A Closed–Form Multifactor Binomial Interest Rate Model

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We have made tremendous progress in the past 20 years in developing interest rate models that exhibit arbitrage-free movements. Many interest rate models have been proposed to value interest rate-contingent claims. These models are widely used in practice, such as in portfolio management, trading, and risk management.

A closed-form binomial model is one popular class of interest rate models. The Ho-Lee [1986] and Hull-White [1990] models are examples. These models are simple to implement and can be used for a broad range of applications. They provide a closed-form solution for the entire discount function at each node point of the binomial lattice. Such closed-form solutions enable us to value interest rate-contingent claims at any future time and state of the binomial lattice, and to value securities whose payoffs depend on the prevailing term structure of interest rates with significant computational efficiency.

So far the closed-form binomial models are all one-factor models. They assume that all interest rates are instantaneously perfectly correlated, allowing the yield curve only one degree of freedom of movement, moving up or down. Such an assumption is inconsistent with empirical observations, which have shown that most yield curve movements are explained by three principal movements: parallel, steepening, and curvature. Litterman and Scheinkman [1991] describe these movements.

The use of multifactor interest rate models is important because one-factor models may erroneously misstate the value of many interest rate-contingent claims. Consider a callable bond. Suppose that the time to expiration of the call provision is short relative to the underlying bond maturity. A one-factor model would overstate the correlation between the stochastic bond price at expiration of the option and the short-term interest rates used to determine the present value of the option payoff. As a result, the one-factor model would tend to discount the payment when the bond is called at a rate lower than is appropriate and overstate the option value.

Similarly, a constant-maturity swap pays a T-period bond rate in exchange for a short-term rate. An erroneous assumption of perfect correlation between the T-period bond rate and the short-term rate would affect the convexity value of the swap.

A spread option is an option on the spread between two interest rates. A one-factor model would significantly understate the spread option value.

Multifactor models are also important in valuing many non-marketable balance sheet items such as demand deposits, where the bank rate depends on the stochastic yield curve shape, and single-premium deferred annuities whose interest payments depend on the insurer's investment returns. Detailed descriptions of such balance sheet items can be found in Ho and Lee [2004].

A multifactor closed-form interest rate model can also improve calibration procedures for interest rate models. Specifically, the interest rate model can be calibrated to a broader set of benchmark securities beyond caps and floors, to also include swaptions of different tenors and option expirations. Using a broader set of benchmark securities for calibration, the interest rate model can better determine the relative value of any contingent claims.

More generally, when an interest rate model can be effectively calibrated to options with early exercise, any contingent claim can be valued relative to a broad range of possible hedging instruments. This is what our proposed model can achieve.

Our multifactor closed-form binomial interest rate model extends the Ho-Lee model [1986, 2004]. The yield curve movements are based on the yield curve movements implied by the benchmark securities market prices. The mean reversion process of the interest rates is deduced from the term structure of volatilities, which is calibrated from the benchmark securities. This approach differs from the Brennan and Schwartz model [1982] whose factors are based on short rates and long rates, where the short rate is assumed to mean-revert to the long rate.

We also describe the relationships between the term structure of volatilities and the yield curve movements. A procedure for calibrating the interest rate model to a set of 70 at-the-money swaptions has an average error of only 1.3%, and the results are stable over monthly observations of two years.

I. ARBITRAGE-FREE TERM STRUCTURE MOVEMENTS

The multifactor recombining binomial model specifies the discount function at each node of the binomial lattice so that these discount function movements are arbitrage-free. We begin with a two-factor model, but generalization to a multifactor model is straightforward.

To introduce some notation, there are $(n + 1)^2$ nodes, denoted by (i, j), $i, j = 0, 1, \ldots, n$, at time n. $P^n_{i,j}(T)$ is the price of a discount bond with time to maturity T in state (i, j) at time n. $P(T)$ is the initial discount function with maturity T that is observed in the market and is an input to the model. We assume that: 1) $P^n_{i,j}(0) = 1$ for all n and all i, j; and 2) $\lim_{T\to\infty} P^n_{i,j}(T) = 0$ for all n and all i, j.

The term structure movement is arbitrage-free if there is no arbitrage opportunity to be gained by holding a particular portfolio of discount bonds at each node point of the binomial lattice. This is the case if the expected return of holding a discount bond of any maturity T over any one period is the one-period return.

Specifically, according to Harrison and Kreps [1979], we require that:

$$P^n_{i,j}(T) = \frac{1}{4}P^n_{i,j}(1)\begin{Bmatrix} P^{n+1}_{i+1,j+1}(\mathrm{T}-1) + \\ P^{n+1}_{i+1,j}(\mathrm{T}-1) + \\ P^{n+1}_{i,j+1}(\mathrm{T}-1) + \\ P^{n+1}_{i,j}(\mathrm{T}-1) \end{Bmatrix} \tag{1}$$

Equation (1) should be satisfied for any $n = 0, 1, 2, \ldots$, $T = 1, 2, \ldots$, and $i, j, = 0, 1, \ldots, n$.

The volatility structure of the model is generated by defining

$$P^n_{i,j}(T) = P^n_{0,0}(T)\alpha_1(n, T)^i \alpha_2(n, T)^j$$

for all $n > 0$, $T > 0$, and all $i, j \geq 0$. The original Ho-Lee model assumes a one-factor model where factor α depends only on T.

By considering the nodes (i, j), $(i, j + 1)$, $(i + 1, j)$, $(i + 1, j + 1)$ at time n and the nodes (i, j), $(i, j + 1)$, $(i, j + 2)$, $(i + 1, j)$, $(i + 1, j + 1)$, $(i + 1, j + 2)$, $(i + 2, j)$, $(i + 2, j + 1)$, $(i + 2, j + 2)$ at time $(n + 1)$ in Equation (1), we deduce a consistency condition that guarantees that the tree is well defined as a recombining tree:

$$\frac{\alpha_1(n, T)}{\alpha_1(n, 1)} = \alpha_1(n + 1, T - 1)$$

$$\frac{\alpha_2(n, T)}{\alpha_2(n, 1)} = \alpha_2(n + 1, T - 1)$$

for all n, $T \geq 1$.

Note that the equations above can be solved with the input data $\delta^1_n := \alpha_1(n, 1)$, $\delta^2_n := \alpha_2(n, 1)$ for $n = 1, 2, \ldots$. The solution is given explicitly by

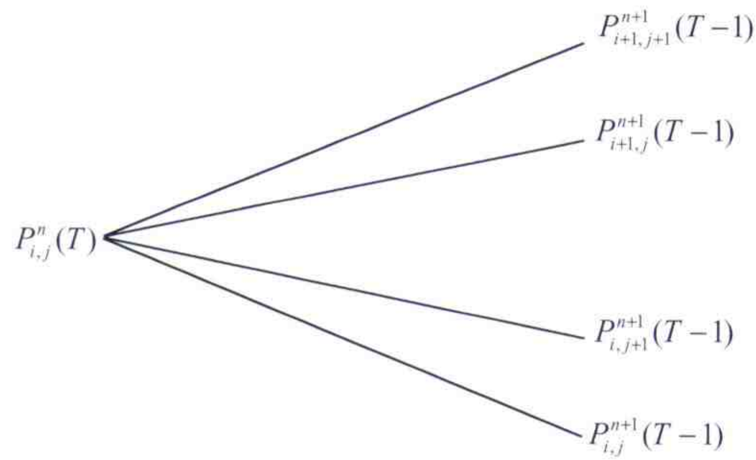
$$\alpha_1(n, T) = \frac{\alpha_1(1, T + n - 1)}{\alpha_1(1, n - 1)} = \delta^1_n \delta^1_{n+1} \ldots \delta^1_{T+n-1} := d^1_{T+n-1,n}$$

$$\alpha_2(n, T) = \frac{\alpha_2(1, T + n - 1)}{\alpha_2(1, n - 1)} = \delta^2_n \delta^2_{n+1} \ldots \delta^2_{T+n-1} := d^2_{T+n-1,n}$$

where we define for $n > m$:

Exhibit 1

Basic Building Blocks of Two-Factor Binomial Tree



$$d^{l}_{n,m} = \delta^{l}_{n}\delta^{1}_{n-1} \ldots \delta^{1}_{m+1}\delta^{1}_{m}$$

and

$$d^{2}_{n,m} = \delta^{2}_{n}\delta^{2}_{n-1} \ldots \delta^{2}_{m+1}\delta^{2}_{m}$$

Exhibit 1 shows the two-factor binomial tree. Each move has the same risk-neutral probability, which is 0.25.

A proposition provides the closed-form solution of any zero-coupon bond at each node point of the arbitrage-free binomial interest rate lattice:

Proposition 1. Closed-Form Solution for Bond Prices

Let $P(T)$ be the initial discount function. Then the arbitrage-free movement of the discount function is given by:

$$P^{n}_{i,j}(T) = 4\frac{P(T+n)}{P(n)}\left(\prod_{k=1}^{n}\frac{1+d^{1}_{n-1,k}}{1+d^{1}_{T+n-1,k}}\right) \times \left(\prod_{k=1}^{n}\frac{1+d^{2}_{n-1,k}}{1+d^{2}_{T+n-1,k}}\right)\left(d^{1}_{T+n-1,n}\right)^{i}\left(d^{2}_{T+n-1,n}\right)^{j} \qquad (2)$$

where we set $d^{1}_{n-1,n} = d^{2}_{n-1,n} = 0$.

Alternatively, Equation (2) can be expressed as

$$P^{n}_{i,j}(T) = \frac{P(T+n)}{P(n)} \times \left[\prod_{k=1}^{n}\left(G^{1}_{Tnk}\right)^{-1}\right]\left(d^{1}_{T+n-1,n}\right)^{i-0.5n} \times \left[\prod_{l=1}^{n}\left(G^{2}_{Tnl}\right)^{-1}\right]\left(d^{2}_{T+n-1,n}\right)^{j-0.5n} \qquad (2')$$

where

$$G^{1}_{Tnk} = \left[\left(d^{1}_{T+n-1,n}\right)^{-0.5} + \left(d^{1}_{T+n-1,n}\right)^{0.5} \times d^{1}_{n-1,k}\right] \Big/ \left(1 + d^{1}_{n-1,k}\right)$$
$$G^{2}_{Tnl} = \left[\left(d^{2}_{T+n-1,n}\right)^{-0.5} + \left(d^{2}_{T+n-1,n}\right)^{0.5} \times d^{2}_{n-1,l}\right] \Big/ \left(1 + d^{2}_{n-1,l}\right)$$

$P^{n}_{i,j}(T)$ is the discount bond with \$1 principal and T time to maturity, at a future time n, in state (i, j). The term structure of volatilities is specified by δ^{1}_{k} and δ^{2}_{k} for $k = 1, \ldots, (n+T)$.

Proof:
By construction, we have:

$$P^{n}_{i,j}(T) = P^{n}_{0,0}(T)\alpha_1(n,T)^{i}\alpha_2(n,T)^{j} = P^{n}_{0,0}(T)(d^{1}_{T+n-1,n})^{i}(d^{2}_{T+n-1,n})^{j}$$

So it suffices to check that

$$\mathrm{P}^{n}_{0,0}(\mathrm{T}) = 4\frac{P(T+n)}{P(n)}\left(\prod_{k=1}^{n}\frac{1+d^{1}_{n-1,k}}{1+d^{1}_{T+n-1,k}}\right)\left(\prod_{k=1}^{n}\frac{1+d^{2}_{n-1,k}}{1+d^{2}_{T+n-1,k}}\right) \quad (2'')$$

We use mathematical induction. For $n = 1$ in Equation (2´´), we have from Equation (1) with $n = i = j = 0$ that:

$$\frac{P(T+1)}{P(1)} = \frac{1}{4}P^{1}_{0,0}(T)\left[\begin{matrix}1+\alpha_1(1,T)+\alpha_2(1,T)+ \\ \alpha_1(1,T)\alpha_2(1,T)\end{matrix}\right]$$

$$= \frac{1}{4}P^{1}_{0,0}(T)\left[1+d^{1}_{T,1}\right]\left[1+d^{2}_{T,1}\right]$$

Hence we get:

$$P^{1}_{0,0}(T) = 4\frac{P(T+1)}{P(1)}\frac{1}{\left[1+d^{1}_{T,1}\right]\left[1+d^{2}_{T,1}\right]}$$

Therefore the formula for $n = 1$ is valid. Now we suppose that the formula is satisfied for n, and we will check the formula for $n + 1$. From Equation (1), we have:

$$P^{n}_{i,j}(\mathrm{T}+1)/\mathrm{P}^{n}_{i,j}(1) = \frac{1}{4}\left\{\begin{matrix}\mathrm{P}^{n+1}_{i+1,j+1}(\mathrm{T})+\mathrm{P}^{n+1}_{i+1,j}(\mathrm{T})+ \\ \mathrm{P}^{n+1}_{i,j+1}(\mathrm{T})+\mathrm{P}^{n+1}_{i,j}(\mathrm{T})\end{matrix}\right\}$$

$$= \frac{1}{4}P^{n+1}_{0,0}(T)\left(d^{1}_{T+n,n+1}\right)^{i}\left(d^{2}_{T+n,n+1}\right)^{j} \times \left(1+d^{1}_{T+n,n+1}\right)\left(1+d^{2}_{T+n,n+1}\right)$$

By induction, we get:

$$P^{n}_{i,j}(T+1)/P^{n}_{i,j}(1) = \frac{P(T+n+1)}{P(n+1)}\left(\prod_{k=1}^{n}\frac{1+d^{1}_{n,k}}{1+d^{1}_{T+n,k}}\right)\times \left(\prod_{k=1}^{n}\frac{1+d^{2}_{n,k}}{1+d^{2}_{T+n,k}}\right)\left(d^{1}_{T+n,n+1}\right)^{i}\left(d^{2}_{T+n,n+1}\right)^{j}$$

where we have used the fact $d_{T+n,n}/d_{n,n} = d_{T+n,n+1}$. Substituting this into the last equation, we get:

$$P^{n+1}_{0,0}(T) = 4\frac{P(T+n+1)}{P(n+1)}\left(\prod_{k=1}^{n}\frac{1+d^{1}_{n,k}}{1+d^{1}_{T+n,k}}\right)\times \left(\prod_{k=1}^{n}\frac{1+d^{2}_{n,k}}{1+d^{2}_{T+n,k}}\right)\frac{1}{\left(1+d^{1}_{T+n,n+1}\right)\left(1+d^{2}_{T+n,n+1}\right)}$$

$$= 4\frac{P(\mathrm{T}+n+1)}{P(n+1)}\left(\prod_{k=1}^{n+1}\frac{1+d^{1}_{n,k}}{1+d^{1}_{T+n,k}}\right)\left(\prod_{k=1}^{n+1}\frac{1+d^{2}_{n,k}}{1+d^{2}_{T+n,k}}\right)$$

because $d_{n,\,n+1} = 0$ (by definition). This is the required formula for $n + 1$, so the proof is complete.

Equation (2) provides the specification of the discount function at each node point of a two-factor binomial model. Extension of the model to any number of factors is straightforward. For an m-factor model, the term structure of volatilities would be specified by m sets of movements $(\delta^{1}_{k}, \delta^{2}_{k}, \ldots, \delta^{m}_{k})$.

$$P^{n}_{i_1,i_2,\ldots,i_m}(T) = 2^{m}\frac{P(T+n)}{P(n)}\left(\prod_{k=1}^{n}\frac{1+d^{1}_{n-1,k}}{1+d^{1}_{T+n-1,k}}\right)\left(\prod_{k=1}^{n}\frac{1+d^{2}_{n-1,k}}{1+d^{2}_{T+n-1,k}}\right)\ldots \left(\prod_{k=1}^{n}\frac{1+d^{m}_{n-1,k}}{1+d^{m}_{T+n-1,k}}\right)\times\left(d^{1}_{T+n-1,n}\right)^{i_1}\left(d^{2}_{T+n-1,n}\right)^{i_2}\ldots\left(d^{m}_{T+n-1,n}\right)^{i_m}$$

It is straightforward to show that this interest rate model is a direct extension of the Ho-Lee model where the instantaneous volatility of the interest rate movements is constant. Specifically, when δ^{1}_{i} is constant for all i, and δ^{2}_{j} is equal to 1 for all j:

$$d^{1}_{n,m} = \left(\delta^{1}\right)^{n-m+1}$$

and

$$d^{2}_{n,m} = 1 \quad (3)$$

If we substitute Equation (3) into Equation (2), we have:

$$P^{n}_{i,j}(T) = 2\frac{P(T+n)}{P(n)}\frac{\left[\begin{matrix}(1+(\delta^{1})^{n-1})\times \\ (1+(\delta^{1})^{n-2})\cdots(1+\delta^{1})\end{matrix}\right](\delta^{1})^{iT}}{(1+(\delta^{1})^{T+n-1})\cdots(1+\delta^{1})^{T}} \quad (4)$$

Equation (4) is the Ho-Lee model.

II. ANALYSIS OF TERM STRUCTURE OF VOLATILITIES

We will use the model to analyze interest rate movements in relation to the term structure of volatilities. Specifically, we first express interest rates in terms of the bond prices. We then study how the yield curve would move, given the term structure of volatilities.

At the initial date, we have the discounted function $P(T)$. We define the yield of a discount bond with maturity T initially as:

$$r(T) = -\frac{\ln P(T)}{T}$$

More generally, we can define the yield curve at each node point to be:

$$\begin{aligned} r_{i,j}^{n}(T) &= -\ln P_{i,j}^{n}(T)/T \\ &= -\frac{1}{T}\ln\frac{P(T+n)}{P(n)} + \frac{\ln\prod_{k=1}^{n} G_{Tnk}^{1}}{T} + \frac{\ln\prod_{l=1}^{n} G_{Tnl}^{2}}{T} \\ &\quad -\frac{\ln d_{T+n-1,n}^{1}}{T}(i - 0.5n) - \frac{\ln d_{T+n-1,n}^{2}}{T}(j - 0.5n) \end{aligned} \tag{5}$$

The spot volatility of term T is the variance of the shifts of the yields for a two-factor binomial model.*

Equation (5) provides a clear depiction of the yield curve movements. It shows that the T-period interest rate at time n and state (i, j) is specified by three terms. The first term

$$-\frac{1}{T}\ln\frac{P(T+n)}{P(n)}$$

is the T-period forward rate at the horizon date n. The second term

$$\frac{\ln\prod_{k=1}^{n} G_{Tnk}^{1}}{T} + \frac{\ln\prod_{l=1}^{n} G_{Tnl}^{2}}{T},$$

called the convexity drift, is discussed later. The third term

$$-\frac{\ln d_{T+n-1,n}^{1}}{T}(i - 0.5n) - \frac{\ln d_{T+n-1,n}^{2}}{T}(j - 0.5n)$$

describes the movements of the yield curve.

The third term shows that state (i, j) represents the yield curve taking i and j steps with step sizes

$$-\frac{\ln d_{T+n-1,n}^{1}}{T}$$

and

$$-\frac{\ln d_{T+n-1,n}^{2}}{T}$$

Therefore, the standard deviation of each of these two binomial movements is half that step size. It follows that the variance of the yield curve movement over each step is the sum of the variances of the two independent binomial movements.

Specifically, we have the forward volatility of term T given by:

$$\sigma^{nf}(T) = \frac{1}{2T}\left[\left(\ln d_{T+n-1,n}^{1}\right)^{2} + \left(\ln d_{T+n-1,n}^{2}\right)^{2}\right]^{0.5}$$

Now we note that the spot volatility of term T is the forward volatility where $n = 1$, and therefore the spot volatility is given by:

$$\sigma(T) = \frac{1}{2T}\left[\left(\ln d_{T,1}^{1}\right)^{2} + \left(\ln d_{T,1}^{2}\right)^{2}\right]^{0.5}$$

Note that since the movement is binomial for each factor, when we take the expectation over the possible states of the world based on risk-neutral probabilities, the third term is zero.

For the convexity drift taking expectations across both sides of Equation (5), we can conclude that the expected yield of a T-period bond over an n-period horizon is the forward rate (for the T periods starting from time n viewed from today) with a drift. The drift D is given by:

$$\begin{aligned} D = \left(\frac{1}{T}\right)\ln\left(\prod_{k=1}^{n} G_{Tnk}^{1}\right) + \\ \left(\frac{1}{\mathrm{T}}\right)\ln\left(\prod_{l=1}^{n} G_{Tnl}^{2}\right) \end{aligned} \tag{6}$$

where

$$G^1_{Tnk} = \left[\left(d^1_{T+n-1,n}\right)^{-0.5} + \left(d^1_{T+n-1,n}\right)^{0.5} \times d^1_{n-1,k}\right] \Big/ (1 + d^1_{n-1,k})$$
$$G^2_{Tnl} = \left[\left(d^2_{T+n-1,n}\right)^{-0.5} + \left(\mathrm{d}^2_{T+n-1,n}\right)^{0.5} \times d^2_{n-1,l}\right] \Big/ (1 + d^2_{n-1,l}) \quad (7)$$

But note that this drift is always positive and increases with volatilities. That is, the expected forward rate of any time period of delivery T and future date n is higher than the forward rate implied by the initial yield curve. And this spread increases with volatilities.

This result is shown by noting $0 < D$, by showing that G^1_{Tnk} and G^2_{Tnl} are greater than 1 for all k and l. Note that $(d^1_{T+n-1,k})^{0.5} < 1$, $0 < d^1_{n-1,k}$, $(d^2_{T+n-1,l})^{0.5} < 1$, and $0 < d^2_{n-1,l}$. Therefore, it follows that:

$$\left[1 - d^1_{n-1,k}\left(\mathrm{d}^1_{T+n-1,k}\right)^{0.5}\right]\left[1 - \left(\mathrm{d}^1_{T+n-1,k}\right)^{0.5}\right] > 0$$
$$\left[1 - d^2_{n-1,l}\left(\mathrm{d}^2_{T+n-1,l}\right)^{0.5}\right]\left[1 - \left(\mathrm{d}^2_{T+n-1,l}\right)^{0.5}\right] > 0 \quad (8)$$

Expanding Equation (8) and simplifying, we get:

$$1 + d^1_{n-1,k}\left(d^1_{T+n-1,k}\right) > \left(d^1_{T+n-1,k}\right)^{0.5}\left(1 + d_{n-1,k}\right)$$
$$1 - d^2_{n-1,k}\left(d^2_{T+n-1,l}\right) > \left(d^2_{T+n-1,l}\right)^{0.5}\left(1 + d^2_{n-1,l}\right)$$

Then it follows that $1 < G^1_{Tnk}$ and $1 < G^2_{Tnl}$.

The results show that expected yield must rise above the forward rate because the expected returns of the bond increase with the volatility of interest rates as a result of the bond convexity. To ensure that the expected returns of the bond are the same as the one-period interest rate for any interest rate volatility, the convexity drift is needed to balance this convexity effect on the bond returns.

III. CALIBRATING MODEL TO PRICES OF BENCHMARK SECURITIES

To implement the n-factor interest rate model, we calibrate it to a set of benchmark securities. We present empirical evidence to support that the two-factor model can fit the market data better than some of the standard one-factor models.

Calibrating the model entails several steps.

1. *Benchmark securities.* The main advantage of using an arbitrage-free interest rate model is to be able to fit the model to the observed term structure of interest rates and a set of interest rate options, called benchmark securities. This calibration approach enables us to value an interest rate-contingent claim relative to the spot curve and the benchmark securities. The spot curve represents the market-determined time value of money. The benchmark securities represent the market pricing of interest rate volatilities. Therefore, if we can calibrate the model to the market time value of money and the market price of interest rate risk, we have a consistent valuation framework.

   Since the multifactor model allows for flexible movements of the yield curve, and the recombining binomial lattice allows for efficient computation, we can use a portfolio of caplets, floorlets, and both American and European swaptions as the benchmark securities.
2. *Specification of the term structure of interest rates.* Any arbitrage-free interest rate model takes the observed spot yield curve as given. To calibrate the model, one methodology is to use the spot swap curve as the input to the model.
3. *Pricing of caplets, floorlets, and European swaptions.* A caplet is equivalent to a put option written on a zero-coupon bond, and a swaption is equivalent to an option (call or put) on a coupon bond. Hence we have a series of zero bond put option prices or coupon bond option prices. We use a closed-form formula for bond option prices, which are completely determined by the initial discount function (which is a model input) and model volatility factors, which are δ^1_n *and* δ^2_n, where $n = 1, 2, 3, \ldots, N$. To determine the volatility factors of the two-factor interest rate model, we use market-observed benchmark securities prices. The two-factor interest rate model assumes that at each instant the yield curve makes two independent movements. Following empirical results on the yield curve movements, we can assume that the movements can be specified by $\ln\delta^1(n) = -2f(n)\sigma^1(n)$ and $\ln\delta^2(n) = -2f(n)\sigma^2(n)$.
4. *Functional form of the term structure of volatilities.* The specific functional form is assumed to be $\sigma(t) = (a + bt)\exp(-ct) + d$ when t is a time-to-maturity. We choose this function to represent the volatility curve because it decays exponentially and, when t is relatively small, the curve may exhibit a hump, depending on the parameters a and b. This configuration of the volatility curve is observed in the market, $\sigma^1(n) = (a_1 + b_1 n)\exp(-c_1 n) + d_1$, and $\sigma^2(n)$

EXHIBIT 2
Market Swaption Volatility Surface

Option Term (years)	Swap Tenor (years) 1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr
1yr	44.40	36.50	32.90	29.70	27.60	25.70	25.40	24.00	23.40	23.30
2yr	31.20	28.80	27.10	25.30	24.20	22.90	23.00	21.80	21.30	21.10
3yr	27.00	25.30	24.30	23.10	22.20	21.30	21.10	20.40	20.00	19.90
4yr	24.00	22.80	21.90	21.20	20.70	19.90	19.70	19.10	18.70	18.60
5yr	22.30	21.30	20.80	19.90	19.40	18.70	18.50	18.00	17.70	17.40
7yr	19.80	19.10	18.50	18.10	17.60	16.90	16.70	16.30	16.00	15.80
10yr	17.40	16.40	15.90	15.60	15.10	14.50	14.50	14.10	13.90	13.70

$=(a_2 + b_2 n)\exp(-c_2 n) + d_2$. The constants a_j, b_j, c_j, d_j, for $j = 1, 2$, are estimated from the benchmark securities market prices.

5. *Calibration*. Caplet, floorlet, and swaption observed market prices give us a set of equations to be satisfied by fitting the model volatility factors δ_n^1, δ_n^2 to the closed-form pricing models of these benchmark securities. Such volatility factors are determined by a non-linear search procedure in determining the parameters a_j, b_j, c_j, d_j $j = 1, 2$, such that the sum of squares of the errors between the observed and model prices of the benchmark securities is minimized.

   The objective function is:

$$\sum_{i=1}^{N}\left(\frac{MarketPrice_i - ModelPrice_i}{MarketPrice_i}\right)^2$$

where N is the number of benchmark securities.

The term structure of interest rates is the U.S. dollar zero swap curve on July 31, 2002. The market swaption volatility surface is given by the Black volatilities quoted in Exhibit 2 (in %). The calibrated yield curve movements are given by the estimated volatility parameters:

$$\sigma_1: a_1 = 0.772079, b_1 = -0.168097, c_1 = 0.498474, d_1 = 0.113347$$
$$\sigma_2: a_2 = 0.023701, b_2 = -0.014509, c_2 = 0.003260, d_2 = 0.245125$$

The first term $\sigma_1(n)$ is a curve that falls from 0.8 to 0.1 and remains level. The second term $\sigma_2(n)$ falls steadily and reaches zero in year 10. The results are consistent with observations that the term structure of volatilities declines with the term.

The valuation errors derived from the estimated swaption prices are given in Exhibit 3.

Results show that the model fits the observed data quite well. Previous work using the Brace, Gatarek and Musiela [1997] model calibrates only to the counter-diagonals of the swap volatility surface, confined only to swaptions where the option term plus the swap tenor is 31 years. When the two-factor model is confined to one factor, the average error exceeds 3%.

Results for tests of the model over 24 months are similar in all periods as shown in Exhibit 4. The average errors are less than approximately 1.5%, reasonably low, especially when the market was quite volatile in this time period. Over the period, the five-year swap rate reached a high of 5.6% and a low of 2.8%; the one-year Black volatility reached a high of 60.8% and a low of 26.1%. Average errors remained quite stable, despite the large changes in both the level and the volatility of rates.

IV. CHARACTERISTICS AND APPLICATIONS OF THE MODEL

The interest rate model has a number of useful characteristics for practical applications. First, it provides us with a closed-form solution for term structures of interest rates at each node. Therefore, we can determine the interest rate yield curve at each node point. The model is thus computationally efficient and minimizes errors in approximation. Compare this to the Black, Derman, and Toy model [1990], which requires numerical construction of the spot rate for each node point on the lattice. In this case, the yield curve at each node point has to be calculated numerically from the binomial lattice of the spot rates, requiring significant computation with the possibility of significant numerical estimation errors.

EXHIBIT 3

Swaption Matrix: Two-Factor Model Errors (in %)

Option Term (years)	Swap Tenor (years) 1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr
1yr	0.55	-1.67	1.35	0.35	-0.11	-1.60	1.97	0.46	1.62	4.47
2yr	-2.91	-1.05	0.29	-0.83	-0.55	-2.01	2.08	0.09	0.84	2.68
3yr	1.00	0.45	1.13	0.11	-0.30	-1.14	0.98	0.46	1.18	3.19
4yr	0.36	-0.77	-1.23	-1.21	-0.55	-1.59	0.18	-0.23	0.13	1.95
5yr	0.61	-0.72	-0.01	-1.54	-1.27	-2.27	-0.70	-0.95	-0.27	0.27
7yr	0.54	-0.35	-0.77	-0.29	-0.47	-2.03	-0.78	-0.92	-0.63	0.12
10yr	3.63	0.61	0.19	0.81	0.00	-1.77	0.41	-0.40	0.00	0.24

EXHIBIT 4

Average Errors over 24 Months (in %)

Date	Average Error	Date	Average Error
1/30/2004	1.76	12/31/2003	1.58
11/28/2003	1.51	10/30/2003	1.75
9/30/2003	1.57	8/29/2003	1.75
7/30/2003	1.22	6/30/2003	1.46
5/30/2003	1.55	4/30/2003	1.73
3/31/2003	1.23	2/28/2003	1.61
1/30/2003	1.61	12/30/2002	1.70
11/29/2002	1.56	10/30/2002	1.69
9/30/2002	1.37	8/30/2002	1.63
7/30/2002	1.12	6/28/2002	1.55
5/30/2002	1.40	4/30/2002	1.49
3/29/2002	1.50	2/28/2002	1.51

Second, the two-factor binomial model takes the initial spot yield curve and initial term structure of volatilities as input data. It does not specify the precise nature of the mean reversion process of the interest rate, but the mean reversion process is instead indirectly specified via the term structure of volatilities. As a result, the model is more flexible in calibration to the observed benchmark securities prices. The Vasicek [1977], Cox, Ingersoll, and Ross [1985], and Hull and White [1990] models, by way of contrast, assume that short-term interest rates follow a certain process. These models may not fit the observed benchmark securities prices satisfactorily.

Third, the two-factor binomial model is a discrete-time and combining model and not a continuous-time and non-combining model. Das and Sundaram [2000] and Grant and Vora [1999] have pointed out that discrete-time models provide computational efficiency for practical implementation. Since we have to convert continuous-time models into discrete-time versions when we implement them for empirical testing or practical purposes, discrete-time models can be used directly. Since the model is recombining, the model is more effective at pricing options (European and American) without using Monte Carlo simulations.

The model can be used to calculate the interest rate sensitivities of a large portfolio. Brace, Gatarek, and Musiela [1997] use a string model that provides flexibility in modeling the yield curve movements, but the valuation of securities requires a Monte Carlo simulation method that often lacks the pricing accuracy of the recombining binomial model.

Fourth, since the model can be calibrated to a large sample of benchmark securities of a broad range of interest rate options, it can provide relative valuation of any contingent claims relative to its hedging instruments or an appropriate spectrum of liquid securities. Therefore, the model can provide more accurate pricing of securities and ratios required for hedging strategies.

Finally, the model is in closed form without any numerical procedures needed to estimate security prices. It is similar to a Black model, where the closed-form solution provides a direct mapping from prices to volatilities. Therefore, it can be used effectively as a standard model in interest rate-contingent claims management. The model's two-factor movements can be used as inputs to value securities that are affected by multifactor movements as Black volatilities are used for the one-factor models.

V. CONCLUSIONS

The multifactor binomial interest rate model we propose is a discrete-time recombining lattice model that provides a closed-form solution for the discount function at each node point. Interest rate movements have the Markovian property that exhibits a mean reversion derived by the term structure of volatilities. The multifactor property enables the model to fit both the current yield curve and the volatility surface at the same time.

Such a model has a broad range of applications in valuing interest rate-contingent claims. It can be calibrated to a large set of benchmark securities of different security types, so it offers a robust methodology to value interest rate-contingent claims.

ENDNOTES

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[*]The variance of a two-factor binomial model has six terms$({}_4C_2)$, each of which represents the spread between the two-factor binomial outcomes. If we replace r^1_{11}, r^1_{10}, by r^1_1, and r^1_{01}, r^1_{00}, by r^1_0, respectively, we will have $(r^1_1(T) - r^1_0(T))^2/4$, which is the spot volatility of a one-factor binomial model.

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