Estimating and Hedging Most Probable Extreme Changes in Multicurrency Term Structures

MATHIEU DIEUDONNÉ AND JEAN-CHRISTOPHE CURTILLET

MATHIEU DIEUDONNÉ
is an associate director of Fixed Income Analysis at WestLB in Düsseldorf, Germany.
mathieu.dieudonne@westlb.de

JEAN-CHRISTOPHE CURTILLET
is executive director and head of Fixed Income Analysis at WestLB in Düsseldorf.
jeanc.curtillet@westlb.de

Extreme changes in term structures can severely impact fixed-income portfolios. This is reason enough to try to estimate the most probable extreme changes in term structures. For a risk manager, modeling the most probable extreme changes in term structures can be seen as a prerequisite to the development of most interest rate risk models. Ever since the amendment of the Capital Accord by the Basle Committee in 1996, which recommends that market risks be covered by capital resources, assessing interest rate risk has gained increasing importance.[1] According to the Amendment to the Capital Accord to Incorporate Market Risks, banks may resort to internal models to estimate market risk and, in particular, interest rate risk. For a fixed-income investor, assessing the most probable changes in term structures is not only useful for measuring interest rate risk but also indispensable for hedging it.

This study presents an innovative approach to tracking the most probable extreme changes in term structures and thus *minimizing* the risk exposure of multicurrency fixed portfolios. More precisely, we define a measure of magnitude of term structure changes. We then identify the most probable changes for any given level of magnitude. History provides evidence of the most probable changes in term structures being linearly related for each possible level of magnitude. As, furthermore, market agitation has a limited impact on this linear relationship, the most probable changes in multicurrency term structures can be deduced from the most probable change in a single reference rate (e.g., the USD 10Y swap rate). Consequently, the interest rate risk of a fixed-income portfolio can be regarded as solely driven by this reference rate.

After having estimated the most probable extreme changes in term structures, we therefore introduce a one-factor interest rate risk model for single as well as for multicurrency fixed-income portfolios. The approach suggested in this study diverges significantly from the various existing VaR models. The key advantage of our approach is that it enables us to convert all portfolio risk positions into equivalent positions in the reference instrument (e.g., the USD 10Y swap). This method not only provides us with a value for the interest rate risk but also suggests a straightforward hedging strategy, since the risk of the portfolio, as defined by our model, can be hedged by taking a single position in the reference bucket with an opposite sign. Nevertheless, our model does not extend to the entire market risk of the portfolio we examine. It concentrates exclusively on interest rate risk and does not account for currency or counterparty risk. This is a crucial point that clearly distinguishes our model from the VaR approach.

In Section I of this study, we introduce a statistical method to estimate the most probable extreme changes in term structures based on principal components analysis (PCA). Then

EXHIBIT 1
Example: Deciles of N_{2bp}

i	d_i
0	0
1	22
2	57
3	91
4	127
5	160
6	206
7	253
8	325
9	369
10	393

data used for the estimations are described in the following section. The results obtained for the EUR term structure and for the main currency term structures observed as a whole are presented in the first part of the "Results" section. In the second part of this section, we investigate the impact of market agitation and time on the most probable extreme changes estimated using our new approach. The final sections are devoted to possible applications of the results. It is explained how, based on realistic assumptions, the estimate obtained for the most probable extreme changes of term structures can be used to compute the interest rate risk of multicurrency fixed-income portfolios. Finally, a hedge strategy of multicurrency fixed-income portfolios against the most probable extreme changes in term structure is described.

MODEL AND EXAMPLE

Modeling the most probable extreme changes in a given term structure is quite an ambitious task. It requires the definition of a term structure model, as well as introducing a measure of the magnitude of the changes in the term structure together with a measure of probability of the changes.

In this study, a term structure is simply regarded as an n-dimensional random vector $R = (r_1,\ldots,r_n)$, where r_i stands for the random interest rate associated with segment i of the term structure we consider. For t in $\{1,\ldots,T\}$, the observed value of R at time t is denoted by $R_t = (r_{1,t},\ldots,r_{n,t})$, where $r_{i,t}$ is the observed value of r_i at time t. Let ΔR be the random vector of daily changes in the term structure. Expressing t in days, the observed value of ΔR at time t is simply $\Delta R_t = R_t - R_{t-1}$. Similarly, we introduce $\Delta r_{i,t} = r_{i,t} - r_{i,t-1}$ the daily change in r_i observed at time t.

In order to estimate the joint distribution of probability of ΔR, we introduce the concept of ρ-neighbor. A change ΔR_j is a ρ-neighbor of ΔR_i if $i \neq j$ and ΔR_j is located within $B(\Delta R_i,\rho)$, where $B(\Delta R_i,\rho)$, is the Euclidian ball[2] centered in ΔR_i with radius ρ. For i in $\{1,\ldots,T\}$, we also introduce $N_\rho(\Delta R_i) = \text{card}\ \{i \neq j / \Delta R_j \in B(\Delta R_i,\rho)\}$, the number of ρ-neighbors of observation ΔR_i.

In other words, $N_\rho(\Delta R_i)$ can be understood as an estimator of the density of ΔR in ΔR_i: the more neighbors that observation ΔR_i has, the higher the density of ΔR in ΔR_i. This mathematical toolkit allows us to approximate the probability distribution of the changes in the term structure, so that the concept of most probable changes makes sense. Finally, we introduce d_i, the i-th decile of N_ρ.

To make these mathematical notations clear, we consider an example of a simplified term structure consisting of only EUR 2Y and 10Y swap rates. Our data set comprises daily data from January 1999 to December 2004. Accordingly, in our example $n = 2$, $T = 1{,}530$, and $\Delta R = (\Delta r_{2Y}, \Delta r_{10Y})$.

We set $\rho = 2$ bp, so that 2 observations with a Euclidian distance of less than 2 bp are regarded as neighbors. Exhibit 1 tabulates the deciles of N_{2bp}, and Exhibit 2 plots the daily changes in EUR 2Y versus EUR 10Y swap rates. In Exhibit 3, we apply the mathematical notations and concepts introduced here to estimate the density of ΔR. More precisely, we plot the curves corresponding to constant density levels (or isodensity curves), that is, curves corresponding to constant numbers of neighbors in a neighborhood of 2 bp for the deciles of N_{2bp}. The isodensity curves clearly indicate an ellipsoidal pattern, which is all the more interesting since, in such a case, covariances or correlations are valid measures for dependencies between variables.[3] Another noticeable pattern of the isodensity curves is that the density of the changes decreases from the center to the periphery of the chart.

We still have to define a measure of magnitude of the term structure changes. Changes of large magnitude will then be regarded as extreme ones. As a measure of magnitude, we simply consider the Euclidian distance in IR^n. Consequently, the sets of changes of the same magnitude are just isomagnitude spheres. In our two-dimensional example, these spheres correspond to circles (see the dotted circle in Exhibit 3). For instance, a 10 bp parallel shift in the term structure or a +10 bp change in the 2Y segment, together with a –10 bp change in the 10Y segment, will be regarded as changes of same magnitude.

Still focusing on our example, if we consider as extreme changes those changes with a magnitude larger than 12.5 bp (i.e., located on or outside the isomagnitude circle with a radius of 12.5 bp), the most probable change is the point on the isomagnitude circle with the highest density, as shown in Exhibit 3, which corresponds

EXHIBIT 2

Example: Daily Changes in the EUR 2Y Versus 10Y Swap Rate

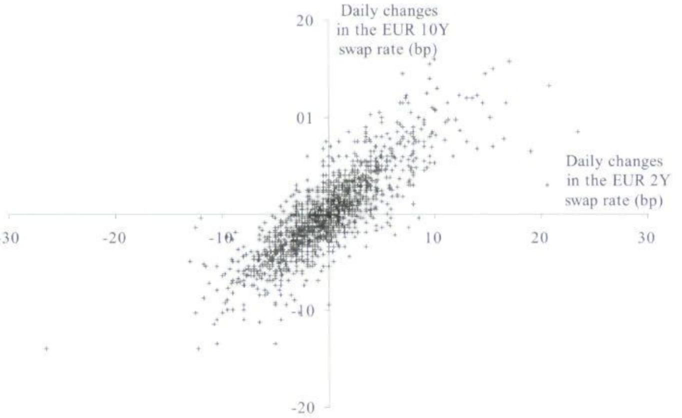


EXHIBIT 3

Example: Empirical Isodensity Curves and Extreme Changes in the EUR Term Structure

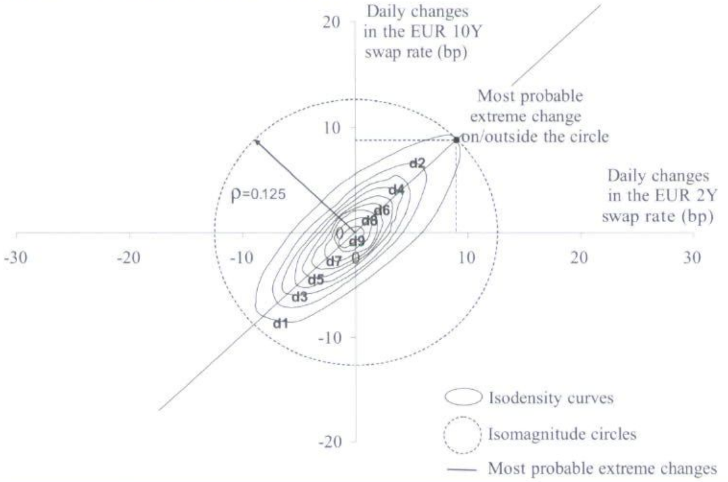
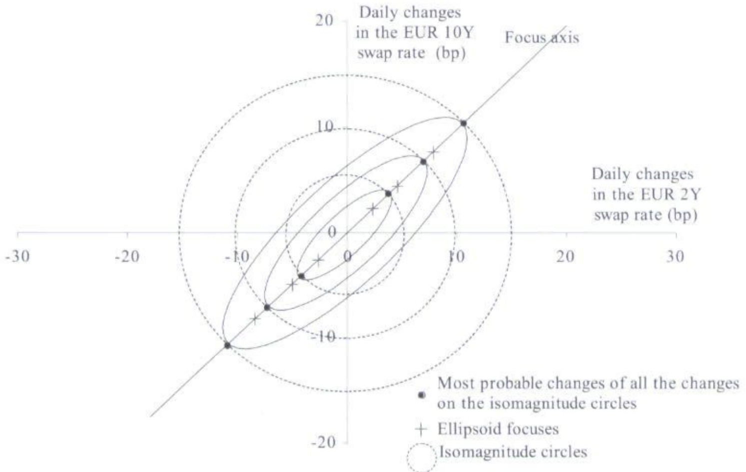


EXHIBIT 4

Example: Isodensity Curves and Extreme Changes in the EUR Term Structure Assuming Symmetrical and Coaxial Ellipsoids



to a parallel shift in the term structure of approximately +9 bp.

Exhibit 3 further provides empirical evidence for all the most probable extreme changes being located along a given line, which shows that the most probable changes corresponding to different levels of magnitude are linearly related. Theoretically speaking, this linear relationship is established if and only if the ellipsoids are centered on 0 with focuses on a unique axis. Exhibit 4 depicts the most probable extreme changes in this theoretical framework.

Based on these assumptions (centered and coaxial ellipsoids), the most probable changes in the term structure among a set of isomagnitude changes are calculated by simply computing the intersections between the axis on which the ellipsoid focuses are located (focus axis) and the isomagnitude spheres (see the points in Exhibit 4). This requires estimating the equation of the focus axis. We can estimate any vector V generating this axis or, for ease of comparison, the unit vector $V*$ that generates this axis and whose first element is positive.[4]

As stated earlier, in the case of elliptical isodensity curves, the use of concepts such as covariances or correlations is theoretically founded. A straightforward approach to estimating the equation of the focus axis (i.e., to estimating $V*$) is therefore to apply a regression model. In a two-dimensional environment, (i.e., if $n = 2$, as in the example), there is no major objection to such a method. In a higher-dimensional framework, however, this may become questionable. The only possible approach using regressions is to run pairwise regressions after defining a reference rate in vector ΔR (e.g., the change in the USD 10Y swap rate). Consider an example where $n = 3$ and $\Delta R = (\Delta r_{2Y}, \Delta r_{10Y}, \Delta r_{30Y})$, with the 10Y swap rate set as reference rate. Two regressions must be run in this case: 1) a regression of the changes in the 2Y swap rate on the changes in the 10Y swap rate and 2) a regression of the changes in the 30Y swap rate on the changes in the 10Y swap rate. If the sensitivity coefficients identified in the two regressions are β_1 and β_2, the focus axis we are interested in is the one-dimensional vector space generated by vector $V = (\beta_1, 1, \beta_2)$ and we have

$$V^* = \frac{\operatorname{sgn}(\beta_1)}{\sqrt{\beta_1^2 + 1 + \beta_2^2}}(\beta_1, 1, \beta_2)$$

Such an approach induces a bias,[5] since the two regressions cannot be run together. As a result, the error terms cannot be simultaneously minimized. Given that this approach is widely used on trading floors, however, there follows a comparison between the regression approach and our model in this article.

A more appropriate method is a PCA, which can identify the axis accounting for the largest part of the variance in an observation set. Assuming coaxial ellipsoids, the focuses of the ellipsoids will be located on this axis, also called the first principal component. This principal component is simply any vector of the eigenspace associated with the largest eigenvalue of the covariance matrix between the time series $\Delta r_1,\ldots,\Delta r_n$, when the covariance matrix is diagonalized in an orthonormal basis. Consequently, our approach will be unbiased if and only if the estimated covariances are free of any bias. As mentioned in our first example, the ellipsoidal pattern of the isodensity curves makes the use of covariances and correlations pertinent. As a result, V^* is the first principal component of the covariance matrix between $\Delta r_1,\ldots,\Delta r_n$ that is normalized to 1 and whose first element is positive.

Furthermore, PCA can provide a statistical measure of the goodness of the estimated axis. The quotient between the largest eigenvalue of the covariance matrix and the sum of its eigenvalues indicates which part of the variance in the data is explained by the first component. In what following, this quotient will be referred to as the Q-coefficient. Moreover, and contrary to regression analyses, PCA allows an interpretation of the residuals based on the remaining components. This is why we will base our estimates of the focus axis on PCA.

In the "Results" section we compare the results obtained for the EUR term structure using both regression and PCA estimators of the focus axis. In addition, we question the stability of our PCA estimator when markets become agitated. This, however, requires market agitation to be defined, which is done in the second part of the "Results" section. Since covariances can vary over time, we also question the stability of our PCA estimator over time by running several PCAs for a range of time periods. This is demonstrated in the last part of the "Results" section.

DATA

The traditional approach to interest rate risk on fixed-income trading floors is to split the various positions, such as delta, gamma, and vega positions, into so-called curve buckets. For each currency, the first buckets consist of the money market rates, followed by the 3-month money market future contracts and then by the swap spot interest rates. Ignoring the spot money market, the USD interest rate market can be split into 20 future buckets (4 contracts a year with delivery in March, June, September, and December over 5 years) and 8 spot swap contracts (6Y, 7Y, 10Y, 12Y, 15Y, 20Y, 25Y, 30Y). For the EUR interest rate market, it is more usual to consider only the first 12 future contracts for liquidity purposes and more spot swap contracts. In this study, we use the same approach. Nevertheless, to eliminate the roll and jump effects on the money market 3M strip curves, we convert them using linear interpolations to 3M forward rate agreement (FRA) curves.

In terms of range of currency, we do not limit this study to the EUR or USD fixed-income markets. GBP, CHF, DKK, SEK, NOK, CAD, JPY, AUD, and NZD term structures are also part of this study. Adding the 3-month future and swap rates of these currencies to the EUR and USD buckets identified earlier we get 204 buckets in 11 currencies.

Our study is based on WestLB data starting between 1999 and 2002, depending on the currency, and ending in December 2004. Since we are interested in daily changes, the quality of the data is of crucial importance. We not only checked our data for correctness but also constructed a sample of data with rates saved at the same time for all curves whenever possible. For the European curves (EUR, SEK, NOK, DKK, CHF, GBP) and the American curves (CAD, USD), rates were saved at 4:15 pm GMT. It was not technically possible to use a similar process for Asian curves (AUD, NZD, JPY) since markets are closed at that time. As a result, R_t contains the European and American curves saved at 4:15 pm GMT at date t, as well as the Asian curves saved at 7:30 am GMT at date $t + 1$.

The reason for introducing this time lag is based on the empirical assumption that most fixed-income market relevant data are published between 9:00 am and 4:00 pm GMT, so that they cannot be reflected in the Asian market on the same day. Nevertheless, data released between 4:15 pm and 7:30 am GMT on the next day will introduce a bias in our observations, since they will be reflected in the Asian curves whereas the European and American curves will not account for this.

Exhibit 5

Single-Currency Framework: Impact of the Estimator on V^* (1999-2004)

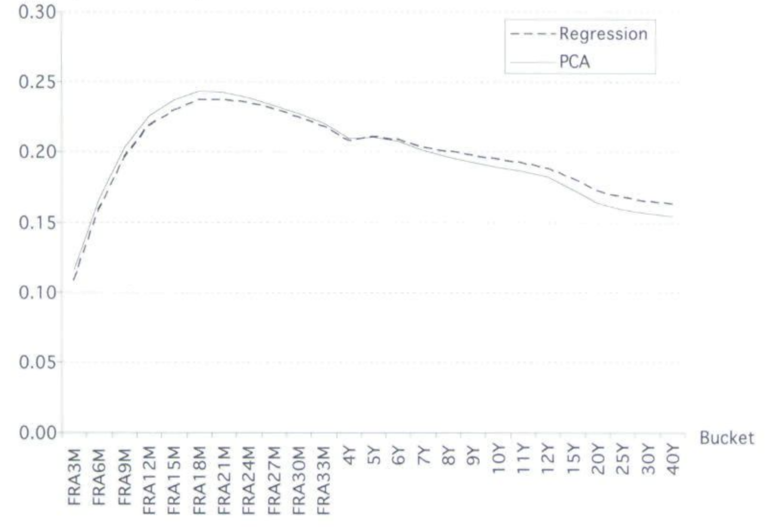
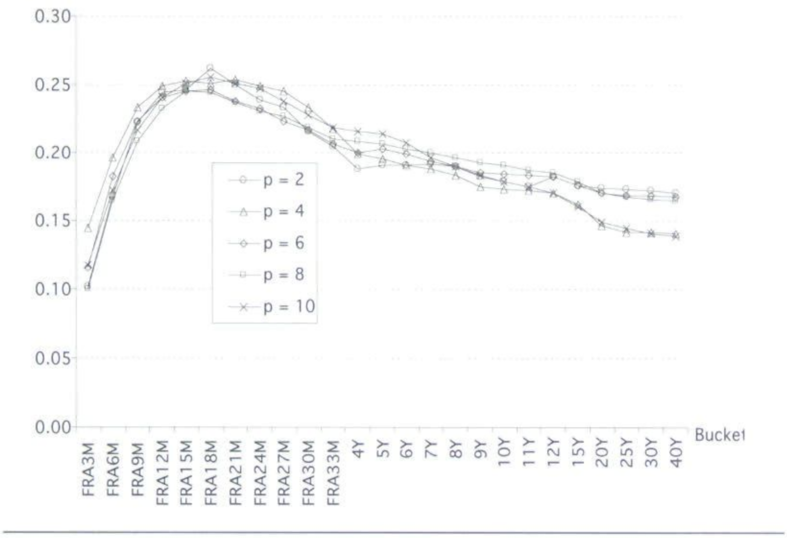


Exhibit 6

Single-Currency Framework: Impact of Term Structure Agitation p on V^* (1999-2004)



RESULTS

Before moving on to the simultaneous changes of several term structures, we begin with the more restrictive single-currency framework, focussing on the EUR term structure.

To gather empirical evidence for the bias of the regression estimator of V^*, we compare estimates obtained using the regression methods to the estimates based on PCA. Exhibit 5 depicts both estimates for the EUR term structure. Compared with the PCA estimate, the regression estimate tends to understate the response of the FRAs to changes in the 10Y swap rate and to overstate the impact of change in the 10Y swap rate on the long end of the swap curve. Furthermore, the R^2 coefficients obtained for the regressions vary from 23 to 99%, so that it is quite difficult to make an overall statement relative to the goodness of the estimate of V^*. On the contrary, the PCA estimate of V^* indicates a Q-coefficient of 94%. This means that 94% of the changes in the EUR term structure can be explained by the vector V^* as estimated using a PCA. Nevertheless, both charts demonstrate a similar pattern. All the buckets of the EUR curve tend to move in the same direction on average and FRAs with expiries longer than 1 year tend to be more volatile than swaps with longer maturities. According to the PCA estimate, a 24 bp change in the 18-month FRA is compensated by only a 15 bp change in the 40Y swap rate.

One of the central aims of this study is to determine whether the most probable extreme changes in term structure are affected by changes in the agitation of the market. We therefore need to introduce a definition for term structure agitation. A term structure can be intuitively understood as agitated at a given date if the changes ΔR_t observed at that date are larger than the changes usually measured. With reference to the example introduced in the first section, this would mean that the changes outside the largest ellipsoid correspond to dates on which the term structure was agitated (*see Exhibit 3*). The definition of term structure agitation must now be mathematically formalized. To this purpose, we introduce the subsets $T_{p,\rho} = \left\{t \in \{1, \ldots, T\} / N_\rho(\Delta R_t) \in \left[d_{p-1}; d_p\right]\right\}$ for p in $\{1,\ldots,10\}$, where d_p denotes the p-th decile of N_ρ. For instance, $T_{1,\rho}$ stands for the 10% of times t (or days) on which ΔR_t had the fewest possible neighbors, and $T_{10,\rho}$ stands for the 10% of times t on which ΔR_t had the greatest number of neighbors. The term structure will be said to be agitated to degree p at time t if t is in $T_{11-p,\rho}$. Once term structure agitation is defined, we can estimate V^* for p in $\{1,\ldots,10\}$. Exhibit 6 depicts the most probable changes in the EUR term structure for various values of p. Here ρ has been set to 10 bp, which corresponds to a maximal parallel shift of 2 bp for 25 buckets. To increase the readability of the chart, only response curves corresponding to even values of

EXHIBIT 7

Single-Currency Framework: Impact of Term Structure Agitation *p* on *Q* (1999-2004)

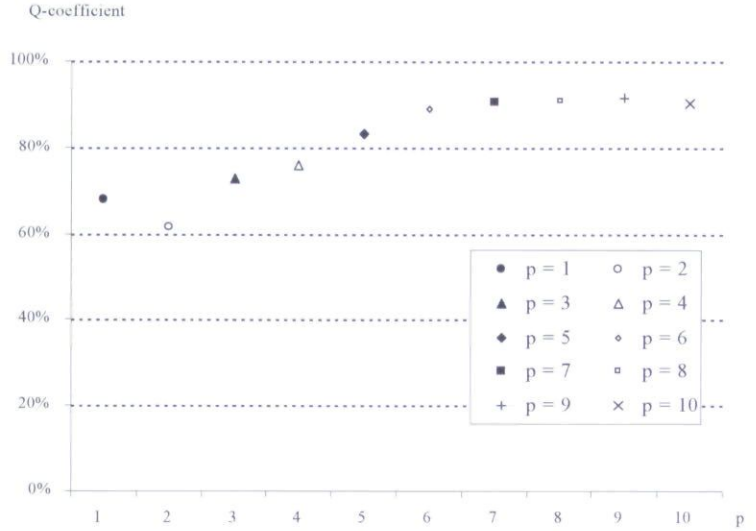
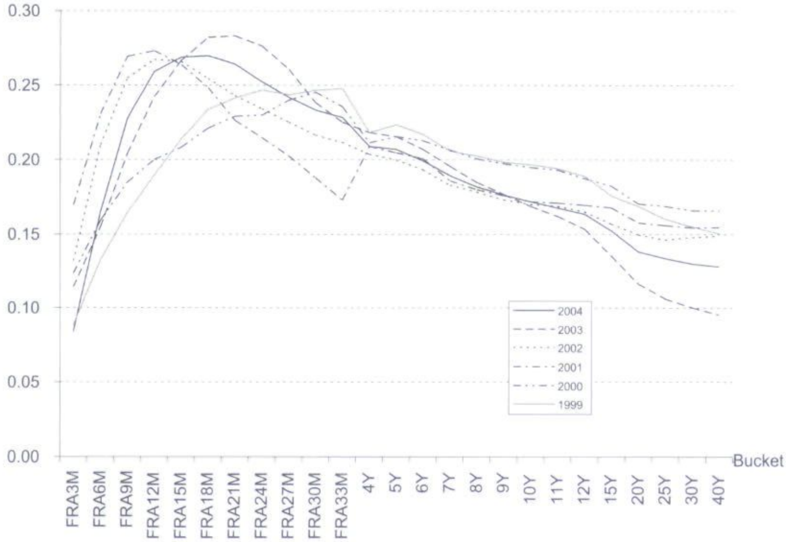


EXHIBIT 8

Single-Currency Framework: Impact of Time on V^* (1999-2004)



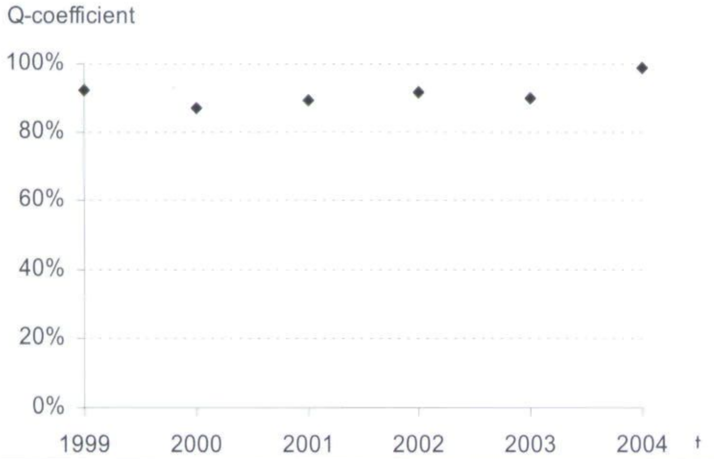
p have been plotted. Contrary to what some market participants often think and according to Exhibit 6, the agitation of the term structure has no major impact on the shape of the most probable extreme changes. It is essential here to note that applying the definition for term structure agitation introduced earlier, each V^* is estimated using a covariance matrix corresponding to the observations located between two ellipsoids. Consequently, the distribution used to estimate the covariance matrix is also ellipsoidal and the covariance estimates are not biased, with the result that our PCA estimator for V^* is unbiased.[6] This is the main advantage of the definition of term structure agitation used here.[7] Interestingly, the Q-coefficient associated with the estimate of V^* tends to increase with market agitation ρ, as illustrated in Exhibit 7. Excepting the Q-coefficient of V^* for quiet markets ($p = 1$), the goodness of the estimates tends to increase with market agitation. This means that the variance explained by V^* increases with market agitation. In other words, when the term structure becomes more agitated, the probability that extreme changes will display the shape estimated by V^* increases.

The impact of market agitation on the most probable extreme changes in the term structure is not the only crucial factor for FI investors and risk managers; the impact of time is at least as essential since it determines how often the model needs to be recalibrated. We therefore divide our observation sample into 6 subsamples corresponding to each year between 1999 and 2004. Consequently, we can observe the most probable extreme changes in the term structure over a complete economic cycle, including increasing and decreasing rate cycles. Exhibit 8 depicts the changes in the shape of V^* over time for p set at 10 and unchanged ρ. Clearly, V^* is deeply impacted by time. Whereas a 17 bp change in the 10Y swap rate corresponded to a 13 bp change in the 40Y swap rate in 2004, the corresponding change in the 40Y was limited to 9 bp in 2003. This confirms the need to recalibrate the model with quite a high frequency. Interestingly, the sensitivity coefficients as estimated using a regression analysis are not more stable. Finally, Exhibit 9 plots the Q-coefficients of V^* between 1999 and 2004. Interestingly, the Q-coefficient remains extremely high over the whole period despite changes in the shape of V^*. Q varies from a minimum of 87% (in 2000) to a maximum of 99% (in 2004).

After investigating the most probable extreme changes in a given term structure, we switch to the multicurrency framework. Once again, we initially focus on the impact of market agitation on the most probable extreme changes and then on the time effect. We extend our analysis to the USD, GBP, CHF, DKK, SEK, NOK, CAD, JPY, AUD, and NZD term structures from the beginning of 2002 to

EXHIBIT 9

Single-Currency Framework: Impact of Time on *Q* (1999-2004)



the end of 2004. The same definition of term structure agitation used for the EUR term structure is applicable for the multicurrency framework. The PCA estimates of V^* for even values of p and for $\rho = 70$ bp (which, for 204 curve buckets, roughly corresponds to a maximal parallel shift of 5 bp) are depicted in Exhibit 10.

Excepting the Asian term structures, the shape of the most probable changes in term structure does not seem to be massively impacted by an increase in market agitation. The unstable results obtained for the Asian markets may be due to the time lag between the European and American markets and the Asian market, which can introduce non-negligible noise in the data set for daily change. Unsurprisingly, changes in the USD term structure are roughly twice as important as changes in the European term structures. This reflects the higher volatility of the USD swap rates compared with the European ones. Changes in the Japanese term structure are on average one-fourth to one-fifth of the daily changes observed in Europe. This is a straightforward consequence of the extremely low level of rates in Japan, where changes are very limited as a result. As regards the goodness of the estimates, the Q-coefficients of the estimates are plotted in Exhibit 11. Once again, the goodness of the model increases with term structure agitation. The interpretation of this phenomenon remains unchanged. When market agitation increases, the probability that extreme changes will display the pattern estimated by V^* increases.

We further investigate the impact of time on the PCA estimates of V^*. To this purpose, we set ρ to 70 bp, as in the previous analysis, but we run the PCA for a term structure agitation p equal to 9 or 10, in order to have enough data to obtain pertinent estimates. Exhibit 12 shows the estimates of V^* for 2002, 2003, and 2004. At first sight, time seems to have a limited impact on V^*. We will nevertheless not conclude here that time has no significant impact on V^*. Looking back at Exhibit 8 and more precisely at the estimates of V^* for the EUR term structure in 2002, 2003, and 2004, we could also have stated that time has a limited impact on V^*. But if we extend the observation period to 1999, 2000, and 2001, it becomes clear that time significantly impacts V^*. Based on the results presented in Exhibit 12, we merely conclude that on a short-term horizon (1 or 2 years), time has a limited effect on the PCA estimates of V^*, which may no longer be true on a longer time horizon.

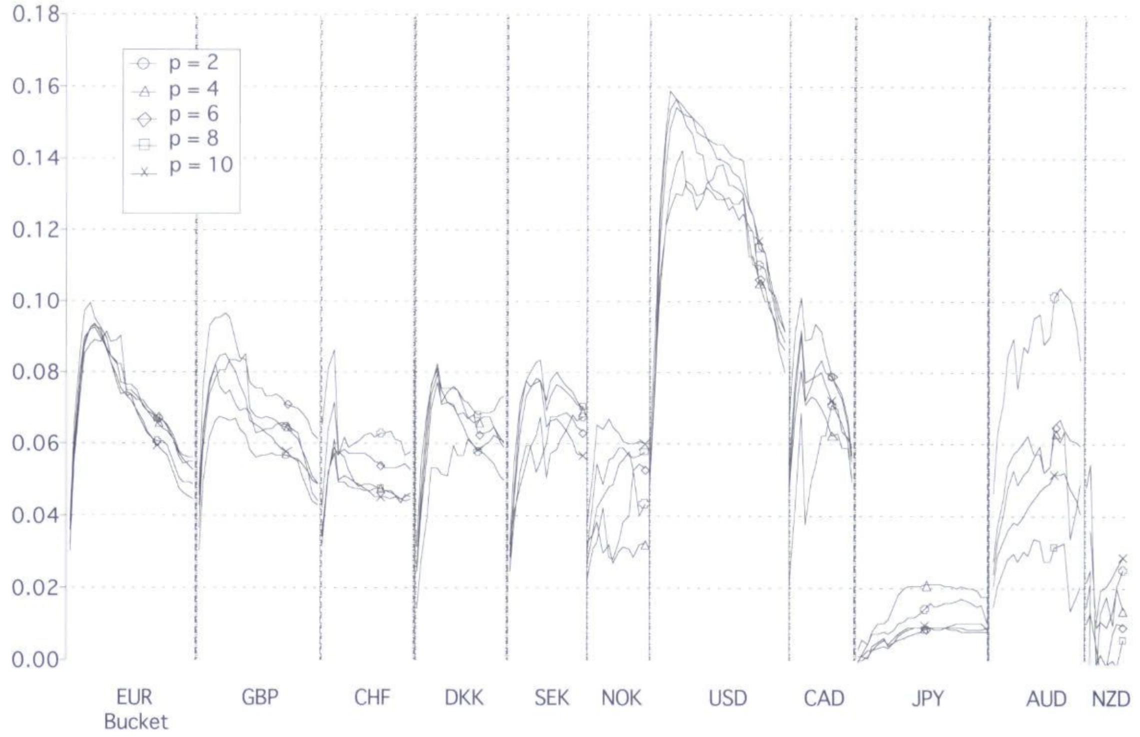
Finally, in Exhibit 13 we plot the Q-coefficients obtained from 2002 to 2004 for the estimates of V^*. They vary from 59% to 78%, which is a further argument for questioning the stability of the estimates of V^* over time.

APPLICATION TO THE ASSESSMENT OF THE INTEREST RATE RISK OF MULTICURRENCY PORTFOLIOS

Assessing the interest rate risk of single- or multicurrency fixed-income portfolios can be seen as a direct application of the estimation of the most probable extreme changes in term structure V^*. Start with a single currency fixed-income portfolio. Its present value is a function of the term structure it is invested in. Yet, a term structure cannot be regarded as simply a series of interest rates. It is also driven by factors such as volatility. This is clearly evident from looking at fixed-income options, whose price at a given time is not solely a function of various interest rates but also depends on the volatilities of these rates. Consequently, the present value P_t of a single-currency fixed-income portfolio at time t can be written as a function f of the time t, of a spot interest rate vector R_t and of a market parameter vector M_t reflecting, among other things, the volatility of the term structure at time t. In other words, $P_t = f(R_t, M_t, t)$. However, if we reduce the market parameter vector M_t to the volatilities of the various interest rates contained in R_t and if we assume that the volatilities of the interest rates are driven by the term structure itself, M_t can be rewritten as a function g of R_t.[8] It follows that P_t can simply be expressed as a func-

EXHIBIT 10

Multicurrency Framework: Impact of Term Structure Agitation *p* on *V (2002-2004)**



tion f^* of R_t and t, that is, $P_t = f(R_t, g(R_t), t) = f^*(R_t, t)$.

In terms of the present value of a multicurrency fixed-income portfolio, it is no longer solely a function of the spot rates R_t and time t but also a function of the various exchange rates represented by vector X_t. As the hypothesis that exchange rates are driven by term structures is hardly sustainable,[9] we will not regard them as a function of R_t. As a result, the price of a multicurrency fixed-income portfolio must be written as a function f^{**} of R_t, X_t, and t. In the following, therefore, the price P_t of a multicurrency fixed-income portfolio will be written as $P_t = f^{**}(R_t, X_t, t)$.

Being interested in the interest rate risk of a fixed-income portfolio means looking at the change ΔP_t in P_t triggered by a change ΔR_t in R_t for an unchanged value of X_t. This change ΔP_t can be written as a function h of ΔR_t, X_t, and t,[10] that is, $\Delta P_t = h(\Delta R_t, X_t, t)$ or, equivalently, $\Delta P_t = h(\Delta r_{1,t}, \ldots, \Delta r_{n,t}, X_t, t)$.

Furthermore, for all i in $\{1,\ldots,n\}$, $\Delta r_{i,t}$ can be written as the change $\Delta r_{k,t}$ in a reference rate r_k times the sensitivity of Δr_i to Δr_k and an error term $\varepsilon_{i,t}$, that is, $\Delta r_{i,t} = g_i(\Delta r_{k,t}) + \varepsilon_{i,t}$ where $g_i(\Delta r_{k,t}) = (V^*_i / V^*_k)\Delta r_{k,t}$ and V^*_i is the i-th coefficient of V^*. Consequently,

$$\Delta P_t = h(g_1(\Delta r_{k,t}) + \varepsilon_{1}, \ldots, g_n(\Delta r_{k,t}) + \varepsilon_{n,t}, X_t, t)$$

Alternatively, we may rewrite

$$\Delta P_t \text{ as } \Delta P_t = h^*(g_1(\Delta r_{k,t}), \ldots, g_n(\Delta r_{k,t}), X_t, t) + \varepsilon_t$$

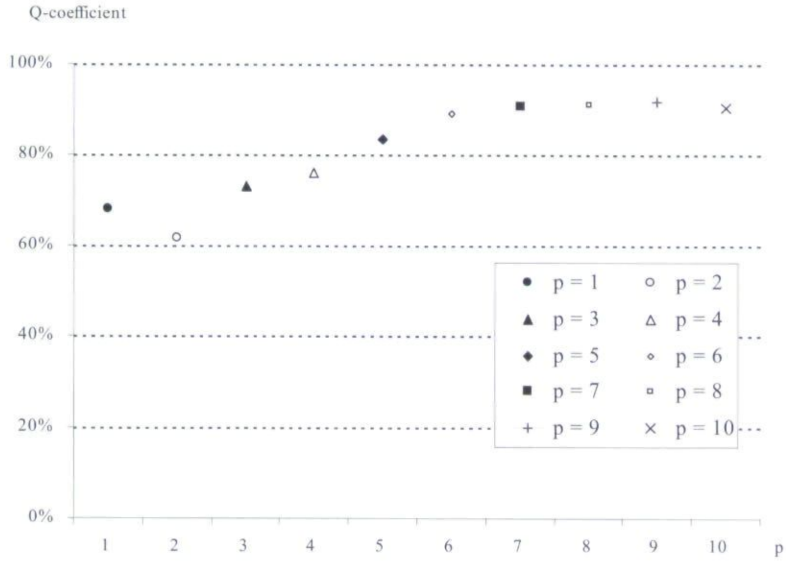
or equivalently, $\Delta P_t = h^{**}(\Delta r_{k,t}, X_t, t) + \varepsilon_t$. If we neglect the ε_t term[11] and regard X_t as a fixed parameter, it is possible to induce the distribution of probability of ΔP from the distribution of probability of Δr_k.

In conclusion, the interest rate risk of a multicurrency fixed-income portfolio can be summed up in the distribution of probability of ΔP, which is almost solely driven by the distribution of probability of Δr_k. If, for instance, we wish to calculate the largest possible loss that can occur in a portfolio in α% of cases as a result of interest rate risk, we simply compute the opposite of the $(1 - \alpha)$-percentile of the distribution of probability of ΔP. In what follows, we will refer to this largest possible loss as interest risk and denote it by $\text{IRR}_t(\alpha)$.

Exhibit 14 provides an example of the interest rate risk $\text{IRR}_t(\alpha)$ for $\alpha = 95\%$, where V^* is estimated using regression and PCA estimators. Since a widespread method

EXHIBIT 11

Multicurrency Framework: Impact of Term Structure Agitation p on Q (2002-2004)



on trading floors is to assume that V^* is collinear to (1, 1, ..., 1), that is, to assume parallel shifts, we also present the results obtained using this method, called here the naive approach. Our example illustrates the interest rate risk of a portfolio made of a 5Y–10Y flattener[12] in the USD swap curve and a 5Y–10Y steepener[13] in the EUR swap curve for $p = 10$, with the reference rate being the USD 10Y swap rate. The flattener trade is decomposed into a USD +1,000,000 delta position in the USD 5Y swap and a USD −1,000,000 delta position in the USD 10Y swap. The steepener trade can be symmetrically split into a EUR −750,000, that is, a USD −1,000,000, delta position[14] in the EUR 5Y swap rate and a EUR +750,000 delta position in the EUR 10Y swap.

We denote the deltas of the above-described positions (expressed in USD) by Δ_{USD5Y}, Δ_{USD10Y}, Δ_{EUR5Y}, and Δ_{EUR10Y}, and the exchange rate between EUR and USD is

EXHIBIT 12

Multicurrency Framework: Impact of Time on V^* (2002-2004)

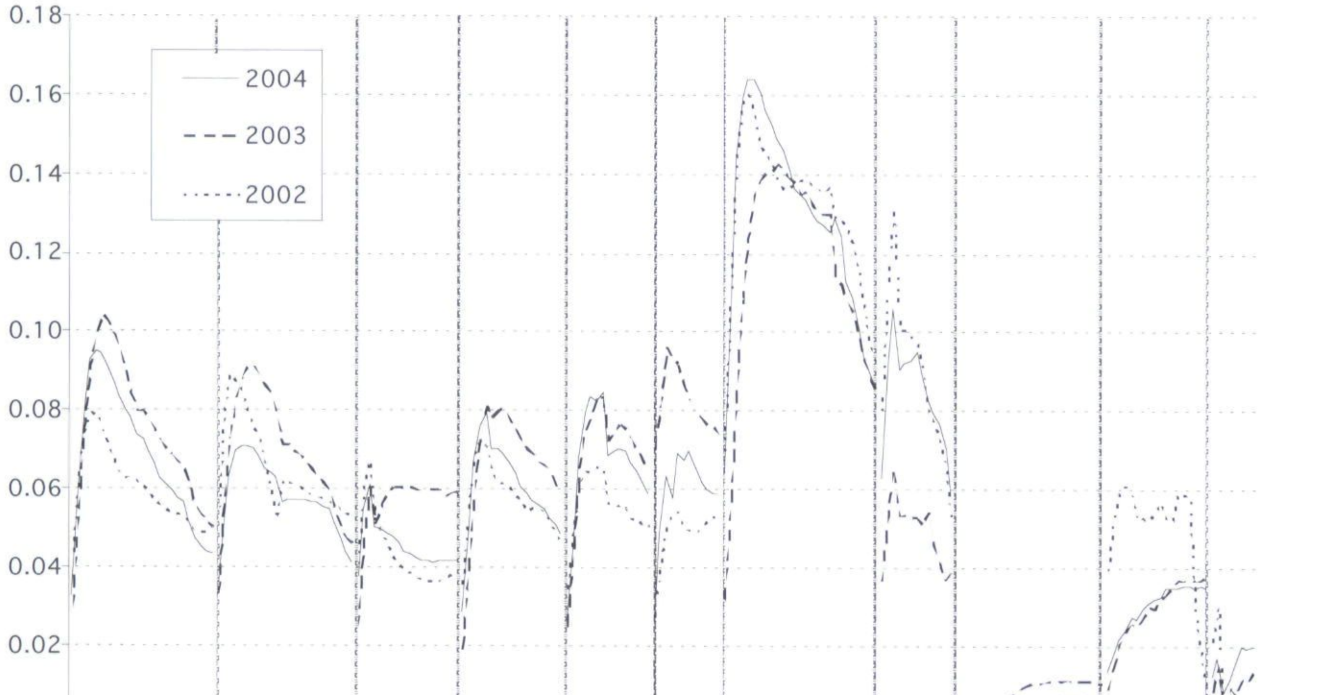
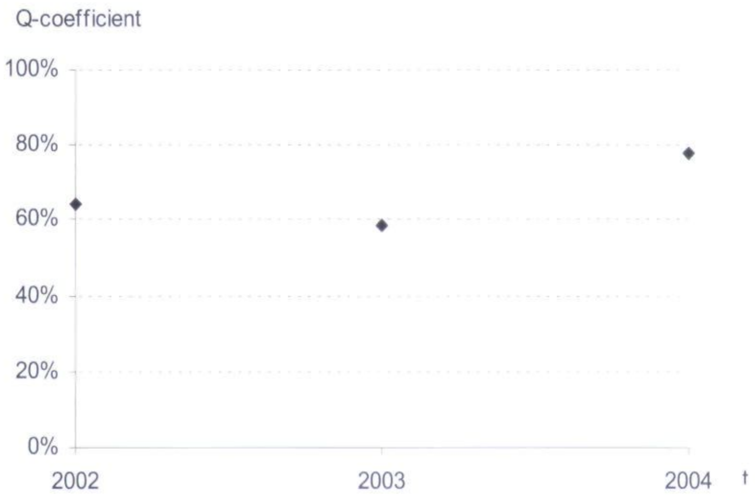


EXHIBIT 13

Multicurrency Framework: Impact of Time on Q (2002-2004)



denoted by X. This gives $\Delta P_t = K\Delta r_{\text{USD10Y},t} + \varepsilon_t$, where

$$K = \Delta_{\text{USD5Y}} \frac{V^*_{\text{USD5Y}}}{V^*_{\text{USD10Y}}} + \Delta_{\text{USD10Y}} \frac{V^*_{\text{USD10Y}}}{V^*_{\text{USD10Y}}} + \Delta_{\text{EUR5Y}} \frac{V^*_{\text{EUR5Y}}}{V^*_{\text{USD10Y}}} X + \Delta_{\text{EUR10Y}} \frac{V^*_{\text{EUR10Y}}}{V^*_{\text{USD10Y}}} X$$

The opposite of the $(1-\alpha)$-percentile of the distribution of probability of ΔP is simply $-K$ times the $(1-\alpha)$-percentile $P_{1-\alpha}$ of the distribution of probability of Δr_{USD10Y}, that is $IRR_t(\alpha) = -K\rho_{1-\alpha}$.[15]

Using the basic approach, that is, assuming that all swap curves move according to a strictly parallel pattern, all positions have an interest rate risk of 11,500,000 since the 5%-percentile of the distribution of the daily changes in the USD 10Y swap rate is roughly −11.5 bp and the 95%-percentile is approximately 11.5 bp. As a result, the flattener and the steepener positions bear a risk of 0. According to the regression approach, an 11.5 bp change in the USD 10Y induces an interest rate risk of USD 612,000 for the flattener position and USD 702,000 for the steepener position. All in all, the portfolio bears a risk of USD 90,000. Alternatively, the PCA approach leads to a risk of USD 1,158,000 for the flattener position, which exceeds that of the regression approach by 89.3%. The steepener position would bear a risk of USD 1,097,000, that is, a 56.3% greater risk than that induced by regression analysis. All in all, the portfolio risk would then sum up to 60,000 USD instead of 90,000 USD.

It is obvious that different approaches lead to different results. Consequently, one should avoid resorting to theoretically ill-founded methods. Since the naive approach lacks a theoretical foundation and the regression method is so constructed that it contains a bias, we recommend using the PCA approach presented in this study.

EXHIBIT 14

Example

Position	FX-rate per USD	Delta in local currency	Naïve Approach V*	Naïve Approach IRR(95%) in ´000 USD	Regression Approach V*	Regression Approach IRR(95%) in ´000 USD	PCA - Most agitated V*	PCA - Most agitated IRR(95%) in ´000 USD	Change
USD 5Y	1.00	+1,000	0.50	+11,500	0.66	+12,112	0.65	+12,658	
USD 10Y	1.00	-1,000	0.50	+11,500	0.62	+11,500	0.59	+11,500	
Flattener				**+0**		**+612**		**+1,158**	**89.3%**
EUR 5Y	0.75	-750	0.50	+11,500	0.32	+5,811	0.36	+7,028	
EUR 10Y	0.75	+750	0.50	+11,500	0.28	+5,109	0.31	+5,931	
Steepener				**+0**		**+702**		**+1,097**	**56.3%**
Portfolio = Flattener & Steepener				**+0**		**+90**		**+60**	**-33.2%**

APPLICATION TO HEDGING OF THE INTEREST RATE RISK OF MULTICURRENCY PORTFOLIOS

Hedging the interest rate risk of a multicurrency fixed-income portfolio means hedging ΔP against changes in Δr_k.

Since $\Delta P_t = h^{**}(\Delta r_{k,t}, X_t, t) + \varepsilon_t$, if we ignore the error term ε_t, regard X_t as a fixed parameter, and apply Ito's lemma to h^{**}, we get $dh^{**}(\Delta r_{k,t}, X_t, t) = \delta d\Delta r_{k,t} + \Theta dt + \frac{1}{2}\Gamma d\Delta r_{k,t}^{\ 2}$ where

$$\delta = \delta(\Delta r_{k,t}, X_t, t) = \frac{\partial h^{**}}{\partial \Delta r_{k,t}}(\Delta r_{k,t}, X_t, t)$$

$$\Theta = \Theta(\Delta r_{k,t}, X_t, t) = \frac{\partial h^{**}}{\partial t}(\Delta r_{k,t}, X_t, t)$$

$$\Gamma = \Gamma(\Delta r_{k,t}, X_t, t) = \frac{\partial^2 h^{**}}{\partial \Delta r_{k,t}^{\ 2}}(\Delta r_{k,t}, X_t, t)$$

Furthermore, $h^{**}(0, X_t, t) = 0$, so that

$$\Delta P_t = h^{**}(\Delta r_{k,t}, X_t, t) - h^{**}(0, X_t, t) + \varepsilon_t$$

On a daily basis, even large changes in Δr_k remain small enough to replace $h^{**}(\Delta r_{k,t}, X_t, t) - h^{**}(0, X_t, t)$ by $\delta \Delta r_{k,t} + \frac{1}{2}\Gamma \Delta r_{k,t}^{\ 2}$. It follows that

$$\Delta P_t \approx \delta \Delta r_{k,t} + \frac{1}{2}\Gamma \Delta r_{k,t}^{\ 2} + \varepsilon_t$$

We now consider a hedge portfolio solely invested in an instrument driven by the reference rate r_k. In what follows, we will take a swap rate as reference and the interest rate swap contract associated with this rate is taken as the hedge instrument.

Denoting the present value of the hedge portfolio by Q_t we get: $\Delta Q_t \approx \delta_Q \Delta r_{k,t} + \frac{1}{2}\Gamma_Q \Delta r_{k,t}^{\ 2} + \varepsilon_t$, where δ_Q and Γ_Q are the derivatives δ and Γ as defined earlier, but for the hedge portfolio.

Ignoring the sensitivity of the floating leg to changes in the reference rate, we get: $\delta_Q = -ND$, where N is the nominal invested in the swap contract and D is the modified duration of the fixed leg of the swap; $\Gamma_Q = NC$ where C is the convexity of the fixed leg of the swap. Consequently, $\Delta Q_t \approx -ND\Delta r_{k,t} + \frac{1}{2}NC\Delta r_{k,t}^{\ 2} + \varepsilon_t$.

The hedge condition can be written as $\Delta P_t + \Delta Q_t = 0$. This is largely satisfied when $\delta - ND = 0$, that is, for a fixed receiver position of nominal $N = \delta/D$ in the reference swap instrument.

Furthermore, the quality of the hedge is optimal when Γ is close to NC and Q is close to 100%.

CONCLUSIONS

PCA proves to be a statistically reliable method of estimating the most probable extreme changes in term structure. Given the assumption of elliptically distributed changes in the term structures we are interested in, the PCA estimator introduced in this study is unbiased. Interestingly, the changes identified using our PCA estimator are relatively insensitive to changes in market agitation. Moreover, the significance of the estimated changes tends to increase with term structure agitation. This means that changes in term structure are more likely to correspond to the estimated changes when market agitation increases. Since interest risk models focus mainly on large changes in term structure, the PCA estimator introduced in this study can prove helpful for the implementation of interest risk models. Furthermore, it can provide fixed-income investors with simple hedge strategies to protect multicurrency portfolios against interest rate risk. However, PCA estimates of the most probable extreme changes in term structure remain quite sensitive to time. Hence, the models using these estimates should be recalibrated on a monthly to yearly basis. The time sensitivity of our estimates is a direct consequence of the sensitivity of the covariance estimates to time.

In this study, we limited our investigations to the interest rate risk of multicurrency fixed-income portfolios, ignoring the currency risk. In other words, we focused on the changes in term structure R_t and ignored the possible changes in cross-currency rates X_t. Nevertheless, there should be no major objection to using the approach suggested in this study to estimate the most probable extreme changes in X_t. Interest rate and currency risks could then be taken into account. Eventually, both risk sources could be integrated into a single value based on a correlation analysis between the reference interest rate and the reference cross-currency exchange rate. If the correlation between these two sources of risk is significantly low, which is likely to be the case on a short-term horizon,[16] both risks could simply be summed, for instance.

ENDNOTES

[1]See Basle Committee ([1996]). Even if the decisions of the Basle Committee are not legally binding, the content of the Capital Accord has been transposed into most national laws. In the European Union, the Capital Adequacy Directive was modified in 1998 to account for market risks.

[2]We call "Euclidian ball" the ball associated with the Euclidian distance in IR^n.

[3]To this purpose, see Embrechts et al. ([1999]).

[4]The concept of a unit vector requires a norm; we use the Euclidian norm in IR^n.

[5]The word "bias" should not be understood in its statistical meaning here, but instead in its usual sense.

[6]The word "unbiased" is to be understood in its common sense here.

[7]A more simple definition of term structure agitation could, for instance, be based on a specific bucket (e.g., EUR or USD 10Y swap). The term structure would then be regarded as agitated on a given day if the change in the rate of this reference bucket is larger than a given quantile of its distribution. Such an approach implies a cut in the ellipsoids and breaches a crucial hypothesis for the pertinence of covariance estimates.

[8]A widespread assumption on fixed-income trading floors is that the volatility σ of a given forward interest rate F_t is inversely proportional to this forward rate; i.e., $\sigma = K/F_t$, where K is a forward specific constant. Since the forward rate F_t is a function of the spot rates R_t, σ can be seen as a function of either F_t or R_t. Consequently and under this assumption, M_t is also a function of R_t; i.e., $M_t = g(R_t)$.

[9]In this study, it would be all the more questionable to assess whether changes in exchange rates are driven by changes in the term structures since we focus on short-term changes.

[10]In theory, the change ΔP_t in the price of the portfolio for constant X_t should also be a function of the interest rates in level, i.e., of R_t. Yet, the level of the interest rates impacts only the carry component of the change in the price, which, on a daily horizon, is negligible compared with the changes in the interest rates, particularly when markets are agitated.

[11]The error term ε_t will be all the more negligible since Q will be close to 100%. As a matter of fact, in such a case, $\Delta r_{i,t} = g_i\left(\Delta r_{k,t}\right)$; i.e., $\varepsilon_{i,t} = 0$ for all i, which means $\varepsilon_t = 0$.

[12]A flattener is a trade betting on a flattening movement of the curve and consists of a short position in the shorter end of the curve (e.g., 5 year) and a long position in the longer end (e.g., 10 year).

[13]A steepener is a trade betting on a steepening movement of the curve.

[14]Assuming 1 USD = 0.75 EUR.

[15]Only true if $K > 0$ else $\mathrm{IRR}_t(\alpha) = -Kp_\alpha$.

[16]A regression analysis between the daily changes in the EUR 10Y swap rate and the changes in the EUR/USD exchange rate returns an R^2 of 9%, which argues for no significant correlation between these two rates (data from January 1999 to December 2004).

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