An Empirical Study of Structural Credit Risk Models Using Stock and Bond Prices

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When option pricing theory was developed some 30 years ago, Black [1985] suggested its most fruitful applications would likely be to the valuation of corporate liabilities. Yet the few attempts to test structural bond pricing models with market prices have not been very successful. Instead, much of the research has shifted focus to the development of reduced-form models, designed to be more empirically tractable. Examples are Jarrow, Lando, and Turnbull [1997], Lando [1998], and Duffie and Singleton [1999].

We argue that the perceived advantage of reduced-form models is more a result of the estimation procedure than of model structure. Estimating a structural model in a similar manner, we find that it compares well to previously tested reduced-form models.

An important objective for structural as well as reduced-form models is to provide term structure of default probabilities for the valuation of credit-risky securities.[1] Structural models obtain default probabilities from a model of the firm's assets and liabilities, and reduced-form models specify the default process (typically, a hazard rate) directly.

While both modeling approaches allow the pricing of bonds, the assets used to estimate the models are different. Structural models are usually estimated using a time series of stock prices, while reduced-form models tend to be estimated using a cross-section of bond prices. All things equal, it is likely that the cross-section of bond prices provides more information about the issuing firm's term structure of default probabilities than a single time series of its stock. This clearly speaks in favor of reduced-form models.

There are two caveats. First, stocks may be more liquid than bonds and thus more informative about a firm's prospects. Second, there may be very few bonds in the cross-section to estimate a model on—in practice, one is often either forced to extend the cross-section to bonds or indexes of the same credit rating, or to include bonds from more than one time, assuming default risk has not changed. See Ronn and Nielsen [1997], Bakshi, Madan, and Zhang [2001], and Elton et al. [2001].

The strengths and weaknesses of either approach are evident. Structural models are at an advantage when there are no similar bonds but instead a highly liquid stock price series is available, while reduced-form models are the only alternative when there is no traded equity, but instead there are an appropriate selection of bonds at hand. We aim to combine the advantages of structural models with those of reduced-form models.

Incorporating bond price information into the estimation procedure of a structural model provides valuable incremental information. With less upside potential, bond prices are more sensitive than stock prices to financial risk. Moreover, the yield spread of a bond, which has a fixed maturity, provides information about the term structure of default probabilities—something that stocks, with infinite "maturity," are less likely to convey.

We estimate a structural model including a bond price and dividend information along with the traditional stock price series. The data used are a sample of 141 U.S. corporate issues totaling 5,594 dealer quotes. The model draws on our tractable building block approach in Ericsson and Reneby [1998] in order to develop an extended version of Leland [1994]. Our econometric methodology is based on a maximum-likelihood approach suggested by Duan [1994], which makes more efficient use of the information in time series of stock prices than the traditional method (see also Duan and Simonato [2002] and Ericsson and Reneby [2002]). To the best of our knowledge, this technique has, to date, not been applied to market data of corporate bonds.

We use the model to predict the yield spread for a given issue between 1 and 50 months out of sample. For the one month-ahead predictions, we find a mean error of −2 basis points so our estimator appears unbiased for practical purposes.

Duffee [1999] provides an empirical study of the reduced-form approach to modeling credit risk. Calibrating a reduced-form model to bond yields, he finds fitting errors similar to our out-of-sample errors. Our model also compares favorably to the results of a study of out-of-sample performance by Bakshi, Madan, and Zhang [2001]. By considering bond portfolios, we show that the bulk of model errors are diversifiable.

Overall, our results enable us to be more optimistic about structural credit risk models than the results of previous studies.

I. MODEL

To implement a structural model, we need pricing formulas for stocks and bonds. To value equity, we use a version of the Leland [1994] model, extended to allow for violations of the absolute priority rule and future debt issues. To value bonds, we discount promised coupons and final repayment with the corresponding risk-adjusted default probabilities and add the value of recovery in default.

Both extension of Leland's equity formula and the prices of bonds are straightforward to achieve using our building block approach in Ericsson and Reneby [1998]. The two building blocks used to price equity are a claim that pays off a dollar in case of default, $G(w_t, t)$, and a complementary claim, $G^{\alpha}(w_t,t)$, that, upon default, pays off a dollar compounded at a rate α. Formulas for both claims are given in the appendix.

The firm's total debt consists of bonds—including the one we want to price—bank loans, accounts payable, and accrued taxes. Clearly, a detailed model of all these payments would not be tractable; we assume that total debt service can be approximated by a continuous coupon C_t that is tax-deductible at a rate $(1 - \tau)$.

The firm's total borrowing is likely to increase as the firm value increases—otherwise, the expected debt-to-equity ratio would tend to zero over longer horizons. We denote the total nominal amount of debt by N_t. In each instant, new debt is issued with face value $\alpha N_t dt$ and corresponding market value $d(\omega_s, S)$. Consequently, the face value of the firm's debt will be growing at a rate $(dN_t = \alpha N_t dt)$.[2] For simplicity we assume a constant risk-free interest rate r and a coupon, $C_t = rN_t$, at which the risk-free value of debt equals its face value.[3]

As in Leland [1994], default is driven by shareholders' endogenous decision to stop servicing debt. This will happen if and when the value of the firm's assets (w_t) reaches the boundary L_t (the formula is given in the appendix). If the firm defaults, bankruptcy costs k are realized and deviations from the absolute priority rule occur; shareholders obtain a fraction ε before the remainder of the assets is distributed to creditors.

The value of the assets follows a geometric Brownian motion under the risk-adjusted measure. It generates cash flow at rate β and has volatility σ:

$$\begin{cases} d\omega_t = (r - \beta)\, \omega_t\, dt + \sigma\, \omega_t\, dW_t \\ \omega_0 = \underline{\omega} \end{cases} \tag{1}$$

The value of equity is determined by future dividends and potential payoffs in financial distress:

$$\mathcal{E}(\omega_t, t) \equiv \begin{cases} E^B\left[\int_t^{\infty} e^{-r(s-t)}\, DIV(\omega_s, s)\, I_{T \nless s}\, ds\right] \\ \qquad + E^B\left[e^{-r(T-t)} \varepsilon_T\, L_T\right] \end{cases} \tag{2}$$

No profits or losses are carried forward. At any given time s, the dividend is determined by the (after-tax) earnings generated by the firm's operations (βw_s) *plus* the inflow from debt issues $[(d(W_s, S)]$ *less* coupon payments $[(1 - \tau)C_s]$:

$$DIV(\omega_s, s) = \beta\omega_s + d(\omega_s, s) - (1-\tau)C_s \tag{3}$$

A negative dividend indicates equityholders are contributing funds to stave off financial distress. This is, of course, in their own self-interest, as the value of expected future gains exceeds that of paying dividends. The value of equity is given by:[4]

$$\begin{aligned}\mathcal{E}(\omega_t, t) &= \omega_t - L_t \cdot G^{\alpha}(\omega_t, t) \\ &\quad -N_t \cdot (1 - G(\omega_t, t)) \\ &\quad +\tau N_t \cdot \frac{r}{r-\alpha} \cdot (1 - G^{\alpha}(\omega_t, t)) \\ &\quad +(1-\varepsilon-k) \cdot L_t \cdot (G^{\alpha}(\omega_t, t) - G(\omega_t, t)) \\ &\quad +\varepsilon L_t \cdot G^{\alpha}(\omega_t, t)\end{aligned}$$

The first line equals the value of receiving all the firm's earnings conditional on no default. The second line is the cost of servicing current debtholders. The third line corresponds to the value of the tax shield. The fourth line is the value of the ability to borrow in the future, using the assets as collateral, and the fifth line is the value of expected payouts to equityholders in reorganization. Setting $\alpha = \varepsilon = 0$, one obtains the Leland [1994] model.

To gain some additional intuition for the equity formula, consider two scenarios. As the value of assets approaches the barrier, G and G^{α} tend toward unity so that lines 1–4 tend toward zero—and equity takes on a value equal to the expected payoff in a reorganization: εL_t. Conversely, as the value of assets increases and the risk of default declines, G and G^{α} tend toward zero and the value of equity approaches the value of assets *less* the value of the now risk-free debt *plus* the tax shield: $\omega_t - N_t + \tau N_t \frac{r}{r-\alpha}$.

Turning to the valuation of a straight coupon bond, two other similar building blocks are used: a dollar-in-default that pays off only if default occurs before T, $G(\omega_t, t|T)$, and a binary option that pays off one dollar at T if there has been no default up to that point, $H(\omega_t, t; T)$. The value of a straight coupon bond with M coupons c paid out at times $\{t_i : i = 1, ..., M\}$ can then be written as

$$\begin{aligned}\mathcal{B}(\omega_t, t) &= \sum_{i=1}^{M-1} c \cdot H(\omega_t, t; t_i) \\ &\quad +(c+P) \cdot H(\omega_t, t; T) \\ &\quad +\psi P \cdot G(\omega_t, t|T)\end{aligned}$$

The value of the bond is equal to the value of the coupons (c), the value of the nominal repayment (P), and the value of the recovery in a default (ψP). Each payment is weighted with a claim capturing the value of receiving one dollar at the particular date.

II. EMPIRICAL PROCEDURE

We apply maximum-likelihood (ML) techniques first proposed by Duan [1994] in the context of deposit insurance. The idea is to use price data from one or several derivatives to deduce the characteristics of the unobserved underlying process. The "derivative" in our setting is the firm's equity and bonds.

In Ericsson and Reneby [2004], we evaluate this maximum-likelihood approach using several theoretical models and compare it to the traditional methods as in Black and Scholes [1973], Merton [1974], Leland and Toft [1996], and Briys and de Varenne [1997]. The main finding is that the ML method is less biased and much more efficient, with standard errors several times lower

The maximum-likelihood estimation will rely mainly on a time series of stock prices, $\mathcal{E}^{obs} = \{\mathcal{E}_i^{obs}\} : i = 1, ..., n\}$. The subscript i is used to index observations, in contrast to subscript t, which refers to time in years. We require the likelihood function of the observed price variable. Defining $f(\cdot)$ as the conditional density for $\mathcal{E}_i^{obs}$ gives us the log-likelihood function for equity:

$$L_{\mathcal{E}}\left(\mathcal{E}^{obs}; \xi, \lambda\right) = \sum_{i=2}^{n} \ln f\left(\mathcal{E}_i^{obs} \middle| \mathcal{E}_{i-1}^{obs}; \xi, \lambda\right) \tag{4}$$

where we let $\xi \equiv \{\alpha, \beta, \sigma\}$ denote the parameters needed to price the bond.

To derive an expression of the density function for equity, we make a change of variables as suggested in Duan [1994]:

$$\begin{aligned}f\left(\mathcal{E}_i^{obs} \middle| \mathcal{E}_{i-1}^{obs}; \xi, \lambda\right) &= g(\ln\omega_i | \ln\omega_{i-1}; \beta, \sigma, \lambda)|_{\omega_i = w(\mathcal{E}_i^{obs}, t_i; \xi)} \\ &\quad \times \left[\left.\frac{\partial \mathcal{E}_i}{\partial \ln\omega_i}\right|_{\omega_i = w(\mathcal{E}_i^{obs}, t_i; \xi)}\right]^{-1}\end{aligned} \tag{5}$$

The equity density is now expressed as a function of g, the conditional density for a normally distributed variable—the log of the asset value. Note that $\mathcal{E} = \mathcal{E}_i(w_i, t_i)$ refers to the equity *formula*, while $\mathcal{E}_i^{obs}$ denotes an observed market value. The function transforming equity to asset value is defined as $w\ (\mathcal{E}_i^{obs},\ t_i;\ \xi) \equiv \varepsilon^{-1}\ (\mathcal{E}_i^{obs},\ t_i;\ \xi)$, the inverse of the equity value function. Hence there is, given ξ, a one-to-one correspondence between the stock price $\mathcal{E}_i^{obs}$ and the implied asset value ω_i.

By inserting (5) into (4) we obtain the log-likelihood of the vector $\mathcal{E}^{obs}$ for a given choice of ξ as:

$$L_{\mathcal{E}}\left(\mathcal{E}^{obs};\xi,\lambda\right) = L_{\ln\omega}\left(\ln w\left(\mathcal{E}_i^{obs},t_i;\xi\right) : i=2...n;\xi,\lambda\right) - \sum_{i=2}^{n} \ln\omega_i \left.\frac{\partial\,\mathcal{E}\left(\omega_i,t_i;\xi\right)}{\partial\,\omega_i}\right|_{\omega_i=w\left(\mathcal{E}_i^{obs},t_i;\xi\right)} \tag{6}$$

It is straightforward to calculate a closed-form solution for $\partial\varepsilon_i/\partial\omega_i$, the delta of the equity formula.

The estimated parameter vector $(\hat{\xi},\ \hat{\lambda})$ is obtained by maximizing Equation (6) with respect to $(\hat{\xi},\ \hat{\lambda})$ and subject to constraints captured by the vector **h**:

$$\max_{\xi,\lambda} L_{\mathcal{E}}\left(\mathcal{E}^{obs};\xi,\lambda\right) \text{ subject to } \mathbf{h}\left(\xi\right)=\mathbf{0}$$

The bond-specific parameters $(c,\ P,\ t_1,\ ...,\ t_M)$ and the risk-free rate (r) are readily observable. We use as a proxy for the risk-free rate constant-maturity Treasury yields interpolated to match the maturity of the corporate bond. The total nominal amount of debt (N_t) is obtained from the firms' balance sheets. The default threshold (L_t) is determined endogenously, as discussed above. We rely on earlier empirical work to determine the APR (Absolute Priority Rule) deviations parameter (ε) and the recovery rate of the aggregate debt $[(1-\varepsilon-k)(L_t/N_t)]$. The recovery rate of the bond (ψ) is based on historical industry and seniority averages.

This leaves us with a parameter vector $\xi = (\alpha,\ \beta,\ \sigma)$ that will be estimated by the ML procedure. Since we allow the estimation procedure to use information in market dividend yields or bond prices, the constraint is:

$$\mathbf{h}\left(\xi\right) = \begin{bmatrix} DIV^{obs} - DIV^{theo}\left(\xi\right) \\ \\ \mathcal{B}^{obs} - \mathcal{B}\left(\omega_{t_n}, t_n;\xi\right) \end{bmatrix}$$

An estimate of the value of assets is obtained using the inverse equity function: $\hat{\omega}_t = w(\ \mathcal{E}_n^{obs};\ t_n;\ \hat{\xi})$. Once we have obtained the pair $(\hat{\omega}_t,\ \hat{\xi})$, it is straightforward to compute the corresponding bond price estimate $\hat{B} = B\ (\hat{\omega}_t;\ t;\ \hat{\xi})$and spread $\hat{\mathcal{S}}$.

III. DATA

Our database consists of bond prices and characteristics, firm balance sheet information, equity prices, and term structure data. As a starting point, we take the FISD (Fixed Income Securities Database) bond database, which includes data for bonds that are part of the Lehman Brothers indexes, or monthly prices for a large array of fixed-income securities. We focus exclusively on corporate bonds for firms based in the U.S. during the coverage of our version of the database, i.e., between January 1994 and February 1998.[5]

To concentrate on credit risk and avoid dealing with option features, we eliminate all callable, convertible, and putable bonds in addition to sinkers. Furthermore, as we want to analyze both time series as well as the cross-sectional aspects of corporate bond yield spreads, we eliminate all bonds with fewer than 12 consecutive monthly observations.[6]

In the end, we are left with 141 bond issues by as many firms. All the bonds we consider are unsecured bullet bonds. The number of monthly price observations on each bond ranges from 12 to 50. In total, there are 5,594 bond price observations.

The FISD database provides all the characteristics of the individual bonds such as coupon, maturity, and ratings, but we also need information about the liability structure of the firm. We use the aggregate nominal debt of the firm as an input to derive the market value of equity and thus give us a model for default probabilities. Our empirical measure of the total debt of the firm is the total liabilities as reported in firms' EDGAR filings. Daily market values for the outstanding equity and information about the risk-free interest term structures over time come from Datastream.

Exhibit 1 lists descriptive statistics for the 141 bonds and issuing firms along with statistics for subgroups across rating and maturity. We split the data into two basic groups, which we refer to as high (rated A or higher) and low (rated BBB or lower) rated issues. The FISD corporate bond data are dominated by investment-grade issues, so speculative-grade bonds—rated BB or lower—make up only slightly more than 10% of the total bond price data at 588 observations.

The average book leverage is 41%, and the average

EXHIBIT 1
Descriptive Statistics for Bond Issues and Issuers

		Market spread	Market price	Coupon	S&P rating	Issue size	Issue age	Maturity	Leverage	Firm size	Stock volatility
All	Mean	89	106.64	8.09	8.0	191,431	3.8	11.8	41.1%	17,725	27.9%
(141 firms, 5595 obs.)	*Std. dev.*	*82*	*9.90*	*1.27*	*2.8*	*117,588*	*2.4*	*7.4*	*17.9%*	*29,412*	*10.5%*
High rating	Mean	58	108.30	7.89	5.7	203,263	4.3	12.6	30.6%	25,155	24.1%
(54 firms, 2270 obs.)	*Std. dev.*	*21*	*10.71*	*1.42*	*1.5*	*126,912*	*2.6*	*7.5*	*12.1%*	*28,638*	*6.3%*
Low Rating	Mean	109	105.50	8.22	9.6	183,337	3.4	11.2	48.3%	12,641	30.5%
(87 firms, 3325 obs.)	*Std. dev.*	*100*	*9.13*	*1.14*	*2.2*	*110,046*	*2.1*	*7.3*	*17.7%*	*28,852*	*12.0%*
Speculative Grade	Mean	229	103.06	8.89	12.9	189,949	2.9	9.0	61.6%	10,048	40.5%
(14 firms, 588 obs.)	*Std. dev.*	*189*	*10.94*	*1.32*	*2.8*	*91,000*	*1.9*	*5.8*	*19.4%*	*8,987*	*16.4%*
Long Maturity	Mean	92	109.79	8.39	7.6	196,602	4.4	17.9	42.2%	23,265	27.0%
(67 firms, 2577 obs.)	*Std. dev.*	*60*	*11.33*	*1.12*	*2.5*	*129,876*	*2.7*	*6.8*	*18.0%*	*37,960*	*10.5%*
Short Maturity	Mean	86	103.95	7.83	8.4	187,024	3.2	6.6	40.2%	13,002	28.7%
(74 firms, 3018 obs.)	*Std. dev.*	*97*	*7.52*	*1.32*	*2.9*	*105,816*	*1.9*	*1.9*	*17.8%*	*18,055*	*10.5%*

equity volatility during the 250 trading days preceding a bond quote is 28%. Bond maturities range from 1 to 30 years and coupon rates from 5.60% to 10.88%. The average numerical rating (8) corresponds to an A– rating. This produces credit spreads of from 10 to 1,589 basis points, with an average of 89.

The recovery rate for individual bonds, ψ, is set as a function of industry and seniority. We categorize the 141 companies in our sample according to the SIC codes used by Altman and Kishore [1996] and assign the bonds recovery rates according to the historical averages they provide. Finally, we set deviations from the absolute priority rule to 5% of the asset value in default.

IV. EMPIRICAL RESULTS

We present our results in several steps. We first attempt to price the bond the following month only (at t_{n+1}). Then we provide an extended version of the same theme; we price the bond during a long period of time (i.e., up to 50 months, or 5 years, after t_n) using only updated market capitalizations, risk-free term structures, and balance sheet data. Finally, we consider the valuation of bond portfolios.

One-Month Out-of-Sample Prediction

The bond price prediction at t_{n+1} is based on the observed stock price and risk-free rate at t_{n+1} and the parameter vector estimated at t_n. This is then repeated for the subsequent month, i.e., estimating the model at t_{n+1} and predicting the bond price at t_{n+2}. By rolling the estimation forward including all available observed prices for all 141 companies, we arrive at a sample size of 5,454 predictions.

The outcome of the test is presented in Exhibit 2. The mean spread error is only –1.7 basis points, corresponding to a 0.1% overprediction of bond prices. The standard deviation of the spread error is 19 basis points. The error is somewhat greater for the speculative-grade bonds, but the relative spread error actually declines from –3.1% for the highest rated issues, to –1.3% for speculative-grade bonds. The correlation between predicted and market spreads is quite high. The average firm time series correlation is 61%.[7]

Duffee [1999] evaluates the reduced-form approach using a square root process for the hazard rate and two additional diffusions to model the risk-free term structure. The model is estimated using an extended Kalman filter on 11 years of monthly data covering 161 firms, although most firms do not appear in every month of the sample (the median is 92 months of a maximum of 132). He estimates his model on a per month, per firm basis (as we do), using a median of 2.47 corporate bond quotes and reports two (in-sample) error measures: the median root mean squared yield error and the median standard deviation of the yield error. Both error measures are 10 basis points and slightly lower for the highest-rated firms.

EXHIBIT 2
One-Month Out-of-Sample Performance

<table>
<tr><th></th><th></th><th colspan="4">Prices</th><th colspan="6">Spreads</th></tr>
<tr><th></th><th></th><th>RPE</th><th>|RPE|</th><th>RMSE(RP)</th><th>Corr(P)</th><th>RSE</th><th>|RSE|</th><th>SE</th><th>|SE|</th><th>RMSE(S)</th><th>Corr(S)</th></tr>
<tr><td>All bonds</td><td>Mean</td><td>0.1%</td><td>0.7%</td><td>1.2%</td><td>96%</td><td>-2.2%</td><td>13%</td><td>-1.7</td><td>11</td><td>15 / 20</td><td>61% / 97%</td></tr>
<tr><td>(141 firms, 5454 obs.)</td><td>Std. dev.</td><td>1.2%</td><td>0.9%</td><td></td><td>6%</td><td>18%</td><td>12%</td><td>19</td><td>15</td><td>13 / -</td><td>24% / -</td></tr>
<tr><td>High Rating</td><td>Mean</td><td>0.1%</td><td>0.5%</td><td>0.7%</td><td>98%</td><td>-3.1%</td><td>12%</td><td>-1.6</td><td>6.7</td><td>8.8 / 9.3</td><td>61% / 92%</td></tr>
<tr><td>(54 firms, 2216 obs.)</td><td>Std. dev.</td><td>0.7%</td><td>0.5%</td><td></td><td>4%</td><td>17%</td><td>12%</td><td>8.9</td><td>6.1</td><td>2.9 / -</td><td>18% / -</td></tr>
<tr><td>Low Rating</td><td>Mean</td><td>0.1%</td><td>0.9%</td><td>1.4%</td><td>95%</td><td>-1.5%</td><td>14%</td><td>-1.8</td><td>14</td><td>19 / 29</td><td>62% / 97%</td></tr>
<tr><td>(87 firms, 3238 obs.)</td><td>Std. dev.</td><td>1.4%</td><td>1.1%</td><td></td><td>7%</td><td>18%</td><td>13%</td><td>23</td><td>19</td><td>15 / -</td><td>27% / -</td></tr>
<tr><td>Speculative Grade</td><td>Mean</td><td>0.2%</td><td>1.6%</td><td>2.4%</td><td>90%</td><td>-1.3%</td><td>14%</td><td>-3.7</td><td>30</td><td>41 / 48</td><td>76% / 97%</td></tr>
<tr><td>(14 firms, 574 obs.)</td><td>Std. dev.</td><td>2.4%</td><td>1.8%</td><td></td><td>10%</td><td>18%</td><td>12%</td><td>45</td><td>34</td><td>26 / -</td><td>18% / -</td></tr>
</table>

RPE: relative pricing error. RMSE: root mean squared error in relative prices (RP) and spreads (S). RSE: relative spread error. SE: spread error. Absolute values are enclosed in | |. Corr(P) and Corr(S) denote the correlation between estimated and actual prices and spreads. Reported figures are calculated as averages across firm means, which differ negligibly from overall averages, except for spread correlations and RMSE. Overall averages are reported after the slash marks.

Although the structure of his study is not directly comparable to ours, we find it encouraging that our out-of-sample errors are of the same size. The median RMSE and median standard deviation of SE are both 11 basis points for our model.

Bakshi, Madan, and Zhang [2001] (BMZ) test a set of reduced-form models with a two-factor model for the risk-free term structure. They extend the basic framework into several models by also including firm-specific variables such as leverage, book-to-market, or stock price. The models are estimated using the firm-specific variable, the risk-free term structure, and at least 12 observations on bonds issued by the firm. Then they predict prices for firms' bonds one month out-of-sample. The predictions are based on the out-of-sample value of the firm-specific variable, the out-of-sample term structure, and the parameter vector estimated in the previous quarter. Hence this framework is comparable to ours.

BMZ report two error measures, the absolute relative pricing error (|RPE|), and the absolute spread error (|SE|). The |RPE| ranges between 1.4% and 1.9% for different model specifications, compared to our average error of about 0.7%. Like us, they find that the errors in both prices and spreads tend to increase as the credit rating worsens. Their average absolute spread error increases from about 24 basis points for AA-rated debt to 32 basis points for BBB debt, providing a mean error of 28 basis points across all bonds. Our average absolute spread error is 11 basis points for all bonds, and reaches 30 basis points for speculative-grade debt.

BMZ do not report percentage spread errors. Our relative spread errors decline as the credit ratings worsen from about 3% to 1%.

It seems that our model compares fairly well to the reduced-form models evaluated by BMZ. One interesting comparison is that they find poorer performance when the stock price is used as a firm-specific variable, i.e., when it incorporates part of the information that is crucial to our model. It thus appears that a structural model for the exact link between stock and bond prices can help improve performance.

Longer-Horizon Out-of-Sample Prediction

To look at bond prices farther out-of-sample, we fix the model's parameters to their initially estimated values and allow only updating of observed quantities such as stock price, book leverage, and term structure data. This is clearly likely to generate larger errors than a more pragmatic approach of reestimating the model for each monthly valuation date. That is, a bond may be actively traded only a short period after issuance, so that only prices around that time can be viewed as informative about the creditworthiness of the firm. This exercise will help us examine model performance under more demanding circumstances.

We estimate parameters exactly as before (incorporating bond price information at t_n), but instead of predicting the bond price at just t_{n+1}, we predict the bond

EXHIBIT 3
Multiple Months Out-of-Sample Performance

		Prices				Spreads					
		RPE	|RPE|	RMSE(RP)	Corr(P)	RSE	|RSE|	SE	|SE|	RMSE(S)	Corr(S)
All bonds	Mean	0.9%	2.6%	3.8%	89%	-18%	47%	-13	38	44 / 57	34% / 77%
(141 firms, 5454 obs.)	*Std. dev.*	*3.7%*	*2.8%*		*18%*	*58%*	*38%*	*56*	*43*	*36 / -*	*46% / -*
High Rating	Mean	0.7%	2.0%	2.7%	95%	-25%	49%	-11	26	30 / 34	28% / 65%
(54 firms, 2216 obs.)	*Std. dev.*	*2.6%*	*1.8%*		*15%*	*56%*	*36%*	*32*	*21*	*15 / -*	*48% / -*
Low Rating	Mean	0.9%	3.0%	4.4%	86%	-13%	46%	-15	46	53 / 77	38% / 77%
(87 firms, 3238 obs.)	*Std. dev.*	*4.3%*	*3.2%*		*20%*	*59%*	*40%*	*68*	*52*	*42 / -*	*44% / -*
Speculative Grade	Mean	2.0%	4.6%	6.3%	78%	-17%	41%	-38	89	99 / 108	57% / 79%
(14 firms, 574 obs.)	*Std. dev.*	*6.0%*	*4.3%*		*19%*	*47%*	*27%*	*120*	*88*	*66 / -*	*34% / -*

RPE: relative pricing error. RMSE: root mean squared error in relative prices (RP) and spreads (S). RSE: relative spread error. SE: spread error. Absolute values are enclosed in | |. Corr(P) and Corr(S) denote the correlation between estimated and actual prices and spreads. Reported figures are calculated as averages across firm means, which differ negligibly from overall averages, except for spread correlations and RMSE. Overall averages are reported after the slash marks.

price at $\{t_{n+j} = 1 ,..., J\}$ where J is equal to the total number of bond price observations available (the prediction horizon). Therefore, we also need the out-of-sample observations on the stock price $\{\varepsilon^{\text{obs}}_{n+j}; i = 1 ,..., J\}$. The prediction at time t_{n+j} is made using $\varepsilon^{\text{obs}}_{n+j}$ and the risk-free interest rate at t_{n+j} (and the parameter vector estimated at t_n).

The results of the test are presented in Exhibit 3. Since our data set provides at least 12 consecutive observations on each bond, we can calculate correlations between the predicted and actual yield spreads over time. The correlation is on average 34% (and 57% in the speculative subsample). This figure suggests that the model indeed captures an important fraction of the actual changes in bond spreads.

The average (across firms and pricing dates) prediction error is now −18% of the total yield spread, indicating the model tends to overprice bonds farther out-of-sample. Compared to the one-month predictions in Exhibit 2, the model's spread predictions are less accurate as well; the average cross-sectional absolute error has increased from 13% of the spread to 47%. Recall, however, that we are now attempting to predict bond prices up to four years after estimation of the model's parameters.

Exhibit 4 allows a visual inspection of the performance of our model for a selection of bonds.

One apparent explanation for overpricing over longer horizons would be that business risk (asset volatility) on average has increased. Indeed, the estimated average asset volatility from the updated estimations increases from 25% to 34% over the sample period. Since we do not reestimate the model when predicting over the longer horizon, this source of increased credit risk will not be captured and spreads will be underestimated.

This explanation is in line with the Campbell and Taksler [2003] finding that stock volatilities have been increasing over this period. Since book leverage has been constant, our results indicate that the cause is an increase in the firm-specific levels of business risks.

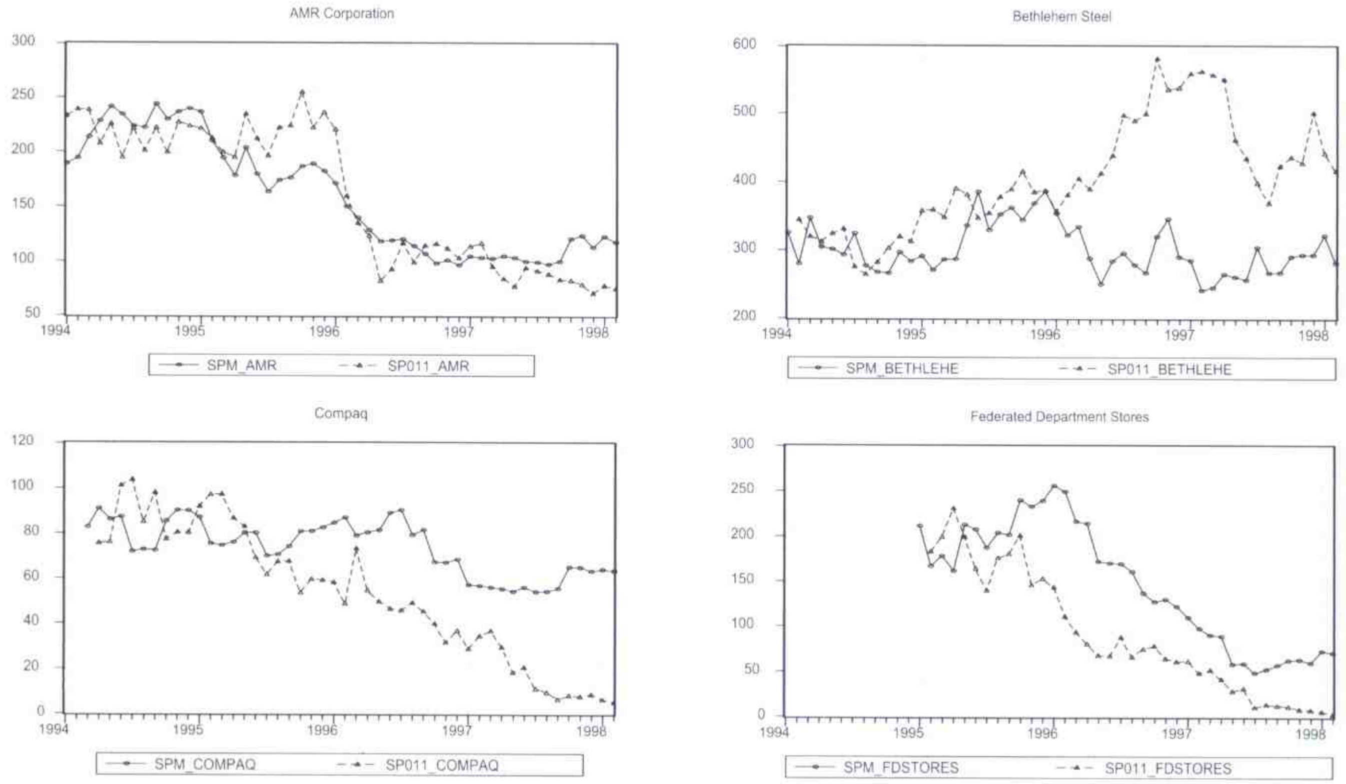
Valuing Bond Portfolios

How well can we expect our model to perform in valuing bond portfolios? If model errors are cross-sectionally independent, the pricing of bond portfolios should turn out to be much more efficient than the pricing of individual bonds.

Exhibit 5 illustrates how the standard deviation of the relative pricing error over time drops as portfolio size increases; the solid line plots a decline from 0.80% to 0.29%. Efficiency is thus improved substantially, even for small portfolios. For example, the standard deviation for a 20-bond portfolio is twice as low as for a single bond. Expanding a portfolio beyond this size has little effect.

We can compare achieved efficiency with the hypothetical efficiency obtained assuming independent errors. In Exhibit 5, the hypothetical efficiency is given by the dotted line, showing a decline to 0.07%. Apparently, a

EXHIBIT 4
Example Multiple Months Out-of-Sample Performance



Y-axis shows spread in basis points. SPM (solid lines) denotes market spread, and SP011 (dashed lines) denotes model spread.

great fraction of the model's errors are diversifiable, but a non-negligible cross-sectional component remains. This is in line with findings by (for example) Collin-Dufresne, Goldstein, and Martin [2001].

V. CONCLUSION

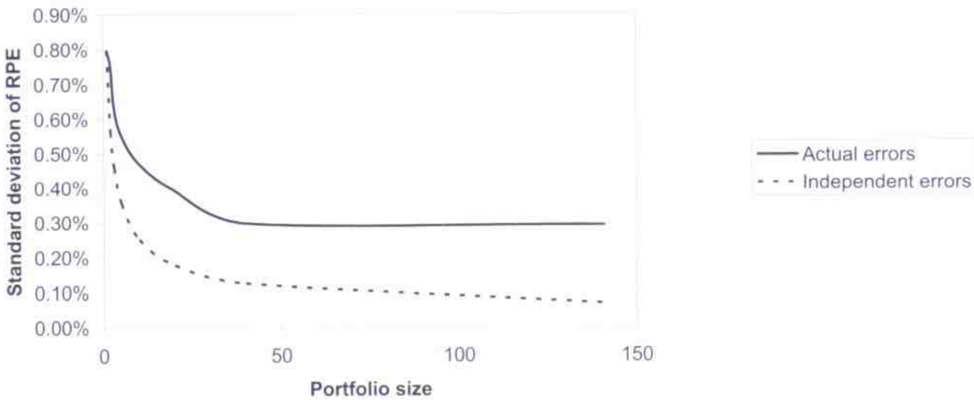
Our structural bond pricing model uses a maximum-likelihood method developed by Duan [1994] for application to a large sample of U.S. corporate bonds. Monte Carlo simulation shows this method to be superior to the commonly used alternative, yet it has to date not been applied to market data. We use as inputs to the estimation stock and bond prices, dividend yields, balance sheet, and risk-free term structure information. In addition, we set the recovery rates for individual bonds as a function of seniority and industry.

The model prices bonds at least as well as reduced-form models, which tend to be viewed as better than structural models in pricing performance. We also find, in a portfolio context, that the greater part of the pricing errors for individual bonds is diversifiable.

If factors other than credit risk affect bond prices, including bond price information in a credit risk model without further risk sources may seem problematic. Any credit risk model—structural or reduced-form—that conditions on bond prices constrains the model to price the bond correctly in-sample. This might effectively overestimate credit risk to compensate for unmodeled risk factors such as liquidity (this effect is acknowledged by Duffee [1999]).

Ideally, a model should of course capture all risk factors. Research to identify these effects and extend traditional purely credit risk-based models to incorporate them is still in its infancy.

EXHIBIT 5
Error Diversification



APPENDIX

Pricing Formulas

Define $G(\omega_t, t)$ as the value of a claim paying \$1 in financial distress:

$$G(\omega_t, t) \equiv E^B\left[e^{-r\mathcal{T}} \cdot 1\right]$$

We let E^B denote expectations under the standard pricing measure. The value of G is given by

$$G(\omega_t, t) = \left(\frac{\omega_t}{L_t}\right)^{-\theta}$$

where:

$$\begin{cases} \theta = \frac{\sqrt{(\mu^B)^2+2r}+\mu^B}{\sigma} \\ \mu^B = \frac{r-\beta-\alpha-0.5\sigma^2}{\sigma} \end{cases}$$

Define $G^\alpha(\omega_t, t)$ as the value of a claim paying $e^{\alpha(\mathcal{T}-t)}$ dollars in financial distress:

$$G^\alpha(\omega_t, t) \equiv E^B\left[e^{-r\mathcal{T}} \cdot e^{\alpha(\mathcal{T}-t)}\right]$$

Its value is

$$G^\alpha(\omega_t, t) = \left(\frac{\omega_t}{L_t}\right)^{-\theta^\alpha}$$

with the constant given by

$$\begin{cases} \theta^\alpha = \frac{\sqrt{(\mu^B)^2+2(r-\alpha)}+\mu^B}{\sigma} \\ \mu^B = \frac{r-\beta-\alpha-0.5\sigma^2}{\sigma} \end{cases}$$

The pricing results follow directly from

$$\begin{aligned} E^B\left[e^{-\rho(\mathcal{T}-t)}\right] &= \int_t^\infty e^{-\rho(s-t)} f^B(X_t; s)\, ds \\ &= e^{-X_t\left(\sqrt{(\mu^B)^2+2\rho}+\mu^B\right)} \end{aligned}$$

The price formulas for the last two building blocks—the binary option $H(\omega_t, t; T)$ and the dollar-in-default with maturity $G(\omega_t, t; T)$—are given below. They include a term that expresses the probabilities (under different measures) that the asset value (ω_T) has not hit the barrier prior to maturity $(\mathcal{T} \nleq T)$—in other words, the *survival probability*. To clarify this common structure, we first state those probabilities in a lemma:

The probabilities of the event $(\mathcal{T} \nleq T)$ (the "*survival event*") at t *under the probability measures* $Q^m := \{B, G\}$ are

$$Q^m(\mathcal{T} \nleq T) = \phi\left(k^m\left(\frac{\omega_t}{L_T}\right)\right) - \left(\frac{\omega_t}{L_t}\right)^{-\frac{2}{\sigma}\mu^m} \phi\left(k^m\left(\frac{L_T}{\omega_t}\right)\right)$$

where

$$k^m(x) = \frac{\ln x + (\alpha + \sigma \cdot \mu^m)(T-t)}{\sigma\sqrt{T-t}}$$

and $\phi(k)$ denotes the cumulative standard normal distribution function with integration limit k.

The probability measure Q^B is the standard risk-adjusted pricing measure, and Q^G is the measure where $G(\omega_t, t)$ is the numeraire. Using this lemma, we obtain the pricing formulas for the building blocks, the binary option and the dollar-in-default claim, in a convenient form.

The price of a down-and-out binary option (with unit payoff at T conditional on no prior default) is:

$$H(\omega_t, t; T) = e^{-r(T-t)} \cdot Q^B(\mathcal{T} \nleq T)$$

with the probability given by the lemma.

The price of a dollar-in-default claim is

$$G(\omega_t, t; T) = G(\omega_t, t) \cdot \left(1 - Q^G(\mathcal{T} \nleq T)\right)$$

with probability given by the lemma.

The value of receiving a dollar if default occurs prior to T must be equal to receiving a dollar-in-default claim with infinite maturity, less a claim where you receive a dollar in default conditional on it *not* occurring prior to T:

$$G(\omega_t, t; T) = G(\omega_t, t) - e^{-r(T-t)} E^B\left[G(\omega_T, T) \cdot I_{\mathcal{T} \nless T}\right]$$

Using a change of probability measure, we can separate the variables within the expectation brackets (see, e.g., Geman, El-Karoui, and Rochet [1995]:

$$\begin{aligned} G(\omega_t, t; T) &= G(\omega_t, t) - e^{-r(T-t)} E^B\left[G(\omega_T, T)\right] \cdot E^G\left[I_{\mathcal{T} \nless T}\right] \\ &= G(\omega_t, t) \cdot \left(1 - Q^G(\mathcal{T} \nleq T)\right) \end{aligned}$$

Value of Debt

The pricing formula for debt issued at t with principal $n = \alpha\, N_t\, dt$ is standard:

$$d(\omega_t, t) = n \cdot (1 - G(\omega_t, t)) + \delta \cdot n \cdot G(\omega_t, t)$$

with the recovery rate determined as:

$$\delta = (1 - \varepsilon - k) \cdot \frac{L_t}{N_t}$$

Equity

Inserting Equation (3) into (2), we obtain:

$$\mathcal{E}(\omega_t, t) \equiv \begin{cases} E^B\left[\int_t^\infty e^{-r(s-t)} \beta \omega_s I_{\mathcal{T} \nless s} ds\right] \\ -E^B\left[\int_t^\infty e^{-r(s-t)} (1-\tau) C_s I_{\mathcal{T} \nless s} ds\right] \\ +E^B\left[\int_t^\infty e^{-r(s-t)} d(\omega_s, s) I_{\mathcal{T} \nless s} ds\right] \\ +E^B\left[e^{-r(\mathcal{T}-t)} \varepsilon_{\mathcal{T}} L_{\mathcal{T}}\right] \end{cases} \tag{A-1}$$

Consider the first line of Equation (A-1):

$$\begin{aligned} & E^B\left[\int_t^\infty e^{-r(s-t)} \beta \omega_s (1 - I_{\mathcal{T}<s})\, ds\right] \\ = \ & \omega_t - \underbrace{E^B\left[\int_{\mathcal{T}}^\infty e^{-r(s-t)} \beta \omega_s ds\right]}_{\text{value at } t \text{ of earnings after default}} = \omega_t - E^B\left[e^{-r(\mathcal{T}-t)} \omega_{\mathcal{T}}\right] \\ = \ & \omega_t - E^B\left[e^{-r(\mathcal{T}-t)} L_{\mathcal{T}}\right] = \omega_t - E^B\left[e^{-r(\mathcal{T}-t)} L_t e^{\alpha(\mathcal{T}-t)}\right] \end{aligned} \tag{A-2}$$

Now split the coupon stream (line 2) into two parts: debt service to current debtholders, and debt service to future debtholders:

$$\begin{aligned} & \int_t^\infty e^{-r(s-t)} (1-\tau) C_s I_{\mathcal{T} \nless s} ds \\ = \ & \int_t^\infty e^{-r(s-t)} (1-\tau) C_t I_{\mathcal{T} \nless s} ds \\ & + \int_t^\infty e^{-r(s-t)} (1-\tau) C_t \left(e^{\alpha(s-t)} - 1\right) I_{\mathcal{T} \nless s} ds \end{aligned} \tag{A-3}$$

(The coupon accruing to debt issued after time t, at time s, is $C_S - C_t = C_t e^{\alpha(s-t)} - C_t$.) The value of future borrowed money [line 3 of Equation (A-1)] must be equal to the value of total debt service to future borrowed money plus the value of payouts in default accruing to future borrowers. Formally:

$$\begin{aligned} & E^B\left[\int_t^\infty e^{-r(s-t)} d(\omega_s, s) I_{\mathcal{T} \nless s} ds\right] \\ = \ & E^B\left[\int_t^\infty e^{-r(s-t)} C_t \left(e^{\alpha(s-t)} - 1\right) I_{\mathcal{T} \nless s} ds\right] \\ & + (1 - \varepsilon - k) \cdot E^B\left[e^{-r(\mathcal{T}-t)} (L_{\mathcal{T}} - L_t)\right] \end{aligned} \tag{A-4}$$

Applying (A-2), (A-3), and (A-4) to Equation (A-1), and cancelling terms, we arrive at:

$$\begin{aligned}\mathcal{E}(\omega_t,t) &= \omega_t - L_t \cdot E^B\left[e^{(\alpha-r)(\mathcal{T}-t)}\right] \\ &\quad - C_t \cdot E^B\left[\int_t^\infty e^{-r(s-t)} I_{\mathcal{T}\nleq s} ds\right] \\ &\quad + \tau C_t \cdot E^B\left[\int_t^\infty e^{-(r-\alpha)(s-t)} I_{\mathcal{T}\nleq s} ds\right]\end{aligned}$$

which, using earlier results, yields the final formula for $r \neq \alpha$. The result for $r = \alpha$ is obtained from taking the limit as $\alpha \to r$.

Default Threshold

The endogenous reorganization barrier is the value of assets at which the value of equity in solvency equals the value of equity in reorganization:

$$\mathcal{E}(\omega_t, t) = \varepsilon \cdot L_t$$

Applying a smooth pasting condition implies that

$$\frac{\partial \mathcal{E}(\omega_t,t)}{\partial \omega_t}\Big|_{\omega_t = L_t} = \varepsilon$$

The optimal barrier can be solved for in closed-form. When $\beta > 0$ and $r \neq \alpha$, it is:

$$L_t\big|_{\beta>0, r\neq\alpha} = \frac{\tau\frac{r}{r-\alpha}\theta^\alpha - \theta}{(\varepsilon-1)\cdot(1+\theta^\alpha) + (1-\varepsilon-k)\cdot(\theta^\alpha - \theta)} N_t$$

As can be seen, the endogenous barrier grows at rate α.

ENDNOTES

[1]We do not consider other default-risky financial instruments such as credit derivatives.

[2]We adopt this assumption in order to obtain closed-form solutions for the equity price, which is crucial for our estimation procedure. Ideally, the debt growth rate would be an endogenous function of the firm value and its environment.

[3]If a firm's debt consists mainly of bonds, this is clearly an underestimation of the market coupon rate (which will compensate for default risk). Yet a non-trivial fraction of the firm's debt is likely to consist of non-interest-bearing debt such as accrued taxes and payments to suppliers. In that case, the overall coupon will lie somewhere between zero and the market coupon on a traded bond.

[4]This formula holds only provided $r \neq \alpha$; the formula for $r = \alpha$ is the limiting case. It does hold for $\beta = 0$.

[5]An important feature of the FISD database is that it distinguishes between trader quotes and matrix quotes. As matrix quotes may not provide a reliable estimate of the price at which a trade could take place, we eliminate them from our sample.

[6]We also eliminate bonds for which it was clear that prices were stale, where data entry errors were apparent, or for which corresponding equity or balance sheet data were not available. Two additional firms were dropped because of convergence problems in the estimation procedure.

[7]For completeness, we also report the overall correlation, which is 97%. Note, however, that this high level is unsurprising, given the nature of the one month-ahead pricing exercise.

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