Improving Discrete Implementation of the Hull and White Two-Factor Model

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Several approaches for modeling the movements of the term structure of interest rates are available. Ho and Lee [1986] were the first to introduce an arbitrage-free single-factor model for the short rate dynamics that matches today's zero-bond market prices. Other authors propose different process specifications to describe interest rate movements involving one or more factors.

A major breakthrough is in Heath, Jarrow, and Morton [1992] (HJM). They derive general properties for an arbitrage-free description of movements of the term structure by looking at the forward rate dynamics. One problem is that the HJM approach does not guarantee the resulting term structure dynamics to be Markovian, and it is expensive to implement in most cases. Second, it assumes a continuum of instantaneous forward rates, a requirement not compatible with real-world markets.

The LIBOR market model for forward rates addresses this second problem. It is a direct modeling approach to describe the joint behavior of a predefined set of discretely compounded forward rates spanning consecutive discrete future time periods. Brace, Gatarek, and Musiela [1997], Jamshidian [1997], and Miltersen, Sandmann, and Sondermann [1997] show how this approach is nested in the HJM approach.

Although it can produce a rich variety of interest rate scenarios, application of the LIBOR model may still be restrictive. The implementation algorithms for the LIBOR model developed so far all use Monte Carlo techniques (e.g., Rebonato [1999], Andersen and Andreasen [2000], Hull and White [2000], Joshi and Rebonato [2001], Longstaff, Santa-Clara, and Schwartz [2001], and Glasserman and Merener [2003]). Monte Carlo simulations have the drawback that they are expensive and difficult to apply to the pricing of American options.[1]

Rebonato [1999] points out that the LIBOR market model can value only derivatives whose payoffs happen on the expiration dates of the modeled forward rates. When numerical efficiency is of crucial importance, the two-factor model proposed by Hull and White [1994b] (HW2 model) appears a valuable alternative, presuming one accepts the underlying assumptions about the short rate and can accommodate the concession of a continuous yield curve. This might be the case if one wants to price American options, which can be exercised at any time until expiration, or continuous time barriers.

The HW2 model is a special case of the HJM approach that has the advantage of being Markovian. It allows for a large variety of possible future yield curves. Discrete implementation involves constructing of a three-dimensional recombining tree.

We examine the accuracy of the discrete implementation of the HW2 model for caplet options as an example. Our evidence suggests that the algorithm's convergence might be weak. We explain in what situations this is the case and provide a modified algorithm that gives up only a little of its numerical efficiency.

I. REVIEW OF THE HULL AND WHITE TWO-FACTOR MODEL

Hull and White [1994b] model the movements of the short rate as:

$$dr(t) = [\phi(t) + u(t) - a \cdot r(t)] \cdot dt + \sigma_r \cdot dz_r \quad (1)$$

where $\phi(t)$ denotes a time-varying parameter that ensures the fit to an observed term structure of interest rates; $u(t)$ stands for the second stochastic function:

$$du(t) = -b \cdot u(t) \cdot dt + \sigma_u \cdot dz_u \quad (2)$$

a and b are mean reversion speeds; σ_r and σ_u are constants representing instantaneous volatilities; and dz_r and dz_u are increments of Wiener processes under the risk-neutral measure with:

$$E\left(dz_r \cdot dz_u\right) = \rho_{ru} \cdot dt \quad (3)$$

Since the stochastic process in (2) affects the drift of (1), a good way to think about u is to view it as a long-term interest rate. Hull and White define a new stochastic function $y(t)$ based on $r(t)$ and $u(t)$:

$$y(t) = r(t) + \frac{1}{b-a} \cdot u(t) \quad (4)$$

where the dynamics of $y(t)$ follow:

$$dy = (\phi(t) - a \cdot y) \cdot dt + \sigma_y \cdot dz_y \quad (5)$$

The variance of the changes in $y(t)$ and the correlation with du are:

$$\sigma_y^2 = \sigma_r^2 + 2 \cdot \frac{\rho_{ru} \cdot \sigma_r \cdot \sigma_u}{b-a} + \frac{\sigma_u^2}{(b-a)^2} \quad (6)$$

and

$$\rho_{yu} = \frac{\sigma_r}{\sigma_y} \cdot \rho_{ru} + \frac{\sigma_u}{\sigma_y \cdot (b-a)} \quad (7)$$

Let (4), (6), and (7) be the synthetic short rate process, synthetic variance, and synthetic correlation, respectively. Hull and White [1994b] develop a three-dimensional tree for the joint movement of y and u. Each node has nine possible successors. The matrix **P** has the nine transition probabilities in each node:

$$\mathbf{P} = \begin{pmatrix} \pi_{uu} & \pi_{um} & \pi_{ud} \\ \pi_{mu} & \pi_{mm} & \pi_{md} \\ \pi_{du} & \pi_{dm} & \pi_{dd} \end{pmatrix} = \begin{pmatrix} \pi_u^y \cdot \pi_u^u & \pi_u^y \cdot \pi_m^u & \pi_u^y \cdot \pi_d^u \\ \pi_m^y \cdot \pi_u^u & \pi_m^y \cdot \pi_m^u & \pi_m^y \cdot \pi_d^u \\ \pi_d^y \cdot \pi_u^u & \pi_d^y \cdot \pi_m^u & \pi_d^y \cdot \pi_d^u \end{pmatrix} \quad (8)$$

if y and u are uncorrelated.

In (8), the superscript refers to the stochastic variables y or u, and the subscripts u, m, and d indicate possible up-, middle-, and down-movements of each factor. For example, π_{dm} is the joint transition probability for a down-movement of y and a middle-movement of u. Each martingale probability is a function of the state of the underlying random variable, the mean reversion speed, and the length of the time step.[2]

These dependencies have been dropped in (8) and in Equations (9) and (10). $\mathbf{P}_{adj}$ is the matrix of martingale transition probabilities adjusted for correlation. Hull and White show that

$$\mathbf{P}_{adj} = \begin{cases} \begin{bmatrix} \pi_{uu} + 5e & \pi_{um} - 4e & \pi_{ud} - e \\ \pi_{mu} - 4e & \pi_{mm} + 8e & \pi_{md} - 4e \\ \pi_{du} - e & \pi_{dm} - 4e & \pi_{dd} + 5e \end{bmatrix}, \text{ if } \rho_{yu} \geq 0 \\ \begin{bmatrix} \pi_{uu} - e & \pi_{um} - 4e & \pi_{ud} + 5e \\ \pi_{mu} - 4e & \pi_{mm} + 8e & \pi_{md} - 4e \\ \pi_{du} + 5e & \pi_{dm} - 4e & \pi_{dd} - e \end{bmatrix}, \text{ if } \rho_{yu} < 0 \end{cases} \quad (9)$$

where

$$e = \min\left[\frac{\pi_{um}}{4}, \pi_{ud}, \frac{\pi_{mu}}{4}, \frac{\pi_{md}}{4}, \pi_{du}, \frac{\pi_{dm}}{4}, \frac{\rho_{yu}}{36}\right]$$

if $\rho_{yu} \geq 0$

$$e = -\max\left[-\pi_{uu}, -\frac{\pi_{um}}{4}, -\frac{\pi_{mu}}{4}, -\frac{\pi_{md}}{4}, -\frac{\pi_{dm}}{4}, -\pi_{dd}, \frac{\rho_{yu}}{36}\right]$$

if $\rho_{yu} < 0$ (10)

The first six terms in (10) avoid negative martingale probabilities but the correlation ρ_{yu} is specified correctly only if $e = |\rho_{yu}/36|$. If one of the six terms is binding, the correlation ρ_{yu} is not correctly matched.

II. CONVERGENCE OF THE ALGORITHM

We start by looking at the convergence properties of the discrete and the analytic solution for the example of a caplet option. A caplet is a call option on a floating spot interest reference rate. The payoff function at time T is

$$\text{Payoff} = k \cdot L \cdot \frac{\max\left[R(T, T+k) - X_C, 0\right]}{1 + k \cdot R(T, T+k)} \tag{11}$$

where $R(T, T+k)$ is the spot rate from time T to $T+k$, X_C is the strike rate, and L is the notional amount on which the caplet is written. The analytic solution for the price of this caplet in the framework of the HW2 model is:[3]

$$\begin{aligned}
\text{Caplet} &= L \cdot (1 + k \cdot X_C) \times \\
&\left[\frac{P(0,T)}{1 + k \cdot X_C} \cdot N(-d_2) - P(0,S) \cdot N(-d_1)\right] \\
d_1 &= \frac{\ln\left(\frac{P(0,S)\cdot(1+k\cdot X_C)}{P(0,T)}\right) + \frac{1}{2}\cdot\sigma_B^2(T,S)}{\sigma_B(T,S)} \\
d_2 &= d_1 - \sigma_B(T,S)
\end{aligned} \tag{12}$$

where $N(\cdot)$ denotes as usual the operator for the standard normal distribution. $P(0, T)$ and $P(0, S)$ are today's prices of zero bonds which pay one monetary unit at times T and $S = T + k$, respectively. The function $\sigma_B(T, S)$ is the volatility of $d[P(t, S)/P(t, T)]$, where $0 \leq t \leq T \leq S$. An analytical representation of $\sigma_B(T, S)$ is in the appendix.

Following Hull and White [1994b], assume that the term structure of interest rates can be represented by the function

$$r(0, x) = 0.08 - 0.05 \cdot \exp(\alpha \cdot x) \tag{13}$$

where $r(0, x)$ is the continuously compounded spot rate for maturity x at time 0, and α is a curvature coefficient. Suppose a yield curve with $\alpha = 0.18$ (similar results can be obtained with different curvature coefficients). Set the parameters $a = 0.47$, $b = 0.38$, $\sigma_r = 0.85\%$, $\sigma_r = 0.99\%$, and $\rho_{ru} = 0.67$. This choice corresponds to Rebonato [1998, p. 308], who obtains these parameters when he calibrates the HW2 model to the German fixed-income market. According to (7), the synthetic correlation associated with these parameters is –0.998.

We use the HW2 tree methodology to price several caplet options on the six-month variable interest rate with notional amount of 100 monetary units and expiration in four years. To test the convergence properties of the algorithm, we consider time steps of the length Δ equal to 0.500, 0.050, 0.025, and 0.010 years. This corresponds to calculating 9, 90, 180, and 450 time steps, respectively. We price caplet options with strikes ranging from 4% to 10%.

Exhibit 1 reports the analytic and the discrete solutions as well as the error percentages, which are given by the difference between the discrete and the analytic value divided by the analytic value.

The discrete solutions and the analytic prices do not converge well. The discrete option prices way overestimate the analytic prices even for the smallest time step considered ($\Delta = 0.01$ years), making them unsatisfactory for most pricing purposes. This result is contrary to the result reported by Hull and White [1994b], who use a different parameter set and find excellent convergence.[4]

This raises a question on the general conditions of parameter sets with good convergence properties versus others that produce poor results. We will show that the reasons for differences are related to the properties of the HW2 algorithm.

III. EXPLANATION OF RESULTS

To obtain the correct correlation ρ_{yu} between y and u in a given state, relationships as follows must hold:

$$\begin{aligned}
e(z_y, z_u) &= \tfrac{\rho_{yu}}{36}, \text{ if } \rho_{yu} \geq 0 \\
e(z_y, z_u) &= -\tfrac{\rho_{yu}}{36}, \text{ if } \rho_{yu} < 0
\end{aligned} \tag{14}$$

Choosing e differently means implicitly changing the short rate volatility σ_r and the correlation coefficient ρ_{ru} for the state considered. The variables σ_r and ρ_{ru}, which are specified as constants in (1) and (3), become state-dependent variables.

While the given synthetic correlation is denoted by ρ_{yu}, the realized synthetic correlation in a specific state (z_y, z_u) is ρ^*_{yu}, which depends on e and is calculated as:

$$\begin{aligned}
\rho^*_{yu}(z_y, z_u) &= 36 \cdot e, \text{ if } \rho^*_{yu} \geq 0 \\
\rho^*_{yu}(z_y, z_u) &= -36 \cdot e, \text{ if } \rho^*_{yu} < 0
\end{aligned} \tag{15}$$

EXHIBIT 1

Analytic versus Discrete Option Values (% error) Using HW2 Tree

Strike	Analytic value	Tree, $\Delta = 0.25$	Tree, $\Delta = 0.05$	Tree, $\Delta = 0.025$	Tree, $\Delta = 0.01$
4.00%	1.3983	2.2817	1.8019	1.6959	1.5835
		(63.17%)	(28.86%)	(21.28%)	(13.24%)
5.00%	1.0399	2.0241	1.5368	1.4068	1.2840
		(94.63%)	(47.78%)	(35.28%)	(23.46%)
6.00%	0.7163	1.7664	1.3002	1.1532	1.0123
		(146.61%)	(81.52%)	(61.00%)	(41.33%)
7.00%	0.4477	1.5088	1.0696	0.9263	0.7776
		(236.99%)	(138.89%)	(106.90%)	(73.69%)
8.00%	0.2490	1.2553	0.8538	0.7212	0.5790
		(404.16%)	(242.91%)	(189.65%)	(132.53%)
9.00%	0.1210	1.0397	0.6909	0.5630	0.4228
		(759.12%)	(470.90%)	(365.21%)	(249.40%)
10.00%	0.0507	0.9182	0.5667	0.4323	0.3005
		(1712.67%)	(1018.67%)	(753.53%)	(493.23%)

The variable e in (15) is given by (10). The implicit realized parameters $\sigma_r^*(z_y, z_u)$ and $\rho_{ru}^*(z_y, z_u)$ in that state then follow by the transformation of (6) and (7):

$$\sigma_r^*(z_y, z_u) = \sqrt{\sigma_y^2 - 2 \cdot \frac{\sigma_y \cdot \sigma_u}{b-a} \cdot \rho_{yu}^*(z_y, z_u) + \frac{\sigma_u^2}{(b-a)^2}} \tag{16}$$

$$\rho_{ru}^*(z_y, z_u) = \rho_{yu}^*(z_y, z_u) \cdot \frac{\sigma_y}{\sigma_r^*(z_y, z_u)} - \frac{\sigma_u}{(b-a) \cdot \sigma_r^*(z_y, z_u)} \tag{17}$$

To explore under which condition $\rho_{yu}^*(z_y, z_u) \neq \rho_{yu}^*(z_y, z_u)$ we find it useful to define (Δ : length of a time step):

$$Y \equiv \frac{a \cdot z_y \cdot \Delta}{\sqrt{2/3}}$$

$$U \equiv \frac{b \cdot z_u \cdot \Delta}{\sqrt{2/3}} \tag{18}$$

where a and b are assumed to be different from 0. For each process r and u, the modeled states $z_{(\cdot)} \in \{z_u, z_u\}$ satisfy:

$$\frac{-\sqrt{2/3}}{a \cdot \Delta} \leq z_{(\cdot)} \leq \frac{\sqrt{2/3}}{a \cdot \Delta} \tag{19}$$

Thus, (18) is a standardization of z_y and z_u on the interval [–1, 1]. This standardization has the advantage that each martingale transition probability can be expressed as a function of Y and U only instead of the respective state of the random variable, the reversion speed, and the length of the time step.

According to (10) and (14), a correlation mismatch is avoided if

$$\begin{aligned} &\rho_{yu} \leq \min\left[9 \cdot \pi_{um}, 36 \cdot \pi_{ud}, 9 \cdot \pi_{mu}, 9 \cdot \pi_{md}, 36 \cdot \pi_{du}, 9 \cdot \pi_{dm}\right] \\ &\quad \text{if } \rho_{yu} \geq 0 \\ &\rho_{yu} \geq \max\left[-36 \cdot \pi_{uu}, -9 \cdot \pi_{um}, -9 \cdot \pi_{mu}, -9 \cdot \pi_{md}, -9 \cdot \pi_{dm}, -36 \cdot \pi_{dd}\right] \\ &\quad \text{if } \rho_{yu} < 0 \end{aligned} \tag{20}$$

In (20) we simplify the notation so that the following holds: $\pi_{uu} \equiv \pi_{uu}(Y, U)$, $\pi_{um} \equiv \pi_{um}(Y, U)$, $\pi_{ud} \equiv \pi_{ud}(Y, U)$, $\pi_{mu} \equiv \pi_{mu}(Y, U)$, $\pi_{mm} \equiv \pi_{mm}(Y, U)$, $\pi_{md} \equiv \pi_{md}(Y, U)$, $\pi_{du} \equiv \pi_{du}(Y, U)$, $\pi_{dm} \equiv \pi_{dm}(Y, U)$, and $\pi_{dd} \equiv \pi_{dd}(Y, U)$, and use these definitions henceforth. The smaller the Δ, the denser the set of Y and U on the interval [–1, 1].

Exhibit 2 depicts the absolute values of the minimum synthetic correlations for a very small time step Δ in the HW2 tree if $\rho_{yu} < 0$ for 10,201 combinations of Y and U. A similar looking graph with the same structure can be calculated for the maximum allowed synthetic correlation if $\rho_{yu} > 0$.

It becomes clear that the convergence properties should be more favorable, the lower the absolute value of the synthetic correlation. The higher the absolute value of the synthetic correlation, the more states reveal distortions. This is why the convergence properties of the parameter set considered are different from the observations of Hull and White [1994b]. The synthetic correlation of –0.998 is close to –1. Hence, there are only a few states that capture the synthetic correlation correctly.

For the great majority of states, the synthetic correlation is noticeably mismatched. Even states that are close to the origin (i.e., $Y = U = 0$) and that are therefore highly probable do not incorporate the correct synthetic correlation for (a, $b > 0$). Consequently, the implicit volatilities and correlations between dr and du diverge considerably from the initially specified σ_r and ρ_{ru} in most states.

Exhibit 3 plots the given parameters for $\sigma_r^*(Y, U)$. It shows that the implied volatilities range from 0.85% (the initially specified value) to 15.09%. This produces substantial overassessment of the option value because of the positive relationship between option value and short rate volatility. For the coefficient of correlation between dr and du, values range between 0.31 and 0.72, compared to the initially specified 0.67.

The synthetic correlation in Hull and White [1994b] is close to 0.124, which implies that the correlation is biased only in states that are unlikely to occur under the martingale measure. Hence the derivatives prices should

EXHIBIT 2

Absolute Values of Minimum Feasible Correlations on Normalized Scale ($\rho_{yu} < 0$)

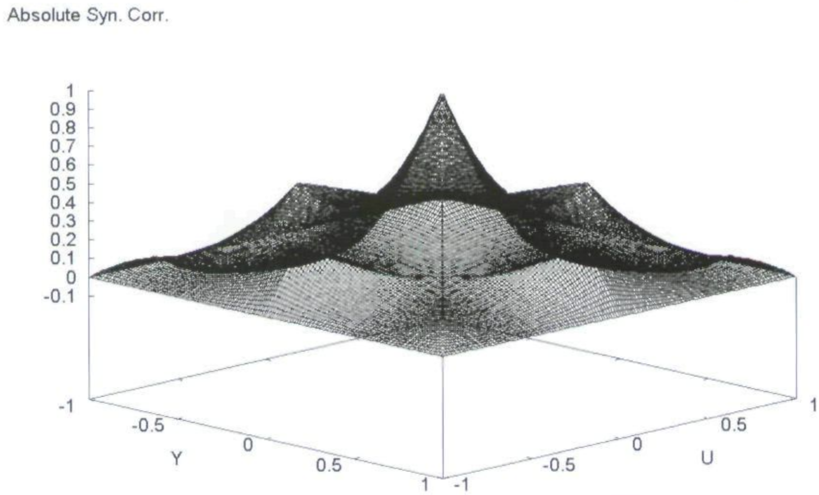
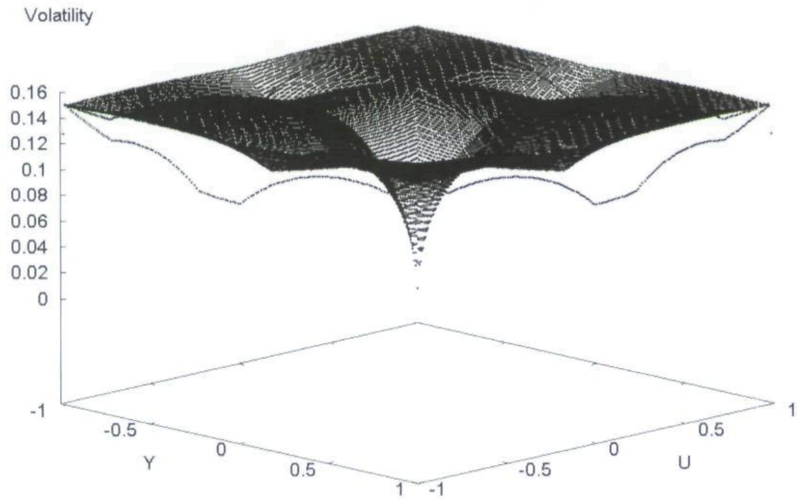


EXHIBIT 3

Implicit Local Volatility for Given Parameter Set and Time Step ($\Delta = 0.025$ Years)



be only slightly distorted. The algorithm converges well for their parameters.

IV. MODIFIED ALGORITHM

Extending the set of possible synthetic correlations in each state must improve the numerical quality of the discrete solutions. We argue that states to which relatively much of the probability mass is assigned are of particular importance. Ideally, the algorithm can be changed so that, in each state, the feasible range for synthetic correlations is $[-1, 1]$. Therefore, we generalize the matrix $\mathbf{P}_{adj}$ of transition probabilities in (10) and obtain $\mathbf{P}_{gen}$:

$$\begin{aligned}
\mathbf{P}_{gen} &= \begin{bmatrix} \pi_{uu} + \epsilon_{uu}e & \pi_{um} + \epsilon_{um}e & \pi_{ud} + \epsilon_{ud}e \\ \pi_{mu} + \epsilon_{mu}e & \pi_{mm} + \epsilon_{mm}e & \pi_{md} + \epsilon_{md}e \\ \pi_{du} + \epsilon_{du}e & \pi_{dm} + \epsilon_{dm}e & \pi_{dd} + \epsilon_{dd}e \end{bmatrix} \\
&= \begin{bmatrix} \pi_{uu} + \epsilon_{uu}e & \pi_{um} - (\epsilon_{uu} + \epsilon_{ud})e & \pi_{ud} + \epsilon_{ud}e \\ \pi_{mu} - (\epsilon_{uu} + \epsilon_{du})e & \pi_{mm} + s_{mm}e & \pi_{md} - (\epsilon_{ud} + \epsilon_{dd})e \\ \pi_{du} + \epsilon_{du}e & \pi_{dm} - (\epsilon_{du} + \epsilon_{dd})e & \pi_{dd} + \epsilon_{dd}e \end{bmatrix} \\
s_{mm} &= (\epsilon_{uu} + \epsilon_{du} + \epsilon_{ud} + \epsilon_{dd})
\end{aligned} \tag{21}$$

Furthermore, we define the set $I = \{uu, um, ..., dd\}$. The transformation in (21) follows due to the requirement that the ϵ coefficients must add up to 0 in each row and column. This guarantees that the drift and variance of dr and du remain unchanged after the adjustment.

According to (21), only four of the nine ϵ coefficients can be varied, i.e., ϵ_{uu}, ϵ_{ud}, ϵ_{du}, and ϵ_{dd}. Hull and White [1994b] determine the jump width in the trinomial tree for Δ_y and Δ_u as $\sqrt{3 \cdot \Delta} \cdot \sigma$, $\sigma \in \{\sigma_y, \sigma_u\}$. Accordingly, generalizing Hull and White's results in (15), the implied synthetic correlation can be written as:

$$\rho^*_{yu}(\epsilon_{uu}, \epsilon_{ud}, \epsilon_{du}, \epsilon_{dd}) = 3 \cdot (\epsilon_{uu} - \epsilon_{ud} - \epsilon_{du} + \epsilon_{dd}) \cdot e \quad (22)$$

To improve the accuracy of the HW2 algorithm, a tuple $(\epsilon_{uu}, \epsilon_{ud}, \epsilon_{du}, \epsilon_{dd})$ has to be found so as to extend the feasible set of ρ^*_{yu} as far as possible. The minimum and maximum synthetic correlation are characterized by the fact that for the given state at least one transition probability becomes equal to zero:

$$\exists \pi_i : \pi_i + \epsilon_i \cdot e = 0, \quad i \in I \quad (23)$$

For each transition probability π_i, $i \in I$, in each state we can solve for the minimum synthetic correlation lb_i, according to:

$$lb_i = \min_{\epsilon_{uu}, \epsilon_{ud}, \epsilon_{du}} \rho^*_{yu}(\epsilon_{uu}, \epsilon_{ud}, \epsilon_{du}, c) \quad (24)$$

$$s.t. \quad \pi_i = 0$$

$$\forall \pi_j : \pi_j \geq 0, \quad j \in \{I \setminus i\}$$

where ρ^*_{yu} is given by (22}, $e = -(\pi_i/\epsilon_i)$, and the constant $c \in \mathbb{R}$ replaces ϵ_{dd} since scaling of the tuple $(\epsilon_{uu}, \epsilon_{ud}, \epsilon_{du}, \epsilon_{dd})$ has no effect on the value of the synthetic correlation.

For each state, we obtain nine optimization results. The minimum feasible synthetic LB for the state is therefore given by

$$LB = \min\{lb_{uu}, ..., lb_{dd}\} \quad (25)$$

The ϵ coefficients are chosen so that the minimum synthetic correlation is equal to LB. Exhibit 4 shows the optimized parameters ϵ_{uu} for different states.

In the case of positive synthetic correlations for each transition probability π_i, $i \in I$, we search for the maximum synthetic correlation ub_i, solving:

$$ub_i = \max_{\epsilon_{uu}, \epsilon_{ud}, \epsilon_{du}} \rho^*_{yu}(\epsilon_{uu}, \epsilon_{ud}, \epsilon_{du}, c) \quad (26)$$

$$s.t. \quad \pi_i = 0$$

$$\forall \pi_j : \pi_j \geq 0, \quad j \in \{I \setminus i\}$$

The maximum feasible synthetic correlation is:

$$UB = \max\{ub_{uu}, ..., ub_{dd}\} \quad (27)$$

For both constraint optimization problems we can derive closed-form solutions by applying Kuhn-Tucker conditions. The necessary computations can be carried out quickly, and there is negligible loss of numerical efficiency in comparison to the original algorithm.

V. OPTIMIZATION RESULTS

We calculate the minimum possible synthetic correlation on a grid of 10,201 combinations of Y and U according to optimization problem (25). For each state, we obtain the corresponding unadjusted transition probabilities to find the optimal ϵ coefficients.

Exhibit 5 summarizes the results. A similar graph can be obtained for the maximum allowable synthetic correlation.

Obviously the modified algorithm clearly allows for much smaller correlations than the original HW2 methodology. Furthermore, on the whole diagonal of the standardized values from $Y = -1$, $U = 1$ to $Y = 1$, $U = -1$, synthetic correlations of -1 are feasible, which enhances pricing accuracy. The smaller the synthetic correlation, the more probability mass of the unconditional measure should be assigned to the set of states along the diagonal.

Although we may observe large distortions for the other states if the synthetic correlation approaches -1, they should have only a minor influence on the option price as they are unlikely to occur under the martingale measure. Similar results are obtained when we solve the optimization problem (28) for the maximum achievable synthetic correlation. The graph looks like the one depicted in Exhibit 5, where along the whole diagonal of standardized values from $Y = -1$, $U = -1$ to $Y = 1$, $U = 1$, synthetic correlations of 1 are feasible.

EXHIBIT 4

Solutions for ϵ_{uu} of Minimization Problem (25) on Mesh of $Y - U$ State-Combinations

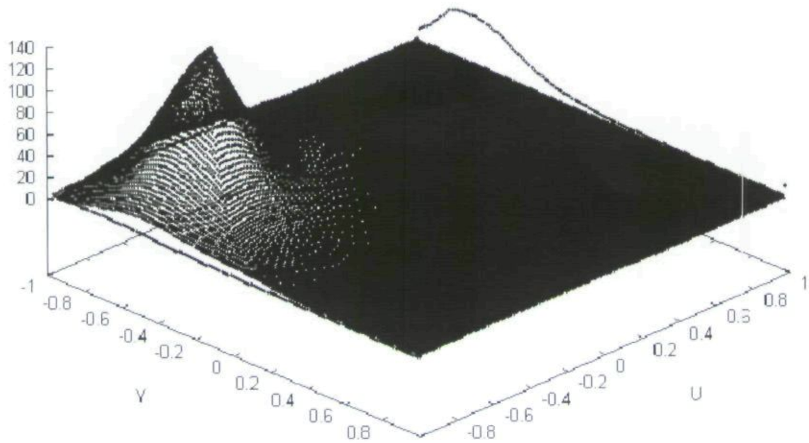
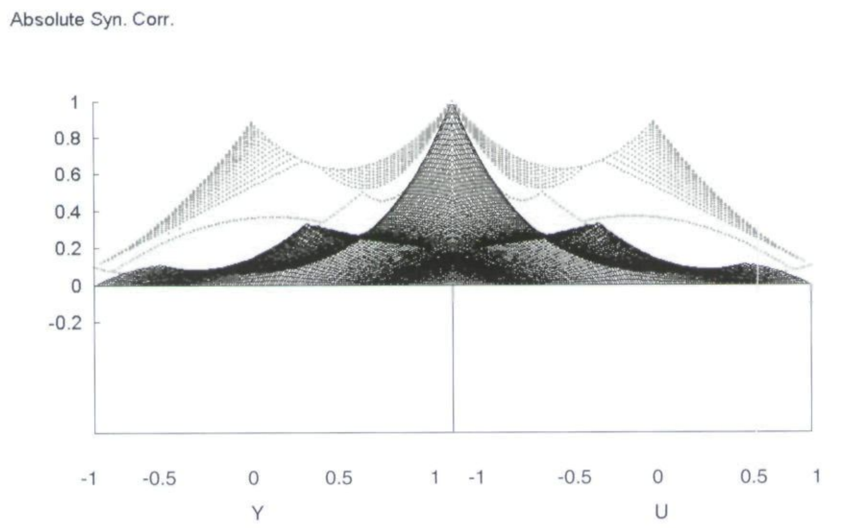


EXHIBIT 5

Comparison of Minimum Feasible Synthetic Correlations



Absolute values (light) in comparison to original HW2 implementation (dark).

Recall that the parameter choices are $a = 0.47$, $b = 0.38$, $\sigma_r = 0.85\%$, $\sigma_r = 0.99\%$, and $\rho_{ru} = 0.67$. If we compare the results for the caplet prices according to the modified algorithm given in Exhibit 6 to those in Exhibit 1, we see a substantial improvement over the traditional HW2 approach. While the valuation error in the HW2 approach for a strike level of, e.g., 5% varies between 23% and 95%, our valuation error drops to between 1% and 18% (depending on the length Δ of the time step considered). Similar results are seen for different strike levels.

This shows that our algorithm gives a better indication of the true price of fixed-income derivative securities.

VI. CONCLUSION

An examination of how well the HW2 tree methodology produces prices that converge with analytic caplet prices shows that the numerical quality of the discrete solutions depends crucially on the parameter set. We explain why the HW2 algorithm converges weakly to the analytic solution if it assumes very low (or high) synthetic correlations.

We modify the algorithm by choosing an optimized tuple of ϵ coefficients for each Y and U value, and a discrete implementation does not lose much of its numerical efficiency because the optimization can be carried

EXHIBIT 6

Analytic versus Discrete Option Values (% error) Using Modified Tree

Strike	Analytic value	Tree, $\Delta = 0.25$	Tree, $\Delta = 0.05$	Tree, $\Delta = 0.025$	Tree, $\Delta = 0.01$
4.00%	1.3983	1.5790	1.4445	1.4122	1.4092
		(12.92%)	(3.30%)	(0.99%)	(0.78%)
5.00%	1.0399	1.2278	1.1248	1.0781	1.0543
		(18.06%)	(8.16%)	(3.67%)	(1.38%)
6.00%	0.7163	0.8768	0.8086	0.7766	0.7403
		(22.41%)	(12.90%)	(8.42%)	(3.35%)
7.00%	0.4477	0.5402	0.5060	0.4916	0.4732
		(20.66%)	(13.02%)	(9.80%)	(5.69%)
8.00%	0.2490	0.3002	0.2879	0.2810	0.2695
		(20.58%)	(15.62%)	(12.85%)	(8.22%)
9.00%	0.1210	0.2138	0.1925	0.1741	0.1473
		(76.67%)	(59.05%)	(43.88%)	(21.69%)
10.00%	0.0507	0.1865	0.1335	0.0978	0.0658
		(268.21%)	(163.54%)	(92.99%)	(29.91%)

out analytically. The results show better convergence of estimates using the modified algorithm.

APPENDIX

Variance of the Bond Price

The total variance $d[P(t, S)/(P(t, T)]$ where $0 \leq t \leq T \leq S$ is:[5]

$$\sigma_B^2(T,S) = \frac{\sigma_r^2}{2a} \cdot B(T,S)^2 \cdot \left(1 - e^{-2aT}\right)$$
$$+\sigma_u^2 \cdot \left(\frac{U^2}{2a} \cdot \left(e^{2aT} - 1\right) + \frac{V^2}{2b} \cdot \left(e^{2bT} - 1\right) - 2 \cdot \frac{UV}{a+b} \cdot \left(e^{T(a+b)} - 1\right)\right)$$
$$+2 \cdot \frac{\rho_{ru} \cdot \sigma_r \cdot \sigma_u}{a} \cdot \left(e^{-aT} - e^{-aS}\right) \times \left(\frac{U}{2a} \cdot \left(e^{2aT} - 1\right) - \frac{V}{a+b} \cdot \left(e^{T(a+b)} - 1\right)\right)$$

where

$$B(t,T) = \frac{1}{a} \cdot \left(1 - e^{-a \cdot (T-t)}\right)$$
$$U = \frac{1}{a \cdot (a-b)} \cdot \left(e^{-aS} - e^{-aT}\right)$$
$$V = \frac{1}{b \cdot (a-b)} \cdot \left(e^{-bS} - e^{-bT}\right)$$

ENDNOTES

The authors are grateful for the comments and suggestions of John C. Hull, Philipp J. Schönbucher, and William T. Ziemba. They also thank participants in the 5th conference of the Swiss Society for Financial Market Research at Basel, the 9th Annual Meeting of the German Finance Association at Cologne, and the workshop "New Directions for Financial Risk Management" of the Center of Financial Studies at Frankfurt, as well as seminar participants at the WHU—Otto Beisheim Graduate School of Management and the TU München (particularly Rudi Zagst) for their comments. The financial support of Dresdner Bank AG is gratefully acknowledged.

[1]See Longstaff and Schwartz [2001] for details on the pricing of American options using Monte Carlo simulations.

[2]See Hull and White [1994a] for determination of the martingale probabilities in a single-factor framework.

[3]See, e.g., Rebonato [1998, p. 305] and Hull and White [1994b, Appendix B.]

[4]See Hull and White [1994b, Exhibit 4]. They assume: $a = 3$, $b = 0.1$, $\sigma_r = 1.00\%$, $\sigma_u = 1.45\%$, and $\rho_{ru} = 0.6$.

[5]See Hull and White [1994b].

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