Taxes, Default Risk, and Credit Spreads

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Traditional term structure models are unable to fully explain corporate bond spreads. They underestimate risk premiums of corporate bonds and more severely so for high-grade bonds (see Elton et al. [2001] and Huang and Huang [2002]). Duffee [1999] and Huang and Huang [2002] document that historical default rates for corporate bonds are significantly lower than would be implied by their credit spreads.

These findings raise some concern about the robustness of these term structure models and specifically whether there are missing factors. Common to all these models is that the effects of personal taxes on bond yields are assumed away (see, for example, Das and Tufano [1996]; Duffie and Singleton [1997, 1999]; Jarrow, Lando, and Turnbull [1997]; and Duffee [1998, 1999]).

It has long been recognized that models that ignore taxes cannot fully explain bond prices. Yawitz, Maloney, and Ederington [1985] and Fons [1987] assume that default probability is stationary and its conditional distribution uniform over time. Their implicit assumption of a buy-and-hold strategy and omission of the term structure of stochastic interest rates results in a relatively simple testable model that relates yields of risky bonds to three elements: the marginal bondholder's tax rate, the probability of default, and the risk-free interest rate on a government bond with the same maturity.

Constantinides and Ingersoll [1984] show that taxes affect a bondholder's trading strategies, giving rise to a timing option value. This value derives from the bondholder's optimal realization of capital gains and losses to maximize tax benefits. Ignoring the value of this tax-timing option biases the estimation of the bond yield and the marginal investor's tax rate.

While all these findings have enhanced our understanding of the effects of taxes on investors' trading strategies and bond yields, default risk is not explicitly considered. We incorporate the effects of taxes, default, and trading strategies into the pricing of corporate bonds. The primary objective is to develop a more general term structure model of defaultable bonds that assesses the potential effects of taxes.

First, we evaluate the effects of taxes on corporate bond spreads by directly accounting for bondholder trading strategies and net returns. This approach is different from the buy-and-hold strategy typically assumed in studies of yield spreads. An explicit consideration of investors' optimal trading strategies allows a more precise assessment of the extent of the yield spread underestimation resulting from tax omission.

Second, we analyze the impact of tax omission on default rate estimation, an element central to bond investment research. Correct measurement of bond default risk is essential in evaluating firm performance and understanding whether investors are fully compensated for the risk they bear. This issue bears on the fundamental question of bond market efficiency. Accurate estimates of default probability are also important for pricing credit risk derivatives and credit swaps and more generally for risk management.

We identify three important elements of tax effects. The first element is state taxes. Corporate bond returns are subject to state taxes as well as federal taxes. We find that state taxes are a key element affecting the yield spread between corporate and Treasury bonds. State taxes have more of an effect on high-grade bonds than low-grade bonds. AAA bonds have a very low default probability. Therefore, the spread between AAA and Treasury bonds is more likely due to state taxes. The default premium is higher for lower-grade bonds, so there is less of an effect of omitting taxes on the total yield spread (in percentage terms).

The second important element is default-induced tax liability and rebates. As default probability increases, bond yield rises, and investor tax liability, including both federal and state taxes, increases. At the same time, the expected government tax rebate increases as default probability increases. The effect of tax liability is partially offset by the effect of the expected tax rebate. On the other hand, the default premium rises as default probability increases. Taken together, the tax premium as a percentage of yield spread declines. Thus, there is less understatement of spreads due to tax omission (in percentage terms) for low-grade bonds.

The third element is the tax-timing option value derived from the investor's optimal return realization strategy. When there is no default, ignoring the timing option value results either in an underestimation of bond prices or an overestimation of yields. When there is default, circumstances become much more complicated. The effect of default risk on the timing option value generally depends on the expected default recovery rate and interest rate volatility.

Our study provides several interesting findings. First, we find that taxes account for a sizable portion of the spread between corporate and Treasury bonds. Term structure models should include taxes in order to help us understand the causes of corporate bond spreads.

Second, the effects of taxes are not linearly additive. Instead, the tax and default effects are interactive. This interactive effect cannot be captured by the constant term in the yield function as some have suggested.[1]

Finally, neglecting taxes results in an overestimation of default probability. When taxes are left out of the term structure model, their effects are factored into the default probability estimate in empirical investigation. The estimation bias increases with maturity and tax rates. As estimates of default probability are essential for pricing bonds and credit risk derivatives, the omission of taxes may give rise to substantial errors in pricing these instruments and testing bond market efficiency.

I. PRICING DEFAULTABLE BONDS UNDER OPTIMAL TRADING STRATEGY

We derive a pricing model for defaultable bonds under an optimal trading strategy. We first set up a discrete-time model to analyze the term structure of taxable coupon bonds with default. The discrete-time example illustrates the fundamental aspect of tax and default effects, and serves as the basis for our later numerical simulations. The discrete-time model is then generalized to a continuous-time model.

Corporate bonds are assumed to be non-callable and have a par value equal to one, a coupon rate κ_t, and a maturity date T. $v(t, T)$ is the time t price of a corporate bond maturing at time $T \geq t$. The short-term risk-free after-tax interest rate is r_t. There is no transaction cost for trading either defaultable or default free bonds.

Corporate bonds are subject to default. The default intensity is λ, and the recovery rate is δ, which indicates the residual value upon default. Both λ and δ are measurable and non-negative.[2]

We adopt the recovery of market value formulation, which specifies the residual value upon default as a fraction of the survival-contingent market value of the bond (see Duffie and Singleton [1999]). Our analysis can be easily extended to either the recovery of treasury or the recovery of face value formulation.[3]

Interest income and short-term capital gains are taxed at the ordinary income rate, while long-term gains are taxed at a lower rate. Let $\tau = \tau_F + \tau_S(1 - \tau_F)$ be the ordinary income tax rate for corporate bond income where τ_F and τ_S are the federal and state income tax rates. To allow for the differential tax treatment of long-term and short-term gains, we introduce a capital gains (or loss) tax function, $\alpha(t_p, t_e)$, which depends on the purchasing time t_p and selling time t_e of the bond. Specifically, we have:

$$\alpha(t_p, t_e) = \begin{cases} 1 & \text{if } t_e - t_p < \text{one year} \\ \alpha_G & \text{if } t_e - t_p > \text{one year} \end{cases} \tag{1}$$

where $0 < \alpha_G < 1$, and the tax rate is set equal to $\alpha\tau$.

Let $L(t)$ be the adjustment due to an amortization of the premium (or discount) per period. For the linear amortization rule, the adjustment will be:

$$L = \left[1 - v(t_p, T)\right]/(T - t_p) \tag{2}$$

which is a non-zero constant when the purchase price $v(t_p, T)$ differs from the principal with a par value of one. The amortization rule in (2) can be extended to other formulas such as the constant-yield method.[4]

For generality, we add a time indicator to the amortization variable, $L(t)$, to allow the amortization to be time-varying. Let $B(t + 1)$ be the basis at time $t + 1$. Then:

$$B(t + 1) = B(t) + L(t + 1) \tag{3}$$

and the initial basis $B(t_p)$ equals the purchase price $v(t_p, T)$ at the purchase time t_p.

Current tax laws grant individual investors an option to time the realization of loss or gain to reduce their tax burden. Under the optimal trading strategy, an investor has to decide whether to sell or hold the bond in each period before maturity, if it does not default. Investors would choose to optimize the value of a defaultable bond, U, which may differ from the market price of the bond, v.

For an investor following the optimal trading strategy, the value of the defaultable bond in a sale is:

$$v(t, T) - \alpha\tau\left(v(t, T) - B(t, t_D)\right) \tag{4}$$

where $B(t, t_D)$ is the basis, and t_D is the time of the most recent trade before current time t. We need a second time indicator, t_D, in the basis variable (and also the amortization variable L) because the basis is reinitialized each time a trade occurs.

The second term in the value function (4) is associated with the tax treatment of gain or loss from the sale by accounting for the amortization. The investor's value of the bond is higher than the market price if there is a capital loss to be realized and lower in the case of a capital gain.

The optimization problem faced by an investor holding the defaultable bond is to maximize the expected discounted value of cash flow, given a probability of default:

$$U(t, T) = \max\left\{U^T(t, T), U^N(t, T)\right\} \tag{5}$$

where $U^T(t, T)$ is the value to the investor trading at time t and $U^N(t, T)$ is the value to the investor not trading at time t. $U^T(t, T)$ and $U^N(t, T)$ can be expressed as:

$$\begin{aligned} U^T(t, T) = E_t^P \; & \left\{\left[-\alpha\tau\left(v(t, T) - B(t, t_D)\right) + \right.\right. \\ & \left.\left. e^{-r_t}\begin{pmatrix} (1 - \tau)\kappa - \tau L(t, t_D) + \\ [U(t + 1, T) | U^T(t, T)] \end{pmatrix}\right] 1_{\{t^* > t+1\}}\right\} \\ & + E_t^P\left\{e^{-r_t}\left[\delta\left[v(t + 1, T) | U^T(t, T)\right]\right.\right. \\ & \left.\left. + \alpha\tau\begin{pmatrix} B(t + 1, t_D) - \\ \delta\left[v(t + 1, T) | U^T(t, T)\right] \end{pmatrix}\right] 1_{\{t^* \in \{t, t+1\}\}}\right\} \end{aligned} \tag{6-A}$$

$$\begin{aligned} U^N(t, T) = E_t^P & \left\{e^{-r_t}\begin{pmatrix} (1 - \tau)\kappa - \tau L(t, t_D) + \\ \left[U(t + 1, T) | U^N(t, T)\right] \end{pmatrix}\right] 1_{\{t^* > t+1\}}\Bigg\} + \\ & E_t^P\left\{e^{-r_t}\left[\delta\left[v(t + 1, T) | U^N(t, T)\right]\right.\right. \\ & \left.\left. + \alpha\tau\begin{pmatrix} B(t + 1, t_D) - \\ \delta\left[v(t + 1, T) | U^N(t, T)\right] \end{pmatrix}\right] 1_{\{t^* \in \{t, t+1\}\}}\right\} \end{aligned} \tag{6-B}$$

where $1_{\{\cdot\}}$ is the indicator function, and t^* is the time of default. The term $\alpha\tau[v(t, T) - B(t, t_D)]$ in Equation (6-A) represents the capital gains tax when a trade occurs at t. The investor chooses to maximize U at time t. The optimal trading rule is complex because the action at time t affects the future value $U(t + 1, T)$.

We assume the investor buys back the same units of bonds right after a sale, and there is no penalty for such action (see Constantinides and Ingersoll [1984]). The investor uses any gain from the sale to purchase the bond and makes up any loss by adding another fund. We concentrate on long-term tax-related trading like Constantinides and Ingersoll [1984], but we incorporate default risk in the valuation of the tax timing option.

Investors are indifferent to holding or selling the bond if their valuation of the bond is exactly equal to the market price:

$$v\left(t_p, T\right) = U(t_p, T) \tag{7}$$

The two boundary conditions for the investor's dynamic programming problem are

$$v(T,T) = 1 \tag{8}$$

$$U(T,T) = 1 \tag{9}$$

At maturity, both the price and the investor's value of the bond are equal to the face value (= 1). Solving (5) subject to the boundary conditions in Equations (7)-(9) yields the bond price, the value of the bond to the investor, and the optimal strategy for the realization of capital gains and losses.

In the continuous-time case, we postulate that short-term interest rates follow a one-factor stochastic process, $dr = (\varphi - \mu r^n)dt + \sigma r^m dw$, where dw is the increment of the scalar Wiener process w, $\phi = 0$, $n = 2$, and $m = 3/2$.[5] To simplify the notation, we denote v_p as the purchase price of the bond $v(t_p, T)$.

The range of interest rates at which the investor's value of the bond exceeds the after-tax proceeds from the immediate sale is referred to as the *continuation region* (see Constantinides and Ingersoll [1984]). In the continuation region, we can rewrite (5) in a continuous differential form:

$$\begin{aligned}
&\frac{1}{2}\sigma^2 r^{2m}\frac{\partial^2 U}{\partial r^2} + \left(\varphi - \mu r^n\right)\frac{\partial U}{\partial r} + \frac{\partial U}{\partial t} + \\
&\left[\delta v - \alpha\tau\left(\delta v - v_p\right) - U\right]\lambda + \\
&(1-\tau)\kappa - \left(\tau - \frac{\partial U}{\partial v_p}\right)\frac{1 - v_p}{T - t_p} - rU = 0
\end{aligned} \tag{10}$$

where λ is assumed to be deterministic. This differential equation represents the after-tax version of the local expectations hypothesis; the expected after-tax rate of return of the bond equals the after-tax single-period interest rate over the interval $\{t, t + dt\}$.

Equation (10) does not have a closed-form solution for bonds with finite maturity, but we can obtain a solution for the taxable consol bond. We then derive an approximate formula of the timing option value for bonds with any finite maturity based on the closed-form solution for the consol.

The analysis can be simplified if we assume a buy-and-hold strategy under which an investor buys the bond and holds it until either maturity or default. The buy-and-hold value of the bond to the investor at time t can be simplified from (6-A) and (6-B) as

$$\begin{aligned}
U(t,T) &= E_t^P\left\{e^{-r_t}\left[(1-\tau)\kappa_t - \tau L(t,t) + U(t+1,T)\right]1_{\{t^*>t+1\}}\right\} \\
&+ E_t^P\left\{e^{-r_t}\left[\delta_t v(t+1,T) + \alpha\tau\left(B(t+1) - \delta_t v(t+1,T)\right)\right]1_{\{t^*\in\{t,t+1\}\}}\right\}
\end{aligned}$$

This equation involves the bond values to both the investor and the market, and it is recursive. To eliminate the bond value to the investor, we can unfold it forward over the life of the bond. This results in a direct discounting of the future cash flow:

$$\begin{aligned}
v(t,T) = E_t^P\Bigg\{&\left[\left[\frac{v(t_m,T)}{\prod_{i=0}^{m-1}\left[1+r_{t_i}\right]} + \sum_{i=1}^{m}\left\{\frac{(1-\tau)\kappa_{t_i} - \tau L(t_i)}{\prod_{k=0}^{i-1}\left[1+r_{t_k}\right]}\right\}\right]1_{\{t^*>T\}}\right. \\
&+ \sum_{i=1}^{m}\left[\frac{\left[\delta_{t_{i-1}} v(t_i,T) + \alpha\tau\left(B(t_i) - \delta_{t_{i-1}} v(t_i,T)\right)\right]}{\prod_{k=0}^{i-1}\left[1+r_{t_k}\right]} + \sum_{k=1}^{i}\left\{\frac{(1-\tau)\kappa_{t_k} - \tau L(t_k)}{\prod_{l=0}^{k-1}\left[1+r_{t_l}\right]}\right\}\right]1_{\{t^*\in\{t,t+1\}\}}\Bigg]
\end{aligned}$$

To obtain the value of zero-coupon bonds, v_o, simply set $\kappa = 0$. This formula can be used to estimate the price under a buy-and-hold strategy for bonds with the given maturity.

Timing Option Values for Consol Bonds

Coupon payment is perpetual, and time to maturity is infinite for consols. To derive the closed-form solution for the tax timing option value, following Constantinides and Ingersoll [1984], we assume symmetric taxation for long- and short-term capital gains.[6]

Under this condition, the pricing formula of the consol bond is (see Appendix A):

$$v = \frac{(1-\tau)\kappa}{r + (1-\alpha\tau)(1-\delta)\lambda - \psi\left(r, \bar{\sigma}^2, \bar{\mu}, \alpha, \tau, \lambda\right)} \tag{11}$$

where

$$\begin{aligned}
&\psi\left(r, \bar{\sigma}^2, \bar{\mu}, \alpha, \tau, \lambda\right) \\
&= r\left(\bar{\sigma}^2 - \bar{\mu}\right)\left[1 + \alpha\tau\frac{\lambda}{[r+\lambda]}\frac{\varsigma}{(1-\varsigma)}\right] + \frac{\alpha\tau}{(1-\varsigma)}r\left[1 - \bar{\sigma}^2 + \bar{\mu}\right]
\end{aligned} \tag{12}$$

ψ captures the timing option value, ζ is the negative root of a quadratic equation linked to the differential Equation (A-10) in Appendix A, $\bar{\sigma} = \frac{\sigma}{1+\varepsilon}$, $\bar{\mu} = \frac{\mu}{1+\varepsilon}$, and $\varepsilon = \lambda/r$. As the value of volatility goes to zero, $|\varsigma|$ goes to infinity and $\psi(r, \bar{\sigma}^2, \bar{\mu}, \alpha, \tau, \lambda)$ approaches zero. Thus, ψ is driven by the variability of stochastic interest rates. The second term in the first set of brackets measures the interactive effect of default intensity with stochastic interest rates. This term has a negative value ($\varsigma < 0$), suggesting a negative effect of default risk on the value of the timing option. The last term in Equation (12) is associated with the effect of optimal trading, which has a positive value.

Comparing v in (11) to the solution for the default-free consol price ($\varepsilon = 0$):

$$v_f = \frac{(1-\tau)\kappa}{\left(1 - \sigma^2 + \mu\right)\left(1 - \dfrac{\alpha\tau}{(1-\varsigma)}\right)r} \tag{13}$$

shows how the probability of default affects the consol price.

On one hand, default risk reduces the bond price in (11) through the default intensity λ or ε ($= \lambda/r$). On the other hand, the combined effect of default and taxes partially offsets the price drop (due to default probability) through $\alpha\tau(1-\delta)\lambda$ because the government subsidizes part of the investor's loss through a tax rebate. As expected, an increase in default probability reduces the bond price.

Note that in the pricing formula for the defaultable consol, the volatility factor, σ, and the drift, μ, are replaced by $\bar{\sigma}$ and $\bar{\mu}$.[7] Thus, the default intensity λ somewhat weakens the sensitivity of the consol price to the volatility and drift of the short-term interest rate.

Setting the capital gains tax rate associated with trading to zero in (11), we obtain the consol price under the buy-and-hold strategy. This is equivalent to deriving the consol price in an economy with zero capital gains tax rates because investors would never pay the capital gains tax:

$$v_B = \frac{(1-\tau)\kappa}{\left\{\left(1 + \varepsilon - \bar{\sigma}^2 + \bar{\mu}\right)\left[1 + \dfrac{\varepsilon\alpha\tau}{1+\varepsilon}\left(\dfrac{\varsigma}{1-\varsigma}\right)\right] - \varepsilon(1-\alpha\tau)\delta\right\}r} \tag{14}$$

The value of the timing option of an early realization of capital losses for a defaultable consol can be measured by the difference between the prices under the optimal trading and the buy-and-hold strategies:

$$\frac{v - v_B}{v} = \frac{\alpha\tau}{\left[1 + \dfrac{\varepsilon\alpha\tau\varsigma}{(1+\varepsilon)(1-\varsigma)} - \dfrac{\varepsilon(1-\alpha\tau)\delta}{\left(1+\varepsilon-\bar{\sigma}^2+\bar{\mu}\right)}\right](1-\varsigma)} \tag{15}$$

For the default-free consol, the value of the timing option can be obtained simply by setting ε to zero:

$$\frac{v_f - v_{f,B}}{v_f} = \frac{\alpha\tau}{\left(1-\varsigma^*\right)} \tag{16}$$

where ς^* is

$$\varsigma^* = \frac{-\left(\sigma^2/2 - \mu\right) - \sqrt{\left(\sigma^2/2 - \mu\right)^2 + 2\sigma^2(1-\tau)}}{\sigma^2}$$

Exhibit 1, Panel A, shows the effect of the default intensity (or $\varepsilon = \lambda(1-\tau)/r$, given r) on the value of the timing option, $v - v_B$.[8] The timing option values are associated with different recovery rates of 0.5, 0.25, and 0 for the solid, dashed, and dotted curves, respectively. The higher the recovery rate, the lower the rate of declines in the timing option value as default intensity increases.

First, the value of the tax timing option declines with default intensity. A higher default intensity means a lower probability of survival for the bond. As a result, the probability of exercising the tax timing option is reduced, and the timing option value declines with default intensity.

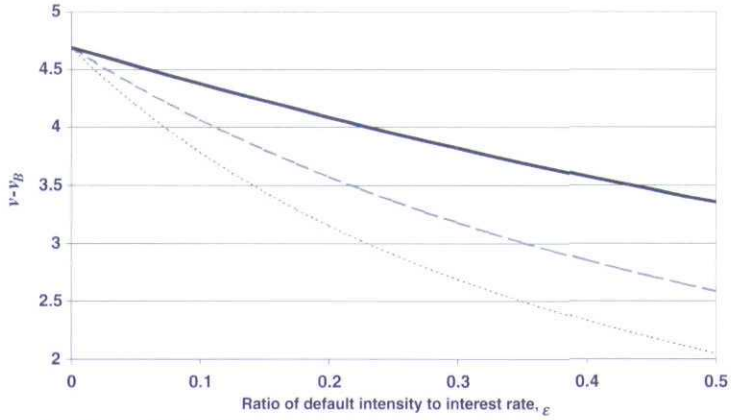
Second, the recovery rate affects the rate of reductions in the timing option value with default intensity.

Exhibit 1, Panel B, shows the effect of the default probability on the relative value of the timing option, $(v - v_B) / v$, where the timing option value is divided by the bond price v. This is the conventional measure of the timing option value. Both the bond value v and the timing option value decline with default intensity. Thus, changes in the relative value of the timing option with respect to default intensity depend on the rates of change in the timing option value and in the bond value v.

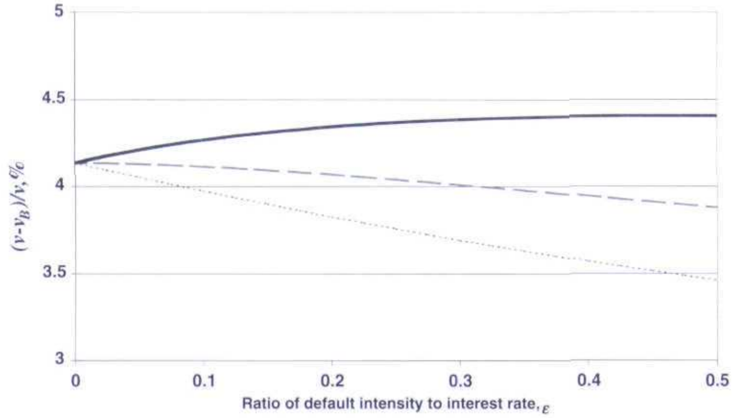
As shown in Panel A, when the recovery rate is high, the timing option value declines less rapidly with default intensity (and, in fact, it declines less rapidly than v). Thus, the relative value of timing options increases with default intensity. Conversely, when the recovery rate is low, the timing option value declines faster than v. As a result, the relative timing option value declines with default intensity.

In general, the timing option value increases with

Exhibit 1
Timing Option Values



A. Value of Timing Option of the Consol



B. Relative Value of Timing Option of the Consol

interest rate volatility. Higher volatility causes more fluctuations in bond price and enhances the chance of optimal trading. Similarly, when default intensity is stochastic, the greater volatility of default intensity will cause more fluctuations in bond price and increase the tax timing option value.

Equation (11) implies that the bond yield

$$Y = \frac{[r + (1 - \alpha\tau)(1 - \delta)\lambda] - \psi\left(r, \bar{\sigma}^2, \bar{\mu}, \alpha, \tau, \lambda\right)}{(1 - \tau)} \tag{17}$$

This yield function can be separated into two components. The first is

$$\frac{r + (1 - \alpha\tau)(1 - \delta)\lambda}{(1 - \tau)}$$

when the recovery rate is zero. This component is equivalent to the yield of defaultable bonds under a buy-and-hold strategy (see Yawitz, Maloney, and Ederington [1985]). The second component of the yield:

$$\frac{-\psi\left(r, \bar{\sigma}^2, \bar{\mu}, \alpha, \tau, \lambda\right)}{(1 - \tau)}$$

is associated with the tax timing option value. The tax timing option increases bond price or reduces yield.

Adjusting Timing Option Value for Finite-Maturity Zeros

Most bonds have a finite maturity, so it is necessary to adjust for the effect of the difference in maturity on the timing option value. For a consol bond, because the timing option value per period, S, is not a function of time, the total option value can be expressed as:

$$v - v_B = \int_t^{\infty} \kappa S e^{-\int_t^u r d\xi} du \tag{18}$$

Using (15), we have

$$S = \frac{\alpha\tau}{\left[1 + \frac{\varepsilon\alpha\tau\varsigma}{(1+\varepsilon)(1-\varsigma)} - \frac{\varepsilon(1-\alpha\tau)\delta}{(1+\varepsilon-\bar{\sigma}^2+\bar{\mu})}\right](1-\varsigma)} \frac{H}{(1+\varepsilon)} \tag{19}$$

where H is defined in Equation (A-9) in Appendix A.

The ratio of the value of a finite-maturity bond compared to its buy-and-hold equivalent can be expressed as

$$f(t,T) = \frac{v(t,T)}{v_B(t,T)} = 1 + \frac{\kappa g(t,T)}{v_B} \tag{20}$$

where

$$g(t,T) = \int_t^T S e^{-\int_t^u r(1-\tau) d\xi} du$$

Using Equations (18)-(20), we can estimate the timing option value for bonds with any maturity. We can calculate the price under the buy-and-hold strategy $v_B(t, T)$. The bond price under the optimal trading strategy $v(t, T)$ will be equal to $f(t, T) \cdot v_B(t, T)$. We use this method to estimate the price and the corporate-Treasury spread below.

II. IMPLICATIONS FOR EMPIRICAL INVESTIGATION

We have shown that bond pricing ignoring taxes is markedly different from pricing including the tax effects. To explore the implications of these effects, we first examine the impact of pricing errors on the estimation of default probability. Then we examine the effects of taxes on the yield spreads of corporate bonds in different rating classes.

Effect of Ignoring Taxes on Estimation of Default Probability

Authors have often assumed away tax effects in deducing the default probability from bond yields (e.g., Jarrow, Lando, and Turnbull [1997]; and Duffee [1999]). Estimates of the default parameter would thus be biased, because the effect of taxes is not a constant term that can be absorbed by the intercept of the regression model.

To assess the potential bias in deducing the default probability from bond yields, we conduct a simulation analysis. We generate a sample of simulated prices of defaultable bonds with known default probability, recovery rate, tax rate, and interest rate from the discrete-time model that accounts for both tax and default effects. At maturity the ex-coupon bond is priced at par, which equals unity. The bond price and its value to the investor at any time prior to maturity can then be obtained through dynamic programming.

Using these simulated prices, we examine errors in estimating default probability when taxes are ignored. The parameter of default probability is calculated from a model that assumes no taxes. Under this restriction, (5) becomes:

$$\begin{aligned} U(t,T) = {} & E_t^P\{e^{-r_t}[\kappa_t + U(t+1,T)][1-\lambda_t]\} + \\ & E_t^P\{e^{-r_t}[\delta_t v(t+1,T)]\lambda_t\} \end{aligned} \tag{21}$$

where λ is the default probability per period, and tax rates are forced to zero. The investor's value function and bond price satisfy conditions as follows:

$$\begin{aligned} & U(T,T) = 1 \\ & U(T,t_p) = v(t_p,T) \end{aligned} \tag{22}$$

Given a bond price from the simulated sample, numerical iterations are performed to obtain the value of the default probability as if there were no taxes. Since our price data are simulated and therefore not subject to random errors, we need only a coupon bond to solve for an unknown default probability, given the values of δ, coupon rate κ, interest rate r, and maturity T.

Exhibit 2 reports the numerical estimates of default probability for bonds with different maturities (in years) and tax rates (τ) but with identical interest rates (r), default probability (λ), and capital gain tax rate factor (α). We examine three scenarios with the true τ [$= \tau_F + \tau_S(1 - \tau_F)$] equal to 15%, 30%, and 50%. The values of the underlying true parameters are $\delta = 0.3$, $\alpha = 0.5$, $r = 0.1$, $\kappa = 0.14$, and $\lambda = 0.05$.[9]

When taxes are ignored, the default probability is generally overestimated, and this upward bias increases with

EXHIBIT 2

Estimates of Default Probability When Taxes are Ignored: Multiple Tax Rates

	$\tau = 0.15$	$\tau = 0.3$	$\tau = 0.5$
Maturity	Estimated Values λ	Estimated Values λ	Estimated Values λ
1	0.05341	0.05751	0.06445
2	0.05342	0.05755	0.06463
3	0.05343	0.05760	0.06483
4	0.05344	0.05765	0.06505
5	0.05345	0.05771	0.06526
6	0.05347	0.05776	0.06549
7	0.05348	0.05782	0.06572
8	0.05349	0.05788	0.06594
9	0.05351	0.05794	0.06617
10	0.05352	0.05800	0.06640
11	0.05353	0.05806	0.06662
12	0.05355	0.05812	0.06685
13	0.05356	0.05818	0.06706
14	0.05357	0.05823	0.06728
15	0.05359	0.05829	0.06748

$\delta = 0.3$, $\alpha = 0.5$, $r = 0.1$, $\kappa = 0.14$, and the true $\lambda = 0.05$.

EXHIBIT 3

Estimates of Default Probability When Taxes are Ignored: Multiple Default Rates

	True Values $\kappa = 0.11$ $\lambda = 0.02$	True Values $\kappa = 0.14$ $\lambda = 0.05$	True Values $\kappa = 0.19$ $\lambda = 0.1$
Maturity	Estimated Values λ	Estimated Values λ	Estimated Values λ
1	0.02191	0.05751	0.1161
2	0.02193	0.05755	0.1161
3	0.02197	0.05760	0.1162
4	0.02201	0.05765	0.1163
5	0.02205	0.05771	0.1164
6	0.02210	0.05776	0.1165
7	0.02215	0.05782	0.1165
8	0.02220	0.05788	0.1166
9	0.02224	0.05794	0.1167
10	0.02230	0.05800	0.1168
11	0.02235	0.05806	0.1169
12	0.02241	0.05812	0.1169
13	0.02246	0.05818	0.1170
14	0.02251	0.05823	0.1170
15	0.02257	0.05829	0.1171

$\tau = 0.3$, $\alpha = 0.5$, $r = 0.1$, and $\delta = 0.3$.

maturity and tax rates. On average, the errors are about 7%, 16%, and 32% when the tax rates are equal to 15%, 30%, and 50%, respectively. Taxes reduce the bond value to investors. When taxes are ignored, their effect is factored into the estimated default parameter. The higher the tax rate, the greater the bias in default probability estimates.

Next, we allow the true default probability to change. We set the true λ values of the bonds equal to 0.02, 0.05, and 0.10 and the corresponding coupon rates κ equal to 0.11, 0.14, and 0.19. Exhibit 3 reports the numerical estimates of default probability for these three scenarios. The remaining parameters are $\tau = 0.3$, $\delta = 0.3$, and values of α and r the same as in Exhibit 2.

The default probability estimates are again biased upward. The bias increases with both maturity and the true (unobserved) default risk. On average, the error ranges from 10% (low λ) to 17% (high λ).

The results in Exhibits 2 and 3 explain why previous research ignoring taxes often overestimates default probability. Duffee [1999], for example, indicates that his estimates of default probability appear to be too high, according to a defaultable bond model with no taxes. This finding is consistent with our simulation results.

Estimating Term Structure Model with Taxes

To examine the effects of taxes on yield spread estimation, we first consider bonds priced at par. From (20), the values of corporate and Treasury bonds can be expressed as

$$1 = f^{cor}(t,T)\left\{\int_t^T \kappa^{cor} v_o^{cor}(t,s)ds + v_o^{cor}(t,T)\right\}$$
$$1 = f^{tre}(t,T)\left\{\int_t^T \kappa^{tre} v_o^{tre}(t,s)ds + v_o^{tre}(t,T)\right\} \qquad (23)$$

The superscripts *cor* and *tre* denote corporate and Treasury bonds, and the tax rate τ is implicitly incorporated in $v_o(t, T)$, which is the zero-coupon bond price under the buy-and-hold strategy. The term $f(t, T)$ is the relative price factor used to adjust the bond value to that under the optimal trading strategy [see Equation (20)].

Since the coupon rate is equal to the yield for par bonds—that is, $Y^{cor} = \kappa^{cor}$ and $Y^{tre} = \kappa^{tre}$—the corporate-Treasury yield spread can be expressed as:

$$Y^{cor} - Y^{tre} = \frac{1 - v_o^{cor}(t,T)}{g^{cor}(t,T) + \int_t^T v_o^{cor}(t,s)ds} - \frac{1 - v_o^{tre}(t,T)}{g^{tre}(t,T) + \int_t^T v_o^{tre}(t,s)ds} \qquad (24)$$

This model can be used to estimate the yield spread given the values of the underlying parameters.[10]

The first important parameter is the default probability. Following the procedure in Elton et al. [2001], we obtain the estimate of default probability from the Standard & Poor's transition matrix. Exhibit 4 shows the evolution of these default probabilities. We also use the recovery rates reported by Altman and Kishore [1998].[11]

We assume the state tax rate is 7.5% and the federal income tax rate is 35.0%. This gives an effective state tax rate of about 4.9% for corporate bonds, which is equal to the average rate suggested by Elton et al. [2001]. The capital gains tax parameter a is set to 0.5. The parameters (μ and σ) of interest rate process, r, are estimated from short-term Treasury rates ($\sigma = 0.04$ and $\mu = 0$).

We use the yield data of Treasuries and corporate bond indexes from January 2001 to February 2004 for empirical estimation. The Treasury data consist of rates for maturities of 1, 2, 3, 5, 10, and 30 years. To partially abstract from the effects of liquidity risk, we use the swap rate data over the same period to estimate the liquidity premium. Swap data provide the one-year LIBOR and swap rates for maturities of 2, 3, 5, 10, and 30 years. The one-month Treasury bill rates are used as a proxy for short rates. All data are monthly.

Treasury bond, corporate bond indexes (BB to AA), and swap data are obtained from Lehman's database. One-month Treasury rates are obtained from the database of the Federal Reserve Bank of St. Louis.

Since index bond yields represent par rates, they can be expressed as an explicit function of the value (or discount factor) of zero-coupon bonds. Denote Y as the par rate and D as the discount factor. Then, we have the relationship (see also Duffie and Singleton [1997]):

$$Y = 2\left[\frac{1 - D_T}{\sum_{i=0}^{2T} D_{i/2}}\right] \tag{25}$$

Given a yield curve, we can calculate the spot rates associated with the discount factor using a standard nonlinear optimization technique.

We use the method proposed by Nelson and Siegel [1987] to estimate spot rates.[12] Details of this estimation procedure are explained in Appendix B. The yield spread is the difference between the spot rate of corporate bonds in a particular rating class and that of the Treasury bond with the same maturity. We estimate the spot rates with maturities up to 20 years from the data.

An additional complication involved in estimating the components of spreads is that the extreme liquidity of Treasury securities makes their yields much lower than

EXHIBIT 4
Evolution of Default Probabilities

Year	AAA	AA	A	BBB	BB	B	CCC
1	0.000	0.000	0.103	0.212	1.209	5.902	22.526
2	0.002	0.017	0.154	0.350	1.754	6.253	18.649
3	0.007	0.037	0.204	0.493	2.147	6.318	15.171
4	0.013	0.061	0.254	0.632	2.424	6.220	12.285
5	0.022	0.087	0.305	0.761	2.612	6.031	10.031
6	0.032	0.115	0.355	0.879	2.733	5.795	8.339
7	0.045	0.145	0.406	0.983	2.804	5.540	7.095
8	0.059	0.177	0.457	1.075	2.836	5.280	6.182
9	0.075	0.210	0.506	1.153	2.840	5.025	5.506
10	0.093	0.243	0.554	1.221	2.822	4.780	4.993
11	0.112	0.278	0.600	1.277	2.790	4.548	4.594
12	0.132	0.313	0.644	1.325	2.746	4.330	4.272
13	0.154	0.348	0.686	1.363	2.695	4.125	4.006
14	0.176	0.383	0.726	1.395	2.639	3.934	3.780
15	0.200	0.419	0.763	1.419	2.581	3.756	3.583
16	0.225	0.453	0.797	1.439	2.520	3.591	3.408
17	0.250	0.488	0.830	1.453	2.460	3.436	3.252
18	0.276	0.521	0.860	1.464	2.400	3.292	3.109
19	0.302	0.554	0.888	1.471	2.341	3.158	2.979
20	0.329	0.586	0.913	1.475	2.284	3.033	2.860

Source: Elton et al. [2001].

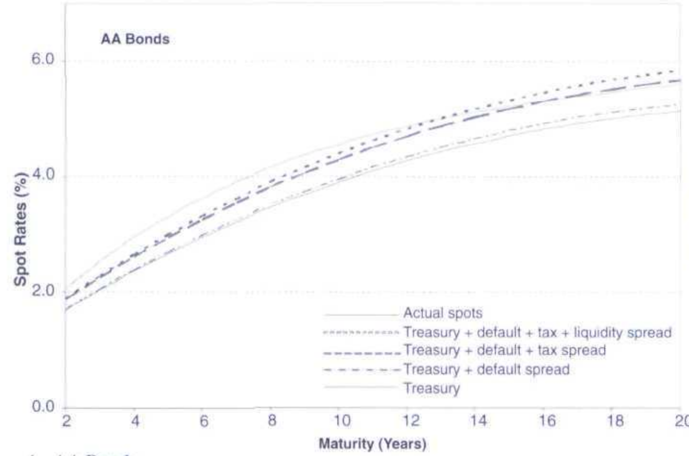
other rates (e.g., swap rates). Longstaff [2004] shows that Treasury bonds can include a significant flight-to-liquidity premium, which in some cases can represent as much as 10% to 15% of the value of the Treasury bond.

Duffie and Singleton [1997] propose an econometric model of the swap yield that assumes the default- and liquidity-adjusted short rate process is a linear combination of independent square root diffusion processes. They focus on the swap rate itself, instead of the decomposition of default and liquidity premiums.

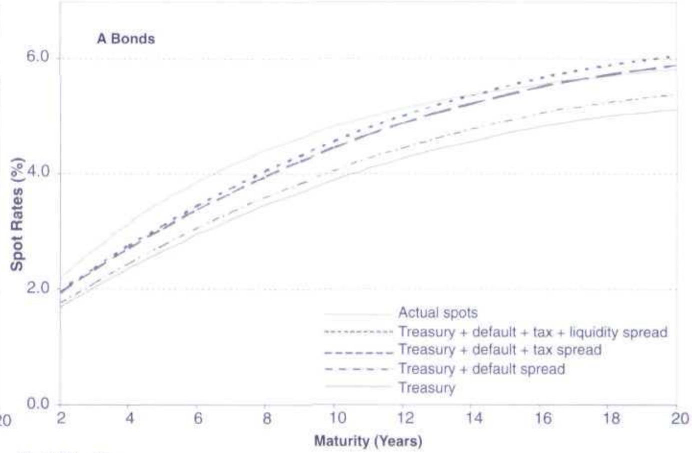
Liu, Longstaff, and Mandell [2002] decompose the swap spread into liquidity and default risk components. They find that the liquidity premium is volatile, and most of the variations in swap spreads are attributable to changes in the relative liquidity of swaps and Treasury bonds. They conclude that if there are credit premiums embedded in swap spreads, they should be interpreted as primarily liquidity premiums. They also suggest that higher swap spreads recently observed in the financial markets are largely due to increased liquidity of Treasury securities. Because liquidity is volatile, one needs to estimate the recent liquidity premium from current market data.

To partially abstract from the effect of liquidity, we estimate the liquidity premium implied by the swap-Treasury yield spread.[13] Like Liu, Longstaff, and Mandell [2002], we model swap yields using a four-factor affine term structure framework. The first three factors are the state variables associated with riskless rates, and the fourth factor represents the dynamics of the swap spread. Instead

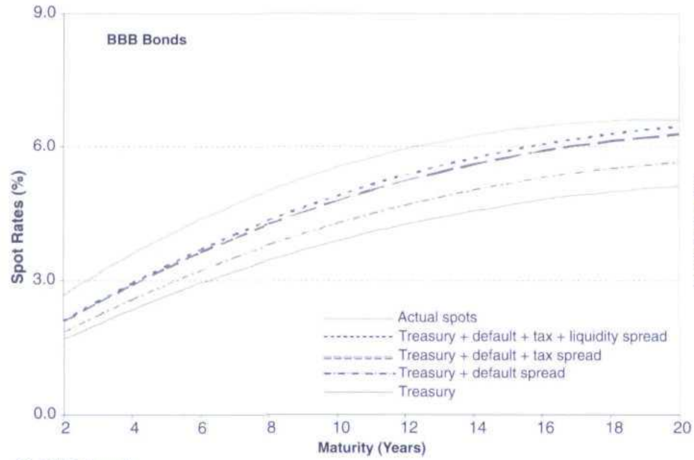
Exhibit 5
Spot Rates



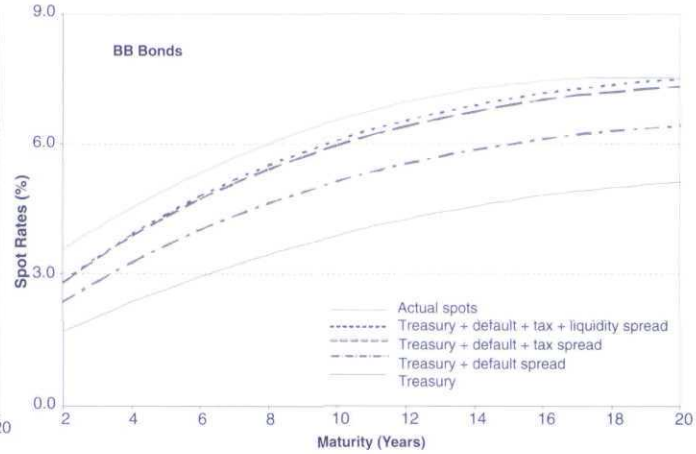
A. AA Bonds



B. A Bonds



C. BBB Bonds



D. BB Bonds

of using the Vasicek process, we assume that the dynamics of the four factors follow a square-root process:

$$ds_{i,t} = \kappa_{si}(\theta_{si} - s_{i,t})dt + \sigma_{si}\sqrt{s_{i,t}}\,dZ^s_{i,t} \quad \text{for} \quad i = 1, 2, 3, 4 \tag{26}$$

where κ_{si}, θ_{si}, and σ_{si}, are constants, and the $dZ^s_{i,t}$ are independent Brownian motions (see Duffie and Singleton [1997]). Under the equivalent martingale measure, these processes can be represented as:

$$ds_{i,t} = (\kappa_{si}\theta_{si} - (\kappa_{si} + \eta_{si})s_{i,t})dt + \sigma_{si}\sqrt{s_{i,t}}\,d\hat{Z}^s_{i,t} \tag{27}$$

where the $d\hat{Z}^s_{i,t}$ are independent Brownian motions, and η_{si} is the market price of risk associated with $d\hat{Z}^s_{i,t}$.

We estimate the state variables and the parameters of the four-factor affine model jointly using the Kalman filtering method (see Duffee [1999] and Jarrow et al. [2003]). To estimate the recent liquidity premium, we fit monthly data for one-year LIBOR, swaps with maturities of 2, 3, 5, 10, and 30 years, and Treasuries of 1, 2, 3, 5, 10, and 30 years jointly over the period of January 2001 to February 2004.

The Kalman filter model fits the data extremely well; the average mean percentage error is only about 3.21%. We obtain estimates of the implied swap spread and the liquidity component of the swap spread from the estimated model parameters using a procedure similar to Liu, Longstaff, and Mandell [2002].[14] The estimated liquidity premium is then added to the default and tax premiums estimated from our term structure model including taxes to explain observed corporate-Treasury spreads.

We obtain the spread estimates for AA, A, BBB, and BB bonds. Exhibit 5 plots the estimated spreads with and without taxes along with the spot yield curves of corporate and Treasury bonds in February 2004. Taxes clearly explain a substantial portion of the yield spread, so adding taxes to the term structure model increases its explanatory power considerably. Ignoring taxes results in substantial underestimation of spreads. The tax spread in percentage terms is higher for higher-grade bonds.

Exhibit 6 summarizes the components of the spread estimated by the model for different bond ratings. The tax premium ranges from 27.6% for BB bonds to 59.4% for AA bonds. Personal taxes explain a higher percentage of the observed spread than default risk does for AA, A, and BBB bonds. For BB bonds, the default risk premium is higher than the tax premium. This is not surprising, given that default risk is relatively high for low-grade bonds. Consistent with the findings of Elton et al. [2001] and Huang and Huang [2002], the default premium appears to account for only a small proportion of the yield spreads for high-grade bonds.

The liquidity premium as a percentage of the corporate-Treasury spread is relatively high for high-grade bonds. For high-grade bonds (e.g., AA), default risk is negligible, so the combined tax and liquidity effects explain most of the yield spread. For BB bonds, the combined tax and liquidity premium in percentage of the yield spread is smaller, partly because the default premium is larger.

Approximation Formulas for Tax Premium

The formula derived from the full model for estimating the tax premium is complex, and it would be better to have a simple approximation to calculate tax spreads. We compare the performance of several approximate formulas.

The first formula we consider is $\tau_S(1 - \tau_F)Y^{tre}$, where τ_S is the marginal state tax rate, τ_F is the marginal federal tax rate, and Y^{tre} is the yield of a Treasury bond with the same maturity as the corporate bond. This formula estimates the effect of state taxes on the corporate bond spread based on the Treasury bond yield Y^{tre}. The second formula is $\tau_S(1 - \tau_F)Y^{cor}$, where Y^{cor} replaces Y^{tre} in the first formula. The first formula $\tau_S(1 - \tau_F)Y^{tre}$ may underestimate the tax effect since in reality state taxes are applied to corporate bond returns instead of Treasury bond returns. The second formula accounts for this difference.

The third formula considered is the state income tax rate multiplied by the corporate bond yield, $\tau_S Y^{cor}$. This is a simplified version of a tax spread formula derived from a single-period discount model (derivation available on request). This formula may provide a better approximation of the tax spread than the second formula $\tau_S(1 - \tau_F)Y^{cor}$ because it takes into account the increase in the federal income tax liability due to an increase in corporate yield to compensate for the state tax burden.

The easiest way to see this is to rewrite $\tau_S Y^{cor}$ as

$$\frac{\tau_S(1 - \tau_F)Y^{cor}}{(1 - \tau_F)}$$

An increase in corporate bond returns by an amount of $\tau_S(1 - \tau_F)Y^{cor}$ over Treasury is not sufficient to cover an investor's additional tax liability, because the incremental return, $\tau_S(1 - \tau_F)Y^{cor}$, is also subject to the federal tax. To be fully compensated, the investor should require an increase in returns by $\tau_S Y^{cor}$, which is equal to $\tau_S(1 - \tau_F)Y^{cor}$ inflated by $1/(1 - \tau_F)$.

We also use the consol formula under the buy-and-hold yield model [the first component of (17)]:

$$\frac{r + (1 - \alpha\tau)(1 - \delta)\bar{\lambda}}{(1 - \tau)}$$

where $\tau = \tau_F + \tau_S(1 - \tau_F)$, and

$$\bar{\lambda} = \frac{1}{T}\int_0^T \lambda_s ds$$

is the average default probability. The after-tax yield of the Treasury bond with the same maturity as the corporate bond is used for r to take the maturity effect into account.

To calculate the tax spread, we first estimate corpo-

EXHIBIT 6

Estimates of Corporate-Treasury Spread Components

Rating	Default Premium	Tax Premium	Liquidity Premium	Total Spread (bp)
AA	11.4%	59.4%	19.0%	54
A	20.7%	45.9%	13.0%	77
BBB	26.9%	28.1%	6.5%	148
BB	45.2%	27.6%	4.0%	244

EXHIBIT 7

Estimates of Tax Premium Using Approximate Formulas

Rating	$\tau_S(1-\tau_F)Y^{tre}$	$\tau_S(1-\tau_F)Y^{cor}$	$\tau_S Y^{cor}$	Buy-and-Hold Formula
AA	34.7%	39.6%	60.9%	62.0%
A	23.8%	28.7%	44.2%	47.3%
BBB	12.0%	16.8%	25.9%	28.6%
BB	7.3%	12.1%	18.7%	27.6%

rate bond yield from this formula. We then force the tax rate (τ) to zero to estimate another yield without taxes. The difference between the first estimate with taxes and the second estimate without taxes represents the tax premium.

Exhibit 7 summarizes the estimates of tax premium using these four simple formulas. The results show that the first and second formulas substantially underestimate the tax premium. The underestimation bias is more serious for the first formula $\tau_S(1-\tau_P)Y^{tre}$. The third formula $\tau_S Y^{cor}$ performs better than the first two formulas, although it tends to underestimate the tax premiums for lower-grade bonds compared to the estimates by the full model in Exhibit 6 (Column 2).

Finally, the last formula

$$\frac{r + (1-\alpha\tau)(1-\delta)\bar{\lambda}}{(1-\tau)}$$

generates higher estimates of tax premium than those estimated by the third formula. Compared to the full model, this formula tends to overestimate the tax premium, particularly for high-grade bonds. This phenomenon is as expected, because the buy-and-hold formula ignores the tax timing option value, which is higher for high-grade bonds. The timing option value has a positive impact on bond price and a negative impact on bond yield. Ignoring this value leads to an overestimation of tax spread.

Overall, the performance of the approximate formulas varies. For higher-grade bonds, $\tau_S Y^{cor}$ does a better job of capturing the tax effect on spreads, while for lower-grade bonds, the buy-and-hold formula appears to be the better choice.

III. CONCLUSIONS

We have examined the effects of personal taxes on the pricing and default risk estimation of defaultable bonds by accounting for investors' trading strategy. We find that a pricing model that does not consider taxes can yield markedly different results from one including tax effects. Neglecting to account for tax effects may result in substantial estimation errors for the default parameter because tax effects are factored into the default probability estimate. Estimation errors are positively related to maturity, bond risk, and tax rates. Failure to account for these tax effects results in an upward-biased estimation of default risk, which could seriously affect the pricing of defaultable bonds and credit risk derivatives.

What is more important, we show that ignoring taxes results in a substantial underestimation of the spread between corporate and Treasury bonds. The estimation bias is higher for high-grade bonds than for low-grade bonds. Estimating the liquidity premium from the swap-Treasury spread, we find that the liquidity premium in percentage of yield spreads can be fairly high for high-grade bonds.

Finally, we compare our estimates of tax premium using the full model with those estimated by approximate formulas. Some of these approximate formulas can provide quite reasonable estimates of the tax premiums embedded in corporate bond spreads if they are used carefully.

APPENDIX A

Pricing of Defaultable Consol Bonds

To include tax and volatility parameters in the pricing of consols, we assume that

$$v = \frac{H\kappa}{r+\lambda} \tag{A-1}$$

where H is the adjustment factor that takes μ, σ, τ, α, and δ into account in the bond pricing. μ and σ are drift and volatility parameters in the stochastic interest rate process (see also Constantinides and Ingersoll [1984]):

$$dr = \mu r^2 dt + \sigma r^{\frac{3}{2}} dw \tag{A-2}$$

In the continuation region, Equation (21) can be rewritten as

$$\frac{1}{2}\sigma^2 r^3 \frac{\partial^2 U}{\partial r^2} + \mu r^2 \frac{\partial U}{\partial r} + \left[\delta v - \alpha\tau\left(\delta v - v_p\right) - U\right]\lambda + (1-\tau)\kappa - rU = 0 \tag{A-3}$$

The boundary conditions are:

$$U\left(t_p\right) = v_p \tag{A-4}$$

$$\frac{\partial U\left(t_p\right)}{\partial r} = (1-\alpha\tau)\frac{\partial v_p}{\partial r} \tag{A-5}$$

and

$$\lim_{v/v_p \to \infty} \frac{\partial U}{\partial v(t_p, T)} = 0 \qquad \text{(A-6)}$$

The condition in (A-6) indicates that the purchase price, v_p, has a negligible effect on the investor's value function when the current bond price v is much higher than the purchase price v_p. Using (A-1), we have:

$$\frac{1}{2}\sigma^2 \frac{1}{(1+\varepsilon)^2} v^2 \frac{\partial^2 U}{\partial v^2} + \left(\sigma^2 \frac{1}{(1+\varepsilon)^2} - \mu \frac{1}{1+\varepsilon}\right) v \frac{\partial U}{\partial v} +$$
$$\left[\delta v - \alpha\tau\left(\delta v - v_p\right) - U\right]\varepsilon - U + \frac{(1-\tau)(1+\varepsilon)v}{H} = 0 \quad \text{(A-7)}$$

where $\varepsilon = \lambda / r$. Assume that ε is constant, and let

$$\bar{\sigma}^2 = \frac{\sigma^2}{(1+\varepsilon)^2} \qquad \bar{\mu} = \frac{\mu}{1+\varepsilon} \qquad \text{(A-8)}$$

We then have the differential equation for a defaultable consol:

$$\frac{1}{2}\bar{\sigma}^2 v^2 \frac{\partial^2 U}{\partial v^2} + \left(\bar{\sigma}^2 - \bar{\mu}\right) v \frac{\partial U}{\partial v} - (1+\varepsilon)U +$$
$$\left(\frac{(1-\tau)(1+\varepsilon)}{H} + \varepsilon\delta(1-\alpha\tau)\right) v + \varepsilon\alpha\tau v_p = 0 \qquad \text{(A-9)}$$

We can solve for H:

$$H = \frac{(1-\tau)(1+\varepsilon)}{\left(1+\varepsilon - \bar{\sigma}^2 + \bar{\mu}\right)\left[1 - \frac{\alpha\tau}{(1-\varsigma)} + \frac{\varepsilon\alpha\tau}{1+\varepsilon}\left(\frac{\varsigma}{1-\varsigma}\right)\right] - \varepsilon(1-\alpha\tau)\delta}$$

where ζ is the negative root of the quadratic equation:

$$\frac{\bar{\sigma}^2}{2}\varsigma(\varsigma - 1) + \left(\bar{\sigma}^2 - \bar{\mu}\right)\varsigma - (1+\varepsilon) = 0 \qquad \text{(A-10)}$$

Substituting H into (A-1), we obtain the bond price

$$v = \frac{(1-\tau)\kappa}{\left\{\left(1+\varepsilon - \bar{\sigma}^2 + \bar{\mu}\right)\left[1 - \frac{\alpha\tau}{(1-\varsigma)} + \frac{\varepsilon\alpha\tau}{1+\varepsilon}\left(\frac{\varsigma}{1-\varsigma}\right)\right] - \varepsilon(1-\alpha\tau)\delta\right\} r} \qquad \text{(A-11)}$$

Equation (A-11) can be rewritten as:

$$v = \frac{(1-\tau)\kappa}{r + (1-\alpha\tau)(1-\delta)\lambda - \psi\left(r, \bar{\sigma}^2, \bar{\mu}, \alpha, \tau, \lambda\right)}$$

where

$$\psi\left(r, \bar{\sigma}^2, \bar{\mu}, \alpha, \tau, \lambda\right)$$
$$= r\left(\bar{\sigma}^2 - \bar{\mu}\right)\left[1 + \alpha\tau \frac{\lambda}{[r+\lambda]} \frac{\varsigma}{(1-\varsigma)}\right] + \frac{\alpha\tau}{(1-\varsigma)} r\left[1 - \bar{\sigma}^2 + \bar{\mu}\right]$$

For a constant interest rate, $\bar{\sigma}^2$ and $\bar{\mu}$ are zero, and ζ goes to infinity.

It can be shown that the investor's value of the defaultable consol bond is:

$$U = v + \alpha\tau\left(v_p - v\right) \qquad v_p \geq v$$
$$= \left\{1 - \frac{\alpha\tau(1+\varepsilon(1-\varsigma))}{(1-\varsigma)(1+\varepsilon)}\left[1 - \left(\frac{v_p}{v}\right)^{1-\varsigma}\right]\right\} v +$$
$$\frac{\varepsilon\alpha\tau}{1+\varepsilon}\left(1 - \left(\frac{v}{v_p}\right)^{\varsigma}\right) v_p \qquad v_p < v \qquad \text{(A-12)}$$

We can identify the effect of default on the investor's valuation of the bond by comparing (A-12) to the value of the default-free consol bond (see Constantinides and Ingersoll [1984]):

$$U_f = v_f + \alpha\tau\left(v_{f,p} - v_f\right) \qquad v_{o,p} \geq v_f$$
$$= \left\{1 - \frac{\alpha\tau}{(1-\varsigma)}\left[1 - \left(\frac{v_{f,p}}{v_f}\right)^{1-\varsigma}\right]\right\} v_f \qquad v_{o,p} < v_f \qquad \text{(A-13)}$$

The investor's value of the default-free consol, U_f, is greater than the value of the defaultable counterpart, U. Intuitively, the event of default interrupts any plan for deferring the capital gains tax. The higher the probability of default, the shorter the horizon for deferring the investor's capital gains taxes, and hence the lower the value of tax deferment (in dollar terms).

APPENDIX B

Estimation of Spot Rates

Using the Nelson and Siegel [1987] methodology, we fit Equations (B-1) to (B-3) below to the yield data of corporate bond index for a given risk class to obtain the spot rate at any time. Let:

$$D(t, \alpha_0, \alpha_1, \alpha_2, \alpha_3) = e^{-r(t, \alpha_0, \alpha_1, \alpha_2, \alpha_3)t} \qquad \text{(B-1)}$$

and

$$r(t, \alpha_o, \alpha_1, \alpha_2, \alpha_3) = \alpha_0 + (\alpha_1 + \alpha_2)\left[\frac{1 - e^{-\alpha_3 t}}{\alpha_3 t}\right] - \alpha_2 e^{-\alpha_3 t} \quad \text{(B-2)}$$

where $D(\cdot)$ is the present value as of current time 0 for a payment that will be received t periods in the future. $r(\cdot)$ is the spot rate at time 0 for a payment to be received at time t, and α_0, α_1, α_2, and α_3 are parameters of the Nelson-Siegel model.

The bond yield can be written as a function of the spot rate:

$$Y = 2\left[\frac{1 - D(T, \alpha_o, \alpha_1, \alpha_2, \alpha_3)}{\sum_{j=0}^{2T} D\left(\frac{j}{2}, \alpha_o, \alpha_1, \alpha_2, \alpha_3\right)}\right] \quad \text{(B-3)}$$

where Y is the yield of a par bond, and $2T$ is the number of semiannual coupon payments. The parameters α_0, α_1, α_2, and α_3 can be estimated from corporate bond yields using the model in (B-3). Following this, the spot rate can be readily calculated from (B-2). We fit Equation (B-3) to AA, A, BBB, and BB index yields obtained from Lehman Brothers' database from January 2001 to February 2004.

ENDNOTES

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[1]See, for example, Duffee [1999].

[2]Bond prices are affected by the default process. The discontinuity in security price changes upon default has often been modeled as a Poisson process (Merton [1971, 1975, 1976]; Jarrow and Rosenfeld [1984]; Jarrow and Madan [1995]), or more generally by a point process (Jarrow and Turnbull [1995]; Duffie and Singleton [1999].

[3]The former specifies the bond residual value as a fraction of the price at the time of default of an otherwise equivalent default-free bond (see Jarrow and Turnbull [1995]), and the latter formulates it as a fraction of the face value of the bond (see Brennan and Schwartz [1980]; Duffee [1998]).

[4]Amortization rules vary by bond issue date. For example, for bonds issued after September 27, 1985, the tax code requires the use of the constant-yield method.

[5]Vasicek's [1977] model of the term structure assumes $n = 1$ and $m = 0$. Cox, Ingersoll, and Ross [1985] assume $n = 1$ and $m = 1/2$. Constantinides and Ingersoll [1984] assume $n = 2$ and $m = 3/2$.

[6]This assumption is quite reasonable if dealers are the marginal investors. Green [1993] has suggested that dealers are likely to dominate the market and become the marginal investors. The assumption of symmetric tax rates renders the value of the consol bond independent of the investor's holding period. The time-independence of the bond valuation allows us to derive an analytical solution for U and v. The issue of premium (discount) amortization is not relevant in the case of consol bonds.

[7]It should be noted that the volatility of the interest rate is not σ, but $\sigma r^{1/2}$.

[8]More precisely, we measure the effect of $\varepsilon = \lambda(1 - \tau)r$ on the timing option value.

[9]These parameter values (except default) are set close to those in Constantinides and Ingersoll [1984] for comparison.

[10]These yield spreads based on yield to maturity are then converted into yield spreads based on spot rates.

[11]The recovery rates are 68.34 (AAA), 59.59 (AA), 60.63 (A), 49.42 (BBB), 39.05 (BB), 37.54 (B), and 38.02 (CCC).

[12]Elton et al. [2001] use the same method for spot rate estimation.

[13]Thanks to Stanley Kon for this suggestion.

[14]We take the Liu, Longstaff, and Mandell [2002] mean ratio of liquidity to default premium (29.3%) to adjust the swap spread to come up with the liquidity spread.

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