A General Model for Hedging Swaps with Eurodollar Futures

RICHARD J. RENDLEMAN, JR.

RICHARD J. RENDLEMAN, JR., is a professor of finance at the Kenan-Flagler Business School of the University of North Carolina at Chapel Hill, NC.
richard_rendleman@unc.edu

Eurodollar futures are used extensively to hedge conventional LIBOR-based interest rate swaps, yet very little has been written on how to actually perform the hedging. Nor does there appear to be any published empirical documentation of how such hedges might perform.

A LIBOR-based interest rate swap is defined as a plain vanilla interest rate swap whose floating payment is equal to LIBOR of a specific maturity times the swap's outstanding notional principal amount. The floating swap payment is reset at the same frequency as the term of the LIBOR instrument.

Kawaller [1988, 1989, 1994] treats a swap as equivalent to a series of Eurodollar futures strips, and determines Eurodollar futures hedging quantities accordingly. His approach to hedging is fairly straightforward when a swap's interest reset dates align with the maturity dates of Eurodollar futures contracts. When the dates do not align, or when the reset dates occur on a frequency different from the three-month Eurodollar futures maturity cycle, Kawaller's methodology requires a best guess estimate of the ex post difference between the LIBOR implied by futures prices and actual LIBOR.

Standard Eurodollar futures contracts mature every March, June, September and December for 40 consecutive maturity months. Contracts are also traded that mature in the four most-nearby months not included in the regular maturity cycle, but they tend to be very illiquid compared to contracts in the regular three-month maturity cycle.

The Kawaller methodology does not provide for the use of futures contracts outside the regular three-month maturity cycle, and applies only to the hedging of newly issued swaps.

Burghardt [2003] provides a comprehensive calculus-based approach to Eurodollar futures-based pricing and hedging of swaps. He separates Eurodollar hedging quantities into two components: a component due to the primary effects of potential interest rate changes, and a much smaller component due to secondary effects. It can be shown that the hedging quantities Kawaller [1994] proposes are equivalent to those associated with Burghardt's primary effects.

Although all Burghardt's examples are based on perfect alignment of futures maturity dates and swap interest reset dates, he suggests that a linear interpolation of cumulative continuously compounded rates implied by the Eurodollar futures curve can be used to deduce cumulative continuous rates and associated present value factors for times that do not align perfectly with futures maturities. He does not use this interpolation methodology in his hedging calculus and examples, however.

Kawaller's approach and Burghardt's hedging examples assume Eurodollar futures rates and forward rates are the same. According to various stochastic interest rate models, the Eurodollar forward rate should be lower than the Eurodollar futures rate, and the extent of the difference should increase as a function of interest rate volatility and contract maturity (see Grinblatt and Jegadeesh [1996], Young [1997, pp. 276-281], Gupta and Subrahmanyam [2000], and Hull [2003, p. 111]).

This difference, sometimes referred to as the *convexity adjustment*, cannot be computed without reference to a specific stochastic interest rate model. Hull recommends a formula derived from the Ho-Lee [1986] model to convert an annual continuously compounded Eurodollar futures rate to a forward rate:[1]

$$\textit{Eurodollar Forward Rate} = \textit{Eurodollar Futures Rate} - \frac{1}{2}\sigma^2\tau_1\tau_2 \qquad (1)$$

where σ is the standard deviation of the change in the annual continuously compounded forward rate in one year, τ_1 is the number of years until the maturity of the futures contract, and τ_2 is the number of years until the maturity of the Eurodollar deposit that underlies the futures contract (or approximately $\tau_1 + 0.25$).[2]

Grinblatt and Jegadeesh [1996] provide numerical examples of the difference between forward interest rates and futures rates for various parameterizations of the Cox, Ingersoll, and Ross [1985] and Vasicek [1977] models. Also, Gupta and Subrahmanyam [2000] provide mathematical expressions for the difference using the Vasicek [1977], Cox, Ingersoll, and Ross [1985], and Hull and White [1990] models.

Burghardt recognizes that Eurodollar futures rates should be adjusted downward for the convexity bias, and provides an example of what these adjustments would have been on June 17, 2002, using a daily report from Carr Futures. He does not show how these adjustments are developed or how they might impact hedging quantities.

The hedging model I propose provides a general framework for dealing with timing differences between swap cash flows and Eurodollar futures maturity dates. It allows for all available futures contracts to be used in hedging, rather than just contracts that are part of the regular three-month maturity cycle, and it reflects the potential difference between Eurodollar futures rates and forward rates. By construction, the model requires no special calculations when the LIBOR value dates of futures contracts do not align with swap reset dates or when overlapping futures contracts are used. Applied to a specific set of hedging examples, the method is shown to be very effective in eliminating the interest rate risks of individual swaps and swap portfolios.

I. SPECIFICATION OF HEDGING MODEL

In Rendleman [2004], I describe a cubic spline-based methodology for interpolating a cumulative continuously compounded zero-coupon yield function associated with the U.S. swap curve. This curve is defined by one-month, three-month, six-month, and one-year LIBOR, by fixed rates for swaps with maturities ranging from two through ten years, and by fixed rates for 15-, 20-, and 30-year swaps. In this specification there are a total of $m = 12$ swaps.

Let $g(t) = s(k_1, k_2, \ldots, k_m)$ denote the interpolated cumulative continuously compounded yield through day t in terms of cubic spline function $s(k_1, k_2, \ldots, k_m)$ estimated in connection with the U.S. swap curve. Then, the (interpolated) present value of \$1 to be received from a swap on day t is $\exp(-g[t])$. In $s(k_1, k_2, \ldots, k_m)$, $k_j = \{x_j, y_j\}$ represents the coordinates of cubic spline knot, j, where x_j denotes time and y_j denotes the cumulative continuously compounded zero-coupon swap yield through the period of time represented by x_j. Although not required by the methodology, the time intervals associated with each value of x_j correspond to the maturity dates of each of the 12 swap contracts.

By construction, the interpolation function, when applied to the pricing of the 12 swaps from which the function is estimated, prices all 12 swaps without error, regardless of the frequency of payments for each individual swap.[3]

Similarly, let $f(t) = z(\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote the interpolated cumulative continuous discount rate through day t associated with the cubic spline function $z(\alpha_1, \alpha_2, \ldots, \alpha_n)$ estimated in terms of the Eurodollar futures curve. The function $z(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is defined by n cubic spline knots $\alpha_1, \alpha_2, \ldots, \alpha_n$, which in turn are defined in terms of adjusted LIBOR, $L_1, L_2, \ldots, L_n$, implied by the prices of n successive Eurodollar futures contracts.

Note that the cubic spline function associated with the U.S. swap curve, $s(k_1, k_2, \ldots, k_m)$, is estimated in terms of m fixed swap rates, while the cubic spline associated with the Eurodollar futures curve is estimated in terms of n Eurodollar futures rates. In the hedging model, it is not necessary that $m = n$ or that the maturity structure of the U.S. swap curve correspond to that of the Eurodollar futures curve.

The key to implementing the hedging model is that the difference between the cumulative continuous futures and forward rates, defined as $h_t = f(t) - g(t)$, is assumed to be independent of the level of $f(t)$. This difference can also be expressed as $h_t = c_t + \varepsilon_t$, where c_t is the cumulative convexity bias through day t, and ε_t represents the difference between estimated values of $f(t)$ and $g(t)$ not due to convexity.

In this specification, no particular stochastic interest rate process is used to define c_t, but it is assumed that $\partial c_t/[\partial f_{(t)}] = 0$ or, alternatively, that $\partial c_t/[\partial f_{(t)}]$ is sufficiently small that it can be ignored in determining Eurodollar futures hedging quantities. Also, if ε_t, the remaining difference between $f(t)$ and $g(t)$, arises because of pricing error in either the Eurodollar futures or swap markets, or if there are timing differences in the observation of prices in the two markets that contribute to the difference between estimated values of $f(t)$ and $g(t)$, such differences are not likely to be functionally related to the level of $f(t)$. Therefore, for computing Eurodollar-based hedging quantities, I assume that $g(t) = f(t) - (c_t + \varepsilon_t)$ and $[dg_{(t)}]/dL_i = [df_{(t)}]/dL_i$.

Although interest rate swaps do not involve exchanges of principal, the mathematics of swap pricing and hedging are greatly simplified if it is assumed that both parties to an interest rate swap exchange the swap's final principal balance at the termination of the swap and also that they make principal payments to one another equal in value to all scheduled principal reductions prior to the swap's maturity date. (Burghardt refers to this approach as the *hypothetical securities method.*)

Under this assumption, the present value, as of a swap's first interest reset date, of all future LIBOR-based floating payments and all principal reductions from the reset date forward is equal to the swap's initial notional principal amount, provided the present value is computed on a Eurodollar-based present value curve. Therefore, to compute Eurodollar futures-based hedging quantities, the swap's entire floating side (including principal reductions), as of the first interest reset date, can be treated as equivalent to a single payment equal in value to the sum of the notional principal amount plus any scheduled LIBOR-based floating payment.

Assume, for example, one is trying to determine Eurodollar futures hedging quantities for a \$100 million LIBOR-based swap with interest reset dates that occur every six months. The first floating payment, based on current six-month LIBOR and due in six months, is \$2 million. Then the swap's entire floating side (including all scheduled principal reductions) can be treated as equivalent to a single payment of \$102 million in six months, provided principal reductions are also included in the swap's fixed payments when computing the swap's value and related hedging quantities.

Let $pv(swap)$ denote the present value of a swap or swap portfolio (book) as a function of its projected cash flows and the present value factors implied by the U.S. swap curve. Using the hypothetical securities method, for a given swap, fixed cash flows are defined as all scheduled fixed cash flows plus principal reductions. Floating cash flows are not projected directly but are captured in their entirety as equivalent to a lump-sum payment, as of the first interest reset date, equal to the sum of the swap's notional value and the first scheduled floating payment.

Denote the cash flow of the swap portfolio on day t as C_t, with positive values of C_t (as defined by the hypothetical securities method) reflecting cash flows to be received, and negative values reflecting cash flows to be paid. Then, the present value of the swap portfolio is defined as follows:

$$pv(swap) = \sum_{t=1}^{T} \exp(-g[t])C_t$$

$$= \sum_{t=1}^{T} \exp(-f[t] + h_t)C_t \tag{2}$$

The mark-to-market value of a single Eurodollar futures contract, i, is

$$mark_i = \left(100 - \frac{1}{4}L_i\right)10,000 \tag{3}$$

where L_i is the (adjusted) quoted LIBOR associated with the futures contract.[4] The value in the parentheses is the mark-to-market value per \$100 par. This value multiplied by 10,000 gives the mark-to-market value of a single Eurodollar futures contract, since the futures price is computed on the basis of \$1 million par value.

Let Q_i denote the number of futures contracts i required to hedge the risk associated with a potential change in adjusted LIBOR, L_i. To hedge this risk, one should choose the quantity Q_i such that

$$\frac{\partial pv(swap)}{\partial L_i} + Q_i\left(\frac{\partial mark_i}{\partial L_i}\right) = 0 \tag{4}$$

Solving Equation (4) for Q_i yields

$$Q_i = -\frac{\partial pv(swap)}{\partial L_i} \div \frac{\partial mark_i}{\partial L_i} \tag{5}$$

From Equation (3), $\partial mark_i / \partial L_i = -2{,}500$. From Equation (2):

$$\frac{\partial pv(swap)}{\partial L_i} = -\sum_{t=1}^{T}\left(\frac{\partial f[t]}{\partial L_i}\right)\exp(-g[t])C_t$$

where T denotes the day of the final swap cash flow. Therefore, substituting for $\partial mark_i/\partial L_i$ and $[\partial pv(swap)]/\partial L_i$ into Equation (5), the quantity of futures contract i required for hedging the swap portfolio is given by:

$$Q_i = -\left(\frac{1}{2,500}\right)\sum_{t=1}^{T}\left(\frac{\partial f[t]}{\partial L_i}\right)\exp(-g[t])C_t \tag{6}$$

In Equation (6), $[\partial f(t)]/\partial L_i$ is estimated numerically by reestimating the Eurodollar futures-based cubic spline function at values of $L_i + 0.001$ and $L_i - 0.001$.[5]

As an aside, when one enters into a LIBOR-based swap, and also when one buys or sells a Eurodollar deposit, the transaction is settled on what is known as the value date, or two business days after the initial transaction date. In $g(t)$, the discount function associated with the U.S. swap curve, the initial date, $t = 0$, is the swap's value date. Therefore, technically, the U.S. swap curve so estimated is a forward curve that begins two business days after the swap trade date.

To maintain consistency, the futures discount function $f(t)$ described later is also estimated for the period starting two business days after the futures trade date. Therefore, if one were estimating both curves today, each would be a forward curve starting two business days from today. Moreover, all swap values calculated from such curves would represent two-day forward values.

Unlike swap values calculated using Equation (2), changes in mark-to-market futures values result in immediate cash inflows and outflows. When a swap position is hedged with Eurodollar futures, there is thus a two-business day difference between futures gains and losses and swaps losses and gains. We ignore interest on this two-day difference when we compute hedging quantities using Equation (6).

II. ESTIMATING CUBIC SPLINE-BASED DISCOUNT FUNCTION FROM EURODOLLAR FUTURES CURVE

The Eurodollar futures contract is a cash-settled contract whose value is based on the implied delivery of a three-month Eurodollar deposit as of the last trading day (maturity date) of the futures contract. The last trading day is the second London business day before the third Wednesday of the maturity month, typically the Monday before the third Wednesday.[6]

Eurodollar futures are available with maturities of up to ten years, although contracts with maturities beyond five years are comparatively illiquid. Regular Eurodollar futures maturity dates are three months apart within a March, June, September, and December maturity cycle. In addition, there are always four additional short-maturity contracts available for trading with maturities one and possibly two months apart.

The Eurodollar futures contract sells at a price, or index value, that reflects a discount from \$100. On the last trading day, the discount is set equal to the three-month British Bankers' Association LIBOR, and a final cash settlement is made on the basis of this rate. Prior to the last day of trading, the discount is set by supply and demand in anticipation of the final settlement price, and reflects the risk-adjusted expected three-month LIBOR to be established on the last day of futures trading.

Contract Details

Exhibit 1 summarizes contract listings and LIBOR implied by settlement prices of all available Eurodollar futures as of the close of trading on January 2, 2001. The information in these contract listings is used to estimate the futures discount function, $f(t)$, as of the swap value date, January 4, 2001.

The date a Eurodollar deposit (or a swap) transaction is executed is called the *fixing date*. Settlement of the deposit takes place on the *value date*, which is two London business days later. For contract 1, a value date of 1/18/01 is shown in column (1). The maturity date of the three-month Eurodollar deposit that underlies any Eurodollar futures contract will be exactly three months after its associated value date, or 4/18/01 in the case of the first contract, unless that date is not a business day in London and in the U.S. In such instances, a modified following business day convention is used to establish the maturity date of the three-month Eurodollar deposit.

Therefore, on January 2, 2001, the value of the first contract listed in Exhibit 1 is based on the anticipated LIBOR to be established on 1/16/2001 for a three-month Eurodollar deposit with a 1/18/2001 value date and 4/18/2001 maturity date. The end-of-day settlement price for this contract was 93.870, which gives an implied

EXHIBIT 1

Estimating Cubic Spline-Based Discount Function from Eurodollar Futures Curve

	Initial LIBOR							Adjusted LIBOR						
Contract	Value Date (1)	Maturity Date (2)	LIBOR (3)	Start Day (4)	End Day (5)	LIBOR Days (6)	Cont. Equiv. (7)	Start Day (8)	End Day (t) (9)	Days (10)	LIBOR (11)	Cont. Equiv. (12)	$g(t)$ (13)	$f(t)$ (14)
									0				0.00000	0.00000
									14				0.00257	0.00257
1	1/18/01	4/18/01	6.130	14	104	90	0.01521	14	104	90	6.130	0.01521	0.01813	0.01778
2	2/22/01	5/22/01	5.925	49	138	89	0.01454	49	138	89	5.925	0.01454	0.02368	0.02323
3	3/21/01	6/21/01	5.760	76	168	92	0.01461	76	168	92	5.760	0.01461	0.02847	0.02781
4	4/18/01	7/18/01	5.640	104	195	91	0.01416	104	195	91	5.640	0.01416	0.03273	0.03193
5	5/16/01	8/16/01	5.500	132	224	92	0.01396	132	224	92	5.500	0.01396	0.03725	0.03624
6	6/20/01	9/20/01	5.335	167	259	92	0.01354	167	259	92	5.335	0.01354	0.04264	0.04120
7	9/19/01	12/19/01	5.190	258	349	91	0.01303	258	349	91	5.190	0.01303	0.05616	0.05409
8	12/19/01	3/19/02	5.280	349	439	90	0.01311	349	439	90	5.280	0.01311	0.06918	0.06721
9	3/20/02	6/20/02	5.270	440	532	92	0.01338	440	532	92	5.270	0.01338	0.08231	0.08073
10	6/19/02	9/19/02	5.380	531	623	92	0.01366	531	623	92	5.380	0.01366	0.09512	0.09424
11	9/18/02	12/18/02	5.455	622	713	91	0.01369	622	713	91	5.455	0.01369	0.10801	0.10778
12	12/18/02	3/18/03	5.575	713	803	90	0.01384	713	803	90	5.575	0.01384	0.12136	0.12162
13	3/19/03	6/19/03	5.565	804	896	92	0.01412	804	896	92	5.565	0.01412	0.13560	0.13590
14	6/18/03	9/18/03	5.615	895	987	92	0.01425	895	987	92	5.615	0.01425	0.14982	0.14999
15	9/17/03	12/17/03	5.665	986	1077	91	0.01422	986	1077	91	5.665	0.01422	0.16400	0.16406
16	12/17/03	3/17/04	5.770	1077	1168	91	0.01448	1077	1168	91	5.770	0.01448	0.17831	0.17854
17	3/17/04	6/17/04	5.745	1168	1260	92	0.01457	1168	1260	92	5.745	0.01457	0.19275	0.19311
18	6/16/04	9/16/04	5.795	1259	1351	92	0.01470	1259	1351	92	5.795	0.01470	0.20706	0.20765
19	9/15/04	12/15/04	5.840	1350	1441	91	0.01465	1350	1441	91	5.840	0.01465	0.22128	0.22215
20	12/15/04	3/15/05	5.940	1441	1531	90	0.01474	1441	1531	90	5.940	0.01474	0.23561	0.23689
21	3/16/05	6/16/05	5.910	1532	1624	92	0.01499	1532	1624	92	5.910	0.01499	0.25051	0.25204
22	6/15/05	9/15/05	5.950	1623	1715	92	0.01509	1623	1720	97	5.953	0.01591	0.26596	0.26779
23	9/21/05	12/21/05	5.990	1721	1812	91	0.01503	1721	1812	91	5.990	0.01503	0.28077	0.28298
24	12/21/05	3/21/06	6.085	1812	1902	90	0.01510	1812	1902	90	6.085	0.01510	0.29523	0.29808
25	3/15/06	6/15/06	6.055	1896	1988	92	0.01536	1896	1993	97	6.058	0.01619	0.30984	0.31327
26	6/21/06	9/21/06	6.095	1994	2086	92	0.01546	1994	2086	92	6.095	0.01546	0.32478	0.32889
27	9/20/06	12/20/06	6.130	2085	2176	91	0.01538	2085	2176	91	6.130	0.01538	0.33928	0.34410
28	12/20/06	3/20/07	6.225	2176	2266	90	0.01544	2176	2266	90	6.225	0.01544	0.35385	0.35954
29	3/21/07	6/21/07	6.195	2267	2359	92	0.01571	2267	2359	92	6.195	0.01571	0.36898	0.37542
30	6/20/07	9/20/07	6.235	2358	2450	92	0.01581	2358	2450	92	6.235	0.01581	0.38383	0.39106
31	9/19/07	12/19/07	6.265	2449	2540	91	0.01571	2449	2540	91	6.265	0.01571	0.39855	0.40660
32	12/19/07	3/19/08	6.360	2540	2631	91	0.01595	2540	2631	91	6.360	0.01595	0.41346	0.42254
33	3/19/08	6/19/08	6.330	2631	2723	92	0.01605	2631	2723	92	6.330	0.01605	0.42857	0.43859
34	6/18/08	9/18/08	6.370	2722	2814	92	0.01615	2722	2814	92	6.370	0.01615	0.44356	0.45457
35	9/17/08	12/17/08	6.400	2813	2904	91	0.01605	2813	2904	91	6.400	0.01605	0.45843	0.47044
36	12/17/08	3/17/09	6.490	2904	2994	90	0.01609	2904	2994	90	6.490	0.01609	0.47336	0.48653
37	3/18/09	6/18/09	6.460	2995	3087	92	0.01637	2995	3087	92	6.460	0.01637	0.48886	0.50309
38	6/17/09	9/17/09	6.495	3086	3178	92	0.01646	3086	3178	92	6.495	0.01646	0.50409	0.51937
39	9/16/09	12/16/09	6.525	3177	3268	91	0.01636	3177	3268	91	6.525	0.01636	0.51922	0.53555
40	12/16/09	3/16/10	6.615	3268	3358	90	0.01640	3268	3358	90	6.615	0.01640	0.53442	0.55195
41	3/17/10	6/17/10	6.585	3359	3451	92	0.01669	3359	3451	92	6.585	0.01669	0.55017	0.56882
42	6/16/10	9/16/10	6.620	3450	3542	92	0.01678	3450	3542	92	6.620	0.01678	0.56562	0.58542
43	9/15/10	12/15/10	6.650	3541	3632	91	0.01667	3541	3632	91	6.650	0.01667	0.58093	0.60190
44	12/15/10	3/15/11	6.740	3632	3722	90	0.01671	3632	3722	90	6.740	0.01671	0.59623	0.61861

The two shaded rows are associated with contracts for which there is a gap between the maturity date of the three-month Eurodollar deposit that underlies the contract and the value date of the Eurodollar deposit associated with the next adjacent contract.

LIBOR of 100 − 93.870 = 6.130%. This rate is shown in column (3). Similar value dates, maturity dates, and implied LIBOR settlement are shown for all 44 Eurodollar contracts available for trading on January 2.

Column (4) shows the number of days from the swap value date, 1/4/2001, until the three-month Eurodollar value date. For contract 1, this period, known as the *stub period*, is 14 days. Column (5) shows the number of days from the swap value date to the maturity date of the three-month Eurodollar deposit that underlies each futures contract, or 104 days for the first contract. Column (6) shows the number of days associated with the underlying

Eurodollar deposit, which for the first contract is 104 – 14 = 90 days.

Column (7) shows the non-annualized continuously compounded rate of interest equivalent to each implied three-month LIBOR. Using an actual/360 day-count, $1 invested in a three-month Eurodollar deposit earning the 6.130% implied LIBOR shown for contract 1 would be worth 1 + 6.130(90/36,000) = 1.015325 as of the deposit's 4/18/01 maturity date.

This represents a continuously compounded rate of return of ln(1.015325) = 0.01521, the amount shown in column (7) for contract 1. For contract 2, the equivalent continuously compounded rate would be ln(1 + 5.925 [89/36,000]) = 0.01454. Continuous returns associated with all other contracts are computed in a similar fashion.

Overlapping Maturities

Note that the maturities of the first six Eurodollar futures contracts shown in Exhibit 1 are only approximately one month apart. This creates an approximate two-month overlap among the three-month Eurodollar deposits that underlie each contract. For example, the three-month Eurodollar deposit associated with the first contract runs from 1/18/01 through 4/18/01. The three-month period for the Eurodollar deposit of the second contract runs from 2/22/01 through 5/22/01. Therefore, the two Eurodollar deposits overlap between 2/22/01 and 4/18/01. A similar pattern is shown for each of the first six contracts.

In all published Eurodollar futures-based hedging applications I know of, the problem of overlapping Eurodollar deposit periods is handled simply by omitting those contracts that are not part of the regular March, June, September, and December maturity cycle. My methodology allows all contracts to be included in estimating the Eurodollar futures curve and in determining appropriate Eurodollar futures hedging quantities.

Beginning with contract 6, all contracts are part of the regular March, June, September, and December maturity cycle. Note, however, that there is also a slight degree of overlap in the three-month Eurodollar deposit periods associated with many adjacent contracts within this cycle. For example, the three-month Eurodollar deposit period for contract 6 runs through 9/20/01, but the three-month period for contract 7 starts a day earlier (9/19/2001). Note also that for contracts 22 and 25, there is a gap between the three-month Eurodollar deposit maturity date and the starting date (value date) of the three-month Eurodollar deposit period for the next adjacent contract. For example, the three-month Eurodollar deposit period for contract 22 ends on 9/15/05, but the three-month deposit period for contract 23 starts on 9/21/05, six days later.

Typically the problems associated with overlap and gaps within the regular March, June, September, and December maturity cycle are handled by assuming that the three-month period for each underlying three-month Eurodollar deposit starts on the LIBOR value date associated with a given futures maturity, and runs through the LIBOR value date of the futures contract that matures three months later. This procedure assumes implicitly that if deposit rates for these adjusted three-month time periods could be observed, they would be the same as those associated with exact three-month time periods.

Adjusting LIBOR and Time Intervals for Maturity Gaps

The procedure for estimating the futures discount function, $f(t)$, requires that all gaps in the three-month Eurodollar deposit maturity structure be filled. I treat gaps such as those illustrated by contracts 22 and 25 as follows.

First, the maturity date of the three-month Eurodollar deposit is adjusted forward to the day (business or otherwise) immediately preceding the value date of the Eurodollar deposit underlying the next adjacent contract. For example, for contract 22, the maturity date of the underlying three-month Eurodollar deposit is moved forward from 9/15/05 to 9/20/05, or, equivalently, from day 1,715 for the 1/4/2001 swap value date to day 1,720. For contract 22, the 1,720 adjusted end day is shown in column (9).

Next, the LIBOR is adjusted to reflect the additional days added to the three-month Eurodollar deposit period and the possible change in yield curve shape over this period. Note that for contract 22 the continuously compounded equivalent rate for the three-month Eurodollar deposit as shown in column (7) is 0.01509, which corresponds to a three-month LIBOR of 5.950% [shown in column (3)].

Let r denote the original unadjusted continuously compounded equivalent rate and r' denote the adjusted rate. Then r' is computed as follows:

$$r' = r\left(\frac{g(t_2) - g(t_0)}{g(t_1) - g(t_0)}\right) \tag{7}$$

where t_0 is the starting day [columns (4) and (8)] of the three-month Eurodollar deposit, t_1 is the unadjusted end day [column (5)], and t_2 is the adjusted end day [column (9)]. Therefore, the ratio of the adjusted continuous rate to the original rate, or r'/r, is assumed to be the same as the ratio of rates for the same time periods implied by the swap curve discount function, $g(t)$.

For example, for contract 22, $g(t_0) = g(1623) = 0.250353$, and $g(t_1) = g(1715) = 0.265155$ (values not shown in Exhibit 1). Also, as shown in column (13), $g(t_2) = g(1720) = 0.265960$. Therefore, applying Equation (7):

$$\begin{aligned} r' &= r\left(\frac{g(t_2) - g(t_0)}{g(t_1) - g(t_0)}\right) \\ &= 0.01509\left(\frac{0.265960 - 0.250353}{0.265155 - 0.250353}\right) \\ &= 0.01591 \end{aligned}$$

This amount is shown in column (12). It is then converted to a LIBOR-equivalent, or 5.953%, which is the value shown in column (11).

Estimating the $f(t)$ Function

The method for estimating the futures discount function, $f(t)$, employs the same type of iterative procedure as described in Rendleman [2004] for estimating the zero-coupon discount function for the swap curve, $g(t)$. I assume that the futures discount function, $f(t)$, can be approximated by a cubic spline using the not-a-knot estimation procedure.[8]

Let $\alpha_j = \{x_j, y_j\}$ denote the spline knot represented by coordinate pair $\{x_j, y_j\}$; $z(\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote the cubic spline function estimated with knots $\alpha_1, \alpha_2, \ldots, \alpha_n$; and $z(\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote the cubic spline function evaluated at x. An appealing property of the cubic spline estimation procedure, and the key to the interpolation method I propose, is that $z(x_j; \alpha_1, \alpha_2, \ldots, \alpha_n) = y_j$, meaning that the function will return the original value of y_j associated with cubic spline knot $\alpha_j = \{x_j, y_j\}$ when evaluated at any x_j value from the set of knots from which the spline is fit.

For the purposes of estimating $z(\alpha_1, \alpha_2, \ldots, \alpha_n)$, two knot points are set to fixed values. The first such point is associated with the initial swap value date, day $t = 0$. On that date, the futures-based cumulative continuous yield is zero. Therefore, the first knot, represented as α_{-1}, is the coordinate pair $\alpha_{-1} = \{x_{-1}, y_{-1}\} = \{0, 0\}$, where x denotes time (in days) and y denotes cumulative continuous yield. The second fixed knot is associated with the futures-based continuous cumulative yield through the value date of the Eurodollar deposit associated with the shortest-maturity futures contract. The futures-based continuous yield is assumed to equal that derived from the U.S. swap curve for the same day.

In Exhibit 1, this date would be 14 days from the initial 1/4/2001 swap value date, and the cumulative continuous yield derived from the swap curve after 14 days is 0.00257. Thus, the second fixed knot, denoted as α_0, is $\alpha_0 = \{14, 0.00257\}$. In Exhibit 1, the x-values for fixed knots α_{-1} and α_0 are shown in the first two rows of column (9), and their corresponding y-values are shown in the first two rows of column (14).

As in Exhibit 1, there are always 44 Eurodollar futures contracts available for trading. Knots 1 through n are associated with these 44 contracts. The x-value of coordinate pair, $\alpha_j = \{x_j, y_j\}$, is set equal to the number of days through the maturity date of the Eurodollar deposit that underlies futures contract j. The y-value, y_j, which represents the cumulative continuous yield through day x_j, is determined by numerical search. The search routine determines a set of y-values for knots $\alpha_1, \alpha_2, \ldots, \alpha_n$ that make the continuous equivalent yields for the Eurodollar deposits underlying each contract computed via the $f(t)$ function equal the *adjusted* continuous equivalent yields as illustrated in column (12) of Exhibit 1.

The final y-values for this function, as computed from futures prices observed on 1/2/2001, are shown in column (14) of Exhibit 1. Although it is not shown in Exhibit 1, all the individual Eurodollar continuous yields computed via the $f(t)$ function fall within 2.3×10^{-9} of the adjusted continuous equivalent rates shown in column (12), and for many of the yields, the difference is on the order of 10^{-17}.

III. HEDGING PERFORMANCE

The hedging model is implemented on a daily basis for calendar year 2001, a particularly volatile year for interest rates. The year began with a slightly inverted yield curve, with one-month LIBOR at 6.55% and the 30-year swap rate at 6.12%.[9] By year's end, one-month LIBOR had fallen to 3.63%, while the 30-year swap rate had actually risen to 6.29%. In September 2001, financial markets were rocked by the tragic events of September 11, and some markets shut down completely in the days immediately following the terrorist attacks.

Ideally, to test the performance of the hedging model, one should observe futures and swap values at the same time. The futures data used to test hedging performance are obtained from the Chicago Mercantile Exchange, and the prices used are daily settlement prices established as of the 3:00PM EST close of trading. Swap rates are taken from the Bloomberg system.

Bloomberg allows historical swaps data to be collected as of 6:00AM, 1:00PM, and 5:30PM of each trading day. Although swap rates established as of the futures closing time of 3:00PM are not available, an average of the 1:00PM and 5:30PM swap rates should provide a reasonable approximation of the swaps rate at 3:00PM. Therefore, to estimate the swap-based discount function, $g(t)$, and to value individual swaps contracts in terms of this function, I use a swap curve based on the average rates of the 1:00PM and 5:30PM data. The methodology for estimating the $g(t)$ function is identical to that described in Rendleman [2004].[10]

Daily futures and swaps data are employed for the 250 days of 2001 for which both the futures market was open and historical data for the complete swaps curve are available from the Bloomberg system. Although the Eurodollar futures market was open on September 12 and 13, the Bloomberg system does not provide swaps data for these dates, so these dates are omitted from the analysis. Thus the analysis assumes that a hedge established on September 11 would have been carried over without revision until September 14, the first full trading day for both markets after the September 11 attacks.

Hedging a Single Swap Contract

I first examine the hedge of a single five-year swap entered into on January 2, 2001, and maintained, with appropriate revisions to Eurodollar futures-based hedging quantities, through December 31, 2001. A five-year swap is chosen as this is the maximum maturity that can be hedged without encountering liquidity problems in long-term Eurodollar futures contracts.[11]

It is assumed one takes a position on January 2, 2001, in a newly issued five-year swap for which a fixed payment is received monthly, and a floating payment, tied to one-month LIBOR, is paid monthly on the same date. The swap's annual fixed rate is 5.88018% (using an actual/360 day-count). 5.88018% is the rate that causes the swap to have zero value at the time of issuance using the $g(t)$ function derived from the U.S. swap curve.

On each of the 250 trading days of 2001, it is assumed that the swap is hedged with Eurodollar futures using hedging quantities established by Equation (6). The following day, the swap is revalued using the $g(t)$ function defined by that day's swap and LIBOR, and a new Eurodollar futures hedge is established (except on December 31 when the position is liquidated). Daily gains and losses are computed for the swap using swap values established by the $g(t)$ function, while gains and losses for futures positions are based on daily changes in futures settlement prices.

As positive cash balances accrue over the 2001 calendar year, interest is assumed to be earned between successive hedging periods at the rate derived from the $g(t)$ function. Similarly, as negative cash balances accrue, interest is assumed to be paid at the same rate.

Exhibit 2 shows how the values of the swap and the hedged swap position would have evolved over the full 2001 calendar year. On November 7, 2001, the swap reached a maximum value for the year of \$7.399 per \$100 of notional principal; it ended the year at a value of \$3.299. The final value of the hedged position, as of December 31, 2001, was \$0.1788 per \$100, or approximately 18 basis points.

We can see that the value of the hedged position stayed centered at approximately zero through mid-August, but after that time the hedged value tended to be slightly positive. There are positive and negative spikes in the value of the hedge immediately following the September 11 attacks. These spikes are most likely due to non-synchronous data problems when interest rates were changing rapidly.[12]

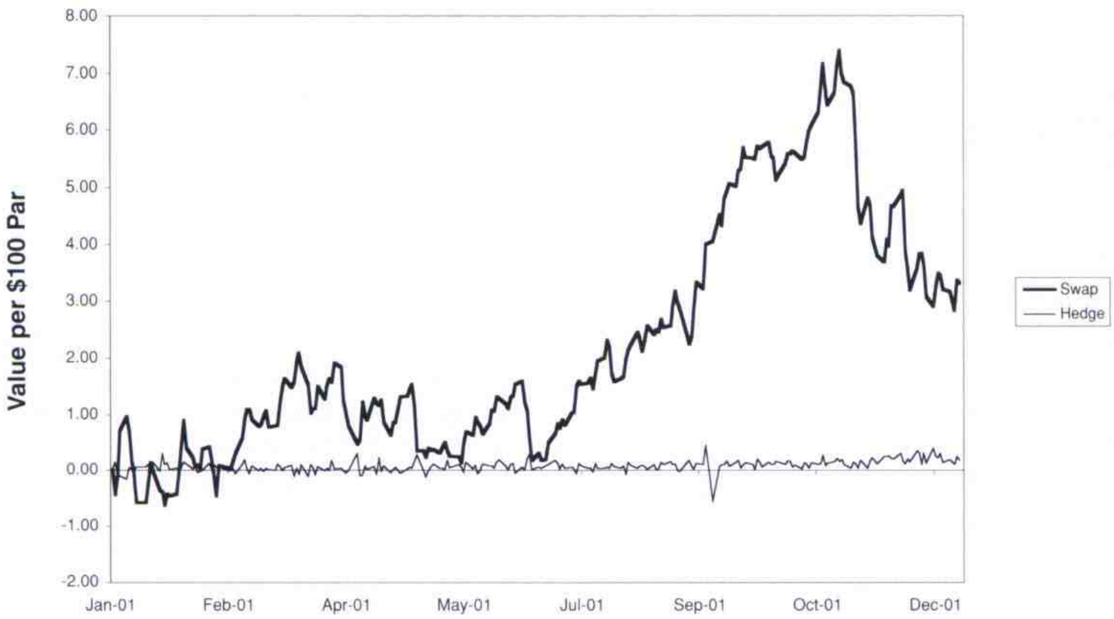
There appear to be many reversals of sign in the value of the hedged position shown in Exhibit 2. To test for the tendency of daily changes in hedged position values to reverse, I run a regression of daily changes in hedged position values against the previous day's change. The regression coefficient is –0.569 with a t-statistic of –10.86 (adjusted $R^2 = 0.322$). Despite attempts to align the futures and swaps data in time, the day-to-day hedging results shown in Exhibit 2 appear to be adversely affected by non-synchronous data problems, implying that hedging performance based on synchronous price data would actually be better.[13]

Column (1) in Exhibit 3 summarizes Eurodollar futures hedging quantities per \$100 million notional principal as of 1/2/2001, the day the hedged swap position is initiated. Note that the hedge calls for short positions in most futures contracts, although two of the shortest-term contracts and two longer-term contracts require relatively small long positions.

Even though the swap that is hedged is scheduled to mature on January 4, 2005, the hedge involves taking posi-

Exhibit 2

Cumulative Performance—Eurodollar Futures Hedge of Five-Year Swap



tions in all contracts maturing after December 2004. (Position sizes are so small for contracts maturing after March 2007 that they do not show up in the exhibit.) These contracts enter into the hedge because a change in their implied LIBOR has an effect on the entire cubic spline-based $f(t)$ discount function, which in turn has an effect on the value of the swap. This effect is small, however, for sections of the function that are not in the immediate vicinity of the respective LIBOR investment periods.

The volume and open interest figures shown in columns (3) and (4) suggest that the futures contracts expiring in April and May would be too illiquid to establish viable futures positions. Even the marked reduction in trading volume, starting with the March 2006 contract, should not indicate a liquidity problem, though, as position sizes for this contract and for those of longer maturity are relatively small.[14]

In an effort to address the liquidity problem, I recompute hedging quantities and the $f(t)$ function omitting contracts that are not part of the regular March, June, September, and December cycle. These hedging quantities are shown in column (2). Starting with the December 2001 contract, the required futures positions for both methods of hedging are practically identical; for shorter-maturity contracts, however, hedging quantities are quite a bit different. When hedging performance using only contracts in the regular three-month maturity cycle is plotted, it is almost indistinguishable from that shown in Exhibit 2 using all contracts, except that the ending value of the hedged swap position is $0.3092 per $100 of notional principal compared to $0.1788 using all contracts.[15]

It is interesting to compare hedging quantities with those computed by the methods proposed by Kawaller [1994] and Burghardt [2003]. Although Kawaller presents a hedging example in which futures and swap-based LIBOR value dates are not aligned, his method requires a best guess estimate of the difference between ex post futures and swap-based LIBOR and is not easy to replicate when futures and swap-based LIBOR dates are not in alignment. Therefore, I have chosen 3/19/2001, the expiration date of the March 2001 Eurodollar futures contract, to compute hedging quantities for a newly issued swap, when futures and swap-based LIBOR value dates are perfectly aligned under Kawaller's method and thus no best guess is necessary.

Computations of hedging quantities using the methods of Kawaller and Burghardt are summarized in the appendix. Not surprisingly, the hedging quantities obtained using their methods are not substantially different from those using my method, provided futures available for hedging under the new method are limited to contracts in the regular three-month maturity cycle.

EXHIBIT 3

Eurodollar Futures Hedging Quantities per $100 Million Notional Principal

		Hedging Quantity			
		Using All Contracts	Using Contracts in 3-Month Cycle	Volume	Open Interest
Contract		(1)	(2)	(3)	(4)
JAN	2001	-71.25	n/a	4,492	46,411
FEB	2001	21.05	n/a	1,700	8,494
MAR	2001	-104.46	-121.22	102,393	541,603
APR	2001	5.73	n/a	0	52
MAY	2001	-4.61	n/a	0	47
JUN	2001	-95.03	-87.31	152,277	454,669
SEP	2001	-94.90	-97.21	142,050	512,857
DEC	2001	-93.95	-93.34	71,079	289,885
MAR	2002	-93.94	-94.10	54,526	277,226
JUN	2002	-91.05	-91.00	63,725	211,552
SEP	2002	-89.55	-89.57	28,632	157,894
DEC	2002	-88.76	-88.75	23,266	118,016
MAR	2003	-88.72	-88.73	13,234	93,619
JUN	2003	-85.96	-85.96	11,280	75,458
SEP	2003	-84.74	-84.74	8,934	72,387
DEC	2003	-84.17	-84.17	9,400	58,025
MAR	2004	-83.21	-83.21	5,013	53,904
JUN	2004	-81.40	-81.40	7,566	53,253
SEP	2004	-79.87	-79.87	4,627	54,012
DEC	2004	-79.19	-79.19	5,138	37,827
MAR	2005	-79.26	-79.26	5,203	31,532
JUN	2005	-80.53	-80.53	5,663	27,940
SEP	2005	-81.96	-81.96	5,353	25,473
DEC	2005	-9.02	-9.02	4,935	12,176
MAR	2006	2.36	2.36	476	12,906
JUN	2006	-0.59	-0.59	476	11,415
SEP	2006	0.16	0.16	476	10,986
DEC	2006	-0.04	-0.04	516	6,372
MAR	2007	0.01	0.01	881	5,816
JUN	2007	0.00	0.00	881	4,506
SEP	2007	0.00	0.00	881	4,105
DEC	2007	0.00	0.00	901	4,964
MAR	2008	0.00	0.00	441	4,143
JUN	2008	0.00	0.00	441	4,015
SEP	2008	0.00	0.00	441	4,008
DEC	2008	0.00	0.00	441	3,011
MAR	2009	0.00	0.00	441	2,523
JUN	2009	0.00	0.00	441	2,466
SEP	2009	0.00	0.00	441	2,357
DEC	2009	0.00	0.00	441	2,073
MAR	2010	0.00	0.00	346	2,309
JUN	2010	0.00	0.00	340	1,810
SEP	2010	0.00	0.00	341	1,659
DEC	2010	0.00	0.00	341	575

Swap's fixed and floating payments are made monthly and are computed on the basis of an actual/360 day-count with the floating payment tied to one-month LIBOR.

Hedging a Swap Book

The performance of a swap book (portfolio) is summarized in Exhibit 4. This particular book would have had a value of zero on January 2, 2001. The duration of the portfolio would have been zero in the sense that the weighted duration of all positive cash flows would equal the weighted duration of all the negative cash flows. Thus, in the context of duration, this portfolio would have been perfectly hedged.

As is well-known, the effectiveness of a duration-based hedge depends critically on parallel yield curve changes. A portfolio such as this swap book that has a large dollar value allocated to fixed-rate positions in very short-term assets would be particularly vulnerable to a significant decline in short-term interest rates compared to long-term interest rates, which is exactly what happened in 2001. Thus, if left duration-neutral but otherwise unhedged, this portfolio would have incurred significant losses during the 2001 calendar year.

Initial Eurodollar futures hedging quantities are summarized in Exhibit 5. Note that most of the quantities for the shortest-term contracts are positive and most for the longest-term contracts are negative, reflecting the relatively large dollar allocation to short positions in very short-term assets.

Exhibit 6 shows how the value of the swap book, hedged and unhedged, would have evolved over the full 2001 calendar year. The swap book, initially worth $0, reached a low of –$2.67 million on December 11, and on December 31 it was worth –$1.56 million. As a hedged book, its value as of December 31 was $13,333. If it is assumed that the hedge was carried out using only futures contracts in the regular March, June, September, and December cycle, a graph of the cumulative value of the hedged position is almost indistinguishable from that shown in Exhibit 6. Under this assumption, however, the December 31 value of the hedged swap book would have been $48,207 rather than $13,333.

Initial futures hedging quantities, when futures contracting is limited to contracts in the regular three-month cycle, are shown in column (2) of Exhibit 5. Except for the shorter-term contracts, these quantities are almost identical to those shown in column (1), which are based on all available futures contracts.

Note that there is a downward spike in hedge values near the end of 2001. This spike occurred on December 26, which is the Boxing Day holiday in the U.K. and a very inactive day in U.S. financial markets. This most likely indicates that swap rates, LIBOR, and Eurodollar futures prices were somewhat out of synch.

IV. CONCLUSION

I have presented a general model for hedging LIBOR-based swaps with Eurodollar futures. Drawing on the cubic spline interpolation methodology in Rendleman [2004], the hedging model automatically reflects:

EXHIBIT 4
Swap Book as of 1/2/2001

	Notional Principal	Fixed Rate	Maturity Date	Payment Interval	Receiver of	Fixed-Side Value	Floating-Side Value	Net Value
1	100,000,000	6.00	7/18/2005	Q	Fixed	101,420,460	-99,958,264	1,462,196
2	19,620,667	8.00	2/13/2006	M	Fixed	21,709,415	-19,570,166	2,139,249
3	170,756,763	5.50	12/15/2003	S	Float	-166,355,348	166,931,445	576,097
4	100,000,000	9.00	1/22/2002	S	Float	-102,846,708	98,669,167	-4,177,542

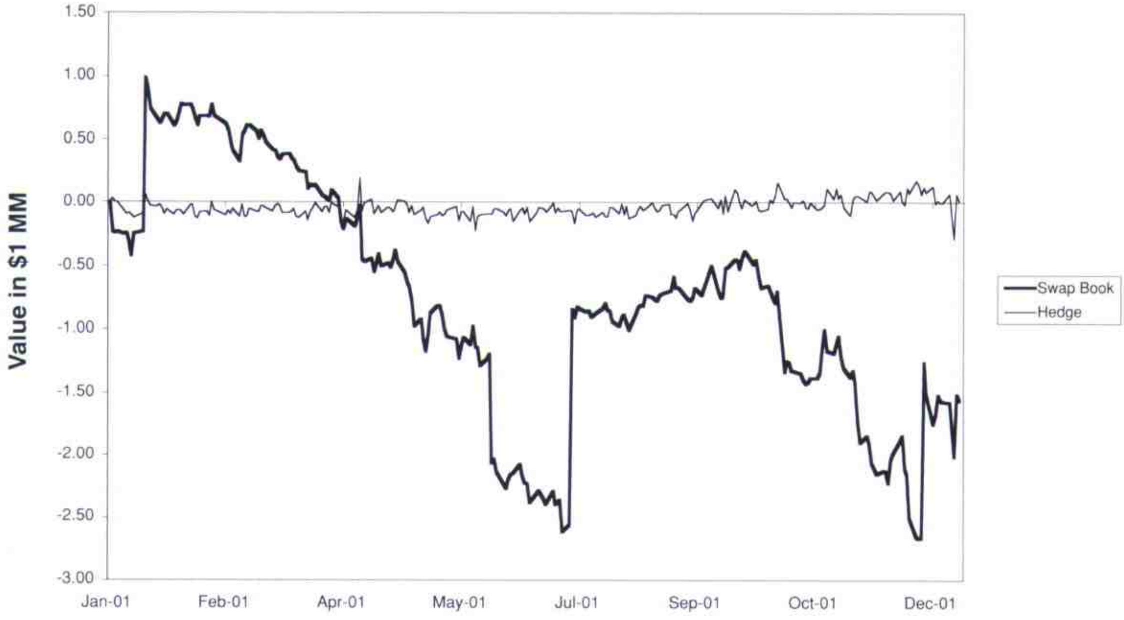
Payment interval: Q = quarterly, M = monthly, S = semiannually. For all swaps, fixed and floating payments are on the same payment interval and occur on the same day. Both floating and fixed payments are calculated on the basis of an actual/360 day-count. Fixed- and floating-side values are computed using the hypothetical securities method.

EXHIBIT 5
Eurodollar Futures Hedging Quantities for Hedging Swap Book

Contract		Hedging Quantity: Using All Contracts (1)	Hedging Quantity: Using Contracts in 3-Month Cycle (2)	Volume (3)	Open Interest (4)
JAN	2001	-14.93	n/a	4,492	46,411
FEB	2001	3.86	n/a	1,700	8,494
MAR	2001	-32.55	-13.00	102,393	541,603
APR	2001	15.10	n/a	0	52
MAY	2001	16.61	n/a	0	47
JUN	2001	131.88	149.67	152,277	454,669
SEP	2001	157.61	154.00	142,050	512,857
DEC	2001	79.71	80.66	71,079	289,885
MAR	2002	40.12	39.86	54,526	277,226
JUN	2002	46.54	46.61	63,725	211,552
SEP	2002	45.48	45.46	28,632	157,894
DEC	2002	43.95	43.95	23,266	118,016
MAR	2003	45.87	45.86	13,234	93,619
JUN	2003	43.37	43.37	11,280	75,458
SEP	2003	42.39	42.39	8,934	72,387
DEC	2003	-104.27	-104.27	9,400	58,025
MAR	2004	-100.66	-100.66	5,013	53,904
JUN	2004	-98.71	-98.71	7,566	53,253
SEP	2004	-97.34	-97.34	4,627	54,012
DEC	2004	-93.39	-93.39	5,138	37,827
MAR	2005	-104.62	-104.62	5,203	31,532
JUN	2005	-41.06	-41.06	5,663	27,940
SEP	2005	-10.41	-10.41	5,353	25,473
DEC	2005	-10.67	-10.67	4,935	12,176
MAR	2006	2.03	2.03	476	12,906
JUN	2006	-0.51	-0.51	476	11,415
SEP	2006	0.14	0.14	476	10,986
DEC	2006	-0.04	-0.04	516	6,372
MAR	2007	0.01	0.01	881	5,816
JUN	2007	0.00	0.00	881	4,506
SEP	2007	0.00	0.00	881	4,105
DEC	2007	0.00	0.00	901	4,964
MAR	2008	0.00	0.00	441	4,143
JUN	2008	0.00	0.00	441	4,015
SEP	2008	0.00	0.00	441	4,008
DEC	2008	0.00	0.00	441	3,011
MAR	2009	0.00	0.00	441	2,523
JUN	2009	0.00	0.00	441	2,466
SEP	2009	0.00	0.00	441	2,357
DEC	2009	0.00	0.00	441	2,073
MAR	2010	0.00	0.00	346	2,309
JUN	2010	0.00	0.00	340	1,810
SEP	2010	0.00	0.00	341	1,659
DEC	2010	0.00	0.00	341	575

E X H I B I T 6

Cumulative Performance—Eurodollar Futures Hedge of Swap Book



- Timing differences between futures maturities and swap payment dates.
- The actual maturities of the three-month Eurodollar deposits that underlie each futures contract. (Usually these maturities are assumed to coincide with the value dates of the futures contracts that mature three months later, but typically the maturities are actually different.)
- The convexity bias defining the difference between Eurodollar futures and forward rates.

The model lets us use futures with overlapping maturities rather than limiting hedging to contracts that are part of the regular March, June, September, and December maturity cycle. Using daily futures prices and swap rates, the model is tested for calendar year 2001 for both a single swap and a swap book, and shown to be very effective in hedging the interest rate risk of LIBOR-based swaps.

The performance of hedges for a number of different swap specifications using all contracts or using only contracts in the regular maturity cycle reveals hardly any difference between the two sets of contracts. None of the performance numbers reflect transaction costs, which should be higher when all contracts are used, since the contracts of irregular maturity are of shorter term and would have to be replaced frequently in any type of long-term hedging strategy. Perhaps this explains the relative lack of liquidity in contracts of irregular maturity as compared to contracts in the regular three-month maturity cycle.

APPENDIX

Computation of Hedging Quantities Following Kawaller and Burghardt

Futures hedging quantities are computed for a newly issued five-year swap on 3/19/2001, the expiration date of the March 2001 Eurodollar futures contract. On this date, futures and swap-based LIBOR value dates are perfectly aligned under the methods of Kawaller and Burghardt, and no interpolation is necessary to compute hedging quantities. The swap requires quarterly fixed and floating payments based on an actual/360 day-count and has a fixed rate of 5.289934%. At this fixed rate, using the $g(t)$ function developed in Rendleman [2004], the present value of the swap's stream of payments per $100 million of notional principal is zero.

The appendix Exhibit summarizes the computation of hedging quantities using the methods of Kawaller and Burghardt and also shows corresponding quantities using my method.

Kawaller's method is built on the calculation of a basis point value (BPV) for each futures contract. Consider the June 2001 futures contract. There are 91 days between the contract's 6/20/2001 LIBOR value date [column (1)] and the 9/19/2001 value date [column (2)] associated with the next contract in the regular March, June, September, and December maturity cycle.

EXHIBIT
Computation of Eurodollar Futures Hedging Quantities for 5-Year $100 Million Par Swap

	Starting Value Date (1)	Ending Value Date (2)	Days Between Dates (3)	Implied LIBOR (4)	BPV (5)	Kawaller Untailed Hedging Quantity (6)	Forward Interest Factor (7)	Cumulative Interest Factor (8)	Kawaller Tailed Quantity (9)	Burghardt Secondary Quantity (10)	Burghardt Total Quantity (11)	Rendleman Quantity (12)
3-mo LIBOR	3/21/01	6/20/01	91	4.880	n/a	n/a	1.0123	1.0123	n/a	n/a	n/a	n/a
JUN 2001	6/20/01	9/19/01	91	4.535	2527.78	-101.11	1.0115	1.0239	-98.75	-0.09	-98.84	-98.11
SEP 2001	9/19/01	12/19/01	91	4.445	2527.78	-101.11	1.0112	1.0354	-97.65	-0.13	-97.80	-96.79
DEC 2001	12/19/01	3/20/02	91	4.635	2527.78	-101.11	1.0117	1.0476	-96.52	-0.18	-96.74	-95.87
MAR 2002	3/20/02	6/19/02	91	4.695	2527.78	-101.11	1.0119	1.0600	-95.39	-0.22	-95.63	-95.91
JUN 2002	6/19/02	9/18/02	91	4.920	2527.78	-101.11	1.0124	1.0732	-94.22	-0.26	-94.48	-93.02
SEP 2002	9/18/02	12/18/02	91	5.125	2527.78	-101.11	1.0130	1.0871	-93.01	-0.28	-93.30	-91.52
DEC 2002	12/18/02	3/19/03	91	5.340	2527.78	-101.11	1.0135	1.1018	-91.77	-0.29	-92.09	-90.71
MAR 2003	3/19/03	6/18/03	91	5.385	2527.78	-101.11	1.0136	1.1168	-90.54	-0.29	-90.86	-90.76
JUN 2003	6/18/03	9/17/03	91	5.470	2527.78	-101.11	1.0138	1.1322	-89.30	-0.28	-89.62	-87.98
SEP 2003	9/17/03	12/17/03	91	5.540	2527.78	-101.11	1.0140	1.1481	-88.07	-0.27	-88.38	-86.79
DEC 2003	12/17/03	3/17/04	91	5.675	2527.78	-101.11	1.0143	1.1645	-86.83	-0.26	-87.14	-86.26
MAR 2004	3/17/04	6/16/04	91	5.670	2527.78	-101.11	1.0143	1.1812	-85.60	-0.24	-85.91	-85.33
JUN 2004	6/16/04	9/15/04	91	5.735	2527.78	-101.11	1.0145	1.1983	-84.38	-0.22	-84.68	-83.55
SEP 2004	9/15/04	12/15/04	91	5.795	2527.78	-101.11	1.0146	1.2159	-83.16	-0.19	-83.46	-82.00
DEC 2004	12/15/04	3/16/05	91	5.910	2527.78	-101.11	1.0149	1.2341	-81.93	-0.17	-82.24	-81.48
MAR 2005	3/16/05	6/15/05	91	5.895	2527.78	-101.11	1.0149	1.2525	-80.73	-0.14	-81.03	-81.12
JUN 2005	6/15/05	9/21/05	98	5.945	2722.22	-108.89	1.0162	1.2727	-85.56	-0.11	-85.87	-84.31
SEP 2005	9/21/05	12/21/05	91	5.995	2527.78	-101.11	1.0152	1.2920	-78.26	-0.07	-78.53	-78.95
DEC 2005	12/21/05	3/15/06	84	6.100	2333.33	-93.33	1.0142	1.3104	-71.23	-0.03	-71.47	-76.11

The swap hedged requires quarterly fixed and floating payments based on an actual/360 day-count. All implied LIBOR are derived from futures settlement prices on 3/19/2001. The three-month LIBOR is the rate fixed on the same date and has an associated value date of 3/21/2001. The swap curve used in connection with hedging quantities is derived from LIBOR and swap rates established on 3/19/2001 with an associated 3/21/2001 value date.

For this contract, the basis point value is computed as $100 million × 0.0001 × 91/360 = $2,527.78 [column (5)]. The value of a basis point change per futures contract is $25. Therefore, Kawaller's untailed hedging quantity is $2,527.78 /$25 = –101.11 contracts [column (6)]. Untailed hedging quantities for the remaining contracts are computed in a similar fashion.

These quantities must be "tailed" to account for timing differences between futures gains and losses and gains and losses on the swap. In general, the tailed quantity equals the untailed quantity divided by the future value of $1 through the interest flow date, which in the case of the June 2001 contract would be 9/19/2001 [column (2)]. The future value of $1 through 6/20/2001 is based on the three-month LIBOR and is computed as 1 + 4.880 × 91/36000 = 1.0123 [column (7)]. Similarly, based on the 4.535 LIBOR implied in the pricing of the June 2001 futures contract, the three-month forward interest factor between 6/20/2001 and 9/19/2001 is computed as 1 + 4.535 × 91/36000 = 1.0115 [column (7)].[16]

Therefore, the future value of $1 through 9/19/2001, labeled the cumulative interest factor in the Exhibit, is the product of these values or 1.0123 × 1.0115 = 1.0239 [column (8)]. Dividing the untailed hedging quantity of –101.11 by the cumulative interest factor of 1.0239 gives a tailed quantity of –98.75 contracts [column (9)]. This compares with –98.11 contracts, shown in column (12), using the hedging methodology I have developed. Similar calculations are made for all contracts through the December 2005 maturity.

Burghardt breaks the hedging quantities for each futures contract into two components. For any contract i, the first component is equal to $(\partial CF_i/\partial F_i)(PV_i/2500)$, where PV_1 is the present value of $1 received as of the payment of swap cash flow i, CF_i is the difference between the i-th fixed and floating swap payments, and F_i is the price of the i-th futures contract.[17] Assuming that the present value is computed from the futures curve without adjusting for convexity, this quantity is equivalent to Kawaller's quantity.

The second component is

$$\frac{1}{2500}\left(CF_i\frac{\partial PV_i}{\partial F_i} + CF_{i+1}\frac{\partial PV_{i+1}}{\partial F_i} + CF_{i+2}\frac{\partial PV_{i+2}}{\partial F_i} + \ldots +\right)$$

Hedging quantities resulting from the second component are shown in the Exhibit, and turn out to be quite small, on the order of –0.20.[18] The total quantity using Burghardt's model [column (11)] is the sum of the first and second components.

The final column shows hedging quantities using my methodology. These quantities tend to be slightly smaller than the quantities computed using Kawaller's and Burghardt's methods. The primary source of difference is that the quantities computed according to the new method reflect a convexity adjustment that is not reflected in the other two quantities. Although Burghardt provides extensive discussion of how convexity bias can affect swap values, he does not incorporate this bias into his hedging model and examples.

Another source of difference is that the new model is based on exact Eurodollar (LIBOR) value dates and maturity dates, while the values computed in the Kawaller and Burghardt models assume that the maturity of successive three-month LIBOR coincides with successive futures maturity dates.

ENDNOTES

The author thanks Don Chance and Ira Kawaller for helpful comments.

[1]Ho and Lee [1986] assume that the continuously compounded forward rate of interest follows a normal distribution with a constant variance.

[2]I use the term *deposit* loosely throughout. In fact, the Eurodollar futures rate is tied to the LIBOR that reflects the rate that the most creditworthy international banks charge on large Eurodollar-denominated loans. The rate that such banks pay on deposits is actually the closely related LIBID rate (see Burghardt [2003, p. 457]).

[3]There is no guarantee that the interpolation method values any financial instruments other than the 12 swaps without error, but when I apply the swap method curves generated by several significantly different parameterizations of the Longstaff-Schwartz [1992] model, I find essentially no difference between theoretically correct present value factors and present value factors determined by the interpolation model. The maximum absolute error in the present value of $1 to be received over a one-day to ten-year maturity range is 0.0007%.

[4]The mark-to-market futures value assumes a discount from 100 based on three-month LIBOR using 90 days as the three-month LIBOR period, regardless of the actual number of days in the period. This results in a discount of 90/360 = 1/4 times the three-month LIBOR.

[5]When the numerical derivatives are computed using smaller increments, there is almost no difference in the resulting hedging quantities.

[6]The last trading day is set so that the third Wednesday of the maturity month will be the value date associated with the three-month Eurodollar instrument used to determine the futures' final settlement price, and the Monday preceding the third Wednesday will be the fixing date.

[7]$36,000 = 360 \times 100$, where 360 is the denominator of the actual/360 day-count, and 100 converts the LIBOR from a percentage rate to a decimal fraction.

[8]The "not-a-knot" method is recommended when one is unsure about the first or second derivatives of the cubic interpolating polynomial at its end-points (see de Boor [2001]).

[9]The 30-year swap rate is the rate reported by Bloomberg as of 5:30 EST on 1/2/2001 for a swap with semiannual fixed payments and quarterly floating payments using a 30/360 day-count.

[10]Although the problem of non-synchronous data is mitigated for swaps by using the average of 1:00 and 5:30 swap rates, the LIBOR that define the front-end of the daily swaps curve are established at the 6:00AM EST LIBOR fixing, nine hours prior to the 3:00 EST futures close.

[11]Daily volume for Eurodollar futures contracts with maturities longer than five years tends to be approximately 10% of the daily volume of contracts of five years or shorter maturity.

[12]To understand how non-synchronous data can cause directional changes in hedged values, consider two identical assets, A and B, with a long position in A hedged by a short position in B. The values of both are $100 at time $t = 0$, $110 at time $t = 1$, $115 at time $t = 2$, and $100 at time $t = 3$. Assume that a hedge is established at time with both assets priced at $100. Also assume that times $t = 1$ and $t = 2$ represent two different times the following day, and the value of A is observed at time $t = 1$ and the value of B at time $t = 2$. In this situation, it will appear that the hedge has lost $5, since the long position in A will appear to have gained $110 – $100 = $10, while the short position in B will appear to have lost $115 – $100 = $15. Assume that on the day following, the values of both securities are observed at time $t = 3$. Then, the long position in A will appear to have lost $110 – $100 = $10 while the short position in B will appear to have gained $115 – $100 = $15, resulting in an apparent net gain of $5 for the day. Thus, even though the hedge is perfect, the non-synchronous price data on the second day causes the hedge to seem to lose value the first day and gain value the second.

[13]One possible source of non-synchronous data arises from the use of LIBOR in defining the front-end of the swaps curve. Since the LIBOR fixing occurs at 6:00AM EST, the LIBOR portion of the swap curve is nine hours behind the Eurodollar futures settlement prices established at 3:00PM. To test for potential problems in using LIBOR, the analysis is rerun using the average of 1:00PM and 5:30PM Eurodollar rates to define the front-end of the U.S. swap curve rather than LIBOR. All results are essentially identical to those reported earlier, including the first-order serial correlation of daily changes in the value of the hedged swap position.

[14]The tendency for volume and open interest to drop sharply for contracts maturing after five years is present throughout the data.

[15]A standard measure of hedging effectiveness is the R^2 resulting from a regression of changes in the value of the hedging instrument against changes in the value of the instrument hedged. Using all contracts, the adjusted R^2 of this regression is 0.808513 compared to 0.808518 using only contracts in the regular three-month maturity cycle. Clearly, in this particular example, there is no significant difference in the effectiveness of the two hedging methods.

[16]Although it is not shown in the Exhibit, the settlement price for the June 2001 contract was 95.465, resulting in an implied LIBOR of 100 − 95.465 = 4.535%.

[17]The cash flows as defined by CF_i are actual projected fixed and floating cash flows rather than the cash flows that would be projected using the hypothetical securities method. Burghardt reports a divisor of 25 rather than 2500. 25 would be the appropriate divisor if the change represented by the remainder of the expression were the result of a basis point change rather than a unit change, but a unit change in the futures price results in a \$2,500 change in value per contract, not \$25.

[18]Secondary hedging quantities will generally be small for a newly issued swap but can be much larger, in absolute magnitude, if the swap's fixed rate is substantially different from the par yield (newly issued rate) for the same maturity. Since Kawaller's hedging quantities are not a function of the swap's fixed rate, I interpret them as applying (approximately) to newly issued swaps only.

REFERENCES

Burghardt, Galen. *The Eurodollar Futures and Options Handbook*. New York: McGraw-Hill, 2003.

Cox, John, Jonathan Ingersoll, and Stephen Ross. "A Theory of the Term Structure of Interest Rates." *Econometrica*, 53 (March 1985), pp. 385-407.

de Boor, Carl. *A Practical Guide to Splines*. New York: Springer-Verlag, 2001.

Grinblatt, Mark, and Narahimhan Jegadeesh. "Relative Pricing of Eurodollar Futures and Forward Contracts." *Journal of Finance*, 51 (September 1996), pp. 1499-1522.

Gupta, Anurag, and Marti G. Subrahmanyam. "An Empirical Examination of the Convexity Bias in the Pricing of Interest Rate Swaps." *Journal of Financial Economics*, 55 (2000), pp. 239-279.

Ho, Thomas S.Y., and Sang-Bin Lee. "Term Structure Movements and Pricing Interest Rate Contingent Claims." *Journal of Finance*, 41 (December 1986), pp. 1011-1029.

Hull, John C. *Options, Futures and Other Derivatives*, 5th ed. Upper Saddle River, NJ: Prentice-Hall, 2003.

Hull, John C., and Alan White. "Pricing Interest Rate Derivative Securities." *Review of Financial Studies*, 3 (1990), pp. 573-592.

Kawaller, Ira G. "Comparing Eurodollar Strips to Interest Rate Swaps." *The Journal of Derivatives*, 2 (Fall 1994), pp. 67-79.

——. "Interest Rate Swaps versus Eurodollar Strips." *Financial Analysts Journal*, 45 (September/October 1989), pp. 55-61.

——. "A Primer on Eurodollar Strips." *Financial Manager's Statement*, 10 (March 1988), pp. 72-75.

Longstaff, Francis, and Eduardo Schwartz. "A Two-Factor Interest-Rate Model and Contingent Claims Valuation." *The Journal of Fixed Income*, 2 (December 1992), pp. 16-23.

Rendleman, Richard J., Jr. "Interpolating the Term Structure from Par Yield and Swap Curves." *The Journal of Fixed Income*, 13 (March 2004), pp. 80-89.

Vasicek, Oldrich. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, 5 (November 1977), pp. 177-188.

Young, Andrew R. "A Morgan Stanley Guide to Fixed Income Analysis." Morgan Stanley and Co., Inc., 1997.