A Poisson Model with Common Shocks for CDO Valuation

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Collateralized debt obligations (CDOs), are credit risk products backed by a pool of debt obligations. They create securities or classes of securities from a portfolio of debt instruments. CDOs can thereby create higher-quality debt from average- or below average-quality debt.

The debt may be bonds, loans, revolving credit facilities, or structured finance obligations, as well as domestic or foreign, or a combination of both. It may also be investment-grade or high-yield or some combination.

When the underlying assets are bonds, the CDO is referred to as a collateralized bond obligation, CBO. When the underlying assets are loans, the CDO is referred to as a collateralized loan obligation, CLO.

A CDO is a very innovative product of the securitization market and is one of the fastest-growing fixed-income sectors. According to a British Bankers' Association credit derivatives report, the portfolio products/CLO market amounted to $262 billion in 2001, and was then predicted to increase to $1248 billion by year-end 2004.

In the late 1980s and early 1990s, CDOs were sold as credit arbitrage investments. They generated huge profits for investors, with returns of 30% to 40% or higher. At that time the spread more than compensated investors for the extra risk. More recently, however, credit spreads have been at historic lows, and barely justify the additional credit risk.

CDOs have survived swings in credit spreads, default rates, and new regulations. The main reason for their survival is that banks use them to free up regulatory capital by securitizing the higher layers of credit risk (Skora [1998]).

CDOs are classified into two general types, market value structures and cash flow structures, although some CDOs have characteristics of both categories. In a market value CDO, the tranches receive payments based essentially on the marked-to-market returns of the collateral pool, as determined in large part by the trading performance of the CDO manager (see Duffie and Garleanu [2001]). In a cash flow CDO, the collateral portfolio is not subjected to active trading by the CDO manager, implying that uncertainty regarding interest and principal payments to the tranches is determined mainly by the number and timing of defaults of the collateral securities.

We concentrate on cash flow CDOs, avoiding an analysis of the trading behavior of CDO managers. All CDOs have an investment manager who manages the collateral.

Payments of interest and principal are redirected to investors through tranches of various credit ratings; the highest tranche is usually rated triple A. Tranches are ordered so that losses in principal or interest of the collateral are absorbed first by the lowest-level tranche and then by the next tranche, and so on. The lowest tranche is the riskiest, and is called the equity tranche. This mechanism for distributing the losses to the several tranches is known as the waterfall.

Losses in CDOs occur when there is a credit event, which is either a default or a credit downgrade of the collateral. The timing and severity of the credit event are critical (*severity* refers to the amount of loss). Credit events are uncertain in size, variable in number, and not independent. Investors benefit, the fewer the credit events and the less the variability in the number of them.

Investors in a particular tranche would like to know the exact distribution of losses to the underlying pool of debt, which depends on the probability of a credit event and the correlation between credit events. Yet correlation as a measure of non-independence is only one number that does not capture the complexity of credit events. It is thus necessary to think about credit ratings in terms of probability of default and correlation of default.

Credit risk has two components: systematic risk and specific risk. Systematic risks, such as industry, country, or economic risk, affect all entities in the market, so if two or more entities are in the same country, industry, or economy they may have non-independent credit risk. Specific risk, on the other hand, is particular to one firm.

I. REVIEW OF CDO VALUATION APPROACHES

The CDO structure can be analyzed from the asset side or the liability side. On the asset side, there is a portfolio of obligors in the asset pool underlying the CDO. On the liability side, we typically have securities issued in the capital market. The two sides of the CDO are connected through the structural definition of the transaction. There are basically five types of securities or tranches created from a portfolio of bonds: equity, junior, mezzanine, senior, and super senior tranches.[1]

There are generally three steps in the valuation of CDOs: modeling multiple default correlation, deriving the loss function distribution, which is a distribution function of the cumulative portfolio default loss at a given time, and valuing tranches (see Bluhm [2003]).

One of the first models of default risk was developed by Merton [1974]. It assumes assets follow lognormal Brownian motion. Such models are easily extended to model portfolio credit risk.

CDO valuation models are classified as structural models or intensity models. Structural models assume defaults occur when the value of the firm falls below the market value of debt that is endogenously computed, and a certain recovery, i.e., the firm value, is paid upon default. Intensity models assume credit events occur exogenously, usually by a Poisson process, and a separately specified recovery is paid. To model multiple default correlation, conditional independence is usually assumed in structural models. Intensity models can provide a more flexible framework to relax the conditional independence restriction.

Structural models describe asset value A_i of the i-th firm asset by a logarithmic Wiener process with constant proportional volatility σ_i:

$$dA_i = \mu A_i dt + \sigma_i A_i dW_i$$

where μ is the expected return on asset value A_i, and W_i is a Wiener process.

The company defaults on its loan if A_i drops below the contractual value of its obliged payment D_i at time T. Denoting by A_{iT} firm value at time T, we thus have

$$P[A_{iT} < D_i] = N(-c_i)$$

where

$$c_i = \frac{1}{\sigma_i\sqrt{T}}(\log A_i - \log D_i + \mu T - \frac{1}{2}\sigma_i^2 T)$$

and N is the cumulative Gaussian distribution function.

If we assume conditional independence for default events, the process W_i can be represented as $W_i = \sqrt{\rho}V + \sqrt{1-\rho}V_i$, where V, V_i, $i = 1, \ldots, n$ are independent standard Gaussian random variables. We get $\mathrm{Var}(W_i) = 1$ and $\mathrm{cov}(W_i, W_j) = \rho$.

Thus ρ can readily be seen as a correlation parameter. The term $\sqrt{\rho}V$ can be interpreted as the i-th firm exposure to a common factor V, such as the state of the economy, and the term $\sqrt{1-\rho}V_i$ represents the firm's specific risk.

Crouhy, Galai, and Mark [2000], Finger [1999], and Vasicek [1997] assume that individual defaults do not significantly affect the risk of a portfolio as long as they can be diversified. This implies that once the common risk factors are fixed, the idiosyncratic risk factors are independent of each other. In other words, except for common shocks, meaningful correlated shocks are ignored under the assumption of conditional independence.

Keep in mind, however, that a CDO is a correlation product. When investors purchase such a product, they are buying correlation risk. Therefore, in order to determine whether investors are getting a fair return, it is necessary to measure the correlation risk.

In practice, in many cases this assumption may be oversimplified.[2] When there are strong and systematic dependences, even a portfolio with a large number of loans can be extremely risky. As Schonbucher and Schubert [2001] have noted, default correlation estimation bias becomes severe when firms are closely interrelated.

Intensity models can provide a more flexible framework to model multiple default correlation (see Davis and Lo [2000], Jarrow and Yu [2001], Kijima [2000], and Schonbucher and Schubert [2001]). The basic idea of intensity models is Poisson arrivals of firm default. For the purpose of deriving a CDO loss function, it will be desirable to assume conditional dependence.

As theoretically defensible as these models are, they are more suitable for evaluating simpler credit products such as default swaps or default baskets, which involve fewer or better-known issuers of underlying bonds than those commonly found in CDOs. When applied to CDO valuation, the number of their parameters makes model calibration very challenging (see Esposito [2002]).[3] There also not be enough market data of default swap spreads for hazard rate estimation.[4]

In other words, application of credit risk models to practical CDO valuation is possible only when enough data are available for default correlation estimation. Therefore, conditional independence is usually assumed for simplicity.

II. PROPOSED CDO VALUATION METHODOLOGY

To avoid the estimation problem of calibrating default correlation among CDO issuers, we propose a technique from actuarial mathematics (see Merino and Nyfeler [2002]). First, we group firms with equal credit ratings into exposure buckets. By grouping firms with equal credit ratings, we reduce the number of model parameters. In our example, the number of parameters declines from 5,051 to 11. Then we use the so-called Poisson model with common shocks, as proposed by Cossette and Marceau [2000], to model the dependence structure among firms of various credit ratings. Finally, the CDO loss function is derived from this model specification.

The model lets us compare conditional independence and dependence assumptions. We use a compound Poisson approach to demonstrate the bias of ignoring dependence.

Grouping Firms into Buckets

The idea behind the intensity model is that credit events can be modeled as a Poisson process, with intensity, or hazard rate, λ, depending on the length of time interval. The general form of the Poisson distribution is:

$$\text{Poisson}\,(k, h) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \text{ for } k = 0, 1, 2, \ldots \tag{1}$$

For a single firm, we are interested only in $k = 0$, i.e., the firm does not default, and $k > 0$, the firm defaults. Then the probability that a company defaults before time t is exponentially distributed; i.e., $P_i(t) = 1 - e^{-\lambda t}$. Default time in intensity models thus can be defined as:

$$\tau = \inf\{t: \int_0^t \lambda(u)du \geq E_1\}$$

where E_1 is a unit exponential random variable independent of λ(u), which is a random hazard rate.

Lando [1998] shows that this default time can be thought of as the first jump time of a Cox process, a generalization of the Poisson process that allows the intensity to be random, with intensity λ(u). That is, hazard rates are linked to the likelihood of default. Through modeling the joint distribution of default risk of the underlying collateral securities, we can determine the risk and valuation of CDOs.

To analyze the credit risk of a CDO portfolio, we often model the total losses over a fixed time period as a random loss variable:

$$L = \sum_{i=1}^{N} Y_i(1 - \delta_i)l_i \tag{2}$$

where Y_i stands for the default indicator, δ_i for the recovery rate, and l_i for the exposure to the i-th firm. The variable Y_i takes the value one with probability $P_i(t)$, and the value zero with probability $1 - P_i(t)$. We always assume the exposures l_i to be constant, and we set each recovery rate equal to one to keep the notation simple. Furthermore, all exposures are assumed to be a predefined base unit of loss, e.g., \$1 million.

The conditional independence framework assumes that, given common factors V representing systematic risk, the indicator variables Y_i are independent. This means that the remaining risk, given the systematic part, is considered purely idiosyncratic, i.e., obligor-specific.

For example:

$$P(Y_i = 1, Y_j = 1|V = \bar{V}) = \\ P(Y_i = 1|V = \bar{V})P(Y_j = 1|V = \bar{V}) \quad (3)$$

where $\bar{V}$ is some common factor return.

Consequently, to implement this conditional independence, we can define the hazard rate λ_i, for firm i in a credit risk model based on a conditional independence framework as:

$$\lambda_i = \lambda_{ii} + \lambda_C \quad (4)$$

where λ_{ii} is subject to firm-specific risk, and λ_C is subject to common risk.

Following the method explored by Esposito [2002], to adapt the conditional dependence framework, we add occurrence of shocks specific to the couples to Equation (4), and define the hazard rate λ_i as:

$$\lambda_i = \lambda_{ii} + \sum_{j \neq i} \lambda_{ij} + \lambda_C \quad (5)$$

The problem with this multivariate setting is the parameter estimation. That is, there are 5,051 model parameters in portfolio for $i = 100$.[5]

According to Merino and Nyfeler [2002], a convenient way to solve this problem is to group firms with equal credit ratings into exposure buckets. Obligors in the same exposure bucket are identified by the indexes B_j: $= \{i \mid Q_{j-1} < Q_i(t) \leq Q_j\}$ for some Q_j, $j = 1, \ldots, v$, and $Q_0 = 0$. Thus, the number of parameters declines from 5,051 to 11 in our example for $v = 4$. We denote by L_j the loss variable of the subportfolio generated by bucket B_j.

Loss Functions and Tranche Valuation

A Poisson model with common shock is used to derive the CDO loss function. That is, we can rewrite the overall portfolio loss variable L in Equation (2) as the sum of the v exposure buckets:

$$L = \sum_{i=1}^{N} Y_i(1 - v\delta_i)l_i = \sum_{j=1}^{v} [\sum_{i \in B_j} Y_i(1 - \delta_i)l_i] = \sum_{j=1}^{v} L_j \quad (6)$$

Following Cossette and Marceau [2000], we denote L_j, $j = 1 \sim 4$ for four classes of credit ratings as:

$$\begin{aligned} L_1 &= L_{11} + L_{12} + L_{13} + L_{14} + L_{1234} \\ L_2 &= L_{22} + L_{12} + L_{23} + L_{24} + L_{1234} \\ L_3 &= L_{33} + L_{13} + L_{23} + L_{34} + L_{1234} \\ L_4 &= L_{44} + L_{14} + L_{24} + L_{34} + L_{1234} \end{aligned} \quad (7)$$

where $L_{ii} \sim Poisson(\lambda_{ii})$ is subject to firm-specific risk, $L_{ij} \sim Poisson(\lambda_{ij})$, $i \neq j$, is subject to shocks specific to the couples, and $L_{1234} \sim Poisson(\lambda_{1234})$ is subject to the common risk. Since the sum of n independent Poisson random variables with parameter λ_k, $k = 1 \sim n$, is Poisson distributed with parameter $\sum_{k=1}^{n} \lambda_k$, we obtain $L_i \sim Poisson(\lambda_i)$ with:

$$\begin{aligned} \lambda_1 &= \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{1234} \\ \lambda_2 &= \lambda_{22} + \lambda_{12} + \lambda_{23} + \lambda_{24} + \lambda_{1234} \\ \lambda_3 &= \lambda_{33} + \lambda_{13} + \lambda_{23} + \lambda_{34} + \lambda_{1234} \\ \lambda_4 &= \lambda_{44} + \lambda_{14} + \lambda_{24} + \lambda_{34} + \lambda_{1234} \end{aligned} \quad (8)$$

The mean and variance of the loss variable L are shown in Appendix A.

By grouping firms into buckets, we reduce the numbers of hazard rate estimations. In this way we avoid an estimation problem. We can then use the Gaussian copula default dependence structure to calculate the default correlation.[6]

Default correlations are calibrated to estimate the hazard rate for the Poisson distribution. The method of estimating Poisson distribution parameters is described in Appendix B. From Equations (B-1) and (B-2), we can estimate λ_{ij}, hazard rates subject to couples-specific risk.

Under such a formulation, we write loss variable L in conditional default dependence and in conditional default independence as

$$L = \sum_{i=1}^{4} L_i = \sum_{i=1}^{4} L_{ii} + \sum_{i,j=1\sim 4}^{i \neq j} L_{ij} + 4L_{1234} \quad (9)$$

and as

$$L^* = \sum_{i=1}^{4} L_i = \sum_{i=1}^{4} L_{ii} + 4L_{1234} \quad (10)$$

Because L and L^* are not the sums of independent Poisson processes, their hazard rates are not simply equal to the sums of individual hazard rates. Therefore, there are no closed-form distribution functions for L and L^*. We use the approach suggested by Duffie and Singleton [1999] to simulate the default frequency of assets before CDO maturity to obtain the frequency distribution of loss variables. The tranches can be valued using the method presented in Appendix C.

Bias Due to the Independence Assumption

Our proposed method has the advantage of facilitating comparison between conditional independence and dependence assumptions. We can use a compound Poisson approach to show that the conditional default independence assumption underestimates the real variance of loss variable by $2\Sigma_{i<j} \lambda_{ij}$.

Under the conditional independent correlation assumption, we also consider four Poisson random variables L^*_i $(i = 1 \sim 4)$, but the definition of L^*_i changes to $L^*_i = L^*_i + L_{1234}$. Under this formulation, L^*_{ii} $(i = 1 \sim 4)$ $\sim Poisson(\lambda^*_i)$ are independent of each other, which is incorrect when there is conditional dependence. Besides, L_i and L^*_i are actually the same thing, $L_i = L^*_i = L^*_{ii} + L_{1234}$. Hence, $\lambda_i = \lambda^*_i = \lambda^*_{ii} + \lambda_{1234}$.

Similarly, the joint moment generating function of $L^*_J = (L^*_1, L^*_2, L^*_3, L^*_4)$ is

$$\begin{aligned} M_{L^*}(t_1,t_2,t_3,t_4) &= E[\exp(t_1L_1 + t_2L_2 + t_3L_3 + t_4L_4)] \\ &= E[\exp(t_1L^*_{11} + t_2L^*_{22} + t_3L^*_{33} + t_4L^*_{44})] \times \\ &\quad E[\exp(t_1 + t_2 + t_3 + t_4)L_{1234}] \end{aligned} \tag{11}$$

Therefore, the joint moment generating function of L^*_J is

$$M_{L^*_J}(t_1, t_2, t_3, t_4) = \exp(A^* + C) \tag{12}$$

with $A^* = \Sigma^4_{i=1}\lambda^*_{ii}(\exp(t_i) - 1)$. Therefore, $M_{L^*}(t) = M_{L^*_J}(t, t, t, t) = \exp(A'' + C'$ with $A'' = (\exp(t) - 1)\Sigma^4_{i=1}\lambda^*_{ii}$. By the same calculation, we have the mean of L^*:

$$\mu_{L^*} = \sum^4_{i=1}\lambda^*_{ii} + 4\lambda_{1234} = \sum^4_{i=1}\lambda_{ii} + 2\sum_{i<j}\lambda_{ij} + 4\lambda_{1234} \tag{13}$$

and the variance of L^*:

$$\sigma^2_{L^*} = \sum^4_{i=1}\lambda^*_{ii} + 16\lambda_{1234}\sum^4_{i=1}\lambda_{ii} + 2\sum_{i<j}\lambda_{ij} + 16\lambda_{1234} \tag{14}$$

Hence, it is shown that the difference between the variance of L and of L^* is $2\Sigma_{i<j}\lambda_{ij}$.

Exhibit 1

Five-Year Horizon PD Correlations among Rating Classes

Rating	A	Baa	Ba
Aa	0.624	0.419	0.519
A		0.692	0.746
Baa			0.738

III. NUMERICAL ILLUSTRATIONS

We present some illustrations to show how our approach works. The CDO portfolios constructed for our illustration comprise issuers categorized into four rating classes tracked by Moody's over the period 1970–2001. The four rating classes are Aa, A, Baa, and Ba. Bonds rated Aa are generally known as high-grade bonds. Ba refers to the low credit quality. 25 firms are in each rating class category, so there are 100 firms included in the CDO assets.

Calibration of Default Probability and Default Correlation

Using the method proposed by Lando [1998], we obtain hazard rates $\lambda_{Aa} = 0.31\%$, $\lambda_A = 0.49\%$, $\lambda_{Baa} = 1.91\%$, and $\lambda_{Ba} = 10.95\%$ for each rating class. Exhibit 1 presents estimated correlations for the five-year probability of default (PD) among rating classes. For example, a correlation of 0.624 between rating Aa and A describes the dependence of default events for a firm with rating Aa and default events for a firm with rating A during the five-year period. The 0.624 correlation between ratings Aa and A is higher than the 0.419 correlation between rating Aa and Baa, which implies that, given that a firm rated Aa defaults in five years, a firm rated A is more likely to default than a firm rated Baa in the same period. The data indicate that there are positive PD correlations among rating classes.

Exhibit 2 shows summary statistics of the PD samples for each rating class over the period 1970-2001. In the column labeled Ba, Mean 10.4% denotes the average default rate, which is the number of firms with rating Ba defaulting in five years divided by the number of all Ba rated firms in five years, out of 28 observations.

There is a clear monotonic relationship between PDs and ratings. The average default rates of firms rated Aa, A, Baa, and Ba are 0.3%, 0.5%, 1.9%, and 10.4%, respectively. Firms rated Ba are the major risk in CDO portfolios. The default rates of lower rating classes also fluctuate more. For example, the default rate of the Ba

EXHIBIT 2

Summary Statistics of PD Samples over 1970-2001

Ratings	Aa	A	Baa	Ba
Mean	0.003175	0.004979	0.019079	0.103704
Median	0.000000	0.000600	0.015450	0.071000
Maximum	0.019300	0.026100	0.054200	0.240500
Minimum	0.000000	0.000000	0.000000	0.025200
Std. Dev.	0.004917	0.007312	0.014084	0.074064
Skewness	1.652719	1.431766	0.862662	0.658584
Kurtosis	5.372600	3.980170	2.945903	1.927395
Jarque-Bera	19.31434	10.68731	3.476281	3.366316
Probability	0.000064	0.004778	0.175847	0.185786
Observations	28	28	28	28

rating class declined from 24.1% in the 1989 recession to 2.5% in 1992 after the economy improved.

Loss Distribution Function

We use Monte Carlo simulation to compute the frequency distribution of CDO loss variables. We follow Duffie and Singleton [1999] in order to simulate correlated default for conditional independence and dependence models. We also perform sensitivity analysis of common factor variables in slump, normal, and boom economy situations.

Assume CDO assets with a nominal amount of $100, and each firm with a nominal amount of $1. If seven firms default before CDO maturity, a five-year period, the CDO asset value is $93. Simulating 2,000 trials with estimated hazard rates, Exhibit 3 plots the frequency distribution. For a clearer presentation we omit the plot of the normal economic conditions.

The expected value of the loss variable is lower in a slumping economy than in boom periods, and the distribution is more concentrated under the conditional independence assumption. Economic circumstances have less of an impact on the loss function.

Tranche Valuation

We evaluate four kinds of CDO tranches: equity, junior, mezzanine, and senior tranches. The super-senior tranche is not included because model risk is insignificant in high-quality tranche prices.

Generally, the equity tranche has 2% of the total bond principal and absorbs all credit losses from the portfolio during the life of the CDO until they have reached 2% of the total bond principal. The junior tranche has 1% of the principal and absorbs all losses during the life of the CDO in excess of 2% of the principal up to a maximum of 3% of the principal. The mezzanine tranche has 4% of the principal and absorbs all losses during the life of the CDO in excess of 3% of the principal up to a maximum of 7% of the principal. The senior tranche has 8% of the principal and absorbs all losses during the life of the CDO in excess of 7% of the principal up to a maximum of 15% of the principal. The maturity of the CDO is five years.

Applying the frequency distribution of loss function we have just obtained, we evaluate four CDO tranches. Results are reported in Exhibit 4. For example, in the senior tranche column, the market value assuming conditional independence and in a boom economy is $94.475, if the nominal amount of the senior tranche is $100. Under the same conditions, the market value of the junior tranche is only $10.65, if the nominal amount of the senior tranche is $100.

The phenomenon that correlation model impact differs by tranche type is consistent with Duffie and Garleanu [2001]. Assuming conditional independence, they find that when there is increased correlation in common factors and firm-specific risk factors, the market value of the senior class drops, and the price of the junior tranche rises. Our results are the same as Hull and White [2004] who use pairwise correlations in their conditional independence assumption.

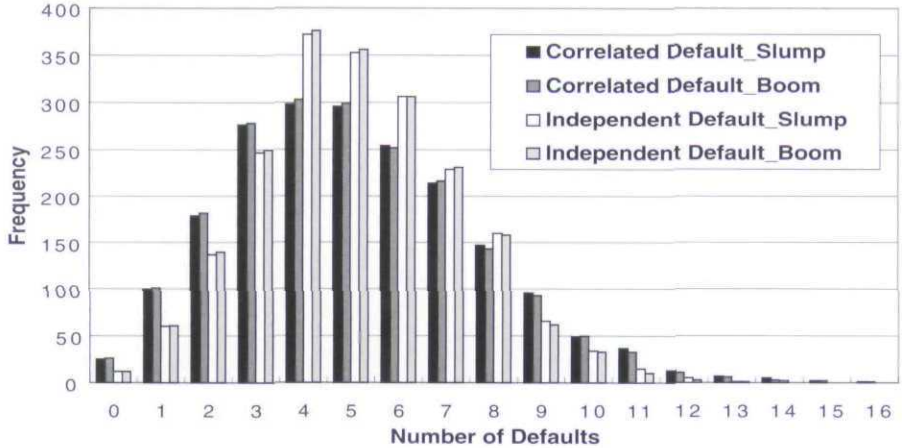
Results

Without having to estimate too many hazard rate parameters, we propose a CDO valuation model to examine the impact of ignoring conditional default dependence for tranche valuation from equity to senior types. We find that the impact differs by tranche type. Low-rated tranche securities, i.e., equity, junior, and mezzanine classes, are underestimated by 1.4% to 4.8% when we ignore conditional dependence. Yet the high-rated tranche security, i.e., the senior class, is overestimated by 3.8% when we ignore conditional dependence.

These biases are caused by underestimation of the variance of the loss function. High credit risk tranche investors benefit from this model risk, while low credit risk tranche investors are hurt by it. This is consistent with Duffie and Garleanu [2001] and Hull and White [2004], who assume conditional independence.

It is worth mentioning that economic conditions have an additional impact on the loss distribution function. In effect, tranche prices differ under different economic situations. For example, under the conditional dependence assumption, when the economy is in a slump, the senior tranche price is 0.415% lower than that in normal economic conditions.

EXHIBIT 3
Frequency Distribution of Loss Function



1. Correlated default boom and correlated default slump denote using a conditional dependent assumption when economic situations are in boom or slump.
2. Independent default boom and independent default slump denote using conditional independent assumption when economic situations are in a boom or slump.

Number of Defaults	Correlated Default_Slump	Correlated Default_Normal	Correlated Default_Boom	Independent Default_Slump	Independent Default_Normal	Independent Default_Boom
0	25	26	26	12	12	12
1	100	101	101	60	60	61
2	179	179	182	137	138	140
3	277	278	279	247	248	250
4	298	299	303	372	374	376
5	296	299	299	353	355	356
6	254	254	252	307	308	307
7	214	215	216	229	232	231
8	147	146	143	160	160	158
9	96	93	93	66	64	62
10	49	50	50	34	32	33
11	37	35	33	15	11	10
12	13	12	11	5	3	3
13	7	6	6	1	1	1
14	5	4	3	2	2	0
15	2	2	2	0	0	0
16	1	1	1	0	0	0

IV. SUMMARY

Using a Poisson model with common shocks, we evaluate CDO tranches under an intensity model framework. First, we estimate hazard rates in our Poisson models. By grouping firms with equal credit ratings, we dramatically reduce the number of hazard rate parameters. The implementation of models assuming conditional dependence can thus be made more efficient.

Second, we use this model to simulate joint default with respect to various dependence structures. Finally, we evaluate four tranches: equity, junior, mezzanine, and senior. The sensitivity analysis is performed under three economic conditions: slump, normal, and boom.

The results show the biases of assuming conditional independence. Ignoring correlation will result in under-

EXHIBIT 4

Comparison of CDO Tranche Prices

CDO Nominal Amounts $100	Equity	Junior	Mezzanine	Senior
Correlated Default Boom	3.825	15.450	51.388	90.650
Correlated Default Normal	3.825	15.300	51.063	90.363
Correlated Default Slump	3.750	15.200	50.800	89.988
Average for Correlated Default	3.800	15.317	51.083	90.333
Independent Default Boom	2.125	10.650	49.988	94.475
Independent Default Normal	2.100	10.500	49.650	94.213
Independent Default Slump	2.100	10.450	49.413	93.763
Average for Independent Default	2.108	10.533	49.683	94.150
Percentage Difference	-1.7%	-4.8%	-1.4%	3.8%

1. *Correlated default normal denotes using a conditional dependent assumption when the economic situation is normal. Independent default normal denotes using a conditional independent assumption when the economic situation is normal.*
2. *Percentage difference denotes price difference between conditional dependent and independent models divided by CDO nominal amounts.*

estimation of the variance of the loss function. This implies that when industries of the issuers are tightly integrated, we have model risk or model misspecification.

Although we have shown theoretically that the variance of the loss function can be underestimated, the bias in tranche prices is not so straightforward because there is no closed-form solution for those prices. Our numerical examples confirm this discrepancy, and highlight the risk of ignoring the conditional default correlations. Appropriate models for pricing CDOs are possible only when default dependence is correctly considered.

APPENDIX A

Derivation of the Mean and Variance of the Loss Variable

Consider four Poisson random variables L_i, $i = 1, \ldots, 4$. We have $L_i \sim Poisson(\lambda_i)$, $i = 1, \ldots, 4$. The joint moment generating function of $L_J = (L_1, L_2, L_3, L_4)$ is:

$$\begin{aligned} M_{L_J}(t_1, t_2, t_3, t_4) &= E[\exp(t_1L_1 + t_2L_2 + t_3L_3 + t_4L_4)] \\ &= E[\exp(t_1L_{11} + t_2L_{22} + t_3L_{33} + t_4L_{44})] \times \\ &\quad E\{\exp[(t_1 + t_2)L_{12}]\} \times E\{\exp[(t_1 + t_3)L_{13}]\} \times \\ &\quad E\{\exp[t_3 + t_4]L_{34}\} \times E\{\exp[(t_1 + t_2 + t_3 + t_4)L_{1234}]\} \end{aligned} \quad \text{(A-1)}$$

Also:

$$\begin{aligned} &E[\exp(t_1L_{11} + t_2L_{22} + t_3L_{33} + t_4L_{44})] \\ &= \exp[\lambda_{11}(\exp(t_1) - 1] \times \exp[\lambda_{22}(\exp(t_2) - 1] \times \\ &\quad \exp[\lambda_{33}(\exp(t_3) - 1] \times \exp[\lambda_{44}(\exp(t_4) - 1] \end{aligned} \quad \text{(A-2)}$$

and

$$E\{\exp[(t_i + t_j)L_{ij}]\} = \exp[\lambda_{ij}(\exp(t_i + t_j) - 1)] \text{ for } i \neq j \quad \text{(A-3)}$$

Also:

$$\begin{aligned} &E\{\exp[(t_1 + t_2 + t_3 + t_4)L_{1234}]\} \\ &= \exp[\lambda_{1234}(\exp(t_1 + t_2 + t_3 + t_4) - 1)] \end{aligned} \quad \text{(A-4)}$$

Therefore, the joint moment generating function of L_J is:

$$M_{L_J}(t_1, t_2, t_3, t_4) = \exp(A + B + C) \quad \text{(A-5)}$$

with

$$\begin{aligned} A &= \sum_{i=1}^{4} \lambda_{ii}(\exp(t_i) - 1) \\ B &= \sum_{i<j} \lambda_{ij}(\exp(t_i + t_j) - 1) \\ C &= \lambda_{1234}(\exp(t_1 + t_2 + t_3 + t_4) - 1) \end{aligned} \quad \text{(A-6)}$$

Following Cossette and Marceau [2000], the moment generating function of L can be written as:

$$M_L(t) = M_{L_1+L_2+L_3+L_4}(t) = M_{L_J}(t, t, t, t) \quad \text{(A-7)}$$

Therefore:

$$M_L(t) = M_{L_J}(t, t, t, t) = \exp(A' + B' + C') \quad \text{(A-8)}$$

with $A' = (\exp(t) - 1)\sum_{i=1}^{4}\lambda_{ii}$, $B' = (\exp(2t) - 1)\sum_{i<j}\lambda_{ij}$,

and $C' = \lambda_{1234}(\exp(4t) - 1)$.

Hence, it is shown that the loss variable L has a compound Poisson distribution. According to the basic property of a moment generating function, we have the mean of L:

$$\mu_L = \sum_{i=1}^{4}\lambda_{ii} + 2\sum_{i<j}\lambda_{ij} + 4\lambda_{1234} \tag{A-9}$$

and the variance of L:

$$\sigma_L^2 = \sum_{i=1}^{4}\lambda_{ii} + 4\sum_{i<j}\lambda_{ij} + 16\lambda_{1234} \tag{A-10}$$

APPENDIX B

Estimating Poisson Distribution Parameters

Because $\mathrm{Cov}[L_i, L_j] = \mathrm{Var}[L_{ij}] + \mathrm{Var}[L_{1234}]$ $(i \neq j)$, for example:

$$\begin{aligned}\mathrm{Cov}(L_1, L_2) &= \mathrm{Cov}\begin{pmatrix} L_{11} + L_{12} + L_{13} + L_{14} + L_{1234}, \\ L_{22} + L_{12} + L_{23} + L_{24} + L_{1234} \end{pmatrix} \\ &= \mathrm{Cov}(L_{12}, L_{12}) + \mathrm{Cov}(L_{1234}, L_{1234}) = \lambda_{12} + \lambda_{1234}\end{aligned} \tag{B-1}$$

Denoting $u_i = L_i/N_i$, we obtain:

$$\begin{aligned}\mathrm{Cov}[L_i, L_j] &= N_i N_j \mathrm{Cov}[L_i/N_i, L_j/N_j] = N_i N_j \mathrm{Cov}(u_i, u_j) \\ &= N_i N_j \sqrt{\mathrm{Var}(u_i)\mathrm{Var}(u_j)}\rho(u_i, u_j)\end{aligned} \tag{B-2}$$

APPENDIX C

Tranche Valuation

Let us consider a tranche of a CDO, where the default payment leg pays all losses that occur on the credit portfolio above a threshold A and below a threshold B where $0 \le A \le B \le \Sigma_{i=1}^{N} M_i$, and M_i is the nominal amount of name i. When $A = 0$, this is usually the equity tranche. If $A > 0$ and $B \le \Sigma_{i=1}^{N} M_i$, we consider mezzanine tranches, and when $B = \Sigma_{i=1}^{N} M_i$, we consider senior or super-senior tranches. To simplify the notation, we use L_{TRi} to denote the cumulative losses on a given tranche:

$$L_{TRi} = (L - A)1_{[A,B]} + (B - A)1_{[B,\sum_{i=1}^{N} M_i]} \tag{C-1}$$

Following Laurent and Gregory [2002], we can write the default payment leg of the given tranche in a discrete sum as

$$E[L_{TRi}] = \sum_{k=A}^{B}(k - A)P(L = k) + (B - A)P(L > B) \tag{C-2}$$

or equivalently as:

$$E[L_{TRi}] = \sum_{k=A}^{B}(k - A)P(L = k) + (B - A)\sum_{k=B+1}^{\Sigma M_i} P(L = k) \tag{C-3}$$

Here, we have assumed that A and B correspond to some possible values of the cumulative loss at maturity. We can then compute the default payment leg of the tranche, since we already know how to compute the probabilities $P(L = k)$ for $k = 1, \ldots, \Sigma M_i$. CDO evaluation occurs in general by means of Monte Carlo simulation. When the discount factor is considered, the price of the default leg is given by

$$E^Q\left[\int_0^T B(0, t)dL_{TRi}(t)\right]$$

where $B(0, t)$ stands for the discount factor at time t, T for the maturity of the CDO, and $L_{TRi}(t)$ for the value of L_{TRi} at time t.

ENDNOTES

[1]A super-senior tranche may have 85% of the principal and absorb all losses during the life of the CDO in excess of 15% of the principal.

[2]In Taiwan, for example, firms in the information technology industry are usually highly integrated, horizontally and vertically. Practitioners in the Taiwan CDO market then need to pay attention to default correlation.

[3]Esposito [2002] proposes a multivariate exponential model, and finds a problem of parameter estimation. If we specify only the first two moments of the distribution, i.e., expected default rate and correlation, we cannot obtain a unique solution for the parameters' value, except for the bivariate case.

[4]Historical firm default data and spread prices in the default swap market are used to estimate default probability and default correlation, but spread prices are rare for small- and medium-sized firms, the major issuers in CDO asset portfolios.

[5]If there are 100 firms, there are 4,950 λ_{ij} parameters, 100 λ_{ii} parameters, and one λ_C parameter to be estimated. We get 5,051 by $\{(100 \times 99) \div 2\} + 100 + 1 = 5{,}051$.

[6]The Gaussian copula model represents the correlation structure between variables that are not normally distributed. It allows the correlation structure of the variables to be estimated separately from their marginal distribution. The joint distribution of default time i is defined as:

$$P(\Phi^{-1}(F_1(\tau_1)) < \Phi^{-1}(F_1(t_1)), ..., \Phi^{-1}(F_n(\tau_n)) < \Phi^{-1}(F_n(t_n)) = \Phi_{n,\rho}(\Phi^{-1}(F_1(t_1)), \ldots, \Phi^{-1}(F_n(t_n)))$$

where F_i is the cumulative distribution function of default time τ_i, $\Phi_{n,p}$ is a multivariate Gaussian vector and covariance matrix equal to ρ, and ϕ is a standard Gaussian distribution function (see Li [2000]).

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