Interpolating the Term Structure from Par Yield and Swap Curves

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Valuing fixed-income securities depends on accurate estimation of the term structure of interest rates, especially when the cash flows of the securities to be valued do not fall on the same dates as the cash flows of the instruments from which the term structure is estimated. Jordan and Mansi [2003] look at interpolation methods for estimating the term structure from U.S. Treasury par yields. Although there are many applications of par yield-based term structure estimation, perhaps the most significant are the valuation of swaps and the calibration of arbitrage-free stochastic term structure models of the Heath, Jarrow, and Morton [1992] class.

Unlike the U.S. Treasury market, where it is highly unlikely one would ever observe a series of bonds that sell exactly at par, pricing services such as Bloomberg provide almost continuous updating of swap curves, which by definition are par yields. For this reason, much of my focus is on estimating the Eurodollar term structure from the swap curve.[1]

If U.S. Treasury bonds and notes paid interest annually, and one could observe the par yields of Treasury securities with maturities of 1, 2, …, n years, one could determine algebraically the present value of $1 to be received as of the end of each of the next n years (and associated zero-coupon yields) from the observed structure of par yields. Treasury bonds and notes pay interest semiannually rather than annually, however. Therefore, the price of a one-year bond is a function of two cash payments; the price of a two-year bond is a function of four cash payments, and so on; the price of an n-year bond is a function of $2n$ cash payments.

There is not enough information in the pricing structure of a series of n Treasury bonds or notes with successive maturities of one to n years selling at par to extract present value factors in either one-year or six-month time increments. A similar problem arises for swaps, since published swap quotes tend to be for swaps with semiannual payments.

If one's purpose in extracting the term structure from a par yield or swap curve is to estimate present value factors in time increments of one year, six months, or less, some type of interpolation method must be employed. As Jordan and Mansi describe, it is standard practice to apply interpolation directly to the par yield curve to infer what par yields would be at the halfway points between annual maturity intervals. Interpolation, if applied to the yields of a series of one-year, two-year, …, n-year Treasury bonds or notes selling at par, would provide estimates of the 0.5-year par yield, the 1.5-year par yield, and so on through the n-0.5-year par yield, and thus would contribute five additional pricing relationships needed to estimate the $2n$ zero-coupon present value factors in successive six-month time increments. Obviously, the solution values for the present value factors so derived will depend upon the interpolation method that is used.[2]

Jordan and Mansi demonstrate the potential accuracy of five interpolation methods applied directly to the par yield curve to obtain interpolated estimates of par yields at the halfway points between successive observed yearly maturities:

- Linear discrete-time interpolation.
- Cubic spline-based discrete-time interpolation.
- The Nelson and Siegel [1987] three-parameter exponential continuous-time model, which provides for level, slope, and curvature in the yield curve.
- The Diament [1993] model, which provides for a three-parameter empirically based yield estimate and two additional parameters to generate humped yield curves.
- The four-parameter model of Mansi and Phillips [2001].

When present value factors computed by these methods are applied to the cash flows of the bonds from which the factors are estimated, the resulting present values do not equal the theoretically correct value of par ($100) except in a flat yield curve environment. Although the present values tend to be close to par, there is inherent pricing error in all five methods.

The interpolation method I develop produces present value factors that, when applied to the cash flows of the bonds or swaps from which the present value factors are derived, give computed security values exactly equal to the original set of security prices. Although there is no guarantee that any present value factor so calculated is individually correct, I can demonstrate that when theoretical term structures are generated by the Longstaff and Schwartz [1992] model, and par yield curves are generated from these term structures, the interpolated term structures and associated present value factors derived from the par yields are almost identical to those generated by the Longstaff and Schwartz model.

I. INTERPOLATION MODEL

Consider a set of m Treasury bonds or notes (or swaps) indexed by i with the maturity in days of security i denoted as t_i, with $t_i < t_{i+1}$ and $t_m = T$. Let $p_i(t_i, c_i, g[T])$ denote a function for pricing security i in terms of its maturity date, t_i; c_i dollars be its annual coupon per $100 par paid in semiannual installments of $c_i/2$ dollars; and $g(T)$ be a continuous present value discounting function evaluated between now (time zero) and T days from now.

Then, the prices of the m securities can be expressed as:

$$
\begin{aligned}
p_1(t_1, c_1, g[T]) &= P_1 \\
p_2(t_2, c_2, g[T]) &= P_2 \\
&\,\,\vdots \\
p_m(t_m, c_m, g[T]) &= P_m
\end{aligned}
\tag{1}
$$

where P_i is the observed market price of security i. When the model is applied to estimating the term structure from the Treasury par yield curve, $P_i \approx \$100$, although there is no requirement in Equations (1) that any of the observed prices equal par.[3]

I assume the discount function, $g(T)$, can be approximated by a cubic spline. Cubic splines are commonly used for fitting a smoothed function to a set of observed $\{x, y\}$ coordinates called *knots*. Let $k_j = \{x_j, y_j\}$ denote the knot represented by coordinate pair $\{x_j, y_j\}$; $s(k_1, k_2, \ldots, k_m)$ denote the cubic spline function estimated with knots $k_1, k_2, \ldots, k_m$; and $s(x; k_1, k_2, \ldots, k_m)$ denote the cubic spline function evaluated at x.

An appealing property of the cubic spline estimation procedure, and the key to the interpolation method I propose is that $s(x_j; k_1, k_2, \ldots, k_m) = y_j$, meaning that the function will return the original value of y_j associated with cubic spline knot $k_j = \{x_j, y_j\}$ when evaluated at any x_j value from the set of knots from which the spline is fit.

In the model, a cubic spline is used to estimate the cumulative continuously compounded discount rate as a function of time.[4] The *not-a-knot* method is employed, which is the recommended method when one is unsure about the first or second derivatives of the cubic interpolating polynomial at its end-points (de Boor [2001]).

Note that m knots are used to estimate the cubic spline function, where m is the number of pricing equations from Equations (1).[5] Although the procedure requires that $x_m = t_m$, that is, the time in days associated with the last knot equals the maturity of the longest-maturity bond from Equations (1), there is no restriction on the other values of x_j other than $x_j > 0$ for $j < m$. With these mild restrictions, the cubic spline is used to interpolate the term structure within the maturity range of the original set of bonds rather than to extrapolate the term structure outside the original maturity range.

EXHIBIT 1

Financial Instruments Used to Value Swaps from U.S. Swap Curve Quotes and LIBOR as of 10/3/2001

Instrument	Description	Rate (%)	Days to Maturity
1.	1-month LIBOR	2.589	31
2.	3-month LIBOR	2.500	94
3.	6-month LIBOR	2.415	182
4.	1-year LIBOR	2.533	367
5.	2-year swap	3.256	731
6.	3-year swap	3.815	1,096
7.	4-year swap	4.197	1,461
8.	5-year swap	4.471	1,826
9.	6-year swap	4.671	2,191
10.	7-year swap	4.833	2,558
11.	8-year swap	4.958	2,922
12.	9-year swap	5.057	3,287
13.	10-year swap	5.145	3,652
14.	15-year swap	5.518	5,479
15.	20-year swap	5.700	7,305
16.	30-year swap	5.785	10,958

Substituting $g(T) = s(k_1, k_2, \ldots, k_m)$ into each pricing relationship in Equations (1), the pricing system can be reexpressed as follows:

$$p_1\left(c_1,\ t_1,\ s\left[k_1,\ k_2,\ \ldots\ k_m\right]\right) = P_1$$
$$p_2\left(c_2,\ t_2,\ s\left[k_1,\ k_2,\ \ldots\ k_m\right]\right) = P_2$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$p_m\left(c_m,\ t_m,\ s\left[k_1,\ k_2,\ \ldots\ k_m\right]\right) = P_m \qquad (2)$$

In this form, there are m non-linear pricing equations expressed as a function of the m spline knots, k_1, k_2, …, k_m. Using a multidimensional Newton-Raphson search as formulated in Press et al. [1992, pp. 372-375], I fix a set of x values associated with m spline coordinates, and solve this system for the y values of the m spline coordinates that cause Equations (2) to hold for all m securities. Once the solution values for the spline knots have been determined, one can use the solution values to estimate the cumulative continuously compounded discount rate and associated present value factor for any time between time zero and time T.

Although Equations (2) do not impose any restriction on the x values of the knots, for the purposes of implementing the model I assume that $x_i = t_i$ for all i. Therefore, the number of days into the future associated with each successive spline knot corresponds to the maturity of each successive bond or swap in Equations (2).

II. NUMERICAL EXAMPLE

A numerical example is based on the U.S. swap curve as reported by the Bloomberg system at 5:30 PM, October 3, 2001. The first four instruments of the swap curve, shown in Exhibit 1, are Eurodollar rates indexed by LIBOR.[6] For pricing purposes, each uses the actual/360 day-count convention. Instruments 5-16 are newly issued swaps with maturities ranging from 2 to 30 years. The rates quoted for each of these instruments are based on semiannual payments using a 30/360 day count and represent the rates that cause the present value of the swaps' fixed payments (plus par at the end) to equal par at the time of issuance.

Using the data in Exhibit 1, the present value of $1 to be received in six months can be determined directly from the six-month LIBOR. Similarly, the present value of $1 to be received in one year can be determined from the one-year LIBOR.

The one and three-month LIBOR, along with the six-month and one-year rates, are used to compute cumulative continuously compounded rates of discount through the end of the first year. These cumulative rates of discount and their associated maturities, along with the cumulative rate of zero as of day zero, form the coordinates for five *fixed* knots in the cubic spline function that is to be estimated. The y coordinates (cumulative continuous discount rates) for the remaining 12 knots are determined as the solution to Equations (2).

Exhibit 2 summarizes the solution values for the cubic spline knots associated with cumulative continuously compounded discount rates estimated in connection with the U.S. swap curve shown in Exhibit 1.[7] The first 5 knots, based on observed LIBOR and a zero starting point, are fixed. The last 12 knot values are solution values to Equations (2) determined by Newton-Raphson search.

Exhibit 3 summarizes the present value calculations for the 12 swaps at the spline knot solution values.[8] The shaded cumulative continuous rates are computed directly from LIBOR and represent fixed y coordinates of the cubic spline. The unshaded boxed rates are y coordinate solution values to Equations (2). All other cumulative continuous rates are interpolated values from the cubic spline function. The present value of the fixed side of the

EXHIBIT 2

Illustration of Spline Interpolation of Cumulative Continuous Discount Rates Associated with U.S. Swap Curve on 10/3/2001

		Spline Knots at Solution		
j	Swap Term (Years)	Days (x_j)	Cumulative Continuous Rate (y_j)	Present Value $1
n/a	n/a	0	*0.00000*	*1.00000*
n/a	n/a	31	*0.00223*	*0.99778*
n/a	n/a	94	*0.00651*	*0.99351*
n/a	n/a	182	*0.01214*	*0.98794*
n/a	n/a	367	*0.02549*	*0.97483*
1	2	731	0.06496	0.93711
2	3	1,096	0.11436	0.89194
3	4	1,461	0.16823	0.84516
4	5	1,826	0.22462	0.79882
5	6	2,191	0.28226	0.75407
6	7	2,558	0.34167	0.71059
7	8	2,922	0.40123	0.66950
8	9	3,287	0.46120	0.63052
9	10	3,652	0.52239	0.59310
10	15	5,479	0.85278	0.42623
11	20	7,305	1.18500	0.30575
12	30	10,958	1.79584	0.16599

Values in italics are fixed and observed directly from LIBOR or from the zero-day starting point.

30-year swap is within $\$10^{-6}$ of \$100.

Although it is not shown in Exhibit 3, the present values for the other swaps are within $\$10^{-9}$ of \$100. With greater computer precision, these errors could be reduced even further.

Exhibit 4 shows a plot of the interpolated daily continuous forward rate curve that is implied by the U.S. swap curve on October 3, 2001. (Each rate is scaled to an annualized equivalent percentage rate by multiplying by 36,500.) Letting PV_t denote the interpolated present value of \$1 to be received on day t, each value plotted in Exhibit 4 is computed as $36{,}500 \times \ln (PV_t/PV_{t+1})$.

Although the interpolation method guarantees that each swap, when evaluated with the present values from the interpolated term structure, will have a fixed-side present value equal to \$100 (within computer precision), there is no way to determine the accuracy of the individual interpolated present value factors without knowing the functional form of the underlying yield curve to begin with.

III. MODEL CALIBRATION

To determine the accuracy of the individual present value factors, I generate four hypothetical yield curves using the stochastic interest rate model of Longstaff and Schwartz [1992]. Using the hypothetical yield curves, I calculate what the fixed swap payments should be for swaps with terms of 2 to 30 years using the same payment dates shown in Exhibit 3. I also compute the cumulative continuous discount rates for 31, 94, 182, and 367 days, corresponding to the observed LIBOR maturities summarized in Exhibits 1 and 2.

I am therefore able to compute hypothetical U.S. swap curves with maturity structures identical to the October 3, 2001 example, but using the Longstaff and Schwartz model to generate the curves rather than actual market quotes. Using the hypothetical U.S. swap curves, I use cubic spline interpolation to generate interpolated cumulative continuous discount rates and associated present value factors and compare these cumulative rates and discount rates to their "known" values from the Longstaff-Schwartz model.

I generate four hypothetical term structures following the Longstaff and Schwartz model, using the model

EXHIBIT 3

Detail of Valuation of Fixed Side of U.S. Swap Curve on 10/3/2001

		Swap Fixed Cash Flows Plus Hypothetical Principal Payment at Maturity											
Swap Term (Years)	Value of Fixed Side of Swap	182 Days	367 Days (1 yr.)	549 Days	731 Days (2 yrs.)	913 Days	1,096 Days (3 yrs.)	1,278 Days	1,461 Days (4 yrs.)	1,643 Days	1,826 Days (5 yrs.)	2,008 Days	2,191 Days (6 yrs)
2	100.0000000	1.628	1.646	1.628	101.619	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	100.0000000	1.908	1.929	1.908	1.897	1.897	101.908	0.000	0.000	0.000	0.000	0.000	0.000
4	100.0000000	2.099	2.122	2.099	2.087	2.087	2.099	2.099	102.099	0.000	0.000	0.000	0.000
5	100.0000000	2.236	2.260	2.236	2.223	2.223	2.236	2.236	2.236	2.236	102.236	0.000	0.000
6	100.0000000	2.336	2.361	2.336	2.323	2.323	2.336	2.336	2.336	2.336	2.336	2.336	102.336
7	100.0000000	2.417	2.443	2.417	2.403	2.403	2.417	2.417	2.417	2.417	2.417	2.417	2.417
8	100.0000000	2.479	2.507	2.479	2.465	2.465	2.479	2.479	2.479	2.479	2.479	2.479	2.479
9	100.0000000	2.529	2.557	2.529	2.514	2.514	2.529	2.529	2.529	2.529	2.529	2.529	2.529
10	100.0000000	2.573	2.601	2.573	2.558	2.558	2.573	2.573	2.573	2.573	2.573	2.573	2.573
15	100.0000000	2.759	2.790	2.759	2.744	2.744	2.759	2.759	2.759	2.759	2.759	2.759	2.759
20	100.0000000	2.850	2.882	2.850	2.834	2.834	2.850	2.850	2.850	2.850	2.850	2.850	2.850
30	100.0000002	2.893	2.925	2.893	2.876	2.876	2.893	2.893	2.893	2.893	2.893	2.893	2.893
Cumulative Continuous Rate		0.01214	0.02549	0.04331	0.06496	0.08883	0.11436	0.14081	0.16823	0.19614	0.22462	0.25320	0.28226
PV Factor		0.98794	0.97483	0.95761	0.93711	0.91500	0.89194	0.86866	0.84516	0.82189	0.79882	0.77631	0.75407

Shaded cumulative continuous rates are computed directly from LIBOR and represent fixed y coordinates of the cubic spline. Unshaded boxed cumulative continuous rates are y-coordinate solution values to Equations (2). All other cumulative continuous rates are interpolated values from the cubic spline.

Swap Term (Years)	3,469 Days	3,652 Days (10 yrs.)	4,932 Days	5,113 Days	5,296 Days	5,479 Days (15 yrs.)	7,123 Days	7,305 Days (20 yrs.)	10,776 Days	10,958 Days (30 yrs.)
2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	2.573	102.573	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	2.759	2.759	2.774	2.728	2.759	102.759	0.000	0.000	0.000	0.000
20	2.850	2.850	2.866	2.818	2.850	2.850	2.866	102.834	0.000	0.000
30	2.893	2.893	2.909	2.860	2.893	2.893	2.909	2.876	2.893	102.876
Cumulative Continuous Rate	0.49148	0.52239	0.75158	0.78503	0.81891	0.85278	1.15246	1.18500	1.76775	1.79584
PV Factor	0.61172	0.59310	0.47162	0.45610	0.44091	0.42623	0.31586	0.30575	0.17072	0.16599

Unshaded boxed cumulative continuous rates are y-coordinate solution values to Equations (2). All other cumulative continuous rates are interpolated values from the cubic spline.

Exhibit 4

Interpolated Forward Rate Curve Implied in 10/3/2001 U.S. Swap Curve

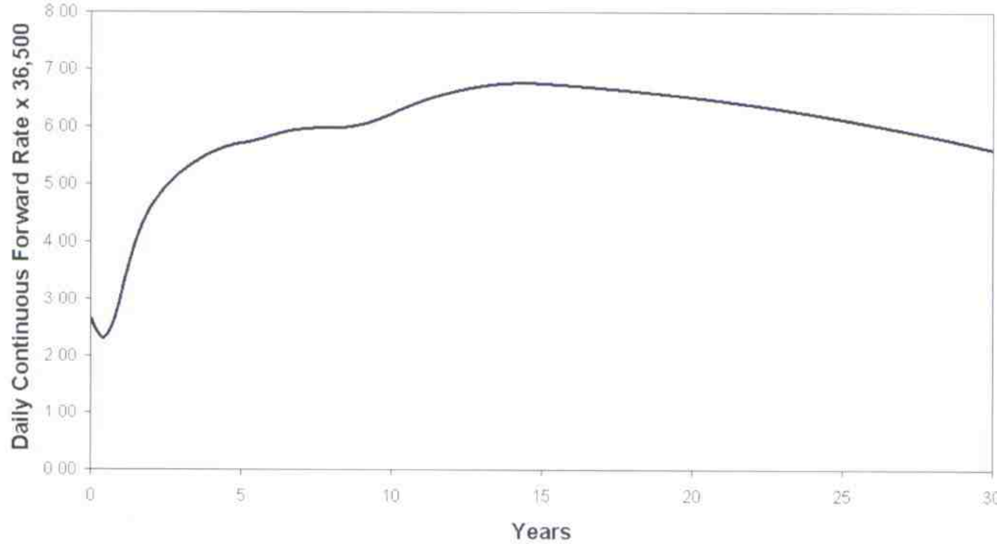


Exhibit 5

Parameter Values for Four Term Structure Curves Generated by Longstaff and Schwartz Model

Parameter	Curve 1	Curve 2	Curve 3	Curve 4
r	0.0550	0.0550	0.0550	0.0550
V	0.0030	0.0070	0.0140	0.0120
α	0.0011	0.0011	0.0030	0.0011
β	0.1325	0.1325	0.1325	0.1325
γ	3.0490	3.0490	3.0490	3.0490
δ	0.0566	0.0566	0.0566	0.0566
η	0.1582	0.1582	0.1582	0.1582
ν	0.3350	0.3350	0.3350	0.3350

parameters summarized in Exhibit 5. The parameters for curve 1 are the same as those estimated empirically in Longstaff and Schwartz [1993]. Parameters for the other curves are chosen to obtain four distinctly different term structure shapes.

Exhibit 6 shows a plot of the known curves of the annualized percentage daily forward rate generated by the four versions of the Longstaff and Schwartz model. Exhibit 7 shows percentage errors in single cash flow present value factors with maturities ranging from 1 day to 30 years when present values are calculated using the interpolated versions of the four curves. Exhibit 8 provides a similar plot for single cash flow maturities of 1 day to 10 years only.

Exhibit 7 shows that the highest absolute percentage errors in present value factors ranging in maturity from 1 day to 30 years are 0.0188%, 0.0445%, 0.0779%, and 0.2034% for curves 1 through 4, respectively. Over the 1-day to 10-year maturity range, the errors are much lower; the highest absolute percentage error drops to 0.0001%, 0.0002%, 0.0005%, and 0.0007% for curves 1 through 4.[9]

Clearly, the greatest interpolation errors occur between years 20 and 30. This is due to the length of time between successive swap maturities at the long end of the maturity spectrum as well as the general nature of cubic spline interpolation.

Cubic spline interpolation is most accurate when the first or second derivative of the spline function at its end-points can be accurately specified in advance. If the end-point derivatives, specified explicitly or implicitly (through the not-a-knot condition), are specified with error, the resulting interpolation will also be subject to its greatest error near the end-points.

EXHIBIT 6

Annualized Daily Forward Rate Curves Generated from Longstaff and Schwartz Model

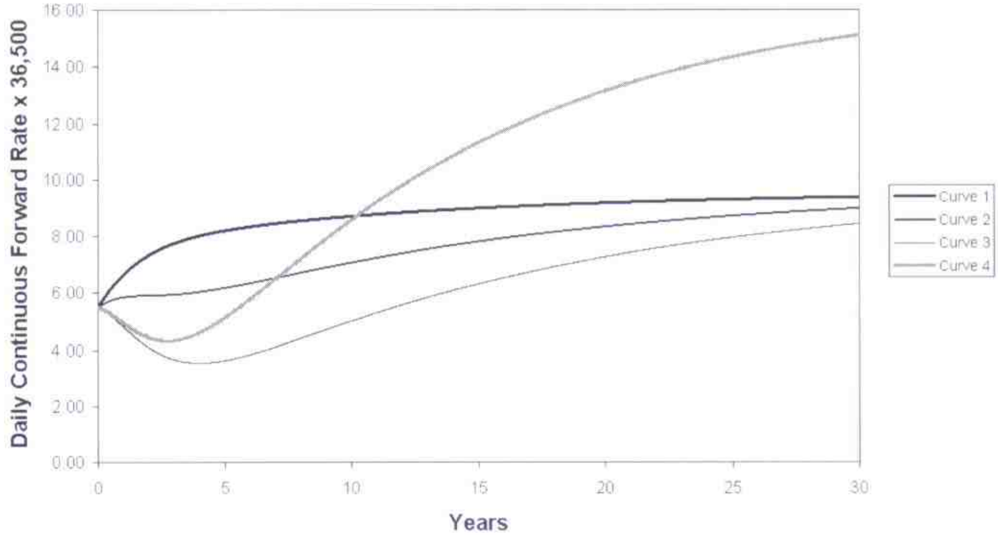
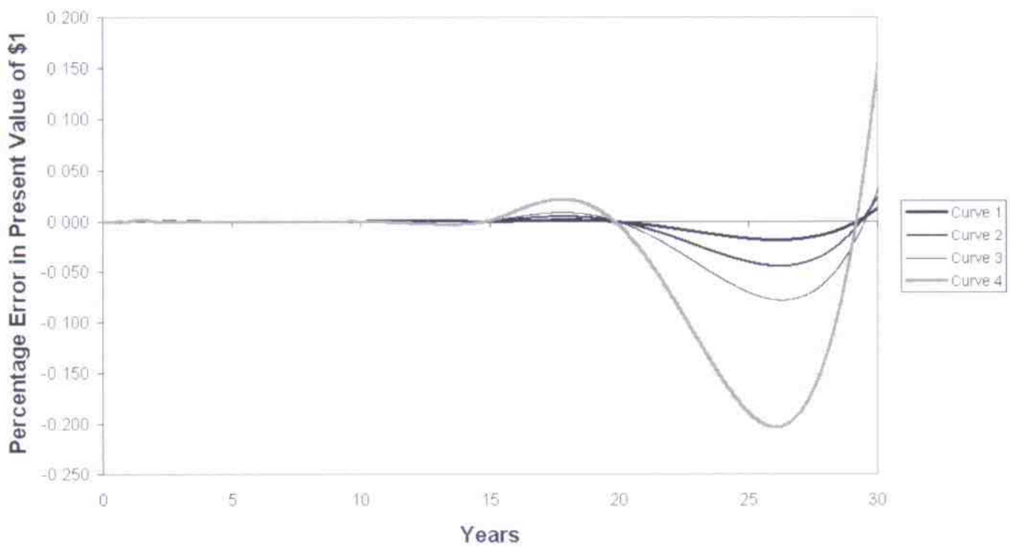


EXHIBIT 7

Percentage Errors Using Four Different Term Structure Curves Generated by Longstaff and Schwartz Model—Years 0–30

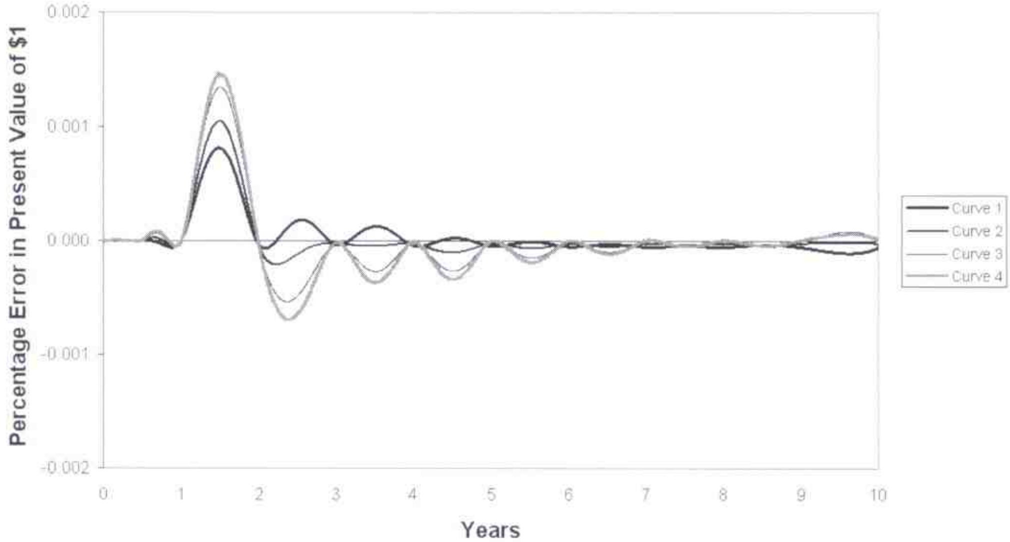


Generally, cubic spline-based interpolation error can work its way out within several knots of the end-points in either direction. In the case of the swap curve, any error associated with the short end of the curve should work its way out after six months to a year, since the initial spline knots are associated with 0-month, 1-month, 3-month, and 6-month LIBOR. Starting at the end of the swap curve and working backward, however, the first several spline knots are associated with 30-year, 20-year, 15-year, and 10-year swap rates. As a result, interpolation error at the front end of the curve can be resolved after only several months into the LIBOR maturity range, while it could take 10 years or more to resolve the error working backward from the back end of the swap maturity range.

We should emphasize that the percentage pricing errors reported apply to the valuation of *single* cash flows. They thus represent the maximum absolute percentage

Exhibit 8

Percentage Errors Using Four Different Term Structure Curves Generated by Longstaff and Schwartz Model—Years 0 through 10



errors in the valuation of a financial asset with a *series* of positive cash flows ranging from 1 day to 30 years in the first instance and 1 day to 10 years in the second.

When a series of cash flows is valued using the interpolated term structure, some cash flows could be subject to positive pricing error, while others could be subject to negative error. Unless all the cash flows are concentrated around the maturity of the cash flow with the maximum absolute percentage pricing error, the positive and negative pricing errors should tend to cancel out, bringing the overall pricing error much closer to zero.

For example, for a financial asset that pays $1 at the end of each day for 30 years, the pricing errors using interpolated present value factors are –0.00088%, –0.00272%, –0.00610%, and –0.00715% for curves 1 through 4, respectively. For a financial asset that pays $1 at the end of each day for 10 years, the respective pricing errors are only 0.000051%, 0.000035%, 0.000010%, and –0.000002%.

IV. SUMMARY AND CONCLUSIONS

Jordan and Mansi [2003] have demonstrated the potential accuracy of five interpolation methods applied directly to the U.S. Treasury par yield curve to obtain interpolated estimates of par yields at the halfway points between successive observed yearly maturities. Using the interpolated halfway point par yield estimates, along with the original set of par yields, they use a bootstrap method to estimate the present value of $1 to be received every six months. Unfortunately, when these solution values are applied to the cash payment streams of the bonds from which they are estimated, the resulting present values only approximately equal the original bond prices.

I develop an entirely different approach to estimating the term structure from the U.S. swap curve. The model takes the prices of m swaps with different but not necessarily equally spaced maturities of $t_1, t_2, \ldots, t_m$ days (with $t_i < t_{i+1}$). When the methodology is applied to the U.S. swap curve as reported by the Bloomberg system, t_1 through t_9 represent the days to maturity associated with swaps with maturities of 2 to 10 years, inclusively, and t_{10} through $t_{12} = t_m$ represent the days to maturity associated with swaps with maturities of 15, 20, and 30 years. LIBOR is used to estimate present value factors and associated cumulative yields over the short end of the swap curve.

I use a multidimensional Newton-Raphson search applied to m cubic spline knot coordinates, $k_j = \{x_j, y_j\}$, where the x value of the coordinate represents days to maturity associated with swap j, and the y coordinate represents the cumulative discount rate through the same number of days, to determine what the cumulative discount rate would have to be, as of the maturity date of each swap, to cause the present value of the fixed payments from each swap (plus principal at the end) to simultaneously equal par.

By construction, the interpolation method causes computed swap values to equal observed swap values. At the same time, there is no guarantee that individual daily present value factors computed along the interpolated curve are accurate. To test the accuracy of the method, I use the Longstaff and Schwartz [1992] model to generate four different yield curves with distinctly different shapes. From each curve I compute what fixed swap rates should be for swaps with the maturity structure as described above. I then apply my methodology to compute an interpolated yield curve (and associated present value factors) from each set of fixed swap rates and compare the interpolated curve to the original known Longstaff-Schwartz curve from which it is generated.

The methodology proves to be extremely accurate in reproducing the original set of Longstaff-Schwartz curves, especially in the 2- to 10-year maturity range. Over this range, the highest absolute error in a single-day present value factor is 0.0007%. Over the entire 1-day to 30-year maturity range, the highest single-day present value error is 0.2034%, and the highest error among the four Longstaff-Schwartz curves associated with a financial asset that pays $1 per day for 30 years is 0.000051%. Therefore, the interpolation method appears to be very robust in providing consistently accurate present value factors over the entire 30-year swap maturity range.

ENDNOTES

The author thanks James Jordan and Patrick Dennis for helpful comments and suggestions.

[1]At the time a swap is issued, its value should equal zero. Although interest rate swaps do not involve an exchange of principal, if a hypothetical principal payment, as of the swap's maturity date, is added to both the fixed and floating sides of the swap, the present value of the floating payments plus the hypothetical principal payment will equal par at the time the swap is issued and at each interest reset date. Therefore, in order for the swap to have zero value at the time of issuance, the present value of its fixed payments plus the hypothetical principal payment at maturity must also equal par. Thus, the fixed rate associated with a newly issued swap can be interpreted as a par yield.

[2]Given the inferred par yields obtained by interpolation, the six-month par yield infers a six-month zero-coupon present value factor. This present value factor, along with the one-year par yield is used to solve for a one-year zero-coupon present value factor. The six-month and one-year present value factors, along with the 1.5-year par yield obtained by interpolation, are used to solve for a 1.5-year zero-coupon present value factor, and so on. This method of solving for present value factors successively in terms of solution values of shorter-term present value factors is known as the bootstrap method.

[3]If we apply the model to bonds whose prices are significantly different from par, those prices could reflect tax and liquidity considerations not reflected in the prices of par bonds. As a result, the term structure so derived could be contaminated by tax and liquidity considerations not reflected in the mathematics of Equations (1). Nevertheless, the mathematics of the model do not require that the bonds sell for par, and in the absence of tax- or liquidity-based pricing effects, the term structure derived from the model should be accurate for any set of original bond or note prices.

[4]The cumulative continuously compounded discount rate through t days is the negative of the natural logarithm of the present value of $1 to be received in t days.

[5]Compared with a natural spline, the not-a-knot method reduces the valuation error in calibration by a factor of approximately $^1/_2$.

[6]The U.S. swap curve, as reported by the Bloomberg system, does not include the one-month LIBOR. I add it to provide an additional data point for estimating the curvature of the front end of the swap curve.

[7]All swap payment dates and LIBOR maturity dates are determined using the *modified following business day convention.* According to the British Bankers' Association (http://www.bba.org.uk/public/libor/41635/4307): "The modified following business day convention states that the maturity date (or swap payment date) is the first following day (relative to the normal payment date) that is a business day in London and the principal financial centre of the currency concerned (United States), unless that day falls in the next calendar month. In this case only, the maturity date will be the first preceding day in which both London and the principal financial centre of the currency concerned are open for business." This is why there are 367 days until maturity for the one-year LIBOR.

[8]Swap interest is paid semiannually and accrued using the 30/360 day-count convention. According to Campbell Harvey, the number of days from $M_1/D_1/Y_1$ to $M_2/D_2/Y_2$ is computed as follows: 1) If D_1 is 31, change D_1 to 30; 2) If D_2 is 31 and D_1 is 30 or 31, then change D_2 to 30; 3) If M_1 is 2, and D_1 is 28 (in a non-leap year) or 29, then change D_1 to 30 (http://www.duke.edu/~charvey/Classes/ba350_1997/pricing/daycount.htm). Then the number of days between payment dates is $360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)$, and the accrued interest amount is the annual interest amount times the number of days so calculated divided by 360. With the modified following business day convention for computing swap payment dates and the 30/360 day-count convention, a swap's fixed payment can be slightly different on its various payment dates. This is why the semiannual fixed swap payments shown in Exhibit 3 are not all the same.

[9]Interpolation was also carried out using a natural cubic spline. With a natural spline, the second derivative of the spline

function is set to zero at both end-points. Using this method, pricing errors at the long end of the 30-year maturity range are approximately twice the size of those reported here, but on the short end the errors are approximately the same.

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