Instantaneous Mean-Variance Analysis of Bond Returns

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No-arbitrage models such as Heath, Jarrow, and Morton [1992], Kennedy [1994, 1997], Duffie and Kan [1996], Goldstein [2000], and Santa-Clara and Sornette [2001] are formulated following a risk-neutral probability measure, and consequently reveal nothing about bond risk premiums. Knowledge of premium amounts could be used to exploit favorable trading opportunities where they exist.

We derive an explicit formula for the instantaneous expected return of any bond. We apply this formula to perform mean-variance analysis on weekly and monthly bond returns, using 15 years of U.S. Treasury data. For monthly returns, we find portfolios whose out-of-sample Sharpe [1996] ratios in annual terms are about 2.5; for weekly returns, portfolio Sharpe ratios exceed 3.0. These are very high numbers (the Sharpe ratio of the S&P 500 during the same period was below 0.4) that would have been indicated arbitrage opportunities, had markets been frictionless.

High Sharpe ratios in the U.S. bond market should not come as any great surprise. Many authors have indicated that the term structure dynamics are inconsistent with the expectations hypothesis with constant risk premiums (Fama [1976, 1984a, 1984b]). In other words, in the U.S. bond market we observe time-varying and often high risk premiums (Dai and Singleton [2002]). Reisman and Zohar [2004a] show the presence of excess yields in the bond market.

Other authors have indicated that future changes in spot rates are predictable (Duffee [2002], Diebold and Li [2002], Reisman and Zohar [2004b]). Accounting for such predictability also contributes to the high Sharpe ratios.

The mean-variance approach we present is based on a formula for the instantaneous risk premiums of bond returns. We show that the instantaneous return at time t on a discount bond $B(t, T)$ that matures in $\tau \equiv T - t$ years satisfies:

$$\frac{dB(t,T)}{B(t,T)} = \left[y(t,\tau) + \tau \cdot \frac{\partial y(t,\tau)}{\partial \tau}\right] dt + \frac{1}{2}\tau^2 d\langle y(t,\tau)\rangle - \tau \cdot dy(t,\tau) \qquad (1)$$

where $y(t, \tau)$ is the spot rate for τ years, $dy(t, \tau)$ is the instantaneous change of the yield with constant time to maturity, and $\langle\cdot\rangle$ denotes quadratic variation.

For conditional mean-variance analysis, the conditional mean and the covariance matrix of the returns need to be determined. Denoting by $\mathcal{F}_t$ the filtration generated by the observed term structure dynamics up to time t, Equation (1) implies that the conditional expected return is given by:

$$\mathbb{E}\left[\frac{dB(t,T)}{B(t,T)}\middle|\mathcal{F}_t\right] = \left[y(t,\tau) + \tau \cdot \frac{\partial y(t,\tau)}{\partial \tau}\right] dt + \frac{1}{2}\tau^2 d\langle y(t,\tau)\rangle - \tau\mathbb{E}[dy(t,\tau)|\mathcal{F}_t] \quad (2)$$

The conditional expectation $\mathbb{E}[dy(t, \tau) | \mathcal{F}_t]$ depends on the modeling of dy.

Diebold and Li [2002] fit an ARMA model for monthly bond yields, while Reisman and Zohar [2004b] fit an ARIMA model for weekly data. Under such modeling, the conditional expectation of dy is analytically defined. It is straightforward to estimated the quadratic variation from the data, so we can determine all terms in (2).

The covariance matrix of the returns is:

$$cov\left(\frac{dB(t,T_1)}{B(t,T_1)}, \frac{dB(t,T_2)}{B(t,T_2)}\right) \quad (3)$$

We simplify by assuming that the covariance matrix is constant, and then it is easily estimated. Clearly more complex models such as GARCH models can be implemented as well.

Using Equations (2) and (3), we find a mean-variance efficient frontier for both monthly and weekly returns. The assumption of a constant covariance matrix is clearly only an approximation, so the frontier that we obtain only approximates the true efficient frontier. The methodology, however, leads us to construct portfolios with very high Sharpe ratios, which are a good fit with what our theory anticipates.

I. RISK PREMIUMS ON BONDS

We first derive an expression for the risk premium of any bond in the real world. Recall that $y(t, \tau)$ denotes the spot rate for τ years at time t. In this definition, the time to maturity τ remains a fixed period. This definition is not unusual; see, for example, Santa-Clara and Sornette [2001]. To this end, we use T and τ interchangeably to denote maturity and time to maturity of the same bond, namely, $T \equiv t + \tau$.

Denote by $B(t, T) = e^{-\tau y(t,\tau)}$ the price at time t of a discount bond that matures at T. Also denote by $\mathcal{F}_t$ the σ-algebra generated by the history of the term structure to time t. Equation (4) below provides a convenient framework for modeling the dynamics of the term structure in both the risk-neutral probability measure and in the real world:

$$-dy(t,\tau) + \eta(t,\tau)dt = \frac{1}{\tau}\left(\frac{dB(t,T)}{B(t,T)} - r(t)dt\right) \quad (4)$$

where

$$\eta(t,\tau) := \frac{1}{\tau}\left[y(t,\tau) - r(t) + \tau\frac{\partial y(t,\tau)}{\partial \tau} + \frac{1}{2}\tau^2\frac{d\langle y(t,\tau)\rangle}{dt}\right] \quad (5)$$

where $\eta(t, \tau)$ is the drift of the process $y(t, \tau)$ in the risk-neutral probability measure; $\tau(\eta(t, \tau) - \mathbb{E}[dy(t, \tau) | \mathcal{F}_t]/dt)$ is the risk premium of the bond $B(t, \tau)$; and $\langle\cdot\rangle$ denotes quadratic variation.

To prove Equation (4), we use Itô's lemma:

$$\begin{aligned}\frac{dB(t,T)}{B(t,T)} - r(t)dt &= -d\left[\tau\hat{y}(t,T)\right] + \frac{1}{2}\tau^2 d\langle y(t,\tau)\rangle - r(t)dt \\ &= [y(t,\tau) - r(t)]dt - \tau\, d\hat{y}(t,T) + \\ &\quad \frac{1}{2}\tau^2 d\langle y(t,\tau)\rangle \end{aligned} \quad (6)$$

where $\hat{y}(t, T)$ is the yield to maturity at time t of a discount bond that matures at $T \equiv t + \tau$. While $\hat{y}(t, T) = y(t, \tau)$ the dynamics of these quantities differ:

$$\begin{aligned} d\hat{y}(t,T) \equiv dy(t, T-t) &= \frac{\partial y(t,\tau)}{\partial t}dt + \frac{\partial y(t,\tau)}{\partial \tau}d\tau \\ &= dy(t,\tau) - \frac{\partial y(t,\tau)}{\partial \tau}dt \end{aligned}$$

and Equation (4) follows.

The right-hand side of Equation (4) represents a zero-cost bond portfolio that is a martingale in the risk-neutral measure; hence $\eta(t, \tau)$ is the risk-neutral drift of y.

The innovation process $dy - \mathbb{E}[dy | \mathcal{F}_t]$ is a martingale in the true world. Hence, the drift of (dB/B) is $r(t) + \tau \cdot (\eta(t, \tau) - \mathbb{E}[dy(t, \tau) | \mathcal{F}_t]/dt)$.

Equation (4) provides a framework for predicting bond returns. It shows that bond returns come from two sources: a deterministic drift term $\eta(t, \tau)$, and a stochastic term $dy(t, \tau)$, which is the future change in the yield. The second term can be predicted using time series modeling, and thus contributes to the overall drift. Even if the yield change dy is not predictable, we can still predict the future change in the bond price using η.

II. MEAN-VARIANCE ANALYSIS

To derive the mean-variance efficient frontier, for the sake of simplicity we restrict the discussion to portfolios of a finite number of bonds. Denoting this number by N, we assume bonds of maturities $T_1, \ldots, T_N$, and times to maturity $\tau_1, \ldots, \tau_N$.

Denote by $\mu_i(t)$ the expected return of bond with maturity T_i given by Equation (2). μ stands for a column vector of all μ_i, and R denotes the covariance matrix of Equation (3). The weights $\omega_i(t)$ are the fraction of unity invested in maturity T_i at time t. The analysis is standard.

Denoting by I a unit column vector of length N, and $\omega(t) := (\omega_1(t), \ldots, \omega_N(t))^T$, these weights must satisfy:

$$\omega^T(t) \cdot \mathrm{I} \equiv \sum_{i=1}^{N} \omega_i(t) = 1 \tag{7}$$

The expected return on portfolio $\omega(t)$ is $\omega^T(t)\mu(t)$, and its variance is $\omega^T(t)R\omega(t)$.

As in standard mean-variance analysis, we seek the portfolio that minimizes the variance when the expected return r is specified. The solution is given by:

$$\omega(t) = \frac{1}{D(t)} \times \left[B(t) \cdot R^{-1}\mathrm{I} - A(t) \cdot R^{-1}\mu(t) + (C \cdot R^{-1}\mu(t) - A(t) \cdot R^{-1}\mathrm{I}) \cdot r\right] \tag{8}$$

where

$$\begin{aligned} A &= \mathbf{1}^T R^{-1}\mu \\ B &= \mu^T R^{-1}\mu \\ C &= \mathbf{1}^T R^{-1}\mathbf{1} \\ D &= BC - A^2 \end{aligned} \tag{9}$$

Substituting (8) into the expression for the variance of the portfolio, we obtain:

$$var(\omega(t)) \equiv \omega^T(t)R\omega(t) = \frac{B - 2Ar + Cr^2}{D} \tag{10}$$

which defines the efficient frontier. Similarly, for a desired standard deviation σ, the expected return on the frontier is given by:

$$r = \frac{2A + \sqrt{4A^2 - 4C(B - D\sigma^2)}}{2C} \tag{11}$$

The minimum-variance portfolio (MVP) is the portfolio on this frontier for which variance is minimized. Its properties are easily derived:

- The MVP's expected return is
  $$m_0 = \frac{A}{C}$$
- The MVP's weights are
  $$\omega_0 = \frac{1}{C}R^{-1}\mathrm{I}$$
- The MVP's variance is
  $$v_0 = \frac{1}{C}$$
- Its covariance with any other portfolio of variance v is
  $$\rho\sqrt{v_0}\sqrt{v} = \frac{1}{C}$$

To compute the Sharpe ratio of a certain asset, we first construct a zero-cost portfolio by selling short the MVP, since within our market it serves as a tradable proxy for the risk-free security. Denote by m the expected return of the given asset and by v its variance. The Sharpe ratio of borrowing at the MVP to finance this asset is independent of the risk-free rate, and is given by

$$\frac{m - m_0}{\sqrt{v + v_0 - 2\rho\sqrt{v \cdot v_0}}} \equiv \frac{m - m_0}{\sqrt{v - v_0}} \tag{12}$$

When the portfolio with expected return m and variance v is efficient, it is not hard to show that this leads to a Sharpe ratio of:

$$\frac{m - m_0}{\sqrt{v - v_0}} \equiv \sqrt{\frac{D}{C}} \tag{13}$$

The result is a single number for all portfolios on the frontier, which makes it a reasonable definition. This definition is equivalent to taking the slope of the asymptote to this frontier, and for assets with $v >> v_0$ this definition is the same as the classic Sharpe ratio in the presence of a risk-free asset. Note, however, that this definition does not apply to single assets, since then the MVP must be taken as the asset itself.

Finally, we analyze the zero-cost portfolio that is obtained by reducing the MVP from any efficient portfolio ω that costs $\mathbf{1}^T\omega \equiv 1$. This portfolio is $\tilde{\omega} \equiv \omega - \omega_0$, and its expected return is $\tilde{m} \equiv m - m_0$. It is not hard to show that

$$\tilde{\omega} = \frac{CR^{-1}\mu - AR^{-1}\mathrm{I}}{D} \cdot \tilde{m} \tag{14}$$

We define the *norm* of a zero-cost portfolio as the size of the long position, which is also equal to the size of the short leg of the portfolio. The long and short parts of portfolio $\tilde{\omega}$ are both equal to half the sum of the absolute values of $\tilde{\omega}$'s components. Therefore the normalized return of all portfolios on the frontier is unique:

$$\frac{2\tilde{m}}{\mathrm{I}^T \cdot |(\tilde{\omega})|} = \frac{2}{\mathrm{I}^T \cdot |(CR^{-1}\mu - AR^{-1}\mathrm{I})/D|} \tag{15}$$

where $|\cdot|$ denotes the vector of absolute values.

III. MEAN-VARIANCE ANALYSIS OF U.S. TREASURY NOTES

We apply the mean-variance framework to determine an instantaneous frontier that is efficient subject to our assumptions. For portfolios on this frontier, we compute the theoretical Sharpe ratio and the Sharpe ratio that is attained in practice. We also demonstrate the long-term profitability of instantaneously efficient allocations. We perform the analysis on both monthly and weekly data, and we obtain high Sharpe ratios in both time frames.

Our data set includes spot rates for three and six months and one, two, three, five, seven, and ten years. The numbers are provided by the U.S. Federal Reserve System on a weekly basis. The data span 15 years between 1986 and 2000. For out-of-sample testing, we use the first five years for fitting the model (1986-1990), and the remaining ten years for testing (1991-2000).

Mean-Variance Analysis of Monthly Returns

To proxy monthly quotes, we sample bond prices once every four weeks. Diebold and Li [2002] and Reisman and Zohar [2004b] indicate there is no evident short-term predictability in such data, so the conditional expectation of dy is equal to the unconditional mean, which we estimate by the historical mean.

Exhibit 1 plots the out-of-sample profit of a zero-cost bond allocation on the frontier. To set a common benchmark for comparison, at any instance we choose the allocation whose yearly return has a theoretical standard deviation of \$10 per year.

Note that the cumulative profit is the sum of monthly gains without discounting. We do not reinvest gains as then the standard deviation would not remain constant in U.S. dollar terms. Instead, one should regard the slope of this plot as the instantaneous rate of profit from the portfolio while restarting every month from a zero-cost normalized position.

From the plot, the average theoretical expected return [based on Equation (2)] is almost \$30 per year. With a theoretical standard deviation of \$10 per year, this implies a theoretical Sharpe ratio of almost 3.0. While the average return in practice is very close to the theoretical prediction, the standard deviation in practice turns out to be \$11.5 rather than \$10.0 per year, so the realized Sharpe ratio is instead 2.41. This high Sharpe ratio would have indicated a statistical arbitrage opportunity had the markets been frictionless.

Exhibit 2 plots the time-dependence of the theoretical Sharpe ratio, which turns out to be quite irregular. Note that since the analysis is intertemporal, dynamic allocations with even higher average Sharpe ratios can be constructed, if we leverage the position when the theoretical Sharpe ratio is high.

The first graph in Exhibit 3 shows the chosen portfolio on the frontier at selected instances, as well as the average of these portfolios for all t. The portfolios do not change much over time, and they all look ragged. Clearly, the holding of opposite positions in successive maturities significantly reduces the variance of the portfolio.

Typically, one should hold the six-month and the two-year notes, while selling short the one-year and the three-month. These portfolios are indicated by the solid lines in Exhibit 3A. In Exhibit 3B, a solid curve represents the term structure at those instances. In these cases, the term structure is sharply upward sloping at the short end, which is quite common, and the steep slope makes the six-month maturity attractive for investment according to Equation (2). The dotted line term structures indicate cases where the slope at the short end is not steep, and then the weights of the short maturities in the corresponding portfolios (the dotted line plots in Exhibit 3A) become small.

Finally, we consider the normalized returns, with norm defined as the size of either the long or short leg of the portfolio. This quantity is interesting since it quantifies the excess return that can be gained in practice, due

EXHIBIT 1
Profit Plots of Zero-Cost Portfolio

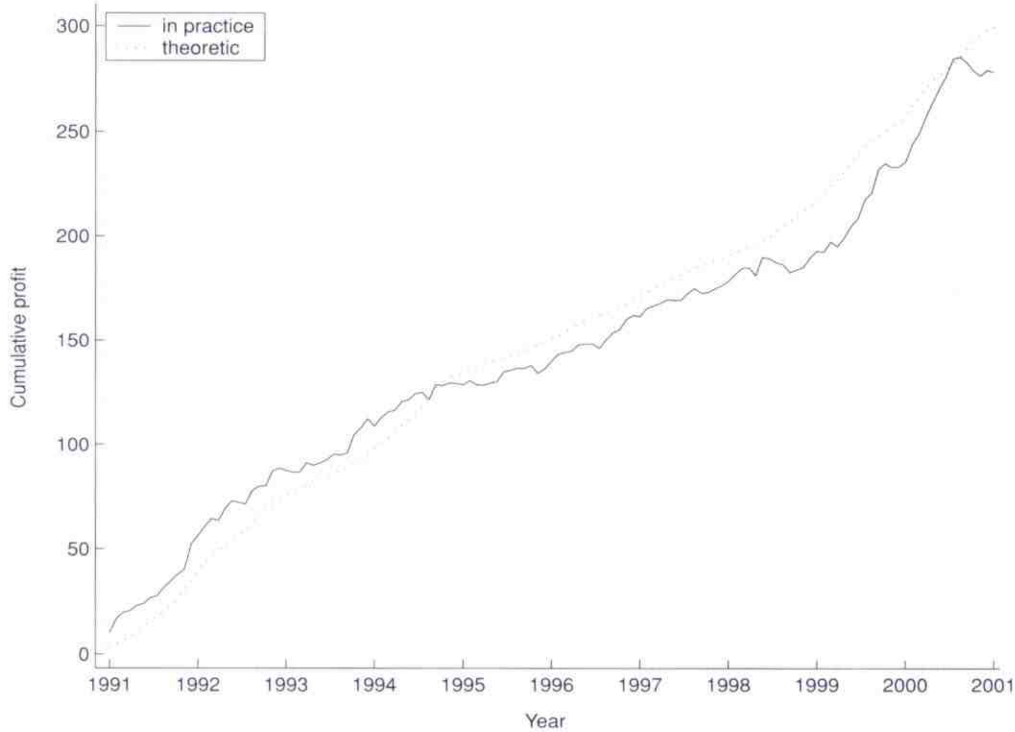


EXHIBIT 2
Time-Dependence of Sharpe Ratio

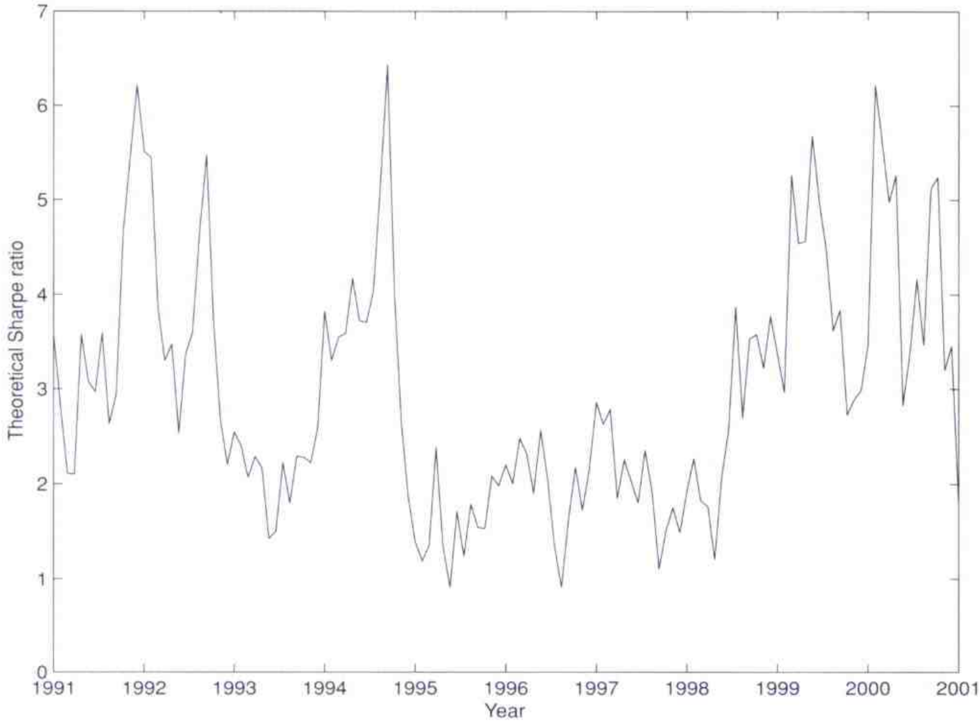
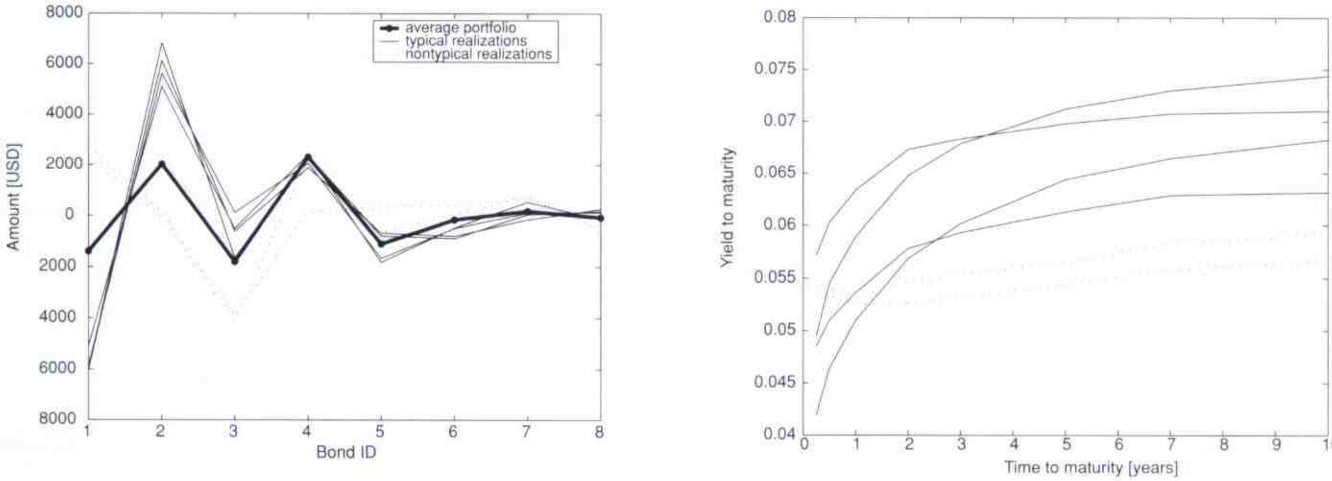


EXHIBIT 3
Resulting Portfolios



A. Portfolios at Selected Times **B. Term Structure at Selected Times**

to limits on borrowing. Since both long and short legs of the average portfolio in Exhibit 3A are approximately \$4,000, and the average return is approximately \$30 per year, on average one expects to gain a normalized return of approximately 0.75%. An accurate computation that averages the instantaneous return normalized by the corresponding instantaneous optimal portfolios produces a normalized return of 0.63%.

Mean-Variance Analysis of Weekly Returns

Our motivation in presenting a weekly mean-variance analysis is twofold. First, we want to show that frequent rebalancing of the portfolio increases the normalized excess return. Second and more important, we demonstrate how predictability can be incorporated in the analysis to enhance the results.

In Reisman and Zohar [2004b], we address the short-term predictability of the term structure of interest rates. There we show that if one models the spot rate via principal component analysis as in Litterman and Scheinkman [1991], $y(t, \tau) \approx \Sigma_{i=1}^{F} u_i(\tau) z_i(t)$ for some $F \geq 2$, the leading factors $z_1(t)$ and $z_2(t)$ have unit roots, and the one-week differences, $dz_1(t)$ and $dz_2(t)$, are well described by an AR(1) model (see Jolliffe [1986] for a review of principal component analysis).

The fitted models for dz_1 and dz_2 based on the data in the years 1986-1990 are (dt is one week):

$$dz_1(t) = 0.26dz_1(t - dt) + \epsilon_1(t)$$

$$dz_2(t) = 0.23dz_2(t - dt) + \epsilon_2(t)$$

Assuming that the noise is zero mean, this model uniquely defines the conditional expectations $\mathbb{E}[dz_i \mid \mathcal{F}_t]$: $\mathbb{E}[dz_1 \mid \mathcal{F}_t] = 0.26dz_1(t - dt)$ and $\mathbb{E}[dz_2 \mid \mathcal{F}_t] = 0.23dz_2(t - dt)$.

Subsequently, we predict $dy(t, \tau)$ by

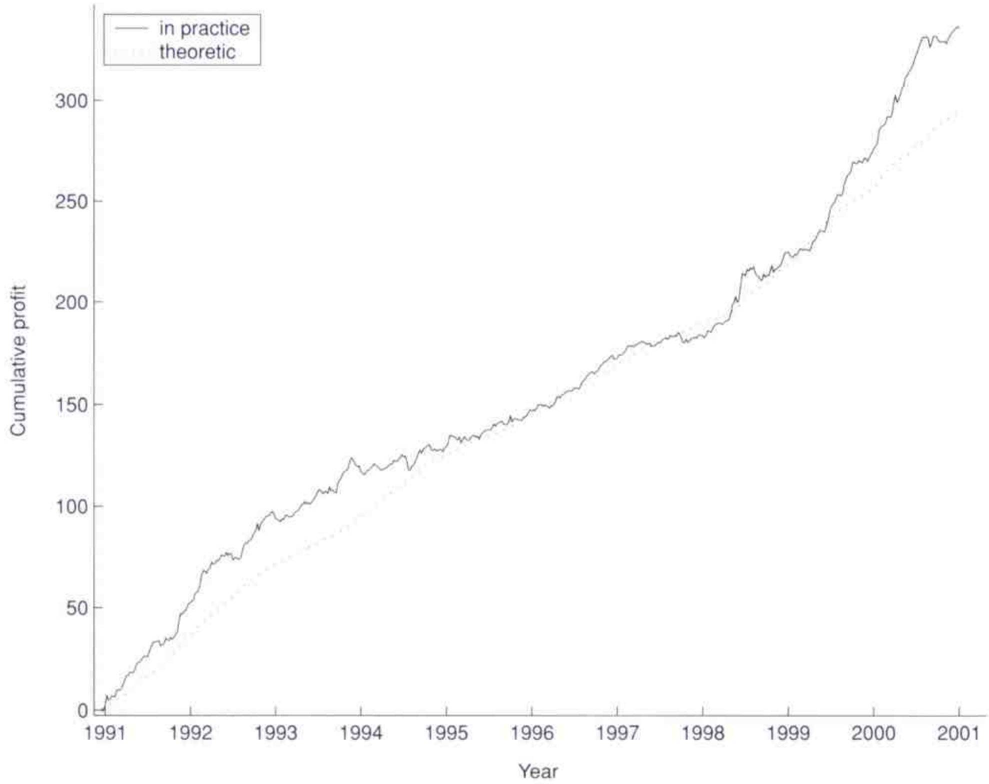
$$\mathbb{E}[dy(t,\tau)|\mathcal{F}_t] = u_1(\tau)\mathbb{E}[dz_1(t)|\mathcal{F}_t] + u_2(\tau)\mathbb{E}[dz_2(t)|\mathcal{F}_t]$$

and substitute this conditional expectation into Equation (2).

Exhibit 4 plots the cumulative profit of a zero-cost allocation that is updated on a weekly basis to have a theoretical constant standard deviation of \$10. Once again the fit with the theoretical prediction is very good, and a very high Sharpe ratio is obtained.

Several differences from the results with monthly data should be noted. First, mainly thanks to the prediction, the Sharpe ratio turns out to be higher. Here we obtain a Sharpe ratio of about 3.08, compared to 2.41 in the case of monthly updates. Second, on average, the normalized return more than doubles, to 1.37% from 0.63%. The cost of this improvement is that the position requires substantial adjustments on a weekly basis.

EXHIBIT 4
Profit Plots of Zero-Cost Efficient Portfolios Updated Weekly



IV. CONCLUSIONS

We have demonstrated a robust method for estimating the risk premiums in bonds, as implied by the term structure. This enables determination of the efficient frontier and construction of portfolios with high Sharpe ratios.

The Sharpe ratio obtained for the U.S. bond market are surprisingly high. The S&P 500 index had a Sharpe ratio of 0.27 over the last 50 years, and about 0.4 in a period that overlaps ours. Hence, Sharpe ratios exceeding 3.0, as we find, are exceptionally high.

Application of this methodology to options markets, futures markets, and other derivative securities has met with striking success. In other words, results of a similar nature are common across other markets.

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