_{t-1}}\right)\Delta y_t P_{t-1}\\ &= -d_{t-1}\,\Delta y_t\; P_{t-1}\end{aligned} \tag{1}$$

where y_t is the yield at time t and $\Delta y_t = y_t - y_{t-1}$. D_{t-1} is the Macaulay duration and d_{t-1} is the modified duration. Let

$$\Delta y_t = z_{t-1} + \varepsilon_t \tag{2}$$

be the stochastic dynamic model that describes the evolution of the yield change where expected yield change is

$$E[\Delta y_t | F_{t-1}] = z_{t-1} \tag{3}$$

and F_t is the set of all information available up to time t. It is assumed that the noise term is distributed normally. That is, $\varepsilon_{t|t-1} \sim N(0,\eta_t\sigma_t)$ where $\eta_t \sim N(0,1)$. Consequently, the expected error and conditional variance of yield changes are respectively given as

$$E\,[\varepsilon_t | F_{t-1}] = 0 \quad \text{and} \quad E\,[\varepsilon_t^2 | F_{t-1}] = \sigma_t^2 \tag{4}$$

It follows that

$$E\left[\Delta P_t | F_{t-1}\right] = -d_{t-1}\; z_{t-1}\; P_{t-1} \tag{5}$$

The conditional variance of price changes in Model I is thus

$$V_I = \operatorname{var}\left(\Delta P_t | F_{t-1}\right) = d_{t-1}^2 \sigma_t^2\; P_{t-1}^2 \tag{6}$$

Note that first-order conditional variance, V_I, does not depend on the conditional mean (z_{t-1}) of Δy_t but only depends on the duration and the conditional variance (σ_t^2) of Δy_t.

Model II: Second-Order Model

The well-known second-order approximation to the change in bond price is given by

$$\Delta P_t = \left\{-d_{t-1}\,\Delta y_t + \tfrac{1}{2}C_{t-1}(\Delta y_t)^2\right\}P_{t-1} \tag{7}$$

We now derive a more accurate model of conditional price variance. Substituting and simplifying, we get

$$\Delta P_t = \left\{\left[\tfrac{1}{2}C_{t-1}z_{t-1}^2 - d_{t-1}z_{t-1}\right] + \tfrac{1}{2}C_{t-1}\varepsilon_t^2 + g_{t-1}\varepsilon_t\right\}P_{t-1} \tag{8}$$

where C_{t-1} is the convexity of the bond and

$$g_{t-1} = C_{t-1}\,z_{t-1} - d_{t-1}$$

Of course, our second-order variance can be obtained by finding the variance of ΔP_t. We illustrate some interesting properties of second-order variance below.

Taking conditional expectations, we readily obtain

$$\Delta P_t = \left\{-d_{t-1}\,\Delta y_t + \tfrac{1}{2}C_{t-1}(\Delta y_t)^2\right\}P_{t-1} \tag{9}$$

Hence,

$$\Delta P_t - E\left[\Delta P_t | F_{t-1}\right] = \left\{\tfrac{1}{2}C_{t-1}e_t + g_{t-1}\varepsilon_t\right\}P_{t-1} \tag{10}$$

where

$$e_t = \varepsilon_t^2 - \sigma_t^2 = (\eta_t^2 - 1)\sigma_t^2$$

This expression is similar to Bollerslev [1986]. The parallel expression of $\Delta P_t - E[\Delta P_t]$ for Model I is merely $-d_{t-1}\,\varepsilon_t P_{t-1}$. Since $\eta_t \sim N(0,1)$, it can be verified that

$$E\left[e_t \mid F_{t-1}\right] = 0 \tag{11}$$

$$E\left[e_t^2 \mid F_{t-1}\right] = \sigma_t^4\left\{E\left(\eta_t^4 \mid F_{t-1}\right) - E\left(2\eta_t^2 \mid F_{t-1}\right) + 1\right\} = 2\sigma_t^4 \tag{12}$$

and

$$E[e_t\varepsilon_t \mid F_{t-1}] = \sigma_t^3\left\{E\left(\eta_t^3 | F_{t-1}\right) - E\left(\eta_t \,| F_{t-1}\right)\right\} = 0 \tag{13}$$

In this context, it is important to note that

$$\operatorname{var}\left(\varepsilon_{t|t-1}^2\right) = 2\sigma_t^4, \quad \operatorname{var}\left(\varepsilon_{t|t-1}\right) = \sigma_t^2,$$

$$\text{and} \quad \operatorname{cov}\left(\varepsilon_{t|t-1}^2, \varepsilon_{t|t-1}\right) = 0$$

Using properties of e_t and ε_t and substituting, we obtain the following expressions for the second-order conditional variance of ΔP_t:

$$V_{\text{II}} = E\left[\left\{g_{t-1}\varepsilon_t + (0.5)\,C_{t-1}e_t\right\}^2\right]P_{t-1}^2 \tag{14a}$$

$$V_{\text{II}} = \left\{\tfrac{1}{2}C_{t-1}^2\ \sigma_t^4 + g_{t-1}^2\ \sigma_t^2\right\}P_{t-1}^2 \tag{14b}$$

In the first expression for V_{II}, the variance of ΔP_t is represented by a squared sum of two error terms: the time-dependent estimation error in the expected change in yield, ε_t, multiplied by g_{t-1}, and the time-dependent estimation error in the variance, e_t, multiplied by $(0.5)C_{t-1}$. Alternatively, the second expression states variance in terms of the second and fourth moments and is more computationally friendly. We also note that V_{II} can be derived by taking the variance of Equation (8).

A number of observations are in order. First, V_{II} depends on z_{t-1}, d_{t-1}, C_{t-1}, σ_t^2, and σ_t^4 whereas V_{I} depends only upon duration and volatility of yield changes. Second, V_{II} reduces to V_{I} when C_{t-1} is zero in V_{II}. Third, if y_t follows a random walk such that z_{t-1} is zero,

$$V_{\text{II}} = \left\{\tfrac{1}{2}C_{t-1}^2\ \sigma_t^4 + d_{t-1}^2\sigma_t^2\right\}P_{t-1}^2 \tag{15}$$

The Difference in Model Variances

One may wish to analyze how Model I conditional price variance (V_{I}) differs from Model II conditional price variance (V_{II}). Consider a portfolio manager deciding whether or not to use the more parsimonious, easily computed, and interpreted Model I. The parsimonious first-order model will be less accurate. Rearranging the above expressions, one can represent the percentage difference (error) in Model I variance compared to Model II variance as R where

$$R = \frac{V_{\text{II}} - V_{\text{I}}}{V_{\text{I}}} = 0.5\,(r_{t-1})^2\ \sigma_t^2 + (r_{t-1})^2\ z_{t-1}^2 - 2r_{t-1}\,z_{t-1} \tag{16}$$

Here r_{t-1} is C_{t-1}/d_{t-1}. For example, assume that a bond portfolio manager has an accuracy tolerance of absolute k. That is, he is willing to use the more parsimonious model if $R < |k|$ but will otherwise use the more accurate Model II.

As an illustrative case, if models for expected changes in yield have no predictive ability, then z_{t-1} is assumed zero and the percentage difference is always positive and briefly stated as $0.5\ (r_{t-1})^2\ \sigma_t^2$. Here, the error of using V_I increases with the squared ratio of convexity to duration and the volatility of interest rates. For short maturities, r_{t-1} is relatively small but the ratio of convexity to duration increases rapidly as maturity increases. If σ_t^2 declines relatively slowly or not at all with increasing maturity, the error of using V_I may well dramatically increase with maturity. In general, the error of using V_I will be greatest for long maturities during times of high yield volatility.

Exhibit 1 illustrates the error of using Model I for long-term U.S. Treasury and United Kingdom bonds during various time periods including those of high volatility. To estimate volatility, we use the historical volatility of change in yield of various time periods. Poon and Granger [2003] find that historical volatility of the most recent trading days frequently does quite well as an estimate of time varying volatility.

In the United States, the 1980's were a time of high volatility. If Model I was used for the period October to December 1981, the error of Model I would have been as much as 37%. A window using the entire first half of the 1980's suggests an error over 20%. For the United Kingdom, the error was almost 18%, using data from 1974 to 1979.

In summary, we have derived alternative models of conditional price variance. The conditional variance of price change in the second-order model is a fairly complex function of duration, convexity, the conditional fourth moment, expected change in yield, and conditional variance of changes in yield. The difference between first- and second-order variance is larger for longer maturities and during times of high yield volatility.

EXHIBIT 1
Percentage Difference in Model I versus Model II

Time Period and Data Frequency	Maturity (Years)	$\left(\frac{C_{t-1}}{D_{t-1}}\right)^2$	σ_t^2	$\frac{V_{II} - V_I}{V_I}$
40 trading days beginning October 27, 1981	30	743.801	0.0010102	0.37561
60 days beginning October 13, 1981	30	621.133	0.0009736	0.30124
1980–1989	30	1,097.931	0.0002599	0.14267
1980–1984	30	1,097.931	0.0003841	0.21088
1985–1989	30	919.235	0.0001359	0.06246
1980–2002	30	1,097.931	0.0001534	0.08421
1990–1999	30	1,434.84	0.0000657	0.04713
125 days (half year) starting October 6, 1981	30	662.6281	0.0007702	0.25511
252 days (full year) starting April 6, 1981	30	766.8497	0.0006471	0.24807
40 trading days, starting August 7, 2003[a]	30	1,112.469	0.0001321	0.07342
60 trading days, starting August 7, 2003[a]	30	2,015.959	0.0001322	0.13305
United Kingdom, 60 months starting October, 1974	20	815.897	0.000441	0.17932

[a]*Uses 20-year volatility to approximate 30-year volatility.*
The difference between Models I and II is shown for different time periods.
Convexity and duration are computed using the semiannual coupons and yields at the beginning of each period.
All data is for 30-year U.S. Treasury bonds except for United Kingdom bonds in last row.

APPLICATIONS OF MODEL II TO RISK/RETURN ANALYSIS

Expected Return for Portfolio Managers

A bond portfolio manager aware of the above theory has an improved ability to predict and analyze bond expected return and expected risk. For example, the portfolio manager should now be aware that convexity enhances expected return through *both* the squared change in expected yield and the variance of expected yield changes in Model II. That is,

$$\frac{\partial E[\Delta P_t]}{\partial C_{t-1}} = \tfrac{1}{2}(z_{t-1}^2 + \sigma_t^2) \tag{17}$$

In this context, convexity is desirable not only when yields are volatile (σ_t^2) but also at times when one has large changes in *predicted* yields *no matter the sign* of predicted yield changes. If both are large, convexity is especially desirable. Also, counter to conventional thought, convexity could significantly increase returns even when volatility of yields is low if large z_{t-1}^2 values are predicted.

Expected Risk for Portfolio Managers

Next consider how V_{II} is affected by time varying conditional yield volatility (σ_t^2). The partial derivative is

$$\frac{\partial V_{\text{II}}}{\partial \sigma_t^2} = \left(C_{t-1}^2\sigma_t^2 + C_{t-1}^2\, z_{t-1}^2 - 2C_{t-1}\, z_{t-1}\, d_{t-1} + d_{t-1}^2\right)P_{t-1}^2 \tag{18}$$

The second derivative is positive[5] so that V_{II} has a minimum value and a U-shape. However, it is extremely hard to believe that the term $-2C_{t-1}\, z_{t-1}\, d_{t-1}$ could be strong enough to result in a negative derivative (sensitivity) when z_{t-1} is positive. If expected changes in yield are zero, then the sensitivity of V_{II} to yield volatility will not be the same as the V_{I} case but will always be enhanced by convexity as given below where the two derivatives are compared. Thus, in this case, those using V_{I} will always underestimate risk arising from yield volatility for bonds with high convexity.

$$\frac{\partial V_{\text{II}}}{\partial \sigma_t^2} = \left(C_{t-1}^2\sigma_t^2 + d_{t-1}^2\right)P_{t-1}^2 > \frac{\partial V}{\partial \sigma_t^2} = \left(d_{t-1}^2\right)P_{t-1}^2 \tag{19}$$

The difference is that V_{II} is more sensitive to σ_t^2 by $C_{t-1}^2\sigma_t^2 P_{t-1}^2$.

Application of Conditional Variance Dependent Upon Convexity for Portfolio Managers

Our analysis of convexity is useful in the context of portfolio managers considering required yield on bonds. Tuckman [2002] and Ilmanen [2000] and Phoa [1997] have produced theory and evidence that bond buyers do or should pay for the positive aspects of convexity by accepting lower yields on longer maturity bonds. The following is an expression Tuckman uses for required yield (γ) for a short period of time (Δt) where i is the risk-free short rate:

$$\gamma = i + d_{t-1}[E(\Delta i)/\Delta t + \lambda] - \tfrac{1}{2}C_{t-1}\,\sigma^2 \tag{20}$$

λ is a risk premium. Consider a case where price variance increases with convexity. This is obviously so when z_{t-1} is zero. Our work suggests that the required yield equation should be adjusted to include somewhat greater required yield for a greater convexity.

Usage by Financial Institution Regulators

Of course financial institutions such as banks and life insurance companies should find the above useful as they choose bonds and assess interest rate risk. Financial institution regulators should also find the above models useful. Regulators such as the Office of the Comptroller of the Currency, the Basel Committee on Banking Supervision, the Federal Reserve and insurance company regulators all require financial institutions to examine their exposure to interest rate risk. One common tool is to estimate the impact of a parallel shift in yield upon net worth. For example, the Federal Reserve's Joint Policy on Interest Rates Risk [SR 96-13] suggests that institutions assume a 200 basis point parallel shift in interest rates as one way to estimate interest rate risk. Our work suggests that such analysis should be supplemented by our models of conditional variance.

CONCLUSION AND SUMMARY

We have developed alternative models for variance of price change that show duration and convexity affect price variance in more complex ways than prior models suggest. For example, the fourth, time varying, moment of yield changes and convexity affect price variance. A first-order model for price variance may give much different results than a second-order model. Portfolio managers using Model I may well have inferior measures of portfolio variance and risk.

Bond portfolio managers aware of our Model II can see that bond variance is more sensitive to yield volatility than simple models would indicate. Our work suggests that convexity may well increase variance of bond prices such that portfolio managers should be compensated for the additional risk. Regulators should consider our Model II as a way to improve assessment of financial institution exposure to interest rate risk.

ENDNOTES

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[1]Also, Mills [1999] finds predictability in the U.K. gilt yields. Pindyck and Rubinfeld [1998] find predictability in U. S. Treasury bond yields as do Jones, Lamont, and Lumsdaine [1998]. Finally, Diebold and Li [2002] and Reisman and Zohar [2004] find price (yield) predictability.

[2]See Bera and Higgins [1993] and Bollerslev, Chou, and Kroner [1992] for extensive reviews of GARCH models used in finance.

[3]The simplistic deterministic equations are Δ Price = −Duration (Δy) P or Δ Price = −Duration (Δy) P + (0.5) Convexity $(\Delta y)^2$ P, where Δy is change in yield and P is price.

[4]D'Antonio and Cook [2004] analyze the effect that different definitions of convexity can have on evaluating bonds.

[5] $$\frac{\partial^2 V_{II}}{\partial(\sigma_t^2)^2} = \left(C_{t-1}{}^2\right) P_{t-1}^2 > 0 .$$

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