Pricing Multiname Default Swaps with Counterparty Risk

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Agents entering into an unfunded swap with a default-risky counterparty must consider the possibility that the counterparty may default before the maturity of the contract. In this case, some or all of the contractual obligations will be left unfulfilled, and there may be a mark-to-market loss. Examples are:

1. A first-to-default protection buyer defaults before maturity and before the default of the reference name.
2. A tranche protection seller defaults before maturity and before the notional exposure is completely eroded by defaults of the reference names.

We present a simple and general methodology for pricing counterparty risk in multiname default swaps, such as n-th-to-default baskets and portfolio loss tranches. The objective is to derive pricing results without having to compute explicit mark-to-market of the swap in each state of counterparty default. In other words, we are looking for a way to price counterparty risk without having to: 1) calculate the conditional distribution of the default times of the surviving reference credits at some future date; and 2) use this conditional joint distribution to perform a forward valuation.

The payoffs of multiname credit derivatives are generally linked to dozens—sometimes hundreds—of credits, and nested valuations are highly impractical. One simplification is to neglect the dependence between the counterparty and the reference credits altogether, which can produce large pricing errors.

To see this, consider two names, each with a 1% default probability over a fixed horizon, and assume that their dependence can be appropriately described by a Gaussian copula model with 20% asset correlation. If we condition on the default of the first name, the probability of default of the second increases from 1% to more than 3%. With 40% asset correlation, which is not unrealistic, the conditional default probability rises to more than 8%. Clearly, to price counterparty risk, one has to take into consideration that positive dependence between the counterparty and the reference names means an above-average number of defaults in the reference portfolio in every state in which the counterparty defaults.

The methodology we describe is general and can be used in conjunction with any underlying default model (structural, reduced-form, or hybrid). Our examples are based on a simple and computationally efficient time-to-default simulation that generates default dependencies among the reference names and the default-risky counterparty by an appropriately calibrated Gaussian copula function. Allowing for different dependence structures does not affect the applicability of the proposed solution.

I. FAIR DEFAULT SWAP SPREAD WITHOUT COUNTERPARTY RISK

We start with a brief description of synthetic loss tranches and review their standard pricing equation when both counterparties are default-free. All the results can be immediately applied to n-th-to-default baskets.

Portfolio loss tranches are popular contracts in which a protection buyer pays a periodic premium to a protection seller who in exchange stands ready to compensate the buyer for a predefined slice of the losses affecting a portfolio of reference names. Exhibit 1 shows the mechanics of a typical mezzanine tranche swap for a portfolio of 100 credits, each with \$10 million notional.

The protection buyer pays an annual premium, generally quoted as a percentage of the tranche's outstanding notional at each payment date. The investor agrees to compensate the protection buyer for losses affecting the reference portfolio above the attachment point of \$50 million, and up to \$150 million, for a total exposure of \$100 million.

Suppose a default-free agent is buying protection on a loss tranche from a default-free counterparty. If we denote by A the dollar exposure provided by the tranche, and by $L(t)$ the cumulative dollar loss on the tranche at time t, we can compute the fair compensation for this exposure as the spread S that solves:[1]

$$SE\left[\sum_{i=1}^{M} B(t_i)(A - L(t_i))\right] = E\left[\int_0^{t_M} B(t)dL(t)\right] \qquad (1)$$

Equation (1) equates the premium leg to the protection leg of the swap. The premium S is paid on the outstanding notional $(A - L(t_i))$ at a set of predetermined dates t_i, $i = 1, 2, 3, \ldots, M$, while protection is paid at the time the loss is incurred. The expectation is taken with respect to the pricing measure, and all payments are discounted using the term structure of risk-free factors $B(t)$.

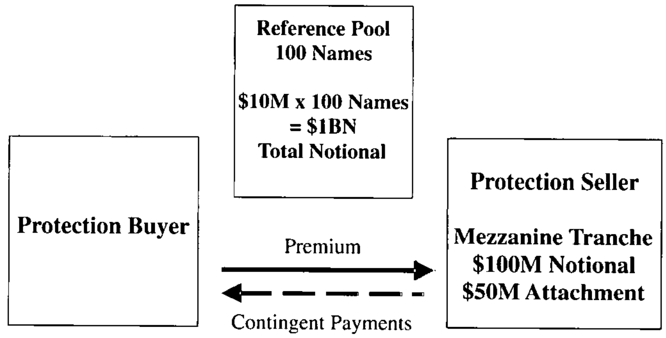
Notice that Equation (1) holds regardless of the assumptions made with respect to the sources of randomness (default times, recoveries, and risk-free rates may all be stochastic) and the model used to generate their (risk-neutral) joint distribution.

To simplify the notation, we use definitions as follows:

$$N(j,k) \equiv \sum_{i=j}^{k} B(t_i)(A - L(t_i)) \text{ for } j \leq k$$

EXHIBIT 1

Mechanics of Typical Tranched Portfolio Swap



indicates the sum of discounted outstanding notional of the tranche between the j-th and the k-th payment dates, and

$$P(u,v) \equiv \int_u^v B(t)dL(t) \text{ for } u \leq v$$

denotes the sum of discounted payments received by the protection buyer between $t = u$ and $t = v$. Notice that the distributions of the random variables N and P depend on the characteristics of the reference credits, such as their market spreads and default correlations, through the dependence on the portfolio loss distribution L.[2]

Using these definitions, Equation (1) can be rewritten as:

$$SE[N(1,M)] = E[P(0,t_M)] \qquad (2)$$

Note that this is the standard pricing equation for an n-th-to-default basket swap if we substitute:

$$(1 - P_n(t_i)) \text{ for } (A - L(t_i))$$

and

$$L_n dP_n(t) \text{ for } dL(t)$$

where $P_n(t)$ is a point process that jumps from 0 to 1 at the time of default of the n-th reference name, and L_n is the loss, given default of the n-th defaulter (notice that, as long as default times are random, L_n is stochastic even if the loss given default of each reference name is assumed to be deterministic).

II. FAIR DEFAULT SWAP SPREAD WITH COUNTERPARTY RISK

We next generalize the pricing of a tranched portfolio swap when one of the counterparties is subject to default risk. For ease of exposition, assume that the default time of the risky counterparty is positively correlated with the default times of the names in the reference set. This is very often the case, as default events clearly have a systematic component. The arguments can be adapted to a general dependence structure of default times.

Default-Free Protection Buyer, Default-Risky Protection Seller

To price a swap between a default-free protection buyer and a default-risky protection seller, we need to specify an assumption regarding the cash flows at the time of default of the risky counterparty. We allow for an asymmetric recovery assumption $\{Q, R; 0 \leq Q \leq R \leq 1\}$, which indicates that the risk-free protection buyer will recover a fraction Q of the marked-to-market value of the trade if this is in her favor, but will pay a possibly higher fraction R of the mark-to-market if this is against her.

Note that we allow for Q and R to be stochastic. Therefore, when we write $Q \leq R$ or $Q = R$, we mean that the (in)equality holds for every element in the state space. We denote by $S_{Q,R}$ the fair spread paid to a default-risky investor with a $\{Q,R; 0 \leq Q \leq R \leq 1\}$ settlement in case of default.[3]

For some necessary definitions, we introduce the additional notation:

τ^* = default time of the default-risky protection seller;

$\tau = \min\{\tau^*, t_M\}$; and

$z = \max\{j \in (0, 1, 2, ..., M): t_j \leq \tau\}$

In words, τ denotes the default time of the counterparty or the maturity of the deal, whichever comes first, and t_z is the last payment date before τ.

CPL is the (discounted) loss on the swap for the protection buyer as a result of the counterparty's default:

$$CPL(S_{Q,R}) = P(\tau, t_M) - S_{Q,R}N(z + 1, M)$$

Finally, we define the (discounted) mark-to-market of the swap for the protection buyer at the time of the counterparty default, *MTM*, as

$$\begin{aligned} MTM(S_{Q,R}) &= E[CPL(S_{Q,R})|\Im(\tau)] \\ &= E[P(\tau, t_M) - S_{Q,R}N(z + 1, M)|\Im(\tau)] \end{aligned} \qquad (3)$$

where the expectation is taken with respect to the pricing measure, and $\Im(\tau)$ denotes the information available at τ.

Equation (3) states that the value of the swap at τ will be given by the difference between the values of the protection leg and the premium leg at τ, and that the valuation is conditional on the information $\Im(\tau)$ available at that time.

The pricing equation for a default swap between a default-free protection buyer and a default-risky protection seller is:

$$\begin{aligned} S_{Q,R}E[N(1, z)] = {} & E[P(0, \tau)] + \\ & E[\min(QMTM(S_{Q,R}), RMTM(S_{Q,R}))] \end{aligned} \qquad (4)$$

Equation (4) equates the present value of the premium paid up to τ to the present value of the protection received up to τ. The second term on the right-hand side indicates the present value of the (possibly negative) amount recovered by the protection buyer in case of default of the counterparty.

Our goal is to price counterparty risk without explicitly calculating the mark-to-market of the swap in each state where the counterparty defaults. This would require, for each state, a derivation of the joint distribution of all stochastic variables conditional on the information set $\Im(\tau)$. For instruments with many underlyings, such as portfolio loss tranches, this would be computationally expensive (see Schönbucher and Schubert [2002]).

It turns out that the task is simple if we impose the symmetric restriction $Q = R$, and that the solution to this symmetric problem paves the way for handling the more general (and more realistic) asymmetric recovery. We analyze these two cases separately.

Symmetric Case: $0 \leq Q = R \leq 1$.

When $0 \leq Q = R = T \leq 1$, Equation (4) simplifies to:

$$S_{T,T}E[N(1, z)] = E[P(0, \tau)] + E[TCPL(S_{T,T})]$$

where we have used the law of iterated expectations and the fact that T is known at τ to conclude that:

$$E[TMTM(S_{T,T})] = E[TCPL(S_{T,T})]$$

Substituting for CPL and collecting terms gives:

$$S_{T,T}E\left[N(1,z)+TN(z+1,M)\right]=$$
$$E\left[P(0,\tau)+TP(\tau,t_M)\right] \qquad (5)$$

Equation (5) shows that the fair spread of this symmetric swap is equivalent to the fair spread on a step-down swap whose payments on both legs jump to T times the contractual payments at the time of default of the risky counterparty.

Once we specify the risk-neutral joint distribution of the stochastic variables in the model, given the information available today, it is straightforward to compute $S_{T,T}$. The expectations in (5) can be obtained by simulating the stochastic variables in the model, calculating discounted premium and protection payments before and after the counterparty default along each path, and averaging across paths.

Notice also that when $Q = R = T = 1$, Equation (5) simplifies to (2), and $S_{1,1}$ is therefore equal to S, the fair swap spread to be paid to a default-free protection seller. The intuition behind this result should be clear; if any amount due in the case of default is fully recovered by the creditor, counterparty risk cannot have any impact on fair pricing.

At the other extreme, when $Q = R = T = 0$, the swap knocks out and there is no additional inflow (or outflow) if the risky counterparty defaults. The fair value of the knockout spread $S_{0,0}$ is determined by:

$$S_{0,0}E\left[N(1,z)\right]=E\left[P(0,\tau)\right]$$

Asymmetric Case: $0 \le Q \le R \le 1$. An agent entering into a swap with a default-risky protection seller is generally concerned about the asymmetry of the potential mark-to-market settlement. It is likely that only a small fraction of a favorable mark would be recovered if the counterparty defaults, while it is possible that a negative mark would have to be paid in full. Collateral agreements can mitigate this problem, but funding the trade may offset the economic rationale for the risky investor. For the pricing to reflect this asymmetry, we need to be able to deal with Equation (4) under the assumption that $Q \le R$.

Solving for $S_{Q,R}$ exactly when $Q \le R$ requires calculation of the mark-to-market of the swap at τ, and this is computationally expensive even with the simplest model. Our strategy, therefore, will be to find bounds for $S_{Q,R}$ and use their midpoint as an estimate; our examples show that the proposed bounds are in fact very tight, so that the resulting maximum estimation error turns out to be very small.

The next two propositions identify an upper bound $S^U_{Q,R}$ and a lower bound $S^L_{Q,R}$, such that:

$$S^L_{Q,R} \le S_{Q,R} \le S^U_{Q,R}$$

Proposition 1: An upper bound $S^U_{Q,R}$ is given by $S_{Q,Q}$.

First, observe that any symmetric solution $S_{T,T}$, $Q \le T \le R$, is an upper bound for $S_{Q,R}$. Intuitively, the default-free protection buyer is better off if:

- The fraction of an unfavorable mark-to-market that she would have to pay in case her counterparty defaults declines from R to T; and
- The fraction of a favorable mark-to-market that she would recover increases from Q to T.

Therefore, the fair spread to be paid to the risky protection seller has to increase.

Next, notice that with positive dependence between the default time of the protection seller and the default times of the reference names, the mark-to-market in the event of default of the risky counterparty is expected to be in favor of the risk-free protection buyer. Thus $S_{Q,Q}$ is the lowest of all upper bounds $S_{T,T}$, $Q \le T \le R$, and therefore the one of interest to us.

Proposition 2: A lower bound $S^L_{Q,R}$ can be computed as the solution to:

$$S^L_{Q,R}E\left[N(1,z)\right]=E\left[P(0,\tau)\right]+$$
$$E\left[\min\left(QCPL\left(S^L_{Q,R}\right),RCPL\left(S^L_{Q,R}\right)\right)\right] \qquad (6)$$

Equation (6) is analogous to (4), with the only difference that we have substituted CPL for MTM. By the law of iterated expectations and Jensen's inequality, we have, for any S:[4]

$$E\left[\min(QCPL(S),RCPL(S))\right]=$$
$$E\left[E\left[\min(QCPL(S),RCPL(S))|\Im(\tau)\right]\right]\le$$
$$E\left[\min\left(QE\left[CPL(S)|\Im(\tau)\right],RE\left[CPL(S)|\Im(\tau)\right]\right)\right]=$$
$$E\left[\min(QMTM(S),RMTM(S))\right]$$

where we have used the fact that Q and R are known at τ, and the observation that, for $0 \leq K_1 \leq K_2 \leq 1$ and $x \in \Re$, $\min(K_1 x, K_2 x)$ is a concave function of x.

From a comparison of Equations (4) and (6), we conclude that:

$$S^L_{Q,R} \leq S_{Q,R}$$

The crucial advantage of our methodology is that these bounds can be obtained very easily, without explicitly calculating the value of the swap in the states when the risky counterparty defaults. The proposed upper bound $S^U_{Q,R} = S_{Q,Q}$ can be computed as the solution to Equation (5). The lower bound, $S^L_{Q,R}$, can be obtained by simulating the stochastic variables in the model, computing discounted premium and protection payments before and after the counterparty default along each path, and numerically solving for the value of $S^L_{Q,R}$ that solves Equation (6), after substituting for CPL:

$$S^L_{Q,R} E[N(1,z)] = E[P(0,\tau)] + E\left[\min\begin{pmatrix} Q\left(P(\tau, t_M) - S^L_{Q,R} N(z+1, M)\right), \\ R\left(P(\tau, t_M) - S^L_{Q,R} N(z+1, M)\right) \end{pmatrix}\right]$$

Window Risk. We have assumed that if the risky investor defaults, it will be when the investor does not owe any money to the protection buyer beyond, possibly, the mark-to-market of the swap. In reality, there is generally a lag between the time a loss affecting the tranche is experienced and the time the protection seller fulfills the associated obligation. If the counterparty defaults during this period, the protection buyer will suffer a direct loss that must be added to the potential cost of replacing the counterparty for fulfillment of the remaining contractual obligations.

If we denote by Δ the length of this window, we can see that the fair spread $S_{Q,R}$ now solves:

$$\begin{aligned} S_{Q,R} E[N(1,z)] = {} & E[P(0,\tau)] + \\ & E\left[\min\left(QMTM(S_{Q,R}), RMTM(S_{Q,R})\right)\right] - \\ & E\left[B(\tau)(1-Q)\left(L(\tau) - L(\tau-\Delta)\right)\right] \end{aligned} \qquad (7)$$

The longer the window, the lower the fair spread will be. Bounds for $S_{Q,R}$ can be computed as explained earlier.

Notice that modeling window risk this way lets us account for counterparty risk when the default-risky protection seller funds the losses not throughout the life of the deal, but only at maturity. In this case, we would need to set $\Delta = \tau$ in Equation (7) and, by definition of cumulative loss, $L(0) = 0$.

Default-Risky Protection Buyer, Default-Free Protection Seller

Analogous arguments can be used to find bounds for the fair spread paid by a default-risky protection buyer to a default-free protection seller. We modify our definitions so that now:

τ^* = default time of the default-risky protection *buyer*;

$\tau = \min\{\tau^*, t_M\}$; and

$z = \max\{j \in (0, 1, 2, \ldots, M): t_j \leq \tau\}$.

A realistic recovery assumption can be represented by $\{R, Q; 0 \leq Q \leq R \leq 1\}$, where we denote by Q the fraction of a favorable mark-to-market that the default-free protection *seller* would recover, and by R the possibly higher fraction of an unfavorable mark that would have to be paid in the event of default of the risky counterparty.

The equation for the fair swap spread $S_{R,Q}$ is now given by:

$$\begin{aligned} S_{R,Q} E[N(1,z)] = {} & E[P(0,\tau)] + \\ & E\left[\max\left(QMTM(S_{R,Q}), RMTM(S_{R,Q})\right)\right] \end{aligned} \qquad (8)$$

Equation (8) equates the present value of the premium paid up to τ to the present value of the protection received up to τ. In addition, the second term on the right-hand side indicates the present value of the amount recovered (or paid) by the protection buyer in the event of default.

Following the reasoning we have established, we can identify bounds $S^L_{R,Q}$ and $S^U_{R,Q}$ such that: $S^L_{R,Q} \leq S_{R,Q} \leq S^U_{R,Q}$.

Proposition 3: A lower bound $S^L_{R,Q}$ is given by $S_{R,R}$.

Proposition 4: An upper bound $S^U_{R,Q}$ can be computed as the solution to:

$$S^U_{R,Q}E[N(1,z)] = E[P(0,\tau)] + E\left[\max\left(QCPL\left(S^U_{R,Q}\right), RCPL\left(S^U_{R,Q}\right)\right)\right]$$

Finally, note that the results obtained so far do not depend on any assumption made with respect to either the sources of randomness or the particular model used to generate their (risk-neutral) joint distribution.

Default-Risky Protection Buyer and Seller

Our methodology can also be applied to two default-risky parties. We have to distinguish between the defaults of the two counterparties, and make assumptions with respect to the mark-to-market recovery for both sides in each of the two default scenarios.

Once the events are clearly spelled out, we can use the same reasoning outlined above, and derive upper and lower bounds for the fair spread. These bounds will generally not be as tight as the bounds derived in the other cases.

III. APPLICATION: PRICING A SYNTHETIC LOSS TRANCHE

We illustrate the application of our methodology by pricing a tranched portfolio default swap. Our goal is to answer several questions:

1. What is the impact of counterparty risk on the fair spread of a synthetic loss tranche?
2. How do fair spreads vary in response to a change in the market-implied default probability of the risky counterparty?
3. How do fair spreads vary as default correlations between the risky counterparty and the reference names in the swap change?
4. How do fair spreads vary as we change assumptions regarding the mark-to-market recovery at the time of default of the risky counterparty?
5. How tight are the bounds for the fair spread of a contract with asymmetric mark-to-market settlement? In other words, what is the maximum error that we incur in order to avoid the computationally expensive valuation of the remaining portion of the swap in each state when the risky counterparty defaults?
6. Using the fair spread of a swap between two default-free counterparties as a benchmark, is there any symmetry between: a) the increase in fair spreads paid by a risky protection buyer to a risk-free protection seller; and b) the reduction in fair spreads paid by a risk-free protection buyer to a risky protection seller?

To apply our methodology, we now have to specify which variables are random, and choose a model to produce their risk-neutral joint distribution. We choose the industry standard and well-documented Gaussian copula framework to link the market-implied distributions of the default times of all defaultable credits, and add the simplifying assumptions of deterministic recovery rates and a deterministic risk-free term structure. Li [2000] shows that this approach to modeling default times can be interpreted as an extension of a one-period Merton model where default times and asset returns share the same correlation matrix.[5]

Note that the general results we have derived hold regardless of the specific distributional assumption for the default times.

Default-Free Protection Buyer, Default-Risky Protection Seller

As we have shown, we can compute bounds for the fair spread of a swap between a risk-free protection buyer and a default-risky protection seller by simulating default times, given only the information available today. The parameters for the base case are:

1. A five-year default swap referring to a pool of 100 names, each with the same notional amount.
2. Each reference credit has a deterministic recovery rate of 35%.
3. Each reference credit, as well as the default-risky protection seller, has a yearly market-implied (risk-neutral) hazard rate of 2%.
4. Asset correlations between any two credits (including the risky counterparty) are all equal to 25%.
5. Q and R are assumed to be deterministic and equal to 35% and 1, respectively. This means that the risk-free protection buyer will recover only 35% of a positive mark-to-market but will pay in full for a negative mark-to-market if the protection seller defaults.
6. Every default in the reference set is immediately covered by the protection seller. In other words, there is no window risk.
7. The risk-free curve is assumed to be deterministic. Risk-neutral cash flows are moved in time using the discount factors implied by the LIBOR curve as of October 2, 2003.

EXHIBIT 2

Fair Spreads in 50,000-Path Monte Carlo Simulation (% std err in parentheses)

	Equity (0–5%)	Mezzanine (5%–10%)	Senior (10%–100%)
Risk-Free Fair Spread	22.84% (0.44%)	6.980% (0.73%)	0.285% (1.16%)
Upper Bound for Risky Fair Spread	22.56% (0.44%)	6.627% (0.77%)	0.250% (1.27%)
Lower Bound for Risky Fair Spread	22.50% (0.44%)	6.576% (0.77%)	0.246% (1.27%)

EXHIBIT 3

Maximum Error Using Midpoint—5 Million-Path Monte Carlo Simulation

	Equity (0–5%)	Mezzanine (5%–10%)	Senior (10%–100%)
Upper Bound for Risky Fair Spread	22.5516%	6.6148%	0.2499%
Lower Bound for Risky Fair Spread	22.4986%	6.5637%	0.2463%
Midpoint	22.5251%	6.5892%	0.2481%
Max Error	2.65 bp	2.55 bp	0.18 bp

Beyond the proposed bounds for the fair spread of this swap, we calculate the spread that would be fair if both counterparties were default-free. We compute these spreads for three tranches of different seniority: an equity piece covering the first 5% of the portfolio losses; a mezzanine tranche exposing the investor to the 5%-10% slice of the losses; and a senior tranche absorbing the remaining defaults.

Exhibit 2 shows the fair spreads for these tranches obtained from a 50,000-path Monte Carlo simulation. The numbers in parentheses indicate the standard errors as percentages of the estimates, and show that 50,000 paths, requiring only a handful of seconds on a standard PC are enough to drive the standard errors to very reasonable levels.

To understand the error introduced solely by the fact that we compute bounds rather than the exact fair spread, we run a 5 million-path simulation to drive the standard errors to insignificant levels. We then compute the maximum error that we can make by estimating the fair spread using the midpoint of the computed bounds.

Exhibit 3 reports the results. It shows that the proposed bounds are quite tight. The adverse effect of the asymmetric recovery is presumably dampened by positive dependence between the default time of the protection seller and the default times of the reference credits, which implies that the mark-to-market of the swap will be in favor of the protection buyer in most states when the risky counterparty defaults.

Sensitivity analysis reveals how the bounds for the fair spreads of the three tranches described change as we vary some of the input parameters. In Exhibit 4, we let the hazard rate of the protection seller vary between 0% and 4%, while keeping the same parameters for the reference portfolio and the same correlations among all credits.

When the protection seller is risk-free, both bounds obviously coincide with the risk-free fair spread. As the hazard rate increases, both bounds drop, and the interval they span becomes wider. Interestingly, the bounds become more convex with the hazard rate of the counterparty as we move up in seniority. Because the senior investor will have to cover portfolio losses in very few scenarios, the marginal effect of an increase in default probability declines.

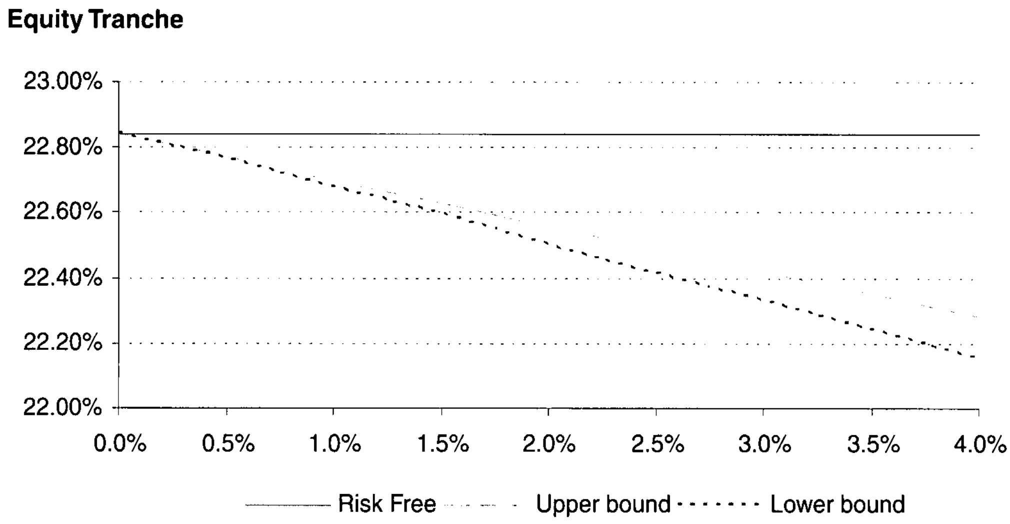
When we set R equal to one, Q measures the level of asymmetry of the mark-to-market recovery process. Exhibit 5 shows the bounds for the fair spreads as we vary Q, while keeping everything else as specified in the base case. With Q equal to one, the bounds once again coincide with the default-free fair spread, since in this case the mark-to-market at τ is always going to be fully paid. As Q drops to zero, the asymmetry increases, and the fair spread paid to the risky protection seller is reduced, to compensate the protection buyer for the higher expected loss in case of default. Both bounds increase almost linearly with Q for all tranches, and, not surprisingly, the interval they span becomes narrower as the asymmetry of the mark-to-market settlement vanishes.

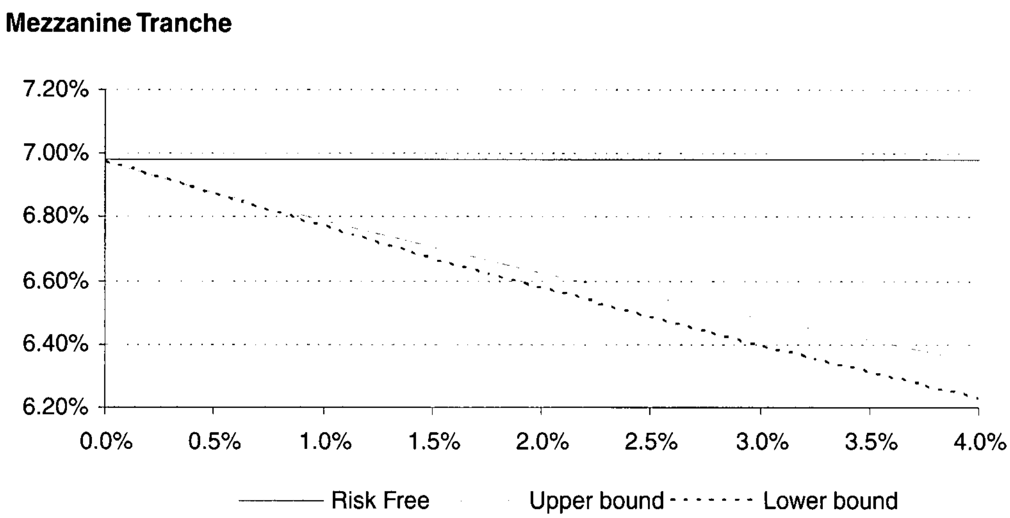
Finally, in Exhibit 6 we analyze the effect of varying the correlation between the risky protection seller and the credits in the reference portfolio, again keeping the remaining parameters (including the correlations among the names in the reference set) at the levels fixed in the base case.

The first thing to notice is that our bounds are not tight at low levels of correlation. When the default time of the counterparty is independent of the default times

EXHIBIT 4

Fair Spreads as Functions of Counterparty Riskiness





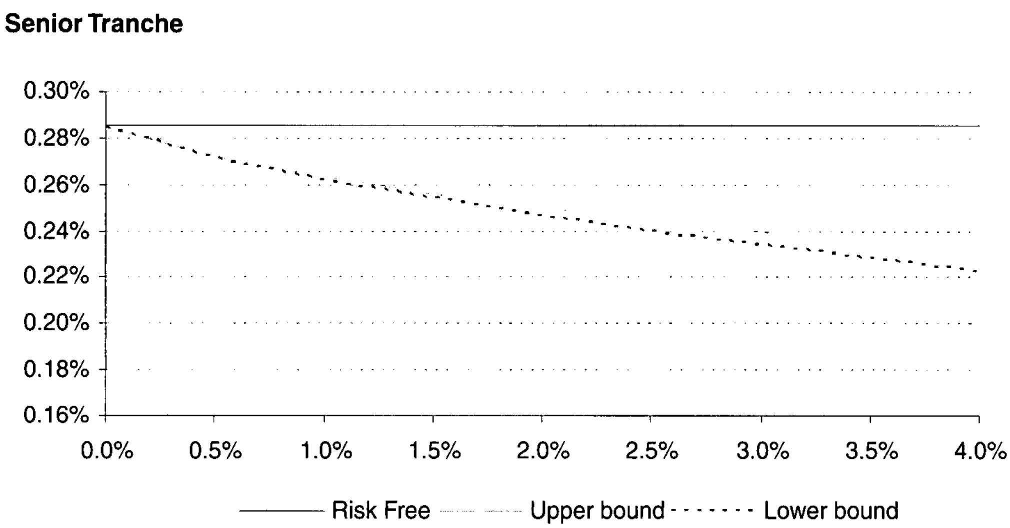


EXHIBIT 5
Fair Spreads as Functions of Asymmetry of Mark-to-Market Recovery

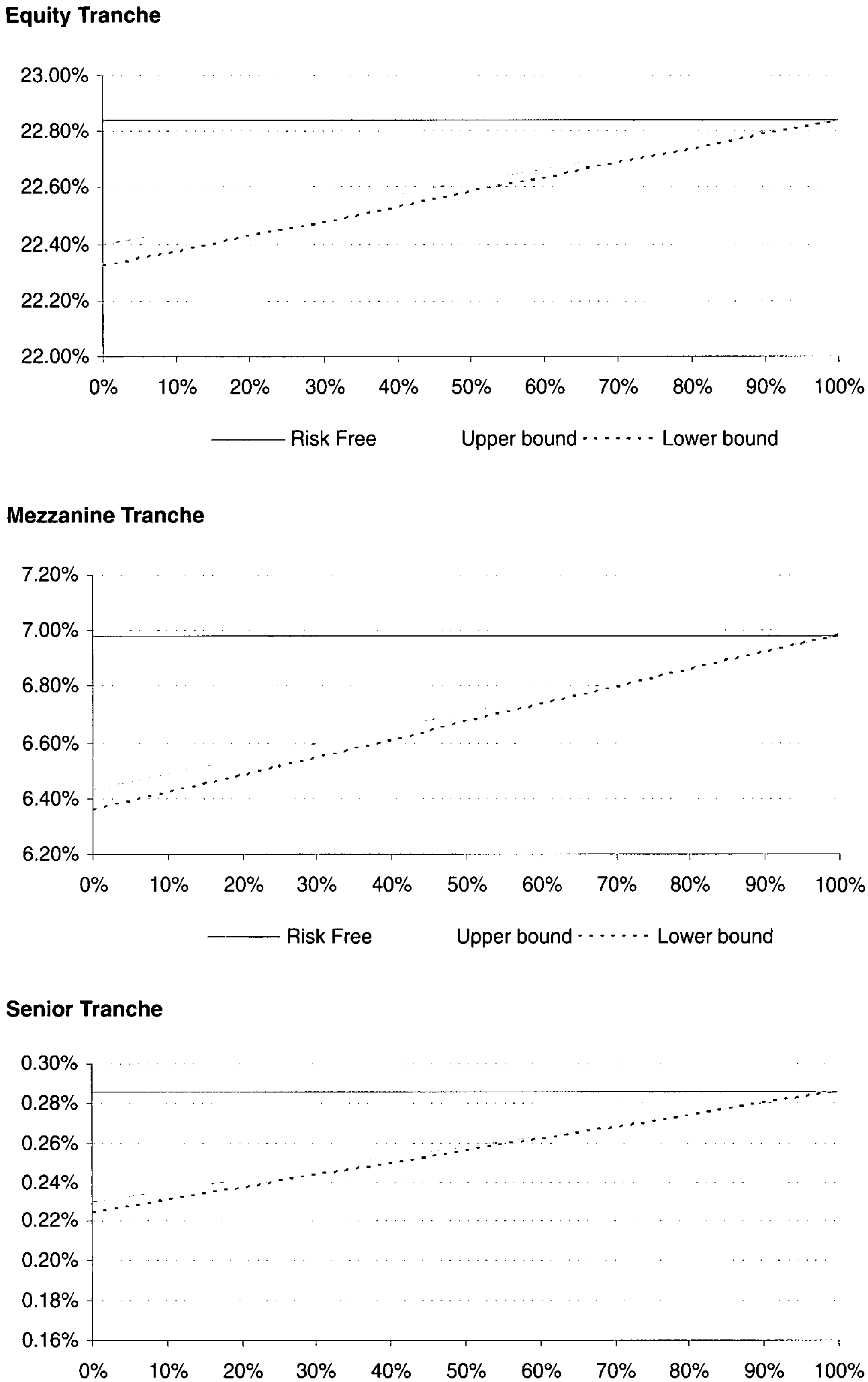
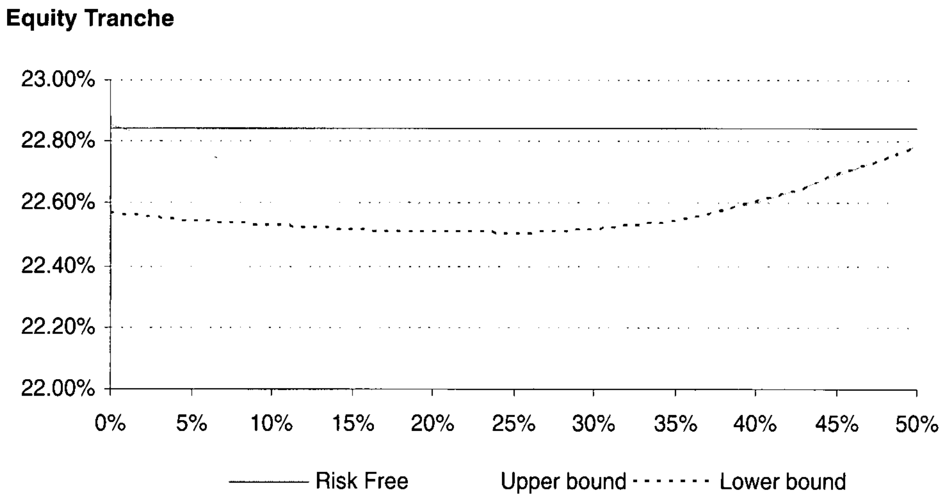
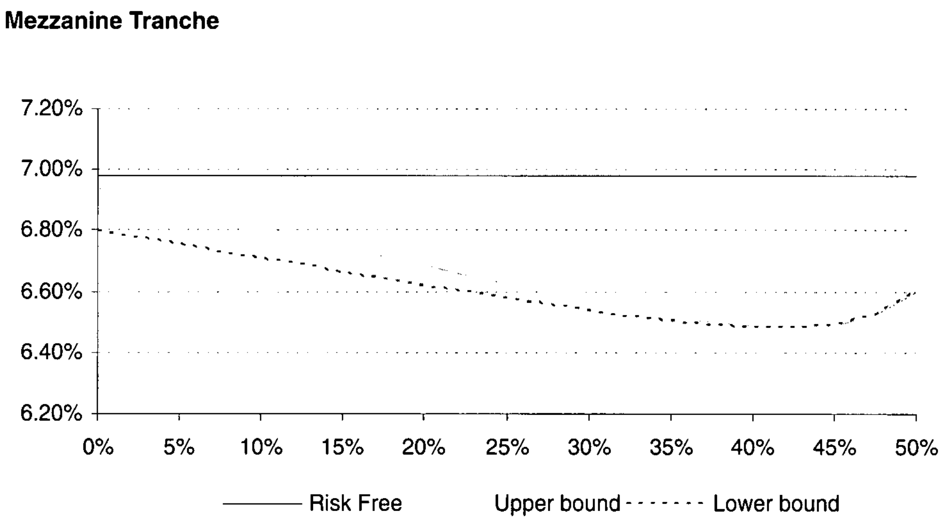


EXHIBIT 6

Fair Spreads as Functions of Correlation Between CP and Reference Credits





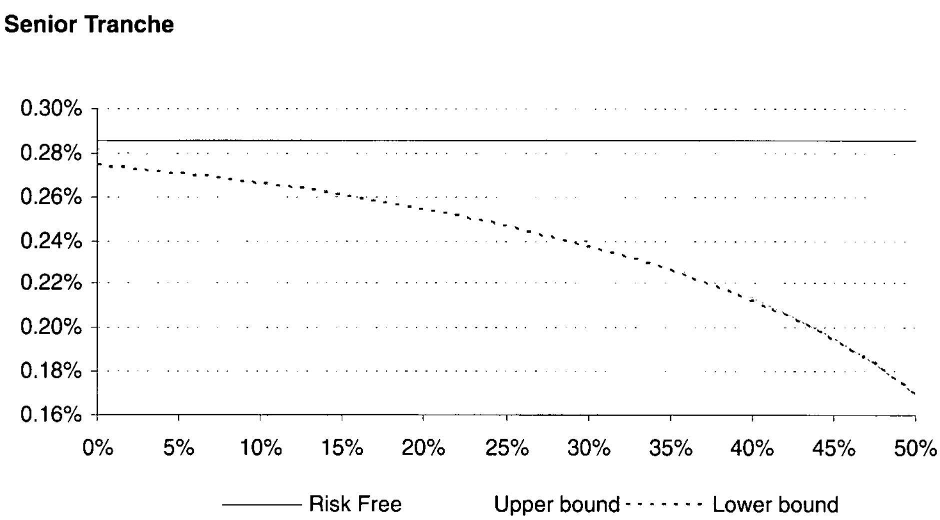


Exhibit 7

Fair Spreads in 5,000-Path Monte Carlo Simulation (% std err in parentheses)

	Equity (0–5%)	Mezzanine (5–10%)	Senior (10–100%)
Upper Bound for Risky Fair Spread	22.89% (0.45%)	7.027% (0.74%)	0.289% (1.29%)
Lower Bound for Risky Fair Spread (=Risk-Free Fair Spread)	22.84% (0.44%)	6.980% (0.73%)	0.285% (1.16%)

of the reference credits, the expected mark-to-market of the swap at τ is close to zero (its exact value will generally depend on the shapes of the hazard rates and the risk-free curve). In this case, the only reason the spread paid to a risky protection seller should be lower than the one paid to a default-free investor is the asymmetry of the mark-to-market settlement. By definition, this effect cannot be captured by our upper bound, which is the fair spread for a (35%, 35%) symmetric swap, and is therefore very close to the risk-free fair spread.

The second thing is that fair spreads of different tranches behave very differently as we vary the correlation between the risky counterparty and the reference credits. At low levels of correlation, a correlation increase makes the expected mark-to-market at τ more favorable for the protection buyer, as more defaults are expected to affect the reference portfolio in the states when the counterparty defaults. Because the protection buyer now expects to lose a fraction $(1 - Q)$ of a higher mark-to-market, it has to pay a lower spread in order to break even. This holds true for tranches of all seniorities.

At high levels of correlation, however, an equity tranche is already exhausted at τ in a significant proportion of the states when the protection seller defaults. When this is the case, the mark-to-market of the equity swap at τ is obviously equal to zero. A further increase in correlation means such scenarios occur more often, and makes the expected mark-to-market at τ less favorable for the protection buyer. Because the protection buyer now expects to lose a fraction $(1 - Q)$ of a lower mark-to-market, the break-even spread has to increase.

The same reasoning holds for the mezzanine tranche, although the fair spread reaches its minimum at a higher correlation level. As far as the senior tranche goes, the fair spread generally declines monotonically with the correlation between the risky counterparty and the reference credits.

One final observation we can draw from Exhibit 6 is that the position of the risk-free protection buyer is more sensitive to an increase in correlation between the risky counterparty and the reference credits for senior tranches than for junior tranches.

Default-Risky Protection Buyer, Default-Free Protection Seller

For a swap between a default-risky protection buyer and a default-free protection seller, we maintain the same base case parameters on the understanding that:

1. It is now the protection *buyer* who has a yearly (risk-neutral) hazard rate equal to 2%.
2. Q = 35% is now the fraction of a favorable mark-to-market recovered by the default-free protection *seller*, while R = 1 is the fraction of a favorable mark-to-market recovered by the default-risky protection *buyer*.

From Proposition 3, the lower bound for the fair spread of this swap is given by $S_{1,1}$, which is also equal to the fair spread between two default-free counterparties. Exhibit 7 shows the bounds for the fair spreads on the same three tranches.

The default riskiness of the protection buyer produces an increase in fair spreads that is significantly smaller than the reduction in fair spreads resulting from the default riskiness of the protection seller. For example, the fair haircut for a 2% hazard rate protection seller on a five-year mezzanine tranche is 35 to 40 bp (Exhibit 2), while the fair surcharge for a 2% hazard rate protection buyer is certainly lower than 5 bp (Exhibit 7).

This difference occurs primarily for two reasons. First, the distribution of the mark-to-market of a loss tranche at any given horizon is strongly asymmetric around zero, as it is bounded on the one side by the outstanding notional of the tranche and on the other by the present value of the remaining premium payments. Therefore, even with independence between the default time of the risky counterparty and the default times of the reference names, the asymmetry of the mark-to-market recovery has a greater impact on the expected loss due to counterparty default when the risky counterparty is the protection seller.

Second, with positive default dependence between the risky counterparty and the reference names in the swap, a risky protection buyer tends to default when the mark-to-market is in its favor, while a risky protection seller tends to default when the mark-to-market is against it.

For these reasons, allowing for the possibility of default of the protection seller has a much more significant impact on fair prices than allowing for the possibility of default of the protection buyer, all else equal. O'Kane and Schloegl [2002] observe the same for single-name default swaps.

IV. SUMMARY

Pricing counterparty risk in multiname default swaps is a challenging task. In principle, one needs to be able to evaluate the mark-to-market of the trade along each path where the counterparty defaults, and precisely at the time the default takes place. This in turn requires calculation of the joint distribution of the default times of the surviving credits, conditional on the information available at the time of default of the counterparty. We suggest a simple methodology that tackles this computational complexity at its root, without imposing any restriction on the choice of the underlying joint default model.

Our two primary results are:

1. If the mark-to-market recovery is symmetric, then nested valuations are unnecessary and the contract can be exactly priced as a step-down swap, a swap whose payments on both legs step down to a fraction of the contractual payments at the time of default of the risky counterparty.
2. If the mark-to-market recovery is asymmetric, as it is more realistic to expect, bounds for the fair swap spread can be derived by means of a simple algorithm that avoids nested valuations.

Using the popular Gaussian copula model as the underlying default mechanism, we apply our methodology to price synthetic loss tranches of different seniorities. The proposed bounds vary as functions of the market-implied default probability of the risky counterparty and the asymmetry of the mark-to-market recovery at default. Most important, these bounds turn out to be surprisingly tight when there is positive dependence between the default time of the risky counterparty and the default times of the reference names in the swap, which is generally the case.

Finally, allowing for the possibility of default of the protection seller has a much more significant impact on fair spreads than allowing for the possibility of default of the protection buyer, all else equal.

ENDNOTES

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[1]To simplify the notation, we do not explicitly consider accrual in the pricing equations.

[2]The risk-neutral expectation of $N(j, k)$ is often called the risky PV01 of a swap with payment dates t_j, $t_j + 1, \ldots, t_{k-1}, t_k$.

[3]We can model a stochastic mark-to-market recovery with $Q \leq R$ if we assume that Q and R are beta-distributed on two non-overlapping supports. Frye [2000] argues that the correlation between defaults and recovery rates may be an important factor in determining the shape of the loss distribution.

[4]An alternative way to look at Proposition 2 is to recognize that, by definition, MTM dominates CPL in the sense of second-order stochastic dominance (see Rothschild and Stiglitz [1970]).

[5]Mashal, Naldi, and Zeevi [2003] provide empirical evidence supporting a fat-tailed generalization of this model. Boscher and Ward [2002] study the consequences of allowing for stochastic recovery rates in this framework.

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