Delivery Options in the Pricing and Hedging of Treasury Bond and Note Futures

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Treasury bond and note futures are widely used in the management of long- and intermediate-term interest rate risk. The Chicago Board of Trade at present offers futures contracts on 30-year Treasury bonds and 10-, 5-, and 2-year Treasury notes. Exhibit 1 provides a summary of the bonds eligible for delivery in connection with each of the four contracts.

Each contract provides for the delivery of bonds or notes (both called bonds hereafter) that are not of the standard maturity. In each contract, control over the decision of the bond to deliver is given to the seller. Presumably, the seller will deliver the bond that provides the most value at the maturity of the contract in relation to its market price. In the vernacular of futures traders, this bond is referred to as the *cheapest to deliver.*

Much has been written on the effect of the delivery option in the pricing of Treasury bond futures. Several of the earliest researchers, including Gay and Manaster [1984, 1991], Boyle [1989], and Hemler [1990], developed pricing models based on the Margrabe [1978, 1982] model for pricing an option to exchange one asset for another. Under this framework, the prices of all Treasury bonds eligible for delivery are assumed to be jointly lognormal. More recent authors have noted that the assumption of joint lognormality is inconsistent with bond prices approaching par at maturity. Moreover, the assumed correlation structure is not likely to reflect the interdependencies among bond prices brought about by changes in interest rates.

Kane and Marcus [1986] estimate the value of the delivery option using Monte Carlo simulations based on term structures and associated volatilities as characterized during the 1978-1982 period. Hegde [1990, 1993] estimates the composite value of all options associated with the Treasury bond futures contract, including the accrued interest option, the wild card option, and the end-of-month option, but does not develop a model to estimate the values of any of the options individually.[1] Gay and Manaster [1986] provide empirical evidence that during the period 1977-1983, the full value of all delivery options was not reflected in Treasury bond futures prices. This research is summarized in Chance and Hemler [1993].

Today it is standard practice to value interest rate-dependent securities and their associated options using either stochastic interest rate models or arbitrage-free stochastic term structure models. With stochastic interest rate models, the values of interest rate-dependent derivative securities and bonds themselves are computed as functions of a specified stochastic interest rate process. With arbitrage-free stochastic term structure models, the values of interest rate-dependent derivative securities are determined in terms of an observed term structure (or a set of observed bond prices) and an estimated volatility structure of forward rates.

Cherubini and Esposito [1995], Bick [1997], Carr and Chen [1997] and Chen and

Yeh [2003] address the problem of pricing the delivery option using the stochastic interest rate models of Vasicek [1977] and Cox, Ingersoll and Ross [1985]. The arbitrage-free stochastic term structure model of Heath, Jarrow, and Morton [1992] provides the analytical framework for pricing the delivery option in Ritchken and Sankarasubramanian [1992, 1995] and Lin and Paxon [1995]. The same analytical framework, modified to incorporate string shocks, is employed by Santa-Clara and Sornette [2001].

Much less has been written on how the delivery option affects the hedging characteristics of the Treasury bond futures contract. Kane and Marcus [1986] use standard regression-based minimum-variance hedge ratios to estimate the hedging effectiveness of the contract over the 1978–1982 period and find the standard deviation of value changes in hedged positions to be approximately half that of unhedged positions. Implicitly, their approach to hedging requires that the pricing effect of the delivery option be reflected in the regression coefficients and that it remain relatively constant over time.

Lin and Paxon [1995] test the effectiveness of hedging German Bunds of various maturities using dynamically rebalanced hedge ratios based on a futures pricing model that incorporates the delivery option. For the period 1987–1991, they find that the presence of the delivery option does not detract from the contract's hedging effectiveness compared to a hypothetical futures contract issued on a single bond without the option to deliver alternative instruments.[2]

I use the arbitrage-free stochastic term structure model of Black, Derman, and Toy [1990] to price the Treasury bond futures contract in terms of the delivery option as a function of various scenarios regarding the level and volatility of interest rates.[3] Analytically, I do not advance the literature beyond other research that values the delivery option in terms of stochastic interest rate and arbitrage-free stochastic term structure models. I do show that the pricing of the delivery option depends critically on the extent to which interest rates differ from 6% (with semiannual compounding], the rate used for computing conversion factors for Treasury bond futures.[4]

If interest rates are significantly above or below 6%, the delivery option has little if any influence on the pricing of Treasury bond futures, and the contract should be priced as if it is a contract on the single bond identified initially as cheapest to deliver. If interest rates are closer to 6%, the delivery option can have a significant effect on the pricing of Treasury bond futures. Moreover, standard duration-based hedging quantities can result in significant hedging error when interest rates are close to 6% but produce little error when interest rates are significantly different from 6%.

I. WHEN THE BOND IDENTIFIED INITIALLY AS CHEAPEST TO DELIVER MUST BE DELIVERED

Assume that n bonds are eligible for delivery in connection with the Treasury bond (or note) futures contract. Each bond pays interest twice a year at an annual rate of C_i per \$100. Let us assume that one semiannual interest payment of $C_i/2$ will be paid prior to the maturity date of the futures contract, and accrued interest of A_i will be due upon the delivery of bond i as of the futures maturity date. Assume that the bond identified initially as cheapest to deliver must be delivered—in other words, there is no switching option. Then, the equilibrium futures price, F, can be determined by computing a futures price, F_i, specific to each eligible bond, i, eligible for delivery that will cause the return from hedging that specific bond to equal the risk-free rate of interest, and then taking the lowest of these F_i values.

Let P_i denote the current full price (quoted price plus accrued interest) of bond i; V_i^* the conversion factor for bond i as of the maturity date of the futures contract; r_1 the risk-free market rate of interest to be earned through the time of the interim coupon payment; and r_2 the risk-free forward market rate of interest between the times of the interim interest payment and the futures maturity date.[5] If the bond is purchased today at a full price of P_i, where P_i equals the bond's quoted price, Q_i, plus accrued interest, and a selling price of $V_i^*F_i$ plus accrued interest of A_i is locked in by taking a short position in the futures contract, the investor will receive two cash flows. The first will be the interim semiannual interest payment of $C_i/2$, which when reinvested at a rate of r_2 will be equivalent to $(C_i/2)(1 + r_2)$as of the futures maturity date. The second will be $V_i^*F_i + A_i$ received as of the futures maturity date.

The solution value for F_i is the value of F_i that causes the present value of the two cash flows to equal the bond's current full price, or

$$P_i = \frac{(C_i/2)(1 + r_2) + V_i^* F_i + A_i}{(1 + r_1)(1 + r_2)} \tag{1}$$

Solving Equation (1) for F_i gives

$$F_i = \frac{P_i(1 + r_1)(1 + r_2) - A_i - (C_i/2)(1 + r_2)}{V_i^*} \tag{2}$$

EXHIBIT 1

Bonds and Notes Eligible for Delivery in Connection with Treasury Bond and Note Futures Contracts

30-year Treasury Bond	U.S. Treasury bonds that, if callable, are not callable for at least 15 years from the first day of the delivery month or, if not callable, have a maturity of at least 15 years from the first day of the delivery month.
10-year Treasury Note	U.S. Treasury notes maturing at least 6-1/2 years, but not more than 10 years, from the first day of the delivery month.
5-year Treasury Note	U.S. Treasury notes with an original maturity of not more than 5 years and 3 months and a remaining maturity of not less than 4 years and 3 months as of the first day of the delivery month. The 5-year Treasury note issued after the last trading day of the contract month will not be eligible for delivery into that month's contract.
2-year Treasury Note	U.S. Treasury notes with an original maturity of not more than 5 years and 3 months and a remaining maturity of not less than 1 year and 9 months from the first day of the delivery month but not more than 2 years from the last day of the delivery month.

Source: Chicago Board of Trade web site, www.cbot.com.

The equilibrium futures price, F, is the minimum of the F_i values calculated for each of the n bonds, or

$$F = \min(F_1, F_2, ..., F_n) \tag{3}$$

If the futures price were higher, the return from hedging the bond for which F_i is the lowest would exceed the risk-free rate of interest, thereby creating an arbitrage opportunity that could not be sustained in an efficient market.

Futures Pricing When There is No Switching Option

When there is no switching option, the futures contract is assumed to mature in exactly one year, and each bond that can be delivered to satisfy the provisions of the contract is scheduled to make its next semiannual interest payment in exactly six months and its second interest payment on the futures maturity date. Therefore, the accrued interest amount as of the futures maturity date is $A_i = C_i/2$.

In the example, only three bonds, numbered 1–3, whose maturities and coupons are summarized in Exhibit 2, are eligible for delivery. The yield curve is assumed to be flat with a market rate of interest of 5% per year with semiannual compounding, or 2.5% every six months. Therefore, $r_1 = r_2 = 0.025$.

The last column of Exhibit 2 shows the futures price for each bond F_i, computed according to Equation (2), that would cause the return from hedging that particular bond to equal the market rate of interest, or in the example, 0.025 every six months. For Bond 3, F_i is computed as follows:

$$F_3^{'} = \frac{P_3(1 + r_1)(1 + r_2) - A_3 - (C_3/2)(1 + r_2)}{V_3^*}$$

$$F_3^{'} = \frac{\$134.086(1.025)(1.025) - (\$8/2) - (\$8/2)(1.025)}{1.20389}$$

$$= \$110.288$$

If the actual futures price were any higher than \$110.288, the return from hedging Bond 3 by selling futures would be higher than 2.5% per six months. Therefore, to prevent arbitrage, the equilibrium futures price must be \$110.288.

Futures Duration

The analysis has shown that Bond 3 will be cheapest to deliver. Without the option to switch delivery to a different bond, the futures contract should be priced as if it is a contract on the third bond, and the interest rate sensitivity of the contract should be directly related to that of the same bond. It is standard practice to measure this interest sensitivity in terms of duration. The duration of the futures contract, D_F, is as follows:

$$D_F = \frac{(V_{CTD}^* F + A_{CTD}^*) D_{CTD}^*}{V_{CTD}^* F} \approx D_{CTD}^* \tag{4}$$

where A_{CTD}^* denotes interest accrued on the cheapest to deliver bond as of the futures maturity date, and D_{CTD}^* denotes the duration of the cheapest to deliver bond measured as of the maturity date of the futures contract.[6] If the futures maturity date coincides with an interest payment date, which is

EXHIBIT 2

Bonds Eligible for Delivery in Connection with One-Year Futures Contract

Bond #	Annual Coupon per $100 Par	Current Maturity (years)	Market Price per $100 Par	Conversion Factor in One Year	Futures Price, F_i, if Bond i is Delivered
1	4.00	28	85.018	0.73422	116.138
2	6.00	21	112.910	1.00000	112.551
3	8.00	17	134.086	1.20389	110.288

Source: Chicago Board of Trade web site, www.cbot.com.

the case in all my examples, $A^*_{CTD} = 0$, and $D_F = D^*_{CTD}$.

The futures duration of Equation (4) would be a correct measure of a contract's interest rate risk under two conditions. First, the contract would have to be priced as if it were a contract on the bond identified initially as cheapest to deliver, with the understanding that this designation could not subsequently be changed. Second, as shown by Cox, Ingersoll, and Ross [1979], the risk-free continuously compounded rate of interest must follow Brownian motion, $dr = bdt + adz$, with constant drift b and variance a^2 for duration to be a sufficient measure of interest rate risk. Under such a process, interest rates can become negative, and bond prices are unbounded functions of maturity, conditions that are not attractive for justifying any type of interest rate risk measure.

Moreover, as shown by Merton [1970] and Cox, Ingersoll, and Ross [1979, p. 59], the price of a zero-coupon bond per dollar of par is $e^{-r\tau-b\tau^2/2+a^2\tau^3/6}$ under this process, where r is the spot rate of interest and τ is the bond's maturity. Since it is unlikely that the entire term structure of interest rates will be consistent with this specific pricing function, the proper use of duration as a measure of interest rate risk is somewhat limited.

II. INCORPORATING THE OPTION TO SWITCH

Incorporating the option to deliver a bond different from that identified initially as cheapest to deliver into the pricing of Treasury bond futures requires specification of a stochastic process for the risk-free rate of interest. Such a process can then be used to deduce corresponding processes for the price of the various Treasury bonds that are eligible for delivery in connection with the futures contract. These bond pricing processes can then be used to price bond futures as a function of the delivery option.

The stochastic process selected should be one that produces a theoretical term structure of interest rates consistent with the observed term structure and thus theoretical bond and note prices consistent with their market prices. The Heath, Jarrow, and Morton [1992] model provides the most general approach to this type of modeling, but its extensive input and computational requirements generally require that its specification be simplified.

The model of Ho and Lee [1986], which can be viewed as the Heath, Jarrow, and Morton model in a very simplified form, assumes a stochastic process for the interest rate of $dr = b(t)dt + adz$, where the time-varying drift, $b(t)$, is determined by the observed term structure. As shown by Cox, Ingersoll, and Ross [1979], if the drift for this process were constant, duration would be a proper measure of interest rate risk for bonds priced according to this process. Therefore, except for its time-varying drift, the Ho-Lee model would be a logical choice for valuing the delivery option and comparing the risk characteristics of the Treasury bond futures contract to duration-based measures. Any difference in risk measures would be due primarily to the delivery option rather than to the assumed stochastic process.

Initially, I developed a model of futures pricing using a binomial version of the Ho-Lee model as a basis for specifying the stochastic process for the risk-free interest rate. To get the most accurate prices possible, I set up the model to allow for a very short binomial time-differencing interval. I expected to observe convergence in the futures price as the time interval was shortened, but the futures price did not converge, and the model produced seemingly irrational prices.

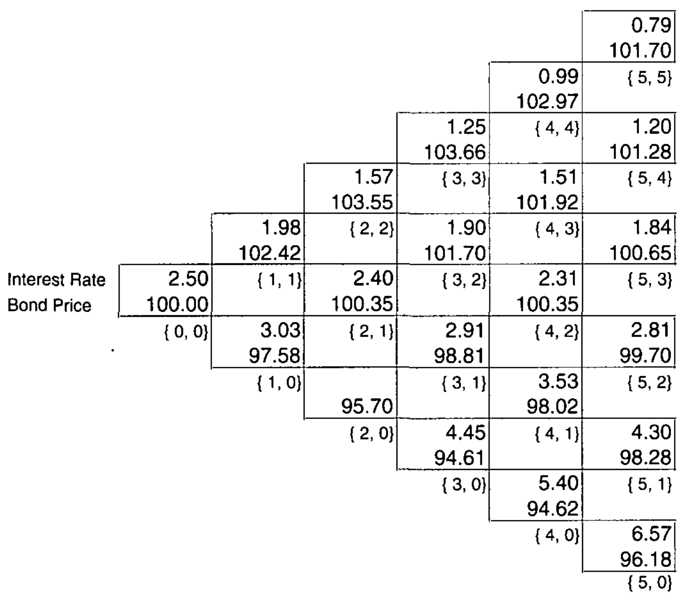
Inspection of model values on the binomial tree revealed that the lack of convergence and the irrational pricing were due to negative interest rates produced by the Ho-Lee model. With a very coarse time-differencing interval, negative interest rates are less likely, but with a very short time interval, the Ho-Lee interest rate tree can become filled with negative interest rates, which then produce anomalous results in pricing the option component of the futures contract. As a result, I abandoned the Ho-Lee model in favor of the Black, Derman, and Toy [1990] (BDT) model.

In the BDT model, the instantaneous risk-free rate of interest is assumed to follow a lognormal process with a time-varying mean and variance. The discrete form of

EXHIBIT 3
Simple Example: Bonds Eligible for Delivery

Bond #	Annual Coupon per \$100 Par	Current Maturity (years)	Market Price per \$100 Par	Conversion Factor in One Year	Futures Price, F_i, if Bond i is Delivered
1	5.00	3	100.000	0.98141	101.894
2	6.00	4	103.585	1.00000	102.754
3	7.00	5	108.752	1.03510	103.536

EXHIBIT 4
Black-Derman-Toy Interest Rate Tree



the BDT model I use assumes that the short-term interest rate follows a multiplicative binomial process with a time-dependent logarithmic mean and variance. The time-dependent mean is determined as a function of the observed term structure of risk-free interest rates. The model thus produces a theoretical term structure that corresponds exactly with the observed term structure. The time-dependent variance must be estimated; it cannot be deduced from observed bond prices.

In an example, I assume that the yield curve is flat, and all risk-free securities yield 5% per year compounded semiannually, or 2.5% every six months. The standard deviation of the natural logarithm of the interest rate in the BDT tree one year from today is estimated to be 0.3 and to remain at 0.3 for all one-year periods in the tree. All interest rate-dependent securities are priced with a risk-adjusted probability, π, associated with an increase in bond prices (or decline in interest rates) from one binomial period to the next equal to 0.5. In addition, the true probability, θ, associated with an increase in bond prices is also 0.5.

The three securities summarized in Exhibit 3 are assumed to be eligible for delivery in connection with the one-year Treasury bond contract, and all are priced to yield 5% per year with semiannual compounding.[7] Exhibit 4 illustrates the pricing of Bond 1, a three-year 5% coupon bond that pays interest semiannually, in terms of the BDT interest rate tree.[8]

Note that the initial interest rate in Exhibit 4 is 2.5% per six months, which corresponds to a rate of 5.0% per year with semiannual compounding or 5.0625% compounded annually. At time 2, or one year from today, three six-month interest rates are possible, 1.57%, 2.40%, and 3.67%. After taking into account that the 2.40% rate will occur twice as often as the other two rates, it is easily shown that the standard deviation of the natural logarithm of the three interest rates is 0.3, the volatility number assumed in constructing the BDT tree.

Each time 5 bond price in Exhibit 4 is simply the present value of the \$100 + \$5/2 = \$102.50 to be received at time 6 discounted at the prevailing six-month interest rate. For example, in state {5, 0}, the price of Bond 1 is

$$\frac{\$102.50}{1.0657} = \$96.18$$

Each time 4 bond price is the risk-adjusted expected value of the bond's value plus accrued interest to be received at time 5 discounted at the prevailing six-month rate. For example, the price for bond 1 in state {4, 0} is

$$\frac{\$98.28(0.5) + \$96.18(0.5) + \$2.50}{1.0540} = \$94.62$$

Prices for Bond 1 in all other states on the tree are computed analogously.

Note that the initial price of \$100.00 for Bond 1 is exactly what one would expect for a 5% coupon bond paying interest semiannually in a market with a flat 5% yield curve with semiannual compounding. Thus, the bond's model and market prices should be equal. Although the detail is not shown, pricing trees for Bonds 2 and 3 are computed the same way.

EXHIBIT 5

Pricing of a One-Year Treasury Bond Futures Contract

Interest Rate / Futures Price

State	Interest Rate	Futures Price
{0, 0}	2.50	101.36
{1, 1}	1.98	103.88
{1, 0}	3.03	98.85
{2, 2}	1.57	105.52
{2, 1}	2.40	102.25
{2, 0}	3.67	95.44

State		Bond 1	Bond 2	Bond 3
{2, 2}	Bond Price	103.5542	108.0567	114.2551
	Conv. Factor	0.9814	1.0000	1.0351
	F*	105.5152	108.0567	110.3809
{2, 1}	Bond Price	100.3506	103.2431	107.7784
	Conv. Factor	0.9814	1.0000	1.0351
	F*	102.2509	103.2431	104.1238
{2, 0}	Bond Price	95.7047	96.4139	98.7898
	Conv. Factor	0.9814	1.0000	1.0351
	F*	97.5171	96.4139	95.4400

Exhibit 5 illustrates the pricing dynamics for the one-year Treasury bond futures contract.[9] For each of the three bonds eligible for delivery, a theoretical market price is shown at time 2 in each interest rate state. Conversion factors for each bond are also shown along with values of F^*, where

$$F^*_{i,\{2,j\}} = \frac{Q^*_{i,\{2,j\}}}{V^*_i}$$

which represents what the futures price must be in state $\{2, j\}$ if bond i is delivered. At this price, the adjusted futures price, $F^*_{i,\{2,j\}} V^*_i$, plus accrued interest will equal the bond's quoted price, $Q^*_{i,\{2,j\}}$ plus the same accrued interest, thereby preventing a riskless arbitrage profit from purchase of the bond in the cash market and immediately delivery for sale through the futures market at a price of $F^*_{i,\{2,j\}} V^*_i$.

The actual futures price in each state at time 2, shown as a shaded value, is the minimum of the various F^* values, or $F^*_{\{2,j\}} = \min(F^*_{1,\{2,j\}}, F^*_{2,\{2,j\}}, F^*_{3,\{2,j\}})$. Otherwise, one could purchase the bond for which $F^*_{\{2,j\}}$ is the lowest, immediately short the futures contract, and deliver the bond at a price above its purchase price.[10] Note that the futures price reflects the delivery of Bond 1 in states {2, 2} and {2, 1}, but Bond 3 is delivered in state {2, 0}.

There are two keys to understanding how the futures price is determined prior to maturity. First, through the process of daily settlement, one who is long the futures contract will receive cash each day equal to the difference between the new settlement price and the previous day's settlement price. For the purpose of determining the futures price, I assume that daily settlement occurs at the end of each binomial period according to the difference between the theoretical futures price at the end and the theoretical price at the beginning. If this difference is positive, the investor receives a positive cash flow. If the difference is negative, the investor must pay a cash amount equal to the difference.

Second, at any time, the contract itself should have no value. Therefore, in each state prior to maturity, the futures price, F, should be the price that causes the risk-adjusted expected profit from the next cash settlement to equal zero. This implies that

$$\left(F_{\{t+1,j+1\}} - F_{\{t,j\}}\right)\pi + \left(F_{\{t+1,j\}} - F_{\{t,j\}}\right)(1 - \pi) = 0$$

and

$$F_{\{t,j\}} = F_{\{t+1,j+1\}}\pi + F_{\{t+1,j\}}(1 - \pi) \tag{5}$$

In words, Equation (5) indicates that the futures price in any state $\{t, j\}$ is the risk-adjusted expected value of the two possible futures prices that can occur at time $t + 1$. Using Equation (5), the futures price in state {1, 1} is $103.88 = 105.52(0.5) + 102.25(1 - 0.5)$, and the other futures prices in the tree are computed in similar fashion.[11]

III. PRICING THE ACTUAL TREASURY BOND FUTURES CONTRACT

To price the actual contract, I use all Treasury bonds issued as of February 2002 with a remaining life of 15 or more years as of the maturity of a hypothetical futures contract maturing in one year. There are 28 such bonds.[12]

To simplify the analysis, all bonds maturing in 2018 are assumed to have exactly 15 years of remaining life as of the maturity of the hypothetical February 2003 futures contract. All bonds maturing in 2019 are assumed to have 16 years of remaining life, and so on. It is assumed that the first semiannual coupon payment for each bond is paid in exactly six months, and the second is paid on the futures maturity date.

Exhibit 6 summarizes coupon and maturities of the 28 bonds. The term structure is assumed to be a flat 5%

EXHIBIT 6

Bonds Used in Analysis of Treasury Bond Futures Contract

Coupon Rate	Initial Maturity (years)	Maturity in One Year (years)
9.125	16	15
9.000	16	15
8.875	17	16
8.125	18	17
8.500	18	17
8.750	18	17
7.875	19	18
8.125	19	18
8.000	19	18
7.250	20	19
7.625	20	19
7.125	21	20
6.250	21	20
7.500	22	21
7.625	23	22
6.875	23	22
6.000	24	23
6.750	24	23
6.500	24	23
6.625	25	24
6.375	25	24
6.125	25	24
5.500	26	25
5.250	26	25
5.250	27	26
6.125	27	26
6.250	28	27
5.375	29	28

per year with semiannual compounding, and the annual standard deviation of the natural logarithm of the interest rate in the BDT tree is assumed to be a constant 0.3. Each binomial period is one-eighth of a year. The longest maturity among the 28 bonds is 29 years. Therefore, $8 \times 29 = 232$ binomial periods are used in the computation of the price dynamics for this particular bond. By the same methodology illustrated in Exhibit 5, the equilibrium futures price for the one-year contract is 109.176 per $100 par.

Exhibit 7 illustrates the standard methodology for determining the cheapest to deliver bond, assuming there is no switching option, even though the futures price of 109.176 reflects the option to switch.[13] Given a futures price of 109.176, Exhibit 7 shows the return from hedging each of the 28 eligible bonds. For each bond, the return from hedging reflects purchasing the bond while simultaneously selling the futures contract with the intention to deliver the same bond, thereby locking in a selling price in one year equal to the adjusted futures price times the bond's conversion factor. The hedging return also reflects receiving half the annual coupon in six months and the other half in one year.

Note that the first bond, with an annual coupon rate of 9.125% and an initial maturity of 16 years, provides a 4.678% return from hedging, the highest among the 28 bonds. This bond would thus be identified as the cheapest to deliver, using the standard analysis that ignores the option to switch. According to the standard analysis, it would not make sense to purchase any of the other bonds while selling futures with the intention of delivering the bond that is purchased, as a higher return from hedging could be achieved by hedging the first bond. This line of reasoning leads to the conclusion that the futures contact effectively becomes a contract on the first bond and should take on its interest rate risk characteristics.

Note that the return from hedging the first bond shown in Exhibit 7 for a full year is 4.678%. This return falls 38.5 basis points below the one-year rate of interest of 5.0625% implied by a flat 5% yield curve with semiannual compounding. Thus, in this particular case, a hedger can lock in an annual return of 4.678% by buying the first bond in Exhibit 7 and selling the futures contract forward, but never trading out of the hedged position, even if it may be optimal to do so. The 38.5 basis points reflects the risk-adjusted expected return associated with the ability to switch into a more profitable bond, bringing the total risk-adjusted expected return from hedging to 4.678% + 38.5 basis points = 5.0625%.

IV. HEDGING WITH ACTUAL TREASURY BOND FUTURES

As numerous textbooks and articles show, if the option to switch is ignored, the duration-based quantity of Treasury bond futures contracts required to hedge the $100,000 face value of a Treasury bond with a current quoted price of Q_S, current accrued interest of A_S, and duration of D_S is given by:

$$N(duration\text{-}based) = \left(\frac{D_S}{D_F}\right)\left(\frac{Q_S + A_S}{F}\right) \tag{6}$$

where D_F is the futures duration as defined by Equation (4), and F is the current futures price.[14] If the bond being hedged is that identified as cheapest to deliver, the hedging quantity becomes

$$N(duration\text{-}based)_{CTD} = \left(\frac{D_{CTD}}{D_F}\right)\left(\frac{Q_{CTD} + A_{CTD}}{F}\right) \tag{7}$$

EXHIBIT 7

Standard Analysis of Cheapest to Deliver Bond
Assuming Bond Identified Initially as Cheapest to Deliver Cannot be Changed

Coupon Rate	Initial Maturity (Years)	Market Price	Conv. Factor		Futures Price		Adjusted Futures Price		Accum. Interest		Locked-in Value	Hedged Return (%)
9.125	16	145.064	1.3063	×	109.176	=	142.612	+	9.239	=	151.851	4.678
9.000	16	143.698	1.2940	×	109.176	=	141.274	+	9.113	=	150.387	4.654
8.875	17	144.027	1.2931	×	109.176	=	141.174	+	8.986	=	150.160	4.258
8.125	18	136.807	1.2245	×	109.176	=	133.689	+	8.227	=	141.915	3.734
8.500	18	141.224	1.2641	×	109.176	=	138.014	+	8.606	=	146.621	3.822
8.750	18	144.168	1.2906	×	109.176	=	140.898	+	8.859	=	149.758	3.877
7.875	19	135.001	1.2047	×	109.176	=	131.522	+	7.973	=	139.495	3.329
8.125	19	138.045	1.2320	×	109.176	=	134.501	+	8.227	=	142.728	3.392
8.000	19	136.523	1.2183	×	109.176	=	133.011	+	8.100	=	141.111	3.361
7.250	20	128.241	1.1406	×	109.176	=	124.523	+	7.341	=	131.864	2.825
7.625	20	132.947	1.1828	×	109.176	=	129.128	+	7.720	=	136.848	2.934
7.125	21	127.434	1.1300	×	109.176	=	123.371	+	7.214	=	130.585	2.472
6.250	21	116.138	1.0289	×	109.176	=	112.330	+	6.328	=	118.658	2.170
7.500	22	133.130	1.1778	×	109.176	=	128.583	+	7.594	=	136.177	2.289
7.625	23	135.640	1.1971	×	109.176	=	130.691	+	7.720	=	138.411	2.043
6.875	23	125.457	1.1061	×	109.176	=	120.761	+	6.961	=	127.722	1.805
6.000	24	113.887	1.0000	×	109.176	=	109.176	+	6.075	=	115.251	1.198
6.750	24	124.302	1.0929	×	109.176	=	119.319	+	6.834	=	126.153	1.490
6.500	24	120.830	1.0619	×	109.176	=	115.938	+	6.581	=	122.519	1.398
6.625	25	123.044	1.0790	×	109.176	=	117.796	+	6.708	=	124.504	1.186
6.375	25	119.499	1.0474	×	109.176	=	114.348	+	6.455	=	120.803	1.091
6.125	25	115.954	1.0158	×	109.176	=	110.900	+	6.202	=	117.101	0.990
5.500	26	107.231	0.9357	×	109.176	=	102.153	+	5.569	=	107.722	0.458
5.250	26	103.615	0.9035	×	109.176	=	98.642	+	5.316	=	103.957	0.330
5.250	27	103.682	0.9019	×	109.176	=	98.463	+	5.316	=	103.779	0.093
6.125	27	116.570	1.0164	×	109.176	=	110.961	+	6.202	=	117.163	0.509
6.250	28	118.728	1.0332	×	109.176	=	112.803	+	6.328	=	119.131	0.339
5.375	29	105.709	0.9157	×	109.176	=	99.976	+	5.442	=	105.418	-0.275

The number of contracts required to hedge the same bond using the option-based approach to valuation is

$$N(options\text{-}based)_{CTD} = \frac{Q^+_{CTD} - Q^-_{CTD}}{r^+ - r^-} \div \frac{F^+ - F^-}{r^+ - r^-} = \frac{Q^+_{CTD} - Q^-_{CTD}}{F^+ - F^-} \quad (8)$$

where Q^+_{CTD} and Q^-_{CTD} denote the bond's quoted prices in the immediate up and down states, respectively; r^+ and r^- denote the immediate up and down state values of the one-period Black, Derman, and Toy interest rate; and F^+ and F^- denote the futures price in the immediate up and down states.[15]

It should be noted that all three hedging quantities can potentially change over the life of the hedge. Therefore, each quantity should be viewed as the initial quantity of a dynamic hedging strategy that incorporates changing hedging quantities over time.

Technically, the duration- and options-based hedging quantities are not directly comparable, since the duration-based quantity assumes an immediate change in interest rates without any passage of time, while the options-based approach assumes that time equal to the length of a single binomial period has elapsed before interest rates can change. Therefore, I also compute a third hedging quantity that represents what the options-based quantity would be if the only bond eligible for delivery were the cheapest to deliver bond identified by a standard analysis such as that illustrated in Exhibit 7, even though the futures price

EXHIBIT 8

Analysis of Futures Hedging Quantities for Hedging Standard Cheapest to Deliver Bond

	Flat Annual Yield with Semiannual Compounding						
Hedging Quantity	3%	4%	5%	6%	7%	8%	9%
Standard duration-based	1.39926	1.39000	1.38512	1.39753	0.92734	0.92097	0.91801
Option-based, ignoring the option to switch	1.39705	1.39137	1.38596	1.38086	0.94322	0.94243	0.94202
Option-based taking account of the option to switch	1.39705	1.38689	1.32747	1.21999	0.97718	0.94494	0.94202

EXHIBIT 9

Analysis of Futures Hedging Quantities for Hedging Standard Cheapest to Deliver Bond in a Flat 6% Yield Curve Environment

	Volatility of 1-Year Interest Rate		
Hedging Quantity	0.1	0.2	0.3
Standard duration-based	1.38521	1.39386	1.39753
Option-based, ignoring the option to switch	1.36517	1.37130	1.38086
Option-based taking account of the option to switch	1.11142	1.16860	1.21999
Futures price	98.96295	98.34901	98.09061

itself reflects the option to switch.

I continue to use the Black, Derman, and Toy model with a binomial time interval equal to one-eighth of a year. Also, I continue to assume that the standard deviation of the natural logarithm of the interest rate in the BDT tree one year from today is 0.3 and remains 0.3 for all one-year periods in the tree. I compute the three hedging quantities for seven different flat yield curve scenarios, however, starting with a 3% yield (compounded semiannually) and going through a 9% yield in increments of 1 percentage point. In the computation of each hedging quantity, the futures price is assumed to reflect the delivery option, but the duration-based hedging quantity and the Black, Derman, and Toy-based quantity, ignoring the switching option, both assume that the contract is a contract on the standard analysis cheapest to deliver bond. Exhibit 8 summarizes the results.[16]

Note that the standard duration-based hedging quantities and those calculated using the Black-Derman-Toy model, ignoring the option to switch, are not substantially different. When interest rates are close to 6%, however, there can be a significant difference between these two hedging quantities and the theoretically correct quantity that takes the delivery option into account. For example, in a 6% interest rate environment, the standard duration-based hedging quantity is 1.39753 contracts, and that computed via the Black, Derman, and Toy model, ignoring the switching option, is 1.38086. Only 1.21999 contracts are required if the switching option is taken into account. Thus, ignoring the switching option could result in an error of 13% to14% in computing the proper hedging quantity.

For interest rate scenarios significantly different from a flat 6% yield curve, the difference in the hedging quantities is not as great. In fact, in both the 3% and the 9% yield environments, the options-based hedging quantities computed with and without the option to change the cheapest to deliver bond are the same. In these environments, it is never optimal to make delivery of any bond other than that originally identified as cheapest to deliver.[17] As interest rates approach 6%, there is more possibility of making a switch, thereby affecting the relative values of the hedging quantities.

Exhibit 9 summarizes the three hedging quantities for a 6% flat yield curve environment for volatilities of 0.1, 0.2, and 0.3. Interestingly, as the volatility declines, there is a greater difference between the theoretically correct options-based hedging quantity that takes account of the switching option and the duration- and options-based quantities that ignore the option to switch. The analysis summarized in Exhibit 10 sheds light on why this is the case.

Exhibit 10 shows the immediate state-dependent hedging effectiveness associated with the standard cheapest to deliver bond using the three hedging methods in a 6% flat yield curve environment (with semiannual compounding).[18] The shaded portions of Exhibit 10 show that there is less potential hedging error from using the incorrect hedging quantity for the low-volatility scenarios than for the higher-volatility scenarios. Clearly, if we were in a 6% flat yield curve environment with no volatility, it

Exhibit 10
Effectiveness of Three Hedging Methods

					Using Standard Duration-Based Hedging Quantity		Options-Based Hedging Quantity Ignoring the Option to Switch		Using Options-Based Hedging Quantity Taking the Option to Switch into Account	
Volatility	State	Futures Value	Futures Profit per Contract	CTD Bond Value	Futures Profit	Hedged CTD Bond	Futures Profit	Hedged CTD Bond	Futures Profit	Hedged CTD Bond
0.1	Up	101.293	2.330	135.426	-3.228	132.197	-3.181	132.244	-2.590	132.835
	Down	96.633	-2.330	130.245	3.228	133.473	3.181	133.427	2.590	132.835
0.2	Up	102.560	4.211	137.757	-5.870	131.887	-5.775	131.982	-4.921	132.835
	Down	94.138	-4.211	127.914	5.870	133.784	5.775	133.689	4.921	132.835
0.3	Up	103.705	5.615	139.685	-7.847	131.839	-7.753	131.932	-6.850	132.835
	Down	92.476	-5.615	125.986	7.847	133.832	7.753	133.739	6.850	132.835

would not matter which bond is identified as optimal (cheapest) to deliver, and since there is no risk to hedging, there would be no hedging error, regardless of how many futures contracts are used to hedge the bond that is identified as optimal to deliver. If volatility is non-zero but low, a specific bond will be identified as optimal to deliver, and thus a specific hedging quantity is required to hedge that particular bond. Since volatility is low, however, determining the hedging quantity with great precision is not as critical as when volatility is high.

It is ironic that the Treasury bond futures contract is designed to make one indifferent as to which bond to deliver in a 6% flat yield curve interest rate environment—yet in this environment determining the proper hedging quantity is the most complex. If we are currently in a 6% flat yield curve environment, it is almost certain that by the time a futures contract matures, the interest rate environment will have changed.

If interest rates end up above 6%, it will be optimal to deliver a long-duration bond; if interest rates end up below 6%, it will be optimal to deliver a short-duration bond; and if the yield curve ends up cutting through 6%, the bond that is optimal to deliver will depend on the yield curve's specific shape. Yet if current yields are substantially different from 6% in either direction, it is unlikely that the bond identified as optimal to deliver will change. Therefore, in these environments, using a standard duration-based approach to hedging should provide reasonably accurate hedging quantities.

V. SUMMARY AND CONCLUSIONS

I have used the Black, Derman, and Toy model to determine the equilibrium Treasury bond futures price in terms of the option to deliver one of many eligible bonds. Using the same pricing model, one can determine the appropriate quantity of Treasury bond futures required for hedging the bond identified initially as cheapest to deliver. If interest rates are close to 6%, the rate used for determining Treasury bond futures conversion factors, the proper hedging quantity can differ significantly from a more standard duration-based quantity that ignores the option to switch delivery to a more optimal bond. If interest rates are significantly different from 6%, though, the option to switch will have little if any value. Therefore, duration-based hedging quantities that ignore the option to switch will be much more accurate when interest rates are significantly different from 6%.

ENDNOTES

[1]The accrued interest option allows the party with a short position in the contract to deliver on any business day of the delivery month. The wild card option, which applies during the first business day through the eighth-to-last business day of the delivery month, allows the short to provide notice of delivery as late as 8:00 PM based on a futures price established at 2:00 PM the same day. The end-of-month option allows the short to make delivery during the last seven business days of the month at the futures price established on the eighth-to-last day.

[2]Lin and Paxon [1995] test the hedging effectiveness of the German government bond (Bund) futures contract traded on LIFFE. The delivery option for this contract is similar in structure to that of the Treasury bond futures contract traded at the Chicago Board of Trade.

[3]In developing the pricing model in this article, I ignore the accrued interest, wild card, and end-of-month options. The model is developed as if delivery can be made on the contract's maturity date only. Chen and Yeh [2003] have shown that the accrued interest option, in the presence of the delivery option, should have no value. Also, they have shown that the incentive for the party

who is short to take advantage of the wild card play is minimal.

[4]Starting with the March 2000 contract, the Chicago Board of Trade switched from conversion factors based on an 8% rate to factors based on a 6% rate.

[5]The conversion factor is the bond's price per $1 par, assuming a yield of 6% per year with semiannual compounding. The amount received upon the delivery of any bond is the futures price times its conversion factor plus accrued interest. Consistent with notation used throughout, variables marked with an asterisk are measured as of the futures maturity date. Therefore, the conversion factor is not the adjustment factor if the bond were delivered today, but instead the factor when the bond is delivered at the maturity of the futures contract.

[6]This form of futures duration is attributable to Clarke [1992] as developed further by Rendleman [1999].

[7]As noted in Exhibit 1, in the actual Treasury bond contract, only bonds with maturities of 15 years or more are eligible to be delivered. The shorter maturities for the bonds in Exhibit 3 are assumed so I can illustrate their valuation using a relatively small binomial tree.

[8]In the tree, each binomial period represents six months. State designations, $\{t, j\}$, denote binomial time, t, and the number of decreases in the interest rate, j, or, equivalently, the number of decreases in bond prices. The interest rate tree is based on a flat term structure of 5% interest per year with semiannual compounding, or 2.5% every six months, and an annual standard deviation of the log of the interest rate of 0.3. The first number shown in each state is the six-month risk-free interest rate stated as a percentage. The second number is the price of the three-year 5% coupon bond per $100 par.

[9]In the tree, each binomial period represents six months. State designations, $\{t, j\}$, denote binomial time, t, and the number of decreases in the interest rate, j. The interest rate tree is based on a flat term structure of 5% interest per year with semi-annual compounding, or 2.5% every six months, and an annual standard deviation of the log of the interest rate of 0.3. The first number shown in each state within the body of the tree is the six-month risk-free interest rate stated as a percentage. The second number is the futures price per $100 par. Also, the bond price, conversion factor, and F^* value are shown for each bond for each state as of the maturity date of the futures contract.

[10]According to the futures contract, the party who is short determines which bond is delivered. Therefore, an arbitrage profit could not be earned with the other two bonds by doing this transaction in reverse, since the party who is long the futures contract has no control over the terms of delivery.

[11]This process implies that the current futures price will equal the risk-adjusted expected maturity value of the futures contract.

[12]The 28 bonds do not include three flower bonds. Also, the 8¾% bonds of May and August 2020 are treated as the same bond.

[13]In Exhibit 7 the market price is the theoretical value from the Black-Derman-Toy model, assuming a flat term structure of interest rates of 5% per year with semiannual compounding, or 5.0625% with annual compounding, and an annual standard deviation of the natural logarithm of the interest rate of 0.3. This is also equivalent to the value obtained by discounting the bond's semiannual cash flows at a rate of 2.5% every six months. Accumulated interest is the future value, as of one year from today, of the first semiannual interest payment, assumed to be received in exactly six months, plus the second semiannual payment. The future value is computed using the six-month forward rate implied by the term structure specification. For example, accumulated interest for the first bond is $(9.125/2)(1.025) + 9.125/2 = 9.239$. The hedged return is the total return from buying a bond at its market price, selling the one-year futures contract, delivering the bond, and receiving the locked-in value one year later. For the first bond, $151.851/145.064 = 1.04678$, or 4.678%.

[14]See, for example, Figlewski [1986, p. 110], Clarke [1992, p. 26], Stoll and Whaley [1993, p. 151], Fabozzi [1996, p. 497], Kolb [1997, p. 207], and Chance [1998, p. 424]. This equation is analyzed in detail in Rendleman [1999].

[15]In this context, "up" and "down" refer to the state of bond prices rather than the state of interest rates. Therefore, the "up" state is the state in which interest rates decline.

[16]In Exhibits 8 and 9, all option-based quantities are computed using theoretical values from the Black-Derman-Toy model, assuming a flat term structure of interest rates and an annual standard deviation of the natural logarithm of the interest rate of 0.3.

[17]If the binomial time-differencing interval approached zero, rather than being set to one-eighth of a year, there would be a low probability that a different bond would be optimal to deliver, and the possibility of such a delivery would have a slight impact on the futures price and hedging quantity.

[18]All option-based quantities are computed using theoretical values from the Black-Derman-Toy model, assuming a 6% flat term structure of interest rates with semiannual compounding.

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