Additional Analytical Approximations of the Term Structure and Distributional Assumptions for Jump-Diffusion Processes

J. BENSON DURHAM

J. BENSON DURHAM is a senior economist at the Board of Governors of the Federal Reserve System in Washington, DC.
j.benson.durham@frb.gov

Jump-diffusion processes for the short rate imply a (second-order parabolic) partial difference differential equation (PDDE) for bond prices that does not necessarily have a closed form solution. In the absence of an analytical answer, one must find numerical solutions to the PDDE or solve an approximate PDDE exactly. This article addresses the literature on the second of these approaches (Ahn and Thompson [1988] and Baz and Das [1996]) in two respects. First, although existing studies focus on a single linearization technique to estimate the PDDE, the following proposes alternative methods that under reasonable parameterizations seem to improve accuracy. Second, closed form solutions, numerical estimates, and closed form approximations of the PDDE each ultimately depend on the assumed distribution of jump sizes, and the current literature only examines the case in which jumps are unimodal and normally distributed. This article explores a broader set of densities, including an exponential/Bernoulli distribution, which permits useful comparisons with an analytical solution, and a Gaussian mixture.

The next section summarizes the literature on linearization techniques to approximate the PDDE. The article then outlines alternative approximations, describes simple parameterizations under the common presumption that jumps follow a (unimodal) Gaussian distribution, and examines alternative distributional assumptions for jumps.

THE BOND PRICING EQUATION AND APPROXIMATIONS UNDER JUMP-DIFFUSION

Given Ito's lemma for jump-diffusion processes, standard arbitrage arguments, and the assumption that the instantaneous short rate, r, follows

$$dr = a(b - r)dt + \sigma dW + Jd\pi(h), \tag{1}$$

where a is the mean reversion coefficient, b is the central tendency of the short rate, t is time, σ is the instantaneous volatility coefficient, dW is the Weiner increment, J is the jump size, $d\pi$ is the increment in a Poisson process with intensity rate h, and $\tau = T - t$ is the time to maturity, the PDDE for bond prices,[1] $P(r, \tau)$, follows

$$0 = \left[a(b - r) - \lambda\sigma\right]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP + hE_t\left[P(r + J, t) - P(r, t)\right] \tag{2}$$

with the boundary condition that $P(r,\tau = 0) = 1$.[2] Given the proposed affine solution, $P(r,\tau) = A(\tau)\exp[-B(\tau)r]$, the expectation term simplifies, and the PDDE becomes

$$0 = \left[a(b-r) - \lambda\sigma\right]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP + hP(r,\tau)E_t\left[\exp\left[-B(\tau)J\right] - 1\right], \quad (3)$$

which can be expressed using the derivatives of the affine solution as two linked ODEs,

$$\frac{\partial B}{\partial \tau} + aB = 1$$

and

$$\frac{\frac{\partial A}{\partial \tau}}{A} = (\lambda\sigma - ab)B + \frac{1}{2}\sigma^2 B^2 + hE_t\left[\exp\left[-B(\tau)J\right] - 1\right].$$

Assumptions regarding the distribution of jump sizes become critical at this juncture. Given the linearity of the expectations operator, the final term in the bond pricing equation can be written as $hP(r,\tau)(E_t\{\exp[-B(\tau)J]\}-1)$, and the expectation, $E_t\{\exp[-B(\tau)J]\}$, is effectively the moment generating function (m.g.f.) of the distribution of jump sizes. For example, one common assumption is that J is normally distributed with mean α and standard deviation γ, and the PDDE given the m.g.f. of a Gaussian distribution follows

$$0 = \left[a(b-r) - \lambda\sigma\right]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP + hP(r,\tau)\left(\exp\left[-\alpha B(\tau) + \frac{1}{2}\gamma^2 B^2(\tau)\right] - 1\right) \quad (4)$$

Notably, different distributional assumptions for jump sizes, and therefore different m.g.f.s, determine the precise form of the PDDE.[3]

Ahn and Thompson [1988] and Baz and Das [1996] employ a linearization technique to produce an exact solution to an approximate PDDE such as (3). They use a two-term Taylor series approximation of the exponential function within the expectation of (3), $\exp[-B(\tau)J]$, to arrive at

$$0 = \left[a(b-r) - \lambda\sigma\right]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP + hPE_t\left[\left(1 - JB + \frac{J^2B^2}{2}\right) - 1\right]. \quad (5)$$

Simplifying, taking expectations, and noting the mean and variance of J by α and γ^2, respectively, the approximate PDDE is then

$$0 = \left[a(b-r) - \lambda\sigma\right]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP + hP\left[-\alpha B + \frac{1}{2}(\mu^2 + \gamma^2)B^2\right]. \quad (6)$$

Given the proposed affine solution and the relevant derivatives, this equation reduces to a system of two ODEs, and the approximate closed form solution for bond prices given the initial data that $A(0) = 1$ and $B(0) = 0$ follows

$$P(r,\tau) = \exp\left\{\begin{array}{l} -\frac{1-e^{-a\tau}}{a}r + \frac{M_1 a + M_2}{a^2}\tau \\ +\frac{M_1 a + 2M_2}{a^3}\left(e^{-a\tau} - 1\right) - \frac{M_2}{2a^3}\left(e^{-2a\tau} - 1\right) \end{array}\right\} \quad (7)$$

where

$$M_1 = -ab + \lambda\sigma - h\alpha \text{ and } M_2 = \frac{1}{2}\left(\sigma^2 + h\left[\alpha^2 + \gamma^2\right]\right).$$[4]

Baz and Das [1996] show that the above expression provides a very close approximation to bond prices vis-à-vis a numerical solution to the pricing equation. This standard approximation does not technically require full knowledge of the distribution of J, but notably, in the case of a Gaussian distribution, the expectation of the two-term Taylor series approximation of $\exp[-B(\tau)J]$ in Equation (3) is not equivalent to the one-term Taylor series approximation of the m.g.f. of a normal distribution in Equation (4). That is, such would imply

$$0 = \left[a(b-r) - \lambda\sigma\right]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP + hP\left[-\alpha B + \frac{1}{2}\gamma^2 B^2\right]. \quad (8)$$

However, of course (6) and (8) are equivalent if $\alpha = 0$, a common assumption in the literature.

AN ALTERNATIVE CLOSED FORM APPROXIMATION UNDER THE GAUSSIAN ASSUMPTION

Although the approximations in Baz and Das [1996] are very close to the numerical estimates, they suggest that "[t]he degree of accuracy of this approach is still an open question" (p. 78). To address this question further, the discussion now turns to an alternative that more explicitly incorporates the specific distributional assumptions for J and employs two-term Taylor series approximations after the expectation (the integral) in the bond pricing equation has been taken.[5] Put differently, the alternative approximates the m.g.f. of the distribution for J. For example, given the Gaussian assumption and its m.g.f., the (approximate) PDDE follows

$$0=\left[a(b-r)-\lambda\sigma\right]\frac{\partial P}{\partial r}-\frac{\partial P}{\partial \tau}+\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}-rP$$
$$+hP\left(-\mu B+\frac{(\mu^2+\gamma^2)B^2}{2}-\frac{\mu\gamma^2B^3}{2}+\frac{\gamma^4B^4}{8}\right) \qquad (9)$$

and by again making use of the initial data and the presumed affine form of the solution, the closed form solution for bond prices using this alternative follows

$$P(r,\tau)=\exp\left\{\begin{array}{l}-\dfrac{1-e^{-a\tau}}{a}r+\dfrac{M_1a^3+M_2a^2+M_3a+M_4}{a^4}\tau \\ +\dfrac{M_1a^3+2M_2a^2+3M_3a+4M_4}{a^5}\left(e^{-a\tau}-1\right) \\ -\dfrac{M_2a^2+3M_3a+6M_4}{2a^5}\left(e^{-2a\tau}-1\right)+ \\ \dfrac{M_3a+4M_4}{3a^5}\left(e^{-3a\tau}-1\right)-\dfrac{M_4}{4a^5}\left(e^{-4a\tau}-1\right)\end{array}\right\}, \qquad (10)$$

where

$$M_1=-ab+\lambda\sigma-h\alpha,\quad M_2=\frac{1}{2}\left(\sigma^2+h\left[\alpha^2+\gamma^2\right]\right),$$
$$M_3=-\frac{1}{2}h\alpha\gamma^2,\quad \text{and}\quad M_4=\frac{1}{8}h\gamma^4.$$ [6]

Interestingly, if $\alpha = 0$, one can arrive at the same approximation in (10) using a somewhat different approach by taking a four-term Taylor expansion of the exponential in (3) instead of the two-term expansion in Baz and Das [1996], as in

$$0=\left[a(b-r)-\lambda\sigma\right]\frac{\partial P}{\partial r}-\frac{\partial P}{\partial \tau}+\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}-rP$$
$$+hPE_t\left[\left(1-JB+\frac{J^2B^2}{2}-\frac{J^3B^3}{3!}+\frac{J^4B^4}{4!}\right)-1\right]. \qquad (11)$$

Of course, in this approximation, one needs the first four moments of the distribution, but the m.g.f. is not explicitly introduced.[7]

To address the relative accuracy of this alternative approximation, Exhibit 1 summarizes bond prices and yields for integer maturity points between 1 and 30 years given a set of parameter values previously used in the literature.[8] Prices and yields based on the numerical estimation, the standard closed form approximations, and the proposed alternative closed form approximations are listed in Columns 2 and 3, Columns 4 and 5, and Columns 7 and 8, respectively. Column 6 reports the absolute value of the difference in basis points between the yields implied by the standard linearization approximation vis-à-vis those implied by the numerical solution. The deviations for this set of parameters average about 0.182 basis points (bp) for each integer maturity point between 1 and 30 years, and the accuracy seems to wane as maturity lengthens. Column 9 reports the same results for the alternative closed form approximation, (10), and indicates that the average absolute deviation is about 0.000018 bp, notably lower than the standard technique, and the accuracy does not steadily increase or decrease as maturity increases. With respect to comparing the two closed form approximations, as Column 10 indicates, on average the alternative approximation is about 2,737 times as accurate as the standard linearization technique across each maturity point. Also, Column 10 indicates that the improvement is largely uniform, although the standard approximation is curiously more accurate with 1 year to maturity. But in general, this analysis suggests that the alternative closed form approximation is quite close to the numerical solution.

Of course, Exhibit 1 does not provide definitive evidence regarding the accuracy of either closed form approximation, because the choice of parameters is ultimately arbitrary. Again, the parameters in Exhibit 1 are consistent with common assumptions made in previous literature to facilitate comparisons. Consideration of the relevant universe of alternatives in this context is not feasible, but perhaps those parameters related to the variance of the short rate process are of particular interest, specifically the relative contribution to total variance from the diffusion

EXHIBIT 1

Jump Size Distribution Assumption: Gaussian

Parameter Assumptions: $a = .1$, $b = .05$, $\sigma = .08$, $\lambda = -.5$, $\lambda_J = 0$, $\alpha = 0$, $\gamma = .01$, $h = 10$, $r = .05$

(1) Maturity	(2) Numerical Solution (Prices)	(3) Numerical Solution (Yields)	(4) Closed-form Approx. (Prices)	(5) Closed-form Approx. (Yields)	(6) |(3)-(5)| (bps)	(7) Alternative Closed-form Approx. (Prices)	(8) Alternative Closed-form Approx. (Yields)	(9) |(3)-(8)| (bps)	(10) (6)/(9)
1	0.934069236	0.068204715	0.934069276	0.068204672	0.000428752	0.934069278	0.068204670	0.000449948	0.953
2	0.846701167	0.083203730	0.846701112	0.083203762	0.000320405	0.846701161	0.083203734	0.000031980	10.019
3	0.750872691	0.095506387	0.750872379	0.095506525	0.001386980	0.750872660	0.095506401	0.000141281	9.817
4	0.655623186	0.105542267	0.655622294	0.105542607	0.003401387	0.655623178	0.105542270	0.000032289	105.343
5	0.566448506	0.113673820	0.566446454	0.113674545	0.007245733	0.566448454	0.113673839	0.000185630	39.033
6	0.486148003	0.120207028	0.486144327	0.120208288	0.012601852	0.486148004	0.120207028	0.000001491	8454.060
7	0.415696372	0.125400023	0.415690473	0.125402050	0.020272123	0.415696340	0.125400034	0.000111923	181.126
8	0.354954352	0.129470761	0.354945834	0.129473760	0.029994767	0.354954291	0.129470782	0.000214002	140.161
9	0.303179960	0.132603192	0.303168628	0.132607344	0.041529478	0.303179931	0.132603202	0.000105782	392.594
10	0.259363390	0.134952515	0.259349119	0.134958017	0.055023606	0.259363378	0.134952520	0.000045569	1207.472
30	0.022733352	0.126130740	0.022701832	0.126176989	0.462489070	0.022733311	0.126130799	0.000596698	775.081
AVERAGE (1-30)					0.182374231			0.000182662	2737.000

parameter, σ, on the one hand, and the contribution of the jump parameter(s)—α, γ, and h if J is normally distributed—on the other. In short, given that the approximations only involve those terms in the PDDE related to jumps, one might expect the closed form solution to the estimated PDDE to be more accurate the smaller the contribution of jumps to total variance of the short rate process. To measure the relative contributions, note that if jumps are normally distributed, the unconditional variance of the jump-diffusion short rate process in (1) follows

$$\frac{\sigma^2 + h\left(\alpha^2 + \gamma^2\right)}{2a}. \tag{12}$$

Therefore, the ratio $\sigma^2/h(\alpha^2 + \gamma^2)$ provides a non-dimensional measure of the relative contributions, and the value of (12) given the chosen parameters in Exhibit 1 indicates that the contribution of diffusion is 6.4 times that of the jumps.[9] Although the jump component in this case is not negligible, perhaps an alternative parameterization that increases its relative contribution would be instructive.

To that end, Exhibit 2 shows the results from a parameterization in which the jump contribution is twice that of the diffusion,[10] with $\sigma = 0.02$, $h = 16$, and the remaining parameter values set to those in Exhibit 1.[11] The results are similar to Exhibit 1, as the alternative closed form approximation is more accurate than the standard linearization technique, as indicated in Column 10. Also, the differences between the approximations and the numerical solutions, as shown in Columns 6 and 9, are somewhat larger than the corresponding differences in Exhibit 1. Even so, the magnitudes of the differences still suggest a high degree of accuracy—the alternative closed form approximation is never even as much as a thousandth of a basis point from the numerical estimate across the term structure. Therefore, at least given this set of parameters, the closed form approximations seem quite close to the numerical solutions, even when jumps contribute more than diffusion to the variance in the short rate.

ALTERNATIVE DISTRIBUTIONAL ASSUMPTIONS FOR JUMP SIZES

The discussion now turns to two alternative assumptions for the distribution of jump sizes. The first follows Das and Foresi [1996], who assume that the absolute value of jump size follows an exponential distribution,[12] with the mean absolute jump size denoted by α^{-1}, and the jump sign follows a Bernoulli distribution, with the probability of a positive jump denoted by w. Notably, as Das and Foresi [1996] show, this assumption produces a closed form solution for bond prices. Therefore, in contrast to Baz and Das [1996], one can adopt this distributional assumption and directly assess the relative accuracy of the numerical and closed form approximations with respect to an

EXHIBIT 2
Jump Size Distribution Assumption: Gaussian

Parameter Assumptions: $a = .1$, $b = .05$, $\sigma = .02$, $\lambda = -.5$, $\lambda_J = 0$, $\alpha = 0$, $\gamma = .01$, $h = 16$, $r = .05$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Maturity	Numerical Solution (Prices)	Numerical Solution (Yields)	Closed-form Approx. (Prices)	Closed-form Approx. (Yields)	|(3)-(5)| (bps)	Alternative Closed-form Approx. (Prices)	Alternative Closed-form Approx. (Yields)	|(3)-(8)| (bps)	(6)/(9)
1	0.946931953	0.054528044	0.946932034	0.054527959	0.000855989	0.946932037	0.054527955	0.000889902	0.962
2	0.890093038	0.058214642	0.890093051	0.058214635	0.000072435	0.890093133	0.058214589	0.000533915	0.136
3	0.832279386	0.061195698	0.832278967	0.061195866	0.001676221	0.832279465	0.061195666	0.000316897	5.289
4	0.775426276	0.063585592	0.775424672	0.063586109	0.005170304	0.775426344	0.063585570	0.000220253	23.474
5	0.720792857	0.065480696	0.720788836	0.065481812	0.011157610	0.720792907	0.065480683	0.000138556	80.528
6	0.669132375	0.066962228	0.669124278	0.066964245	0.020169058	0.669132374	0.066962228	0.000003710	5437.033
7	0.620834572	0.068098660	0.620820569	0.068101883	0.032222468	0.620834587	0.068098657	0.000033852	951.873
8	0.576038273	0.068947647	0.576016315	0.068952412	0.047649736	0.576038272	0.068947647	0.000000513	92878.259
9	0.534715729	0.069557780	0.534683774	0.069564420	0.066403176	0.534715668	0.069557793	0.000125263	530.109
10	0.496734188	0.069970023	0.496690491	0.069978820	0.087971389	0.496734185	0.069970024	0.000006530	13472.396
30	0.142383314	0.064974415	0.142067596	0.065048410	0.739948509	0.142382921	0.064974507	0.000920713	803.669
AVERAGE (1-30)					0.291876010			0.000382830	4457.384

analytical solution. In other words, one can test whether numerical solutions to the exact PDDE are more accurate than closed form solutions to an approximate PDDE.

The second distribution assumption, which the existing literature on jump-diffusion processes and ATSMs does not consider, directly addresses the prospect that, in most notable contract to the Gaussian case, jump sizes are bimodally distributed with modes sufficiently distant from the origin. Although this distributional assumption for J does not produce a closed form solution, this section compares the accuracy of two alternative closed form approximations vis-à-vis the numerical solution, similar to the analysis in the previous section.

The Exponential Distributional Assumption for J

Rather than theoretical notions about the true distribution of jump sizes, consideration of the alternative distribution from Das and Foresi [1996] is driven more by the fact that it produces an analytical solution for bond prices which, in turn, allows us to compare the accuracy of numerical solutions with closed form solutions to the approximate PDDE. Given the m.g.f. of an exponential distribution and the probability of a positive jump,[13] w, we can derive the moments necessary to calculate both the standard linearization technique as well as a version of the alternative. Given this information, the bond pricing equation using the latter approximation is

$$0 = \left[a(b-r) - \lambda\sigma\right]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP + hP\left(-\frac{(2w-1)B}{\alpha} + \frac{B^2}{\alpha^2} - \frac{(2w-1)B^3}{\alpha^3} + \frac{B^4}{\alpha^4}\right) \quad (13)$$

and the solution follows (10) (with the same restrictions) but with

$$M_1 = -ab + \lambda\sigma - h\frac{(2w-1)}{\alpha},\ M_2 = \frac{\sigma^2}{2} + \frac{h}{\alpha^2},$$
$$M_3 = -h\frac{(2w-1)}{\alpha^3},\ \text{and}\ M_4 = \frac{h}{\alpha^4}.$$[14]

Again, the standard linearization only considers the first two moments of the distribution, and the solution also follows (10) but with

$$M_1 = -ab + \lambda\sigma - h\frac{(2w-1)}{\alpha},\ M_2 = \frac{\sigma^2}{2} + \frac{h}{\alpha^2},\ M_3 = 0,$$
and $M_4 = 0$.

Exhibit 3 examines the accuracy of the numerical solution and the two alternative closed form approximations. Columns 2 and 3 summarize the closed form solution from Das and Foresi [1996] in terms of bond prices and yields, respectively, given an arbitrary but reasonable set of parameters that match the assumptions in Exhibit 1 and their parameterization as closely as possible. Columns 4 and 5, Columns 7 and 8, and Columns 10 and 11 list

EXHIBIT 3

Jump Size Distribution Assumption: Jump Size (Exponential), Jump Sign (Bernoulli)

Parameter Assumptions: $a = .1$, $b = .05$, $\sigma = .08$, $\lambda = -.5$, $\lambda_J = 0$, $\alpha = 200$, $h = 10$, $r = .05$

(1) Maturity	(2) Closed-form Solution (Prices)	(3) Closed-form Solution (Yields)	(4) Numerical Solution (Prices)	(5) Numerical Solution (Yields)	(6) |(3)-(5)| (bps)	(7) Closed-form Approx. (Prices)	(8) Closed-form Approx. (Yields)	(9) |(3)-(8)| (bps)	(10) Alternative Closed-form Approx. (Prices)	(11) Alternative Closed-form Approx. (Yields)	(12) |(3)-(11)| (bps)	(13) (6)/(12)	(14) (9)/(12)
1	0.933997016	0.068282036	0.933997034	0.068282016	0.000196531	0.933997015	0.068282037	0.000010597	0.933997016	0.068282036	0.000000000	426040.409	22972.824
2	0.846214110	0.083491433	0.846214122	0.083491426	0.000072620	0.846214085	0.083491448	0.000144221	0.846214110	0.083491433	0.000000008	8840.552	17557.045
3	0.749516426	0.096109015	0.749516468	0.096108997	0.000184047	0.749516286	0.096109077	0.000622925	0.749516426	0.096109015	0.000000076	2436.504	8246.568
4	0.653009465	0.106540914	0.653009530	0.106540889	0.000251077	0.653009025	0.106541082	0.001684882	0.653009465	0.106540914	0.000000333	754.743	5064.794
5	0.562338466	0.115130271	0.562338517	0.115130253	0.000181843	0.562337473	0.115130625	0.003531050	0.562338466	0.115130272	0.000000998	182.139	3536.785
6	0.480464234	0.122167081	0.480464412	0.122167020	0.000618196	0.480462417	0.122167712	0.006304027	0.480464233	0.122167082	0.000002355	262.497	2676.804
7	0.408495115	0.127896475	0.408495117	0.127896474	0.000006128	0.408492232	0.127897483	0.010084814	0.408495114	0.127896475	0.000004714	1.300	2139.541
8	0.346384594	0.132525697	0.346384668	0.132525670	0.000269145	0.346380465	0.132527187	0.014898757	0.346384591	0.132525698	0.000008374	32.139	1779.093
9	0.293443572	0.136229991	0.293443577	0.136229989	0.000017346	0.293438099	0.136232063	0.020725446	0.293443569	0.136229992	0.000013598	1.276	1524.123
10	0.248683202	0.139157547	0.248683230	0.139157536	0.000113423	0.248676361	0.139160298	0.027509605	0.248683197	0.139157550	0.000020586	5.510	1336.323
30	0.015234401	0.139473307	0.015234401	0.139473307	0.000000992	0.015223829	0.139496445	0.231379809	0.015234381	0.139473350	0.000433622	0.002	533.598
AVERAGE (1-30)					0.000068720			0.091259127			0.000142405	14618.630	2714.428

prices and yields from the corresponding numerical solution, standard linearization technique, and the alternative closed form approximation, respectively. Column 6 indicates the absolute difference in yields between the closed form solution and the numerical estimates expressed in basis points. On average from 1 to 30 years to maturity, the numerical approximation is within about 0.000069 bp of the actual solution. Column 9 lists the corresponding differences using the standard linearization technique, which on average is about 0.091 bp across the maturity spectrum. Also, Column 12 lists the absolute difference in yields between the alternative closed form approximation and the actual solution, which averages about 0.00014 bp from 1 to 30 years to maturity. Therefore, in general, the absolute accuracy of both closed form approximations seems favorable using an alternative distributional assumption for J. Moreover, with respect to the relative accuracy of the methods, Column 13 shows that on average the alternative closed form approximation is actually more precise than the numerical solution. Also, as indicted in Column 14 and consistent with Exhibits 1 and 2, the alternative is clearly more accurate than the standard linearization technique on average and for each maturity point.

With respect to an alternative parameter set, consider again the relative contribution of the diffusion and jump volatility parameters to the total variance of the short rate. In the case of the exponential distribution, the unconditional variance follows

$$\frac{\sigma^2 + h\left(\dfrac{2}{\alpha^2}\right)}{2a} \tag{14}$$

and given the set of parameters in Exhibit 3, the diffusion contribution is 12.8 times that of the jumps. Similar to the Gaussian assumption discussed previously, the case in which the jump contribution is twice that of the diffusion component—which again obtains with $\sigma = 0.02$, $h = 16$, and holding γ constant—is perhaps instructive. As Columns 9 and 12 in Exhibit 4 suggest, the approximations are somewhat less accurate when compared with the corresponding estimates in Exhibit 3. Nonetheless, the alternative is never even as much as seventy thousandths of a basis point from the closed form solution, and as Column 13 shows, the estimates are on average closer than the numerical solution, albeit not for those at 6 and 30 years to maturity.

Therefore, even when jumps contribute twice as much as diffusion to variance in the short rate, the alternative closed form approximation produces very accurate estimates under certain reasonable cases, and its comparative precision suggests that use of the numerical solution as an absolute benchmark is potentially questionable. That is, one cannot in principle be sure whether approximate solutions to the exact PDDE produce more precise estimates than exact solutions to an approximate PDDE.[15]

The Bimodal Gaussian Mixture Distributional Assumption for J

So far, the discussion of alternative distributional assumptions has not appealed to intuition about the true form of the density. In fact, the common presumption in the literature explored in Exhibits 1 and 2 that J has an expected value of zero and follows a normal distribution seems problematic. Simply put, if J is both centered around

EXHIBIT 4

Jump Size Distribution Assumption: Jump Size (Exponential), Jump Sign (Bernoulli)

Parameter Assumptions: $a = .1$, $b = .05$, $\sigma = .02$, $\lambda = -.5$, $\lambda_J = 0$, $\alpha = 200$, $h = 16$, $r = .05$

(1) Maturity	(2) Closed-form Solution (Prices)	(3) Closed-form Solution (Yields)	(4) Numerical Solution (Prices)	(5) Numerical Solution (Yields)	(6) |(3)-(5)| (bps)	(7) Closed-form Approx. (Prices)	(8) Closed-form Approx. (Yields)	(9) |(3)-(8)| (bps)	(10) Alternative Closed-form Approx. (Prices)	(11) Alternative Closed-form Approx. (Yields)	(12) |(3)-(11)| (bps)	(13) (6)/(12)	(14) (9)/(12)
1	0.946814828	0.054651741	0.946814818	0.054651751	0.000101212	0.946814826	0.054651742	0.000016958	0.946814828	0.054651741	0.000000001	71159.182	11922.551
2	0.889274056	0.058674909	0.889274057	0.058674908	0.000009841	0.889274014	0.058674932	0.000230754	0.889274056	0.058674909	0.000000014	711.046	16672.487
3	0.829875535	0.062159849	0.829875528	0.062159852	0.000028293	0.829875287	0.062159949	0.000996681	0.829875535	0.062159849	0.000000121	233.181	8214.244
4	0.770486148	0.065183400	0.770486149	0.065183400	0.000001985	0.770485318	0.065183670	0.002695812	0.770486148	0.065183400	0.000000533	3.725	5059.827
5	0.712443352	0.067810975	0.712443389	0.067810965	0.000101969	0.712441340	0.067811540	0.005649680	0.712443352	0.067810975	0.000001598	63.823	3536.185
6	0.656659352	0.070098314	0.656659352	0.070098314	0.000002001	0.656655378	0.070099323	0.010086443	0.656659351	0.070098314	0.000003768	0.531	2676.603
7	0.603716397	0.072092962	0.603716400	0.072092961	0.000009084	0.603709578	0.072094575	0.016135702	0.603716393	0.072092963	0.000007542	1.205	2139.461
8	0.553948203	0.073835512	0.553948261	0.073835498	0.000130368	0.553937640	0.073837895	0.023838011	0.553948198	0.073835513	0.000013399	9.730	1779.063
9	0.507506434	0.075360654	0.507506495	0.075360641	0.000133923	0.507491288	0.075363971	0.033160714	0.507506424	0.075360657	0.000021757	6.155	1524.112
10	0.464413165	0.076698068	0.464413203	0.076698060	0.000082633	0.464392724	0.076702470	0.044015368	0.464413150	0.076698071	0.000032938	2.509	1336.312
30	0.075044381	0.086322520	0.075044378	0.086322521	0.000012782	0.074961082	0.086359540	0.370207694	0.075044225	0.086322589	0.000693796	0.018	533.597
AVERAGE (1-30)					0.000029764			0.146014603			0.000227849	2406.431	2315.326

the origin and Gaussian, the most common jump (i.e., the mode) would not seem to be a jump (i.e., a non-local change) in any meaningful sense of the word. Rather, the assumption that the average jump size is zero only seems tenable if jump size is bimodally distributed, as (large) positive and negative values cancel when taking the average (or forming expectations).

The assumption in Das and Foresi [1996] that the absolute value of jump size follows an exponential distribution and that jump sign follows a Bernoulli distribution is an improvement in this regard, with the component means at $-\alpha^{-1}$ and α^{-1} under the assumption that $w = 0.5$. However, another desirable feature of any possible distribution for jump sizes would seem to be that the modes are of considerable distance from the origin, given the fundamental motivation to incorporate the possibility of sizeable asset price movements. In addition, jumps might also be symmetrically distributed around the means, which is not a feature of exponential distributions.

Therefore, a useful alternative would be bimodal with jumps symmetrically distributed around modes of sufficient distance from the origin.[16] In this regard, consider a Bernoulli mixture of Gaussians, as in

$$f(J) = \frac{w}{\sqrt{2\pi}\gamma_1}\exp\left[-\frac{(J-\alpha_1)^2}{2\gamma_1^2}\right] + \frac{(1-w)}{2\sqrt{2\pi}\gamma_2}\exp\left[-\frac{(J-\alpha_2)^2}{2\gamma_2^2}\right], \quad (15)$$

where $f(J)$ is the total density, w is the weight assigned to the first component Gaussian distribution, α_1 and α_2 are the component means, and γ_1^2 and γ_2^2 are the component variances. Also, the m.g.f. for this distribution, $m_J(\theta)$, follows

$$m_J(\theta) = w\exp\left[\mu_1\theta + \frac{1}{2}\theta^2\gamma_1^2\right] + (1-w)\exp\left[\mu_2\theta + \frac{1}{2}\theta^2\gamma_2^2\right]. \quad (16)$$

Returning to the fundamental PDDE of bond prices for jump-diffusions and remembering that the expectation resembles the m.g.f., where $-B(\tau)$ is the constant, the PDDE in the case of the Gaussian mixture becomes

$$0 = \left[a(b-r) - \lambda\sigma\right]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP$$
$$+hP(r,\tau)\begin{bmatrix} w\exp\left[-\mu_1 B + \frac{1}{2}B^2\gamma_1^2\right] \\ +(1-w)\exp\left[-\mu_2 B + \frac{1}{2}B^2\gamma_2^2\right] - 1 \end{bmatrix}, \quad (17)$$

which can only be solved numerically.[17]

Turning to the approximations of the PDDE, recall that the standard linearization technique requires only the first and second moments of J, and therefore the PDDE follows

$$0 = \left[a(b-r) - \lambda\sigma\right]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP$$
$$+hP(r,\tau)\begin{bmatrix} -\left[w\alpha_1 + (1-w)\alpha_2\right]B \\ +\frac{\left[w(\alpha_1^2+\gamma_1^2) + (1-w)(\alpha_2^2+\gamma_2^2)\right]}{2}B^2 \end{bmatrix}. \quad (18)$$

The solution to this equation again follows (10) but with

$$M_1 = -ab + \lambda\sigma - h[w\alpha_1 + (1-w)\alpha_2],$$

$$M_2 = \frac{1}{2}(\sigma^2 + h[w(\alpha_1^2 + \gamma_1^2) + (1-w)(\alpha_2^2 + \gamma_2^2)]),$$

$M_3 = 0$, and $M_4 = 0$.[18] With respect to the corresponding alternative approximation that considers the two-term Taylor series approximation of the m.g.f. of the distribution, (16), the PDDE follows

$$0 = \left[a(b-r) - \lambda\sigma\right]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP$$
$$+hP(r,\tau)\left\{\begin{array}{l} -\left[w\alpha_1 + (1-w)\alpha_2\right]B \\ +\frac{1}{2}\left[w\left(\gamma_1^2 + \alpha_1^2\right) + (1-w)\left(\gamma_2^2 + \alpha_2^2\right)\right]B^2 \\ -\frac{1}{2}\left[w\alpha_1\gamma_1^2 + (1-w)\alpha_2\gamma_2^2\right]B^3 \\ +\frac{1}{8}\left[w\gamma_1^4 + (1-w)\gamma_2^4\right]B^4 \end{array}\right\}, \tag{19}$$

where the solution again follows (10) but with

$$M_1 = -ab + \lambda\sigma - h[w\alpha_1 + (1-w)\alpha_2],$$

$$M_2 = \frac{1}{2}(\sigma^2 + h[w(\gamma_1^2 + \alpha_1^2) + (1-w)(\gamma_2^2 + \alpha_2^2)]),$$

$$M_3 = -\frac{1}{2}h[w\alpha_1\gamma_1^2 + (1-w)\alpha_2\gamma_2^2], \text{ and}$$

$$M_4 = \frac{1}{8}h[w\gamma_1^4 + (1-w)\gamma_2^4].$$

Similar to Exhibit 1, Exhibit 5 shows prices and yields based on the numerical estimation, the standard linearization technique, and the alternative closed form approximation in Columns 2 and 3, Columns 4 and 5, and Columns 7 and 8, respectively. The chosen parameters follow those in Exhibits 1 and 3 as closely as possible; the component distributions of the Gaussian mixtures have means of 60 and −40 bp, with standard deviations of 15 and 10 bp, respectively, and the probability of a positive jump is 40%. As Column 6 indicates, the average absolute difference across the maturity spectrum between the numerical solution and the standard linearization approximation is about 0.18 bp, and the precision seems to decrease with maturity. Also, Column 9 indicates that the alternative closed form approximation is on average within about 0.15 bp of the numerical solution, and Column 10 indicates that the alternative is consistently more precise than the standard method but, at least given this set of parameters, does not provide much greater accuracy.[19]

With respect to alternative parameterizations, the unconditional variance of the short-rate process under the Gaussian mixture assumption follows

$$\frac{\sigma^2 + h\left[w\left(\alpha_1^2 + \gamma_1^2\right) + (1-w)\left(\alpha_2^2 + \gamma_2^2\right)\right]}{2a} \tag{20}$$

EXHIBIT 5

Jump Size Distribution Assumption: Gaussain Mixture

Parameter Assumptions: $a = .1$, $b = .05$, $\sigma = .08$, $\lambda = -.5$, $\lambda_J = 0$, $\alpha_1 = .006$, $\gamma_1 = .0015$, $\alpha_2 = -.004$, $\gamma_2 = .001$, $w = .4$, $h = 10$, $r = .05$

(1) Maturity	(2) Numerical Solution (Prices)	(3) Numerical Solution (Yields)	(4) Closed-form Approx. (Prices)	(5) Closed-form Approx. (Yields)	(6) |(3)-(5)| (bps)	(7) Alternative Closed-form Approx. (Prices)	(8) Alternative Closed-form Approx. (Yields)	(9) |(3)-(8)| (bps)	(10) (6)/(9)
1	0.933961543	0.068320016	0.933961609	0.068319946	0.000707134	0.933961606	0.068319949	0.000673842	1.049
2	0.845975278	0.083632571	0.845975544	0.083632414	0.001574248	0.845975504	0.083632437	0.001337272	1.177
3	0.748851687	0.096404777	0.748852695	0.096404328	0.004487965	0.748852535	0.096404399	0.003774745	1.189
4	0.651729871	0.107031278	0.651732328	0.107030335	0.009423686	0.651731934	0.107030487	0.007912785	1.191
5	0.560330319	0.115845763	0.560334967	0.115844104	0.016591326	0.560334227	0.115844368	0.013948318	1.189
6	0.477695296	0.123130368	0.477702571	0.123127829	0.025381651	0.477701396	0.123128239	0.021282488	1.193
7	0.405000382	0.129123896	0.405010730	0.129120245	0.036502031	0.405009071	0.129120831	0.030647478	1.191
8	0.342245780	0.134028268	0.342259201	0.134023367	0.049016585	0.342257044	0.134024154	0.041140383	1.191
9	0.288768415	0.138014472	0.288784779	0.138008175	0.062962581	0.288782147	0.138009188	0.052835263	1.192
10	0.243589169	0.141227221	0.243608181	0.141219416	0.078044571	0.243605119	0.141220673	0.065474340	1.192
30	0.012501667	0.146063110	0.012516727	0.146022979	0.401313488	0.012514268	0.146029528	0.335817191	1.195
AVERAGE (1-30)					0.179641423			0.150461970	1.188

EXHIBIT 6

Jump Size Distribution Assumption: Gaussain Mixture

Parameter Assumptions: $a = .1$, $b = .05$, $\sigma = .02$, $\lambda = -.5$, $\lambda_J = 0$, $\alpha_1 = .006$, $\gamma_1 = .0015$, $\alpha_2 = -.004$, $\gamma_2 = .001$, $w = .4$, $h = 31$, $r = .05$

(1) Maturity	(2) Numerical Solution (Prices)	(3) Numerical Solution (Yields)	(4) Closed-form Approx. (Prices)	(5) Closed-form Approx. (Yields)	(6) |(3)-(5)| (bps)	(7) Alternative Closed-form Approx. (Prices)	(8) Alternative Closed-form Approx. (Yields)	(9) |(3)-(8)| (bps)	(10) (6)/(9)
1	0.946813334	0.054653318	0.946813434	0.054653212	0.001062380	0.946813425	0.054653223	0.000959176	1.108
2	0.889263445	0.058680874	0.889264293	0.058680398	0.004767323	0.889264162	0.058680471	0.004032697	1.182
3	0.829843310	0.062172793	0.829846785	0.062171397	0.013955096	0.829846234	0.062171618	0.011744114	1.188
4	0.770417818	0.065205572	0.770426852	0.065202641	0.029314439	0.770425409	0.065203109	0.024630644	1.190
5	0.712324541	0.067844331	0.712342796	0.067839206	0.051254263	0.712339878	0.067840025	0.043060938	1.190
6	0.656477452	0.070144489	0.656508715	0.070136552	0.079369294	0.656503709	0.070137823	0.066661887	1.191
7	0.603461496	0.072153292	0.603509244	0.072141989	0.113028317	0.603501577	0.072143804	0.094879204	1.191
8	0.553613333	0.073911099	0.553680605	0.073895910	0.151883803	0.553669790	0.073898352	0.127467579	1.192
9	0.507087788	0.075452349	0.507176829	0.075432840	0.195084647	0.507162499	0.075435980	0.163689961	1.192
10	0.463909922	0.076806488	0.464022085	0.076782313	0.241748614	0.464004004	0.076786210	0.202780896	1.192
30	0.074116961	0.086737029	0.074394125	0.086612610	1.244190704	0.074348824	0.086632914	1.041152182	1.195
AVERAGE (1-30)					0.556422124			0.465965820	1.190

and the relative contribution of diffusion is just over 25 times that of jumps given the parameter values in Exhibit 5. Exhibit 6 considers an alternative parameterization in which jumps contribute about twice as much as the diffusion component to total variance, with $\sigma = 0.02$, $h = 0.31$, and the remaining parameters as in Exhibit 5.[20] Similar to Exhibits 2 and 4, the results generally suggest that both closed form approximations become somewhat less accurate the more jumps contribute to total variance in the short rate. For example, Column 9 indicates that the alternative closed form approximation is on average within about 0.466 bp of the numerical solution across the maturity spectrum and just over a basis point at 30 years. However, the absolute degree of precision is roughly comparable.

A few simplifying assumptions regarding the Gaussian mixture seem intuitive. That is, perhaps the component distributions of J are simply symmetric reflections across the origin, with equal but opposite means and equal variances. Under this formulation, $\alpha_2 = -\alpha_1$, $\gamma_2 = \gamma_1$, and $w = 0.5$, and the solution again follows (10) but with

$$M_1 = -ab + \lambda\sigma, M_2 = \frac{1}{2}(\sigma^2 + h[\gamma_1^2 + \alpha_1^2]), M_3 = 0,$$

and

$$M_4 = \frac{1}{8}h\gamma_1^4.$$

Interestingly, this bimodal Gaussian mixture implies an approximate PDDE that is no more complex than that under the assumption that J is normally distributed. Also, this "restricted" Gaussian mixture requires three fewer parameters to estimate than the full model and the same number as the model in which J is Gaussian and unimodal.

How Critical Is the Distributional Assumption for Jumps?

How sensitive are estimates for bond yields to the distributional assumption for jumps? One imperfect way to address this issue is to assume a constant mean and variance for jump size but alter the specific form of the distribution and compare the corresponding closed form approximations for yields derived from the two-term Taylor series approximations of the m.g.f. in the bond pricing equation. Toward that end, Column 2 in Exhibit 7 shows the term structure of interest rates given the parameter assumptions in Exhibit 1, which again stipulate that jumps are normally distributed with a zero mean and a standard deviation of 100 bp. Alternatively, Column 3 shows the results with the same mean and variance as in Column 2, but jumps follow the exponential distribution.[21] Similarly, Column 4 shows the term structure assuming jumps follow a bimodal mixture of Gaussians,[22] and Column 5 follows its "restricted" form where $\alpha_2 = -\alpha_1$, $\gamma_2 = \gamma_1$, and $w = 0.5$.[23]

EXHIBIT 7
Alternative Distributional Assumptions for Jumps

Parameter Assumptions: $a = .1$, $b = .05$, $\sigma = .08$, $\lambda = -.5$, $\lambda_J = 0$, $\alpha = 0$, $\gamma = .01$, $h = 10$, $r = .05$

(1) Maturity	(2) Gaussian (Yields)	(3) Exponential (Yields)	(4) Gaussian Mixture (Yields)	(5) Restricted Gaussian Mixture (Yields)	(6) |(2)-(3)| (bps)	(7) |(2)-(4)| (bps)	(8) |(2)-(5)| (bps)
1	0.068204670	0.068204668	0.068204694	0.068204671	0.000021196	0.000242376	0.000009273
2	0.083203734	0.083203705	0.083203912	0.083203746	0.000288425	0.001783383	0.000126186
3	0.095506401	0.095506276	0.095506954	0.095506455	0.001245699	0.005526912	0.000544993
4	0.105542270	0.105541933	0.105543472	0.105542417	0.003369098	0.012016760	0.001473981
5	0.113673839	0.113673133	0.113675990	0.113674147	0.007060103	0.021513576	0.003088795
6	0.120207028	0.120205768	0.120210434	0.120207579	0.012603343	0.034064835	0.005513963
7	0.125400034	0.125398018	0.125404991	0.125400916	0.020160200	0.049564714	0.008820088
8	0.129470782	0.129467804	0.129477562	0.129472085	0.029780764	0.067803170	0.013029084
9	0.132603202	0.132599060	0.132612053	0.132605014	0.041423695	0.088504781	0.018122867
10	0.134952520	0.134947022	0.134963655	0.134954925	0.054978037	0.111358544	0.024052891
30	0.126130799	0.126084610	0.126196232	0.126151007	0.461892372	0.654328235	0.202077913
AVERAGE (1-30)					0.182233444	0.281566146	0.079727132

Given this particular set of parameters and holding constant the mean and variance for jumps, the implied term structure displays marginal differences across distributional assumptions. For example, the average difference between the unimodal Gaussian and the exponential, Gaussian mixture, and "restricted" Gaussian mixture is only about 0.18, 0.28, and 0.08 bp, respectively, as shown in Columns 6, 7, and 8. Exhibit 8 considers an alternative set of parameters where the intensity of jumps increases fivefold, with $h = 50$, but the standard deviation of jumps is 50 bp, half the size assumed in Exhibit 7.[24] With average absolute differences of about 0.06, 0.18, and 0.06 bp across the term structure for the three distributional assumptions, Columns 6, 7, and 8 again suggest that alternative distributional assumptions do not seem to have a pronounced effect.

One should not necessarily make strong inferences about the relative importance of distributional assumptions for J from this flawed exercise. In particular, while one must hold the first and second moments constant across distributional assumptions to facilitate comparisons, variance has less meaning without the full context of the distribution. For example, the assumption that the variance of jumps is 100 bp, the stipulation in Exhibit 1, may seem sensible in the context of a unimodal normal distribution centered around the origin, but, say, a Bernoulli mixture of Gaussians with the same (overall) variance might not have much intuitive appeal. Moreover, there is no substantive reason to knowingly use the "wrong" distribution.

CONCLUSIONS

Incorporating the possibility of non-local changes in the short rate (or other latent factors) considerably complicates the derivation of solutions for bond prices. The preceding analysis expands on existing linearization techniques in two ways. First, this article proposes an alternative that seems to provide considerable gains in accuracy. Second, the analysis considers a richer set of jump distributions. While previous literature exclusively assumes that jumps are normally distributed, this article considers an exponential/Bernoulli density and a bimodal Gaussian mixture. The former assumption is particularly useful in that the PDDE has a closed form solution that enables comparison of the relative accuracy between numerical estimates and the approximate closed form solutions. Although the choice of parameters is ultimately an arbitrary exercise, the results suggest that the alternative closed form approximation, rather than the standard linearization

EXHIBIT 8
Alternative Distributional Assumptions for Jumps

Parameter Assumptions: $a = .1$, $b = .05$, $\sigma = .08$, $\lambda = -.5$, $\lambda_J = 0$, $\alpha = 0$, $\gamma = .005$, $h = 50$, $r = .05$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Maturity	Gaussian (Yields)	Exponential (Yields)	Gaussian Mixture (Yields)	Restricted Gaussian Mixture (Yields)	|(2)-(3)| (bps)	|(2)-(4)| (bps)	|(2)-(5)| (bps)
1	0.068165989	0.068165988	0.068166003	0.068165989	0.000006624	0.000142401	0.000006624
2	0.083059911	0.083059902	0.083060016	0.083059920	0.000090133	0.001056683	0.000090133
3	0.095205211	0.095205172	0.095205540	0.095205250	0.000389281	0.003298451	0.000389281
4	0.105043264	0.105043158	0.105043985	0.105043369	0.001052843	0.007216019	0.001052843
5	0.112946284	0.112946063	0.112947583	0.112946505	0.002206282	0.012987996	0.002206282
6	0.119228183	0.119227789	0.119230249	0.119228576	0.003938545	0.020661140	0.003938545
7	0.124153704	0.124153074	0.124156722	0.124154334	0.006300063	0.030184757	0.006300063
8	0.127946116	0.127945185	0.127950260	0.127947046	0.009306489	0.041440047	0.009306489
9	0.130793691	0.130792396	0.130799117	0.130794985	0.012944905	0.054264031	0.012944905
10	0.132855159	0.132853441	0.132862006	0.132856877	0.017180637	0.068468357	0.017180637
30	0.119502826	0.119488392	0.119543991	0.119517260	0.144341366	0.411650592	0.144341366
AVERAGE (1-30)					0.056947951	0.176027195	0.056947951

technique, produces more accurate estimates of analytical solutions than numerical methods under some specifications. This finding is insensitive to whether the Gaussian or Poisson disturbances contribute more to the overall variance of the short rate. With respect to other alternative assumptions, the Gaussian mixture follows the intuition that the jump size is bimodally distributed and that the modes are sufficiently distant from zero.

Beyond alternative distributional assumptions, additional relaxations of the model seem instructive. For example, h may be time varying. Anecdotal evidence suggests that most jumps do not occur at truly random times because monetary policy announcements and economic news releases are largely scheduled, and the jump intensity would seem to be very different during the (instantaneous) interval over which markets process any new information on monetary policy or the economy, compared to "quiet times" during which no announcement or release is on the calendar.[25] In addition, given the presumed mean reversion in the short-rate process, jump intensity might be a (deterministic or otherwise) function of r and b, and h might increase as the distance between b and r increases.[26] Moreover, w might be a function of b and r, as positive (negative) jumps might be more likely if the short rate is substantially less (greater) than its central tendency. Of course, these considerations would richen jump-diffusion models but would additionally complicate practical application.

ACKNOWLEDGMENTS

With no implication whatsoever, the author thanks Sanjiv Das, Henrik Rasmussen, and seminar participants at the Federal Reserve Board for helpful comments and suggestions on this project. The views presented are solely those of the author and do not necessarily represent those of the Federal Reserve Board or its staff.

ENDNOTES

[1]Given that the equation includes an expectation, previous studies also refer to the PDDE as a partial integrodifferential equation (PIDE).

[2]See Baz and Das [1996], Das and Foresi [1996], or Das [2002] for a more complete derivation of the PDDE.

[3]A couple comments are noteworthy before examining linearization approximations. First, one can easily relax the assumption that jump risk can be diversified away. Following Baz and Das [1996] and Das and Foresi [1996], one can readily account for systematic jump risk in the approximations out-

lined below by substituting $h(1 - \lambda_j)$ for h. Sensitivity analyses along these lines similar to Exhibits 1–6 produce trivial differences in (approximate) bond yields. Second, although the existence of closed form solutions depends critically on the distributional assumption for J, one can easily solve for the Fourier transform of the bond price using the transform of the PDDE, (2), and the derivation of the solution does not depend on the presumed affine form for bond prices.

[4]Note that in order for the condition $\lim_{\tau\to\infty} P(r,\tau) = 0$ to be met, $M_1 a + M_2 < 0$.

[5]Of course, one need not limit the approximations to two terms in practice.

[6]In order for the condition that bond prices approach zero as the time to maturity goes to infinity, i.e., $\lim_{\tau\to\infty} P(r,\tau) = 0$, the parameters must satisfy $M_1a^3 + M_2a^2 + M_3a + M_4 < 0$.

[7]Then again, one is likely to use the m.g.f., if available, to derive higher moments.

[8]The parameter values follow those in Exhibit 5 from Baz and Das [1996]. The numerical approximations use the NDSolve command in Mathematica, which follow a modified Runge-Kutta procedure.

[9]More generally, the unconditional variance of the short rate process is $(\sigma^2 + hE_t[J^2])/2a$.

[10]Johannes [2004] finds that jumps account for more than half (almost two-thirds) of the condition variance of interest rate changes at low (high) levels.

[11]The condition $M_1a^3 + M_2a^2 + M_3a + M_4 < 0$ still holds for this parameterization.

[12]Johannes [2004] considers the case in which jumps are log-normal.

[13]The m.g.f., $m_J(\theta)$, of an exponential distribution follows $m_J(\theta) = \alpha/\alpha - \theta$, and the nth moment of the distribution follows

$$\left.\frac{\partial^{(n)}}{\partial t} m(t)\right|_{t=0} = \frac{n!}{\alpha^n}.$$

Therefore, following the distribution in Das and Foresi [1996], the expected value of jumps follows

$$w\frac{1}{\alpha} + (1-w)\left(-\frac{1}{\alpha}\right).$$

[14]This approximation broadly follows (11).

[15]More generally, higher-order Taylor expansions would result in approximate PDDEs that would of course be more accurate than the two-term approximations used in the alternative linearization technique.

[16]Insofar as the short rate is a policy rate such as the federal funds rate in the United States, which Das [2002] uses as a proxy, one could argue that jump sizes follow a discrete multinomial distribution. However, a pure jump process, as opposed to a jump-diffusion, might better characterize the target funds rate process. See Piazzesi [2005] for an analysis of jumps and the target federal funds rate.

[17]One could entertain the possibility that the weight between the two distributions is functionally dependent on the distance between r and b. That is, negative (positive) jumps might be more likely if r is sufficiently above (below) the mean.

[18]Notably, one does not obtain the same result if one takes the one-term Taylor series approximation of the m.g.f. of the Bernoulli mixture of Gaussians. In this case, $M_2 = \frac{1}{2}(\sigma^2 + h[w\gamma_1^2 + (1-w)\gamma_2^2$.

[19]Again, given no closed form solution, one ultimately cannot judge the accuracy of the numerical approximation.

[20]The precise ratio of the diffusion to jump contributions to total variance given the parameterization in Exhibit 5 (Exhibit 6) is 25.098 (0.506009).

[21]In order for jumps to have a standard deviation of 100 bp for the exponential distribution, we must have $\alpha = 141.421$.

[22]For the Bernoulli mixture of Gaussians, the parameters follow those in Exhibit 5, but in order for the distribution to match the variance of the unimodal Gaussian assumption, we must have $\gamma_1 = 0.00901436$ and $\gamma_2 = 0.00851436$. (The difference in standard deviations of the two component distributions is fixed at 5 bp.) Notably, given means of 60 and -40 bp, standard deviations of about 90.1 and 85.1 bp, respectively, seem excessive in that the component distributions for both positive and negative jumps have considerable mass on the opposite side of the origin.

[23]In order for jumps to have a standard deviation of 100 bp for the "restricted" bimodal Bernoulli mixture of Gaussians, we must have $\gamma_1 = 0.00866025$.

[24]Lowering the stipulated value of the variance of J to 50 bp produces considerably more intuitive assumptions for the Bernoulli mixture of Gaussians, as $\gamma_1 = 0.00126954$ and $\gamma_2 = 0.000769536$. Therefore, there is negligible mass on the opposite side of the origin for the component distributions for positive and negative jumps. (However, to meet the restriction in the case of the "restricted" Bernoulli mixture of Gaussians, we must have $\gamma_1 = 0$.) Also, for the exponential distributions, $\alpha = 282.843$.

[25]For example, Piazzesi [2005] models jump intensity differently on days when the Federal Open Market Committee (FOMC) meets. Also, the results in Johannes [2004] suggest that jumps occur only when announcements contain significant unexpected components.

[26]Irrespective of the longer-run central tendency, b, Johannes [2004] finds that jumps are more likely at higher interest rate levels, and he models intensity as a function of the level of the short rate.

REFERENCES

Ahn, C.M., and Thompson, H. E. "Jump-Diffusion Processes and the Term Structure of Interest Rates." *Journal of Finance*, 43 (1988), pp. 155-174.

Baz, J., and Das, S. R. "Analytical Approximations of the Term Structure for Jump-Diffusion Processes: A Numerical Analysis." *Journal of Fixed Income*, 6 (1996), pp. 78-86.

Das, S.R. "The Surprise Element: Jumps in Interest Rates." *Journal of Econometrics*, 106 (2002), pp. 27-65.

Das, S.R., and Foresi, S. "Exact Solutions for Bond and Options Prices with Systematic Jump Risk." *Review of Derivatives Research*, 1 (1996), pp. 7-24.

Johannes, M. "The Statistical and Economic Role of Jumps in Continuous-Time Interest Rate Models." *Journal of Finance*, 59 (2004), pp. 227-260.

Piazzesi, M. "Bond Yields and the Federal Reserve." *Journal of Political Economy*, 113 (2005), pp. 311-344.