A Simple Exponential Model for Dependent Defaults

Kay Giesecke

Kay Giesecke
is a visiting assistant professor of operations research at the School of Operations Research and Industrial Engineering at Cornell University in Ithaca, NY.
giesecke@orie.cornell.edu

Credit risk refers to the risk of incurring losses due to unexpected changes in the credit quality of a counterparty or issuer. Whether because issuers are subject to common general economic factors or because firms are directly linked, the credit quality changes of several issuers are often correlated. Investors holding positions with numerous counterparties such as financial institutions are therefore exposed to the aggregated risk of losses due to correlated credit event arrivals.

The effective measurement and management of this aggregated credit risk is one of the core businesses of a financial institution. Credit risk measurement involves estimating the distribution of aggregated losses. Credit derivatives, which allow holders to isolate and trade credit risk by providing a payoff upon a credit event arrival with respect to a reference entity, allow the active management of credit risk. *Multiname* credit derivatives have payoffs contingent on the credit quality of a number of reference entities. Collateralized debt obligations (CDO) allow the restructuring of portfolios of credit-risky securities. They involve prioritized tranches whose cash flows are linked to the performance of a pool of debt instruments.

The estimation of aggregated loss distributions in credit risk measurement and the valuation of multiname credit derivatives and CDOs requires a model for the joint default behavior of numerous credit-risky securities such as bonds or loans. Intensity-based models, which model the stochastic structure of default by an exogenous intensity or conditional default arrival rate, have been successfully extended to multifirm settings.

Duffie and Garleanu [2001] assume that an individual firm's default intensity is composed of some idiosyncratic component and some systematic component affecting all considered firms. Duffie and Singleton [1998] present several computationally efficient models relying on intensity processes that exhibit common or correlated jumps. Schoenbucher and Schubert [2001] impose an assumed default dependence structure directly on a generic stochastic intensity model for individual firms. Giesecke [2002] underpins a multifirm intensity model with a structural asset-based model, where firms' assets and default thresholds are correlated.

We study here a simple and tractable intensity-based model for correlated default times that is in the spirit of Duffie and Garleanu [2001]. The model is based on the idea that a firm's default is driven by idiosyncratic as well as other regional, sectoral, industry, or economywide shocks whose arrivals are modeled by independent Poisson processes.

In this approach, a default is governed again by a Poisson process, and default times are jointly exponentially distributed. Various important quantities such as moments, default correlations, conditional default probabilities, and joint default probabilities can be calculated explicitly. A closely related approach is in Lindskog and McNeil [2001], who consider default losses of different types.

The complete non-linear default dependence structure can be described by the copula of the default times. In the exponential model the *exponential copula* arises naturally. This explicitly available copula facilitates the calculation of non-linear default correlation measures and lends itself to flexible and efficient simulation algorithms for correlated default arrival times for pricing and risk management purposes. These algorithms can be used to generate default times with an exponential dependence structure but arbitrary marginal default distribution. This means one can combine any given single default probability model with the exponential correlation model discussed here.

The parameters of the model can be calibrated from individual default swaps or bond price data (for pricing purposes) as well as data provided by rating agencies (e.g., Moody's diversity score) and often-implemented credit risk management solutions (e.g., KMV's CreditMonitor).

I. EXPONENTIAL MODEL

We consider the model first in a two-firm case, and then extend to the general multivariate case.

Bivariate Case

We begin by deriving the basic properties of the model with two firms labeled 1 and 2. The idea of the approach is to let defaults of firms be driven by firm-specific as well as economywide shock events.

Suppose there are Poisson processes N_1, N_2, and N with respective intensities λ_1, λ_2, and λ.[4] We interpret λ_i as the idiosyncratic shock intensity of firm i, while λ is the intensity of a macroeconomic or economywide shock affecting both firms simultaneously.

We define the default time τ_i of firm i by

$$\tau_i = \inf\{t \geq 0: N_i(t) + N(t) > 0\}$$

meaning that a default takes place completely unexpectedly if either an idiosyncratic or a systematic shock strikes the firm for the first time. Note that there is a positive probability of a simultaneous default of both firms. Thus firm i defaults with intensity $\lambda_i + \lambda$, and we have:

$$s_i(t) = P[\tau_i > t] = P[N_i(t) + N(t) = 0] = e^{-(\lambda_i+\lambda)t}$$

The expected default time and the default time variance are given by:

$$E[\tau_i] = \frac{1}{\lambda_i + \lambda}$$

$$\mathrm{Var}[\tau_i] = \frac{1}{(\lambda_i + \lambda)^2}$$

The joint survival probability is found to be:

$$\begin{aligned} s(t, u) &= P[\tau_1 > t, \tau_2 > u] \\ &= P[N_1(t) = 0, N_2(u) = 0, N(t \vee u) = 0] \\ &= e^{-\lambda_1 t - \lambda_2 u - \lambda(t \vee u)} \\ &= e^{-(\lambda_1+\lambda)t - (\lambda_2+\lambda)u + \lambda(t \wedge u)} \\ &= s_1(t)s_2(u)\min(e^{\lambda t}, e^{\lambda u}) \end{aligned} \tag{1}$$

This *bivariate exponential distribution* is well known in reliability modeling; see, for example, Marshall and Olkin [1967]. The function s has an absolutely continuous and singular component, which can be calculated explicitly. The moment-generating function of s is available analytically as well.

There is a unique function $C^\tau : [0, 1]^2 \rightarrow [0, 1]$, called the *survival copula* of the default time vector (τ_1, τ_2) such that joint survival probabilities can be represented as

$$s(t, u) = C^\tau[s_1(t), s_2(u)]$$

The copula C^τ describes the complete non-linear *default time dependence structure*. As C^τ couples exponential marginals s_i with the exponential joint survival function s, we will call C^τ the exponential copula. In the reliability literature this copula is also known as the Marshall-Olkin copula. See Nelsen [1999] for background information.

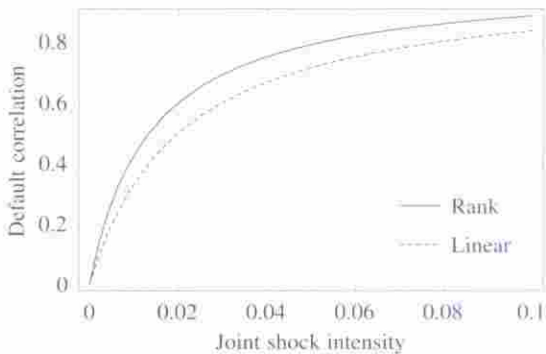
Letting s_i^{-1} denote the inverse function of s_i and defining:

$$\theta_i = \frac{\lambda}{\lambda_i + \lambda}$$

as the ratio of joint default intensity to the default intensity of firm i, we obtain:

$$C^\tau(u, v) = s[s_1^{-1}(u), s_2^{-1}(v)] = \min(vu^{1-\theta_1}, uv^{1-\theta_2})$$

EXHIBIT 1
Rank and Linear Default Correlation



The parameter vector $\theta = (\theta_1, \theta_2)$ controls the degree of dependence between the default times; we write $C^\tau = C^\tau_\theta$. If the firms default independently of each other ($\lambda = 0$ or $\lambda_1, \lambda_2 \to \infty$), then $\theta_1 = \theta_2 = 0$, and we get $C^\tau_\theta(u,v) = uv$, the product copula. If the firms are perfectly positively correlated and they default simultaneously ($\lambda \to \infty$ or $\lambda_1 = \lambda_2 = 0$), then $\theta_1 = \theta_2 = 1$, and $C^\tau_\theta(u,v) = u \wedge v$, the Fréchet upper-bound copula.

Hence:

$$uv \leq C^\tau_\theta(u, v) \leq u \wedge v \qquad (2)$$
$$\theta \in [0, 1]^2$$
$$u, v \in [0, 1]$$

meaning that in our model defaults can only be positively related.

Besides the survival copula C^τ there is a unique function $K^\tau [0,1]^2 \to [0,1]$ such that the bivariate distribution function p of τ can be represented as

$$p(t, u) = P[\tau_1 \leq t, \tau_2 \leq u] = K^\tau(p_1(t), p_2(u))$$

where $p_i(t) = P[\tau_i \leq t] = 1 - s_i(t)$ is the distribution function of τ.

The (usual) copula K^τ and the survival copula C^τ describe in an equivalent way the dependence between the default times. Noting that $s(t, u) = 1 - p_1(t) - p_2(u) + p(t, u)$, we see that these copulas are related via:

$$\begin{aligned} K^\tau(u, v) &= C^\tau(1-u, 1-v) + u + v - 1 \\ &= \min([1-v][1-u]^{1-\theta_1}, [1-u][1-v]^{1-\theta_2}) + u + v - 1 \end{aligned}$$

Equation (2) suggests a partial ordering on the set of copulas as function-valued default correlation measures. A scalar-valued measure such as rank correlation can perhaps provide more intuition about the degree of stochastic dependence between the defaults. Spearman's rank correlation ρ^S is simply the linear correlation ρ of the copula K^τ given by:

$$\begin{aligned} \rho^S(\tau_1, \tau_2) &= \rho[p_1(\tau_1), p_2(\tau_2)] \\ &= 12\int_0^1\int_0^1 K^\tau(u, v)du dv - 3 \\ &= \frac{3\theta_1\theta_2}{2\theta_1 + 2\theta_2 - \theta_1\theta_2} \\ &= \frac{3\lambda}{3\lambda + 2\lambda_1 + 2\lambda_2} \end{aligned} \qquad (3)$$

showing that ρ^S is a function of the copula K^τ only. Notice that the same result obtains when we compute ρ^S with C^τ instead of K^τ.

While Spearman's $\rho^S(\tau_1, \tau_2)$ measures the degree of monotonic default dependence, linear default time correlation given by

$$\rho(\tau_1, \tau_2) = \frac{\lambda}{\lambda + \lambda_1 + \lambda_2} \qquad (4)$$

measures the degree of linear default time-dependence only. Clearly $\rho \leq \rho^S$, and linear default correlation underestimates the true default dependence.

Exhibit 1 plots both measures as functions of the joint shock intensity λ. We fix $\lambda_1 = \lambda_2 = 0.01$, which corresponds to a one-year default probability of about 1% when firms are independent. If λ is zero, then firms default independently and $\rho = \rho^S = 0$. With increasing λ, the joint shock component of the default risk dominates the idiosyncratic component, and the default correlation increases.

The relationships among ρ^S, idiosyncratic intensities, and joint shock intensities is shown in Exhibit 2, where ρ^S is plotted as a function of $\lambda_1 = \lambda_2$ for varying λ. Rank default correlations decline with idiosyncratic default risk, because with increasing $\lambda_1 = \lambda_2$ the idiosyncratic risk component dominates the joint shock component of default risk. If $\lambda_1 = \lambda_2 = 0$, then only the joint shock matters; if this shock occurs, firms default simultaneously and the rank default correlation is one.

Summarizing, we have $\rho^S, \rho \in [0, 1]$, where $\rho^S = \rho = 0$ if $\lambda = 0$ or $\lambda_1, \lambda_2 \to \infty$ (C^τ is the product copula),

EXHIBIT 2

Rank Default Correlation as Function of Idiosyncratic Shock Intensity and Varying Joint Shock Intensity

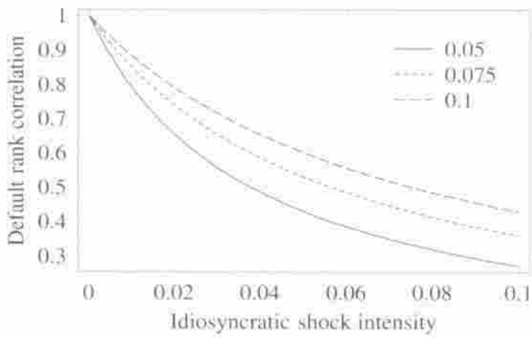
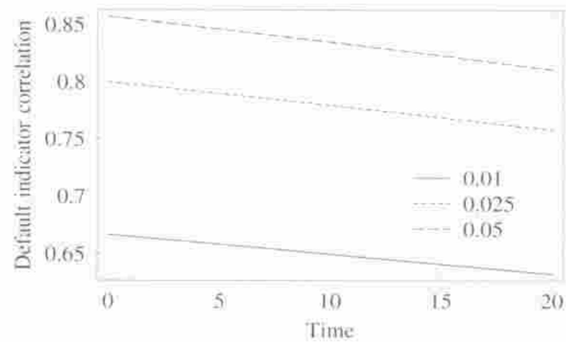


EXHIBIT 3

Default Indicator Correlation over Time—Varying Joint Shock Intensity



and $\rho^S = \rho = 1$ if $\lambda \to \infty$ or $\lambda_1 = \lambda_2 = 0$ (C^τ is the Fréchet upper-bound copula).

Another common default correlation measure is the linear correlation of the default indicator variables:

$$\rho(1_{\{\tau_1 \le t\}}, 1_{\{\tau_2 \le t\}}) = \frac{s(t,t) - s_1(t)s_2(t)}{\sqrt{p_1(t)s_1(t)p_2(t)s_2(t)}} \tag{5}$$

Exhibit 3 shows $\rho(1_{\{\tau_1 \le t\}}, 1_{\{\tau_2 \le t\}})$ over time for varying degrees of joint shock intensities λ (again we set $\lambda_1 = \lambda_2 = 0.01$). In our model, default indicator correlation declines with time. This is different from the structural model of Zhou [2001], where the indicator correlation is hump-shaped.

Conclusions drawn on the basis of linear default time correlation $\rho(\tau_1, \tau_2)$ and linear default indicator correlation $\rho(1_{\{\tau_1 \le t\}}, 1_{\{\tau_2 \le t\}})$ should be interpreted with care. Both are covariance-based, and hence are only the natural dependence measures for joint elliptical random variables; see Embrechts, McNeil, and Straumann [2001]. Neither default times nor default events are joint-elliptical, so these measures can lead to severe misinterpretations of the true default correlation structure. ρ^S is by contrast defined on the level of the copula C^τ, and therefore does not share these deficiencies.

General Multivariate Case

We now extend the default model to the general multivariate case with $n \ge 2$ firms. The default of an individual firm is driven by some idiosyncratic shock as well as other sectoral, industry, country-specific, or economy-wide shocks. These

$$m = \sum_{k=1}^{n} \binom{n}{k}$$

shocks are governed by independent Poisson processes that are numbered consecutively. To indicate whether a non-firm-specific shock leads to a default of some given firm, we introduce a matrix $(a_{ij})_{n \times m}$, where $a_{ij} = 1$ if shock $j \in \{1, 2, \ldots, m\}$, modeled through the Poisson process N_j with intensity λ_j, leads to a default of firm $j \in \{1, 2, \ldots, n\}$, and $a_{ij} = 0$ otherwise.

For example, for $n = 3$ firms, a full specification of the model would involve:

$$(a_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Other specifications can be adapted to the case at hand. For example, if economywide shock events can be excluded, one would set $a_{i7} = 0$ for $i = 1, 2, 3$, which corresponds to bivariate dependence only.

According to this default specification, we get

$$\tau_i = \inf\{t \ge 0: \sum_{k=1}^{m} a_{ik} N_k(t) > 0\}$$

meaning that firm i defaults with intensity $\sum_{k=1}^{m} a_{ik}\lambda_k$ and

$$s_i(t) = \exp(-\sum_{k=1}^{m} a_{ik}\lambda_k t)$$

Arguments similar to those leading to (1) yield the joint survival function:

$$s(t_1, \ldots, t_n) = P[\tau_1 > t_1, \ldots, \tau_n > t_n]$$

$$= \exp(-\sum_{k=1}^{m} \lambda_k \max(a_{1k}t_1, \ldots, a_{nk}t_n)) \qquad (6)$$

Joint default probabilities p and the associated copula K^τ can be obtained by standard arguments from s:

$$p(t_1, \ldots, t_n) = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} s(v_{1i_1}, \ldots, v_{ni_n}) \qquad (7)$$

where $v_{j1} = t_j$ and $v_{j2} = 0$. The exponential survival copula associated with s can be found via $C^\tau(u_1, \ldots, u_n) = s[s_1^{-1}(u_1), \ldots, s_n^{-1}(u_n)]$. Fixing some $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, the two-dimensional marginal copula is given by:

$$C^\tau(u_i, u_j) = C^\tau(1, \ldots, 1, u_i, 1, \ldots, 1, u_j, 1, \ldots, 1)$$

$$= \min(u_i^{1-\theta_i} u_j, u_i u_j^{1-\theta_j})$$

where we define, analogously to the bivariate case:

$$\theta_i = \frac{\sum_{k=1}^{m} a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^{m} a_{ik} \lambda_k}$$

$$\theta_j = \frac{\sum_{k=1}^{m} a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^{m} a_{jk} \lambda_k}$$

as the ratio of the joint default intensity of firms i and j to the default intensity of firm i or j, respectively. In analogy to (3), we can now compute Spearman's rank default time correlation matrix $(\rho_{ij}^S)_{n \times n}$ by

$$\rho_{ij}^S = \frac{3\theta_i\theta_j}{2\theta_i + 2\theta_j - \theta_i\theta_j} \qquad (8)$$

Extensions

We note at least two extensions of the basic setup. Instead of assuming that shock incidents lead to immediate default events, shocks may not be necessarily fatal. In the bivariate case, suppose an idiosyncratic shock leads to a default of firm i only with a prespecified probability q_i. An economywide shock leads to a default of both firms with probability q_{11}, to a default of firm 1 only with probability q_{10}, and to a default of firm 2 only with probability q_{01}. The exponential default time distribution is:

$$s(t, u) = e^{-\gamma_1 t - \gamma_2 u - \gamma(t \vee u)}$$

where $\gamma_1 = \lambda_1 q_1 + \lambda q_{10}$, $\gamma_2 = \lambda_2 q_2 + \lambda q_{01}$, and $\gamma = \lambda q_{11}$. Of course, with $q_1 = q_2 = q_{11} = 1$, we obtain the model in Equation (1). The general case is analogous. Note, however, that in the non-fatal model interpretation, there are potentially many model parameters, which makes the model calibration very challenging.

Another extension concerns the variability of intensities over time. In our Poisson setup, under risk-neutrality the term structure of model-implied credit yield spreads is flat (the local yield spread is in fact given by the constant intensity itself). In practice, however, credit spreads often vary substantially over time.

To capture these effects, in a first step the Poisson framework can be generalized to deterministically varying intensities, i.e., to inhomogeneous Poisson shock arrivals. The intensity function may, for example, be assumed to be piecewise constant, which is a reasonably flexible approximation in certain applications. In the bivariate case, $s_i(t) = \exp(\int_0^t [\lambda_i(r) + \lambda(r)]dr)$ and

$$s(t, u) = e^{-\int_0^t \lambda_1(r)dr - \int_0^u \lambda_2(r)dr - \int_0^{t \vee u} \lambda(r)dr}$$

In a second step, we can extend to general stochastic intensities. Such models would capture, in a realistic way, the stochastic variation in the term structure of credit spreads. One needs, however, a large and reliable database to calibrate the parameters of such a stochastic intensity model. Now $s_i(t) = E[\exp(\int_0^t [\lambda_i(u) + \lambda(u)])]$ and

$$s(t, u) = E[e^{-\int_0^t \lambda_1(r)dr - \int_0^u \lambda_2(r)dr - \int_0^{t \vee u} \lambda(r)dr}]$$

For an extensive study of various correlation models with stochastic intensities, see Duffie and Singleton [1998].

II. SIMULATING CORRELATED DEFAULTS

Let us consider a four-step algorithm, which generates default arrival times with an exponential dependence structure C^τ while allowing for arbitrary marginal default time distributions, which we take as given.

1. Simulate an m-vector $(t_1, \ldots, t_m)$ of independent exponential shock arrival times with given parameter vector $(\lambda_1, \ldots, \lambda_m)$ where $\lambda_k > 0$. This is done by drawing, for $k \in \{1, 2, \ldots, m\}$, an independent standard uniform random variate U_k and setting:

$$t_k = -\frac{1}{\lambda_k}\ln U_k$$

Indeed, $P[t_k > T] = P[-\ln U_k/\lambda_k > T] = P[U_k \le e^{-\lambda_k T}] = e^{-\lambda_k T}$.

2. Simulate an n-vector $(T_1, \ldots, T_n)$ of joint exponential default times by considering, for each firm $i \in \{1, 2, \ldots, n\}$, the minimum of the relevant shock arrival times:

$$T_i = \min\{t_k : 1 \le k \le m,\ a_{ik} = 1\}$$

3. Generate a sample $(v_1, \ldots, v_n)$ from the (survival) default time copula C^τ by setting, for $i \in \{1, 2, \ldots, n\}$:

$$v_i = s_i(T_i) = \exp(-T_i \sum_{k=1}^{m} a_{ik}\lambda_k)$$

4. In order to generate an n-vector $Z = (Z_1, \ldots, Z_n)$ of correlated default arrival times with given marginal survival function q_i and exponential default dependence structure C^τ, set, for $i \in \{1, 2, \ldots, n\}$:

$$Z_i = q_i^{-1}(v_i)$$

provided that the inverse q_i^{-1} of q_i exists.

We can base our default time simulation on survival marginals q_i from a model completely different from the exponential shock model already developed. One could use a general stochastic intensity-based model or a structural model for q_i, or use some estimated survival function. To account for different types of idiosyncratic default risk, we can also choose different survival marginals q_i for different firms. The algorithm then generates correlated default times Z_i with these, given q_i and an exponential dependence structure:

$$P[Z_1 > t_1, \ldots, Z_n > t_n] = C^\tau(q_1(t_1), \ldots, q_n(t_n))$$

In the special case where $q_i = s_i$ for all i, the simulated default time vector Z is jointly exponentially distributed. That is, Z has exponential marginals and an exponential copula C^τ, and its joint survival function is given by Equation (6). In that case, $Z_i = T_i$, and we have to perform steps 1 and 2 only.

From the methodology, this approach bears some interesting similarities to the approach followed by Schoenbucher and Schubert [2001]. In their approach, the survival marginals are given by some stochastic intensity model, while one is free to choose the default copula joining marginals and joint default distribution (the copula is not directly prescribed by their intensity model). In our approach, the copula of the default times is prescribed by the model structure, while one is free to choose appropriate marginals. The exponential marginals would be only one possible choice, although a natural one.

III. ESTIMATING SHOCK INTENSITIES

A first step in estimating the model parameters is to specify the shock impact matrix (a_{ij}). In a second step, the relevant (at most m) shock intensities must be estimated. Here one may be interested in risk-neutral intensities for derivatives pricing purposes, or real-world objective intensities for risk management purposes.

To deal with the limited appropriate data available for the estimation of higher-order shock intensities, it might be reasonable to specify a model (a_{ij}) with bivariate (trivariate) dependence only, i.e., a model where at most two (three) simultaneous defaults are possible. Another assumption to account for limited data is to set higher-order shock intensities equal for shocks leading to more than one (two, three, ...) defaults. The simplest model to account for default correlation in a credible way would then rely on bivariate dependence with equal intensities for the joint defaults of two firms (we refer to this as *bivariate symmetric dependence*). We would then have to estimate $n + 1$ intensities.

Risk-Neutral Intensities

Liquid bond price or corporate credit default swap quotes can be used to back out risk-neutral survival probabilities $s'_i(t)$ of each firm i for some fixed horizon t (see Hull and White [2000] for the basic procedure).

We then have the relations:

$$-\frac{1}{t}\ln s'_i(t) = \sum_{k=1}^{m} a_{ik}\lambda_k \text{ for } i = 1, 2, \ldots, n \tag{9}$$

If quotes for a range of different bond or swap maturities t are available, we can obtain a system of m equations in the m unknown intensities λ_k. If this system does not admit a unique solution, we can choose the λ_k via some least squares minimization procedure, for example.

If price observations are available only for some particular date t, the relations (9) alone are not sufficient to determine all intensities uniquely; higher-order survival probabilities are needed in addition. If prices of appropriate products (such as first-to-default swaps) are quoted, one can back out risk-neutral higher-order survival probabilities s'. Liquid quotes are rarely available, however.

Bivariate joint survival probabilities s' can be derived from single-firm default swap values if the protection-selling swap counterparty is itself subject to default. With suitable assumptions on the model structure (a_{ij}) and the associated shock intensities, we can then exploit the relation:

$$-\ln s'(t_1, \ldots, t_n) = \sum_{k=1}^{m} \lambda_k \max(a_{1k}t_1, \ldots, a_{nk}t_n),\ t_i \geq 0$$

in addition to (9) in order to determine the shock intensities λ_k.

Objective Intensities

Real-world (objective) survival probabilities $s_i(t)$ of individual firms can be estimated from historical default data. Without a proprietary database, one can use long-run average values of credit rating classes, which are commonly provided by credit rating agencies. Equation (9) is then used with s_i in place of s_i'. Data on joint defaults, however, are much too sparse to obtain reliable statistical estimates for joint default probabilities, so one has to resort to other sources of suitable information.

Suppose Moody's supplies a diversity score d for the n-firm portfolio under consideration. This is most common if the underlying portfolio is the collateral pool of some collateralized debt obligation (CDO). That is, the original portfolio of n correlated firms is considered to be equivalent to a portfolio of d independent firms with the same survival probability $q(t)$ but a notional principal of (n/d) times the original notional.

The diversity score d is determined so that the first two moments of the distribution of the number of defaults in these two portfolios

$$\sum_{i=1}^{n} 1_{\{\tau_i > t\}} \text{ and } \frac{n}{d}\sum_{i=1}^{d} 1_{\{\sigma_i > t\}} \tag{10}$$

are equal for some fixed horizon t, say, one year. The $1_{\{\sigma_i > t\}}$ are independent and identically distributed Bernoulli random variables with success probability $q(t)$. This yields the relations:

$$nq(t) = \sum_{i=1}^{n} s_i(t)$$

$$\frac{n^2}{d} q(t)(1 - q(t)) =$$

$$\sum_{i=1}^{n}\left(s_i(t)(1 - s_i(t)) + \sum_{j=1, j\neq i}^{n} (s_{ij}(t, t) - s_i(t)s_j(t))\right)$$

Given the diversity score d, one can now use these equations to obtain an estimate of the sum of the pairwise joint survival probabilities s_{ij} for firms i and j. Assuming a model (a_{ij}) with bivariate symmetric dependence, we are then able to estimate the shock intensity leading to a simultaneous default of two firms.

Credit risk management software packages such as KMV's Portfolio Manager often provide pairwise default event correlations in the form of (5). Rating agencies provide default correlation matrices for industries. Erturk [2000] and Nagpal and Bahar [2001] estimate these correlations using historical default data from Standard & Poor's. For estimates of rating category correlations from Moody's data, see Carty [1997].

Such correlations let us calibrate a model with (not necessarily symmetric) bivariate dependence via (5). More consistent would be the use of rank default correlations (3), which would let us estimate a model (a_{ij}) that is fully specified (i.e., a model beyond bivariate symmetric dependence).

IV. APPLICATIONS

We describe two applications of the model.

Default Distribution

In credit risk management, the exponential default model can be used to measure the aggregated default risk associated with some portfolio of credits. The most comprehensive measure of that aggregated risk is the distribution of total losses due to defaults.

We consider such a loss distribution under the simplifying assumption that there is zero recovery in case of default. The default loss L_t at some fixed horizon t is then equal to the number of the defaulted firms $L_t = n - M_t$, where:

$$M_t = \sum_{i=1}^{n} 1_{\{\tau_i > t\}}$$

is the number of firms that are still operating at t. The distribution of M_t can be computed directly from the joint survival probabilities.

By standard arguments we find

$$P[M_t = k] = \sum_{i=k}^{n} \binom{i}{k} (-1)^{i-k} \sum_{J \subset \{1,\ldots,n\}, |J|=i} P[\bigcap_{j \in J} \{\tau_j > t\}]$$

where the $|J|$-dimensional marginal joint survival probability

$$P[\bigcap_{j \in J} \{\tau_j > t\}]$$

is directly available from (6) for all $J \subset \{1, \ldots, n\}$. This can be simplified if the firms in the portfolio are homogeneous and symmetric, i.e., if the default time vector is exchangeable:

$$(\tau_1, \ldots, \tau_n) \stackrel{d}{=} (\tau_{z(1)}, \ldots, \tau_{z(n)})$$

for any permutation $z(1), \ldots, z(n)$ of indexes $(1, \ldots, n)$ (by $\stackrel{d}{=}$ we mean equality in distribution).

Such a portfolio structure is a reasonable approximation for a well-diversified retail or small and medium sized exposures (SME) portfolio (or at least subportfolios of such portfolios). We then denote by π_k the k-th dimensional survival probability:

$$\pi_k(t) = P[\tau_{i_1} > t, \ldots, \tau_{i_k} > t],$$
$$\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\},\ 1 \le k \le n$$

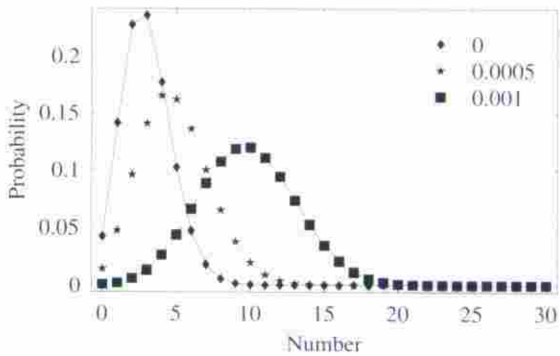
which is the probability of survival of any arbitrarily selected subgroup of k firms by horizon t.

We then have simply

$$P[M_t = k] = \sum_{i=0}^{n-k} (-1)^i \frac{n!}{i!k!(n-k-i)!} \pi_{k+i}(t)$$

To illustrate the effect of default correlation on the default distribution, let us consider an exponential model (a_{ij}) with symmetric bivariate dependence, where firms default individually with intensity λ, and shocks leading to a default of any two firms arrive simultaneously with intensity $\bar{\lambda}$. Then:

EXHIBIT 4
Distribution of Number of Defaults—Varying Joint Shock Intensity



$$\pi_k(t) = \exp(-(\lambda k + \bar{\lambda}[\binom{k}{2} + k(n-k)])t)$$

Exhibit 4 graphs the distribution of the number of defaults $P[L_t = k] = P[M_t = n - k]$ for a ten-year time horizon with $n = 30$ firms, where the joint shock intensity $\bar{\lambda}$ is varied. We hold the one-year default probability fixed at 1%.

It is clear that the expected number of defaults $E[L_t]$ increases with the systemic shock probability. This effect is shown in Exhibit 5. It is interesting that the variance of the number of defaults is hump-shaped; the variance increases with $\bar{\lambda}$ up to a certain point only; after that point it declines with $\bar{\lambda}$.

The joint shock intensity $\bar{\lambda}$ starts to dominate the idiosyncratic risk component λ after it has reached some critical level. Then a default is mainly due to a systemic shock, and there is a greater probability of a systemic default of all firms, while the probability of few defaults vanishes. This means that for increasing $\bar{\lambda}$ all the default probability mass will be shifted toward the right end-point of the default distribution (*see Exhibit 6*). For sufficiently high $\bar{\lambda}$, it is almost certain that all firms default before the horizon, and the variance vanishes.

First-to-Default Baskets

The exponential model can also be applied to the valuation of multifirm credit derivatives. Commonly traded are k-th-to-default basket swaps, which pay some specified amount if at least k defaults in the reference bond basket occur before the maturity of the contract.

As an example, we consider a binary first-to-default swap, which involves the payment of one unit of account

EXHIBIT 5

Expected Defaults and Variance of Total Defaults as Function of Joint Shock Intensity

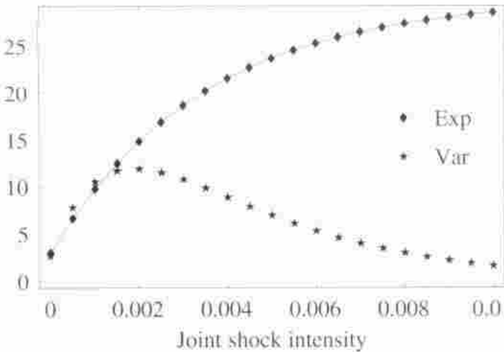
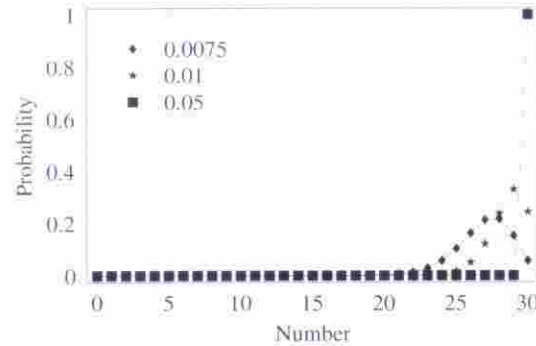


EXHIBIT 6

Distribution of Number of Defaults—Varying Joint Shock Intensity



upon the *first* default in the reference portfolio in exchange for a periodic payment (the swap spread). The swap spread is paid up to the maturity T of the swap or the first default, whichever is first. The index set of the reference portfolio is $\{1, 2, \ldots, n\}$.

Let us denote by $\tau = \min_i(\tau_i)$ the first-to-default time. We have

$$P[\tau > t] = \exp(-t\sum_{k=1}^{m}\lambda_k \max(a_{1k}, \ldots, a_{nk}))$$

Assuming that P is some risk-neutral probability, the value c of the contingent leg of the swap is at time zero given by

$$c = E[e^{-\int_0^{\tau} r_s ds} 1_{\{\tau \le T\}}]$$

where $(r_t)_{t\ge 0}$ is the riskless short rate. Supposing for simplicity that $r_t = r > 0$ for all t, we get

$$\begin{aligned} c &= \int_0^{\infty} e^{-ru} 1_{\{u \le T\}} P[\tau \in du] \\ &= \Lambda\int_0^T e^{-(r+\Lambda)u} du = \frac{\Lambda}{r+\Lambda}(1 - e^{-(r+\Lambda)T}) \end{aligned}$$

where $\Lambda = \sum_{k=1}^{m}\lambda_k \max(a_{1k}, \ldots, a_{nk})$. If the (constant) swap spread R is paid at dates $t_1 < t_2 < \ldots < t_j = T$, then the fee leg has a value of

$$f = \sum_{i:t_i \le T} E[e^{-\int_0^{t_i} r_s ds} R 1_{\{\tau > t_i\}}] = R\sum_{i:t_i \le T} e^{-(r+\Lambda)t_i}$$

where for the second equality we invoke the assumption of constant short rates. We neglect any accrued swap spread here.

The value of the fee leg paid by the protection buyer compensates the protection seller for paying one unit of account upon the first default in the reference portfolio. The swap spread R is therefore such that $c = f$ at inception of the contract ($t = 0$).

For illustration, we assume that the reference portfolio has a bivariate symmetric dependence structure with idiosyncratic shock intensity λ and joint shock intensity $\bar{\lambda}$. Symmetry and homogeneity are quite realistic for first-to-default baskets. Then

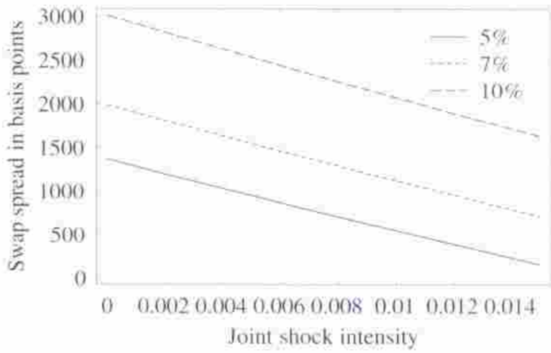
$$\Lambda = \lambda n + \bar{\lambda}\binom{n}{2}$$

Supposing that $n = 5$, $r = 0$, $T = 1$, $t_1 = 0.5$, and $t_2 = 1$ (i.e., semiannual coupon payments), in Exhibit 7 we plot the swap spread R as a function of the joint shock intensity $\bar{\lambda}$ for varying one-year default probabilities (while increasing $\bar{\lambda}$, we reduce λ so that the default probability remains constant). Since the likelihood of a payment by the protection seller increases with individual firm default probabilities, the spread widens with these default probabilities. The spread narrows with $\bar{\lambda}$, which is also intuitively clear; for increasing (positive) default correlation, the probability of multiple defaults increases and the degree of default protection provided by a first-to-default swap is diminished. With zero correlation the premium is at its maximum, because the likelihood of multiple defaults is at its minimum (given that negative correlation is excluded).

Finally note that we can use the distribution of the number of defaults to analyze general k-th-to-default baskets.

EXHIBIT 7

Swap Spread as Function of Joint Shock Intensity—Varying Individual Default Probability



V. CONCLUSION

A thorough understanding of the joint default behavior of credit-risky securities is essential for credit risk measurement as well as the valuation of multiname credit derivatives and collateralized debt obligations. We study a simple and tractable intensity-based model for correlated defaults, in which default arrival times are jointly exponentially distributed. The model can be easily implemented since all relevant quantities are available in closed form.

Simulation of dependent default times is shown to be straightforward as well.

ENDNOTES

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[1]All random variables are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Depending on the specific application, P is the physical probability (risk management setting) or some risk-neutral probability (valuation setting).

REFERENCES

Carty, Lea. "Moody's Rating Migration and Credit Quality Correlation 1920-1996." Special Comment, Moody's Investors Service, 1997.

Duffie, Darrell, and Nicolae Garleanu. "Risk and Valuation of Collateralized Debt Obligations." *Financial Analysts Journal*, 57(1) (2001), pp. 41–59.

Duffie, Darrell, and Kenneth J. Singleton. "Simulating Correlated Defaults." Working paper, Stanford University, 1998.

Embrechts, Paul, Alexander J. McNeil, and Daniel Straumann. "Correlation and Dependence in Risk Management." In M. Dempster, ed., *Risk Management: Value at Risk and Beyond*. Cambridge: Cambridge University Press, 2001.

Erturk, Erkan. "Default Correlation Among Investment-Grade Borrowers." *The Journal of Fixed Income*, 9(4), pp. 55–60.

Giesecke, Kay. "Successive Correlated Defaults: Pricing Trends and Simulation." Working paper, Cornell University, 2002.

Hull, John, and Alan White. "Valuing Credit Default Swaps I: No Counterparty Default Risk." *The Journal of Derivatives*, 8 (2000), pp. 29–40.

Lindskog, Filip, and Alexander McNeil. "Common Poisson Shock Models: Applications to Insurance and Credit Risk Modeling." Working paper, RiskLab, ETH Zurich, 2001.

Marshall, Albert W., and Ingram Olkin. "A Multivariate Exponential Distribution." *The Journal of the American Statistical Association*, 62 (1967), pp. 30–44.

Nagpal, Krishan, and Reza Bahar. "Measuring Default Correlation." *Risk*, 14(3) (2001), pp. 129–132.

Nelsen, Roger. *An Introduction to Copulas*. New York: Springer-Verlag, 1999.

Schönbucher, Philipp, and Dirk Schubert. "Copula-Dependent Default Risk in Intensity Models." Working paper, Universität Bonn, 2001.

Zhou, Chunsheng. "An Analysis of Default Correlation and Multiple Defaults." *Review of Financial Studies*, 14(2) (2001), pp. 555–576.