Default Probability Dynamics in Structural Models

JOHN HUND

JOHN HUND
is an assistant professor of finance at the A.B. Freeman School of Business at Tulane University in New Orleans, LA.
jhund@tulane.edu

Merton [1974] develops a structural framework for the pricing of corporate debt subject to default by noting the analogy between the payoff to equityholders in a levered firm and the boundary condition of a call option. When firm value follows a geometric diffusion, and under assumptions consistent with the Black-Scholes option pricing model, the differential equation for the equity value corresponds to the expression for a call option on the value of the firm, with an exercise price equal to the face value of the (discount) bond issue. Since bondholder value is the difference between the total firm value and the equity value, the pricing of corporate bonds follows directly.[1]

Kim, Ramaswamy, and Sundaresan [1993] and Duffie and Lando [1999] have noted the difficulty structural models have in generating the magnitude of spreads observed in credit markets. Recent empirical work by Huang and Huang [2002] has also focused on the inability of such models to explain a large portion of this spread. Very little work addresses the volatility dynamics implied by this approach. With the popularity of credit portfolio management tools based on structural models (primarily Moody's/KMV), it is critical to understand the volatility process that default probabilities in these models follow.

I argue that the model generates smoothness in the probability of default that is not a feature of the market; this smoothness is a direct consequence of the model's fundamental assumptions. Portfolios whose credit risk is generated by structural models will be less volatile than the actual market. I highlight the discrepancies between model volatility and actual volatility using an example drawn from the liquid market for Brady bonds.

This article explores in detail the default probability dynamics implied by such an approach, with an emphasis on the behavior of the volatility of the default probability under the model. It is worth noting that the structural models of Leland [1994] and Leland and Toft [1996] differ in complexity from the Merton model primarily due to the endogeneity and optimality of financing choice. The separation implied by the derived time-homogeneous structure of the corporate debt issue (and corresponding default boundary) means that the dynamics of default are very similar to those discussed here, once the financing decision has been made and a face value of debt (and maturity in Leland and Toft [1996]) is chosen.[2]

I. RISKY DEBT AS AN OPTION

To fix notation and motivate the problem, we recall that under the Merton framework the value of the firm, V, evolves through time according to

$$dV = \mu V dt + \sigma V dW \tag{1}$$

where μ is the (instantaneous) expected return on the firm, σ is the (instantaneous) standard deviation of firm value, and dW is a standard Brownian motion. The bond issue is characterized by its face value, B, and maturity, T.[3]

The price of the bond at time τ is given by $F(V, \tau)$ and the value of the equity at time t is given by $f(V, \tau)$, where time τ is defined as the remaining time until maturity; i.e., $\tau = T - t$. Default risk in this setting is the event that, at maturity, the firm is worth less than what is owed to the bondholders, i.e., that $V_T < B$. In this event, it is optimal and possible (because of limited liability) for the equityholders to transfer the firm to the bondholders. Thus, at maturity, the payoff to bondholders is

$$F(V, 0) = \min (V, B) \tag{2}$$

and using the fact that $f(V, \tau) = V - F(V, \tau)$:

$$f(V, 0) = \max (0, V - B) \tag{3}$$

which is the payoff to a call option on the value of the firm, struck at the face value of the debt.

For any European option, put-call parity requires that the difference between the price of a put and a call struck at the same level equals the difference between the value of the underlying and the discounted value of the strike price, which in this context yields

$$f(V, \tau) - P = V - Be^{-r\tau} \tag{4}$$

where P is the price of a put option struck at B.

Rearranging yields the familiar fundamental result that the value of risky debt is equal to the value of riskless debt less the value of a put option on the firm:

$$F(V, \tau) = Be^{-r\tau} - P \tag{5}$$

The intuition behind this result is usually that the bondholders are short a put option on the firm to the equityholders, since in the event that the firm is worth less than its fixed liabilities the equityholders can "put" the firm to the bondholders without threat of financial recourse. In the Merton model, Equation (5) leads directly to the analytical representation for the price of a risky discount bond:

$$F(V,\tau) = Be^{-r\tau}\left\{\Phi\left[h_2\left(d,\sigma^2\tau\right)\right] + \frac{1}{d}\Phi\left[h_1\left(d,\sigma^2\tau\right)\right]\right\} \tag{6}$$

$$d = \frac{Be^{-r\tau}}{V} \tag{7}$$

$$h_1\left(d,\sigma^2\tau\right) = -\frac{\frac{1}{2}\sigma^2\tau - \ln(d)}{\sigma\sqrt{\tau}}$$

$$h_2\left(d,\sigma^2\tau\right) = -\frac{\frac{1}{2}\sigma^2\tau + \ln(d)}{\sigma\sqrt{\tau}} \tag{8}$$

Merton [1974, Equation (12.16)] expresses the price of the risky bond in units of the risk-free bond for convenience:

$$\frac{F(V,\tau)}{Be^{-r\tau}} = Q\left(d,\sigma^2\tau\right) = \Phi\left[h_2\left(d,\sigma^2\tau\right)\right] + \frac{1}{d}\Phi\left[h_1\left(d,\sigma^2\tau\right)\right] \tag{9}$$

II. STRUCTURAL MODEL PROBABILITIES OF DEFAULT

It is less widely known that Equation (9) is exactly the survival probability of the bond issue, and corresponds directly to later structural models (Equation (15) in Leland and Toft [1996]) as well as more generically to the hitting time distribution of a Brownian motion (see Harrison [1990]) or the functional form of the cumulative distribution for the inverse Gaussian density (see Chhikara and Folks [1989] and Seshadri [1993]).[4] The relationship on the left-hand side of Equation (9) also forms the basis for the most widely used "back of the envelope" credit derivative pricing model where term structures of default probabilities are calculated as one minus the ratio of bootstrapped (and usually smoothed) term structures of zero-coupon risky and risk-free bonds.[5]

The probability of default in the Merton model is just:

$$\begin{aligned} H\left(d,\sigma^2\tau\right) &= 1 - Q\left(d,\sigma^2\tau\right) &(10)\\ &= \Phi\left[-h_2\left(d,\sigma^2\tau\right)\right] - \frac{1}{d}\Phi\left[h_1\left(d,\sigma^2\tau\right)\right] \\ &= \frac{P}{Be^{-r\tau}} &(11)\end{aligned}$$

which is a function of V (through d) and τ.[6]

$H(d, \sigma^2\tau)$ satisfies the axioms of probability, since it is continuous and bounded between 0 and 1, as the denominator of Equation (11) is recognizable as the asymptotic upper bound on the payoff of a put option. As V increases (the put is increasingly out of the money), $H(d, \sigma^2\tau)$ goes to zero; as V declines (the put is increasingly in the money), the default probability goes toward 1.0.

III. DEFAULT PROBABILITY DYNAMICS: TWO DERIVATIONS

By Itô's lemma, the dynamics of the default probability are:

$$dH = \left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial V} rV + \frac{1}{2}\frac{\partial^2 H}{\partial V}\sigma^2 V^2\right) dt + \frac{\partial H}{\partial V}\sigma V dW \quad (12)$$

The first derivative with respect to firm value is:

$$\begin{aligned}\frac{\partial H}{\partial V} &= -\frac{\partial Q}{\partial d}\frac{\partial d}{\partial V} \\ &= -\left(-\frac{\Phi(h_1)}{d^2}\right)\left(-\frac{Be^{-r\tau}}{V^2}\right) \\ &= \left(-\frac{\Phi(h_1)}{Be^{-r\tau}}\right)\end{aligned} \quad (13)$$

and with respect to time:

$$\begin{aligned}\frac{\partial H}{\partial t} &= -\frac{\partial Q}{\partial \tau}\frac{\partial \tau}{\partial t} \\ &= \frac{\partial\left[\Phi(-h_2) - \frac{1}{d}\Phi(h_1)\right]}{\partial \tau}(-1) \quad (14) \\ &= \frac{\partial\left[\Phi(-h_2) - \frac{Ve^{r\tau}}{B}\Phi(h_1)\right]}{\partial \tau}(-1) \\ &= \left[\phi(h_1)\frac{V}{Be^{-r\tau}}\frac{\sigma}{2\sqrt{\tau}} - \frac{rV}{Be^{-r\tau}}\Phi(h_1)\right](-1) \\ &= -\phi(h_1)\frac{V}{Be^{-r\tau}}\frac{\sigma}{2\sqrt{\tau}} + \frac{rV}{Be^{-r\tau}}\Phi(h_1)\end{aligned} \quad (15)$$

where ϕ is the standard normal density function given by:

$$\frac{\partial \Phi(x)}{\partial x} = \phi(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$$

and the second derivative of H with respect to V is

$$\begin{aligned}\frac{\partial^2 H}{\partial V^2} &= -\frac{1}{Be^{-r\tau}}\frac{\partial \Phi(h_1)}{\partial V} \\ &= -\frac{1}{Be^{-r\tau}}\frac{\partial \Phi(h_1)}{\partial h_1}\frac{\partial h_1}{\partial V} \\ &= -\frac{1}{Be^{-r\tau}}\phi(h_1)\left(-\frac{1}{V\sigma\sqrt{\tau}}\right)\end{aligned} \quad (16)$$

Substituting Equations (13), (14), and (16) into Equation (12) and simplifying yields:

$$\begin{aligned}dH = &\left(-\phi(h_1)\frac{V}{Be^{-r\tau}}\frac{\sigma}{2\sqrt{\tau}} + \frac{rV}{Be^{-r\tau}}\Phi(h_1) + \right. \\ &\left.\left(-\frac{\Phi(h_1)}{Be^{-r\tau}}\right) rV\right) dt + \\ &\left(\frac{1}{2}\left(-\frac{1}{Be^{-r\tau}}\right)\phi(h_1)\left(-\frac{1}{V\sigma\sqrt{\tau}}\right)\sigma^2 V^2\right) dt + \\ &\left(-\frac{\Phi(h_1)}{Be^{-r\tau}}\right)\sigma V dW^*\end{aligned} \quad (17)$$

$$= \frac{1}{Be^{-r\tau}} \times$$

$$\left[\begin{array}{c}\left(-\frac{\phi(h_1)V\sigma}{2\sqrt{\tau}} + \Phi(h_1)rV - \Phi(h_1)rV + \frac{\phi(h_1)\sigma V}{2\sqrt{\tau}}\right)dt \\ + \left(-\frac{\Phi(h_1)}{Be^{-r\tau}}\right)\sigma V dW^*\end{array}\right] \quad (18)$$

$$= \frac{-\Phi(h_1)\sigma V}{Be^{-r\tau}}dW^* \quad (19)$$

$$= \frac{\Delta_p \sigma V}{Be^{-r\tau}}dW^* \quad (20)$$

where Δ_p is the delta of the put option struck at B, and dW^* is the Brownian motion under the risk-neutral measure.

The result in Equation (20) shows that under the risk-neutral measure, the default probability in the Merton model is a martingale. Intuitively, this is a consequence of the fact that the put gains value under the risk-neutral measure at r, yet the default probability is the put value discounted by the price of a discount bond, rendering it driftless.[7]

The same result can be derived directly, by noting that the drift term in dH is exactly identical to much of the Black-Scholes partial differential equation (PDE) plus a correction for the discounting. Specifically, using Equation (12):

$$
\begin{aligned}
dH &= \left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial V} rV + \frac{1}{2}\frac{\partial^2 H}{\partial V}\sigma^2 V^2\right) \times \\
&\quad dt + \frac{\partial H}{\partial V}\sigma V dW \\
&= \frac{1}{Be^{-r(T-t)}}\left(\frac{\partial P}{\partial t} - rP + \frac{\partial P}{\partial V} rV + \frac{1}{2}\frac{\partial^2 P}{\partial V}\sigma^2 V^2\right) \times \\
&\quad dt + \frac{\partial P}{\partial V}\sigma V dW \\
&= \frac{1}{Be^{-r(T-t)}}\left(\Theta_p - rP + \Delta_p rV + \frac{1}{2}\Gamma_p \sigma^2 V^2\right) \times \\
&\quad dt + \Delta_p \sigma V dW
\end{aligned} \tag{21}
$$

Because

$$
\begin{aligned}
H &= \frac{P}{Be^{-r(T-t)}} \\
\frac{\partial H}{\partial t} &= \frac{e^{r(T-t)}}{B}\left[\frac{\partial P}{\partial t} + (-1)\, rP\right]
\end{aligned} \tag{22}
$$

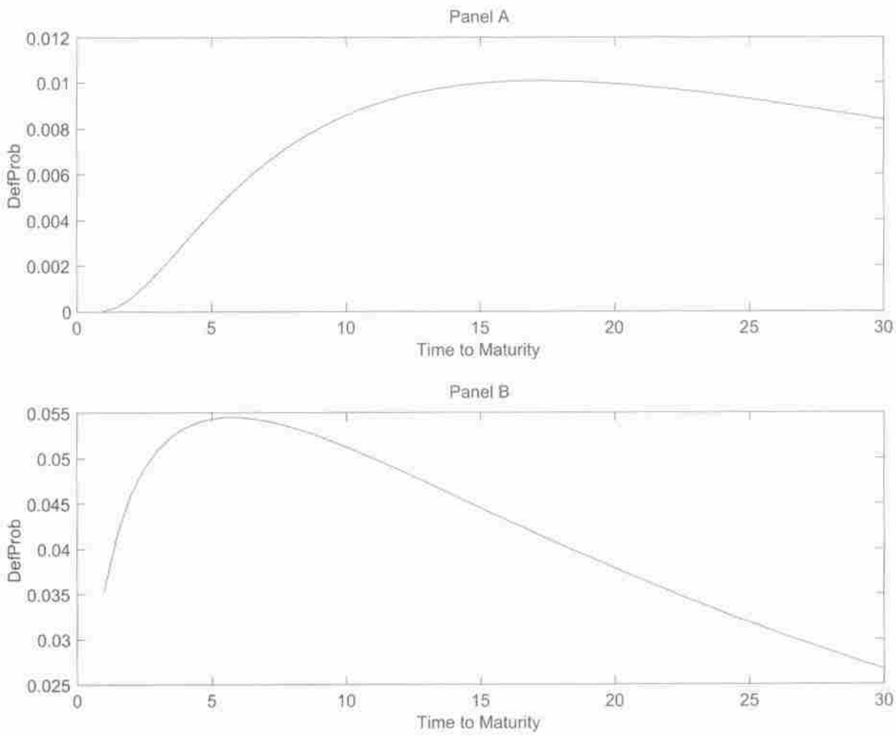
The Black-Scholes PDE relates the greeks in Equation (21) to rP, thus leading directly to Equation (20).[8]

IV. DEFAULT PROBABILITY SENSITIVITIES

The probability of default increases with time to maturity for firms with low leverage, and rapidly increases and then declines for highly levered firms (*see Exhibit 1*). This term structure behavior is consistent for all structural models of default, and is an artifact of properties associated with random walks and returns to the origin (see Feller [1968]).

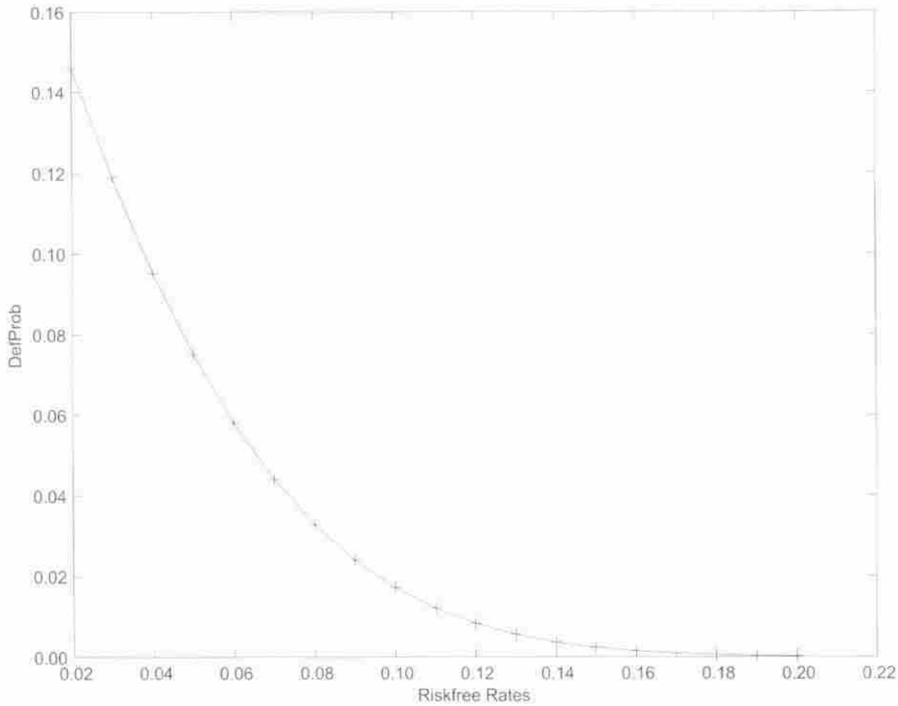
Economically, firms with little debt are more likely

EXHIBIT 1
Default Probability Term Structures



Term structures of default probabilities for low-leverage (Panel A) and high-leverage (Panel B) companies with debt maturities ranging from 1 to 30 years. Firm value is set at 100; the risk-free rate is 7%; asset volatility is 20%; and the boundary for low leverage is 55 and for high leverage 95.

EXHIBIT 2
Default Probability Risk-Free Rate Sensitivity



Sensitivity of default probability of risky discount debt under the Merton model. Debt has maturity of 10 years; firm value is set at 100; the default boundary is 90, and asset volatility is 20%.

to default, the longer the time available for bad outcomes; firms that are very close to default either default quickly, or continue to survive, evidence that the value of the firm has moved farther away from default. Structural models all have zero probability of default over the next instant, which is immediately apparent from Equation (22), as out-of-the-money options are worthless immediately prior to expiration.[9]

Structural default probabilities decline unambiguously with the risk-free rate (*see Exhibit 2*), an implication empirically tested and confirmed in Longstaff and Schwartz [1995] and Duffee [1998]. As is discussed there, increases in the risk-free rate increase the drift away from the boundary under the risk-neutral measure, and concomitantly reduce the probability of default.[10]

Volatility of Default Probability Process

The volatility of the default probability process, σ_H, is given by

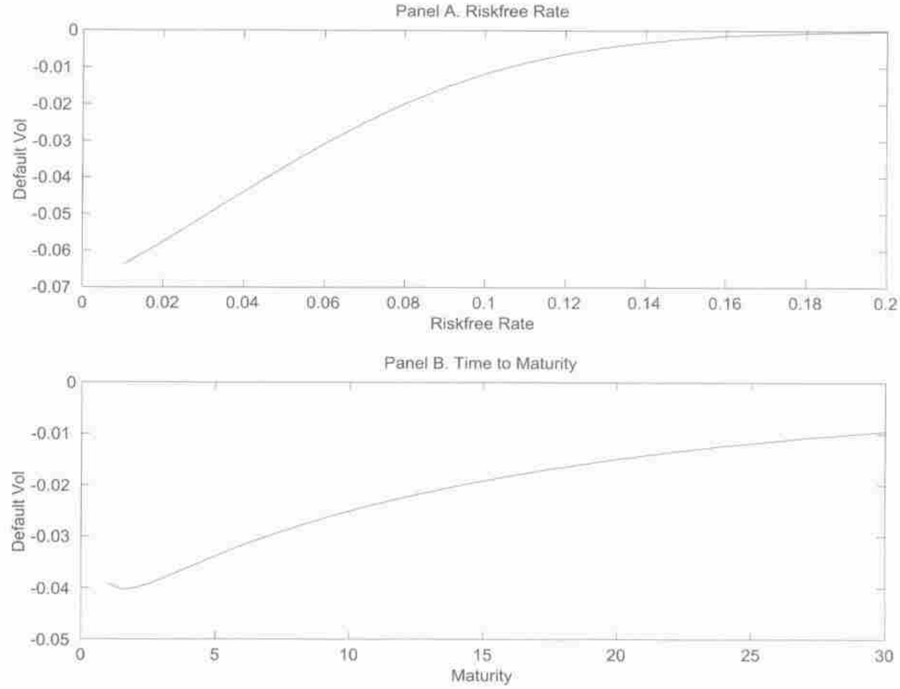
$$\sigma_H = \frac{\Delta_p \sigma V}{Be^{-r\tau}} \tag{23}$$

The most important insight apparent in Equation (23) is that the volatility of the default probability in the Merton model is bounded from above by the volatility of the firm's assets, since $\Delta_p \in [-1, 0]$.[11] This boundedness is exactly why structural models have difficulty generating sufficient spread volatility to empirically match that observed in credit markets, especially for medium- to highly rated firms (since the debt of these low-leverage firms corresponds to deep out-of-the-money puts).

Intuitively, default probability volatility is positively related to the volatility of the firm's assets, since more volatile firms are more likely to move closer to the default boundary. Exhibit 3 shows that volatility declines with time to maturity and with the risk-free rate.

The firm's asset volatility is scaled by the delta of a put option, and thus the effect of greater asset volatility increases as the value declines. As the delta becomes more negative (greater in absolute magnitude), however, V declines in magnitude, so there are offsetting effects. The effect is more pronounced in the other direction; as V increases away from the boundary, Δ_p declines rapidly to zero, offsetting the rise in V. Default volatility thus has a parabolic shape as can be seen in Exhibit 4.

EXHIBIT 3
Sensitivity of Volatility to *r* and *T*



Sensitivity of default volatility in the Merton model to a range of risk-free rates (Panel A) and times to maturity (Panel B). The firm value is 100; the debt face value is 90; asset volatility is 20%; the time to maturity (Panel A) is 10 years, and the risk-free rate (Panel B) is 7%.

Maximal Volatility Point

The maximum value of σ_H is reached

$$\frac{\partial \sigma_H}{\partial V} = \frac{\sigma}{Be^{-r\tau}}\left[\gamma_p V + \Delta_p\right] = 0 \tag{24}$$

when the percentage gamma (also called dollar gamma) of the put is equal to the delta of the option. This point is always below the boundary value B, where the left tail of the (increasing) gamma function crosses the (decreasing) negative of the put delta function.

Exhibit 5 graphs the maximum condition defined by Equation (24). The precise location of the maximal volatility point depends on the asset volatility, the remaining time until maturity, and the risk-free interest rate.

Exhibit 6 depicts the path of the maximal volatility point as risk-free rates increase. As rates increase, the volatility maximum tends toward low firm values, because as default becomes less and less likely, the chance of reaching high-volatility states is also diminished.

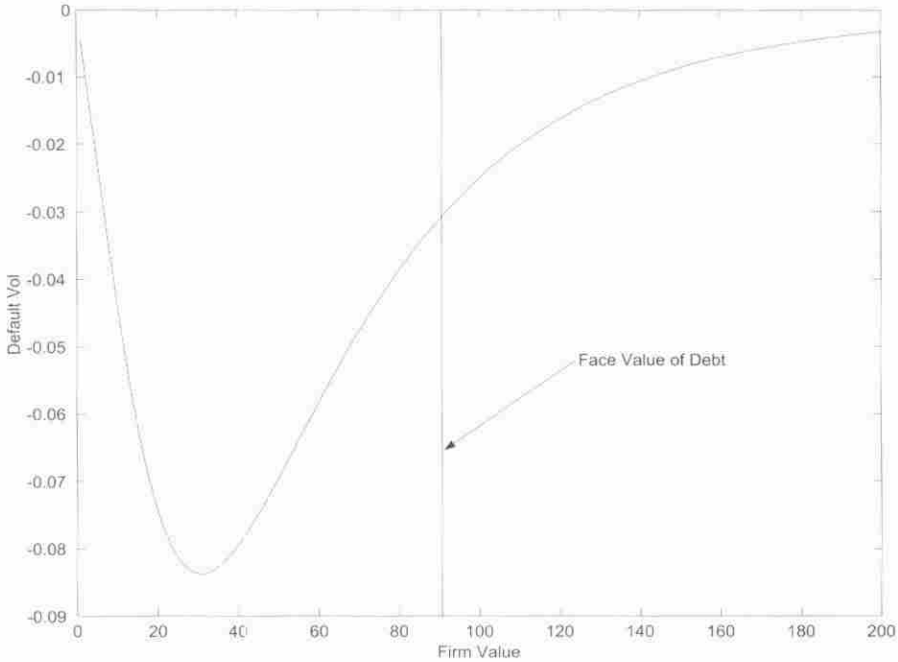
Exhibit 7 shows the movement in the maximal volatility point as a function of time to maturity. As time to maturity shortens, volatility is maximized at asset values closer to the money (i.e., the face value of the debt) due to the rapidly increasing peakedness of option gamma with respect to diminishing time to maturity.

Changes in asset volatility more ambiguously affect the location where default probabilities reach their maximum volatility. At very low asset volatilities, the maximum is closer to the money, but it declines as asset volatility increases. Eventually, asset volatility reaches the state that the maximal point tends toward the face value (the strike price). This reinforcing effect implies that very risky firms will reach their maximum volatility (which is high) near the money, as the put delta widens to reflect the increasing probability of hitting a wide range of states. This behavior is graphed in Exhibit 8.

V. SIMULATION OF DEFAULT PROBABILITIES

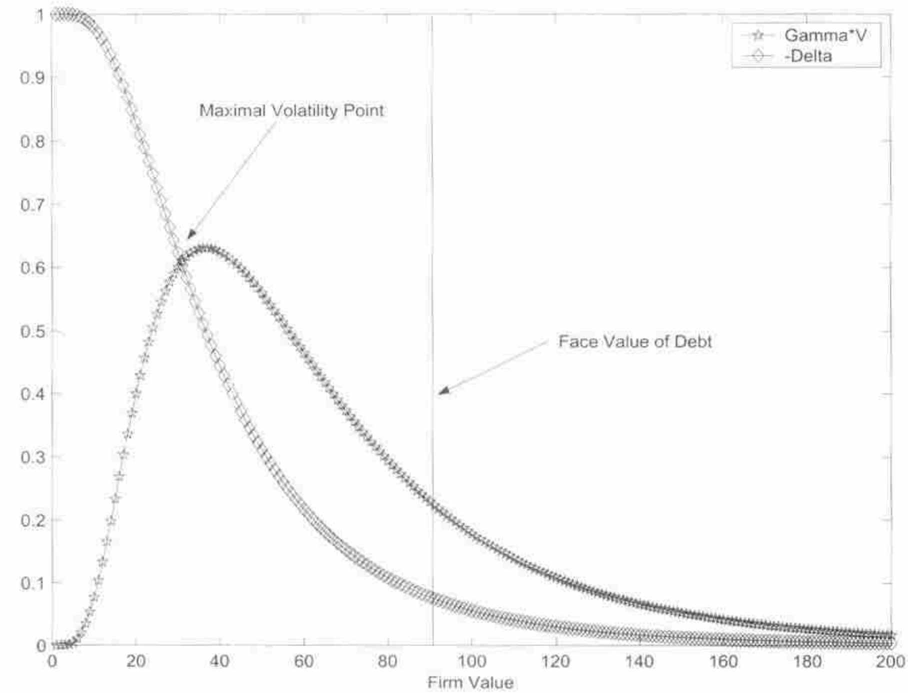
Simulation of the dH process shows dramatically how difficult it can be to generate reasonable levels of credit spread volatility from the Merton model, for both high- and low-leverage firms. The underlying firm process (the same for both paths) represents a rather unlikely

EXHIBIT 4
Default Volatility of Merton Model



Default volatility of risky discount debt with maturity of 10 years and a face value of 90 over a range of firm values. The risk-free rate is 7%; asset volatility is 20%. An increase in firm value reduces the probability of default, so volatility is maximized at its highest absolute value.

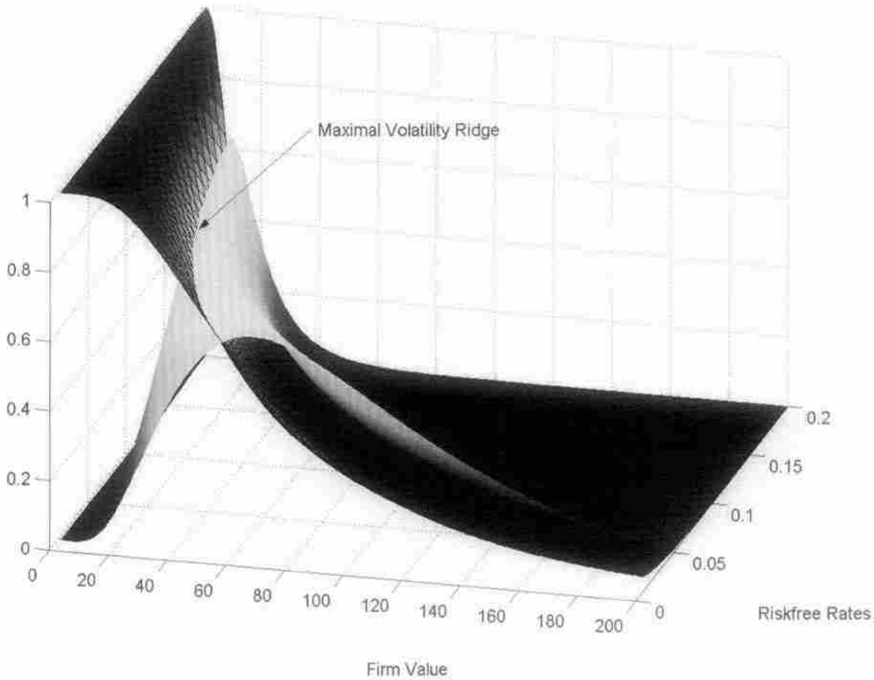
EXHIBIT 5
Maximal Volatility Point Determination



Relationship that determines the point of maximal volatility for the default probabilities in the Merton structural model; i.e., where dollar gamma equals the negative of the put delta. Face value of debt is 90; the risk-free rate is 7%; time to maturity is 10 years, and asset volatility is 20%.

EXHIBIT 6

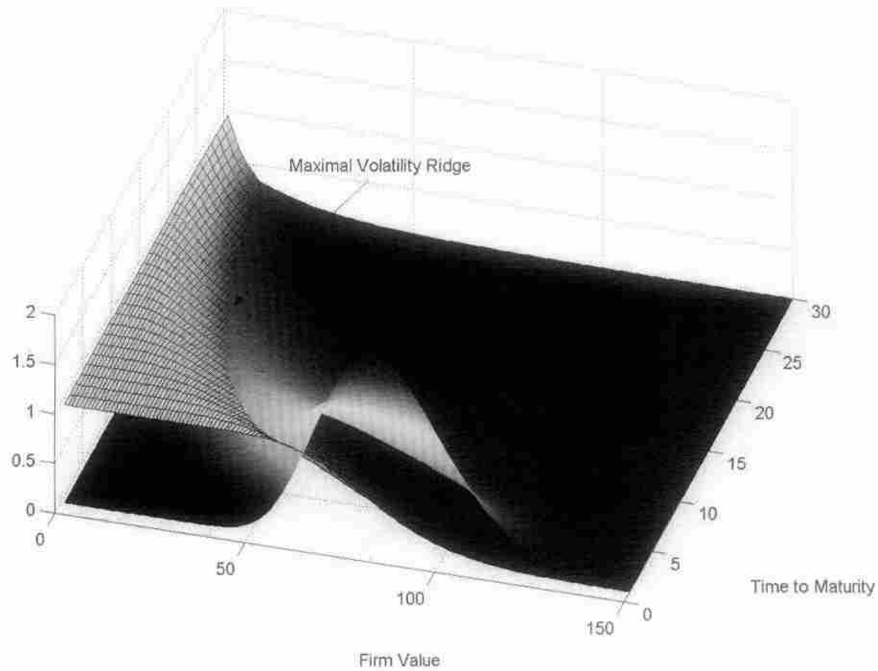
Change in Maximal Volatility Point: Risk-Free Rate



Decline in maximum volatility point as risk-free rates increase, showing the intersection of the dollar gamma and negative put delta functions over a range of firm values and risk-free rates. The face value of debt is 90; the time to maturity is 10 years; and asset value is 20%.

EXHIBIT 7

Change in Maximal Volatility Point: Time to Maturity



Decline in maximum volatility point as maturity increases, showing the intersection of the dollar gamma and negative put delta functions over a range of firm values and times to maturity. The face value of debt is 90; the risk-free rate is 7%; and asset value is 20%.

(and dire) trajectory, and even though there is a positive drift of 7% (the risk-free rate) the process finishes after ten years at 60, from a beginning value of 100. This is primarily due to a rather high asset volatility of 20% that in some sense swamps the positive drift.[12]

The initial maturity of the debt is 20 years, and the face value is 90 for the highly levered case and 55 for the low-leverage case. Exhibits 9 and 10 plot the high and low leverage cases, and Exhibit 11 depicts summary statistics for the paths.

Consistent with Equation (20), neither the differenced default probabilities (not shown) nor the credit spreads display a drift or evidence of mean reversion.[13] Changes in credit spreads are very symmetric, with nearly no skewness, and maximum values are very similar to minimum values. As expected, higher-leverage firms have higher volatility, but not markedly so for such a dramatic difference in leverage. Overall, the smoothness in both processes is very evident.

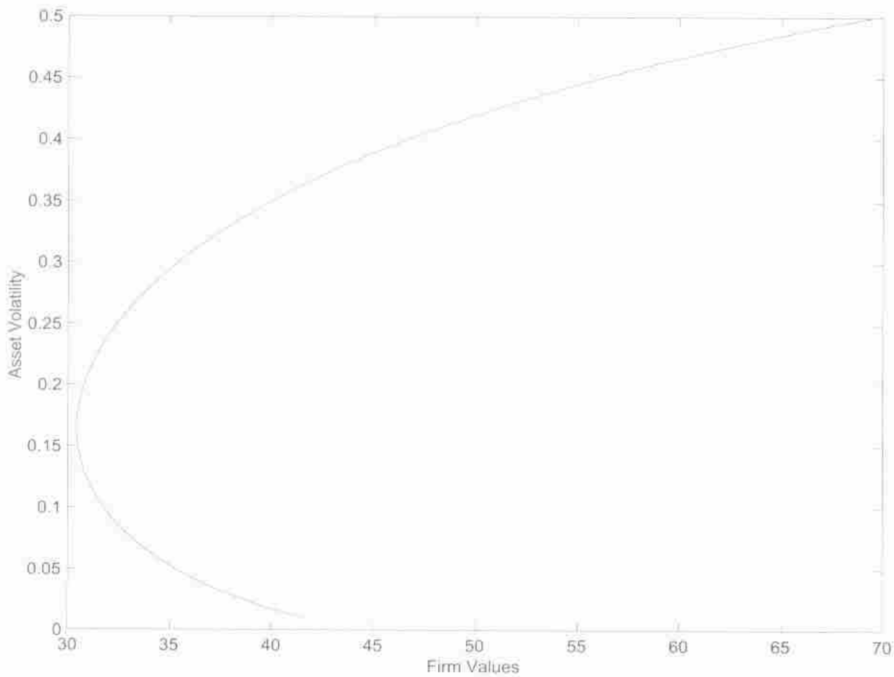
The plots of differenced credit spreads in Panels B of Exhibits 9 and 10 point to an interesting property of structural credit models that is at odds with empirical observation in emerging sovereign markets. Volatility is more clustered (shows more time variation) for low-leverage firms than for high-leverage firms, due to the greater percentage moves in the put delta far out-of-the-money. The prevalence of differential heteroscedasticity across leverage levels in corporate debt data is an open empirical question that may merit closer examination.

VI. STRUCTURAL DEFAULT DYNAMICS AND EMERGING MARKET DATA

It is difficult to observe volatility in the price of corporate bonds. Data are usually unavailable at a high enough frequency, given the relative thinness of the market. This problem is exacerbated by poor credit quality; often the worse the credit, the more difficult to get consistent high-frequency price data.

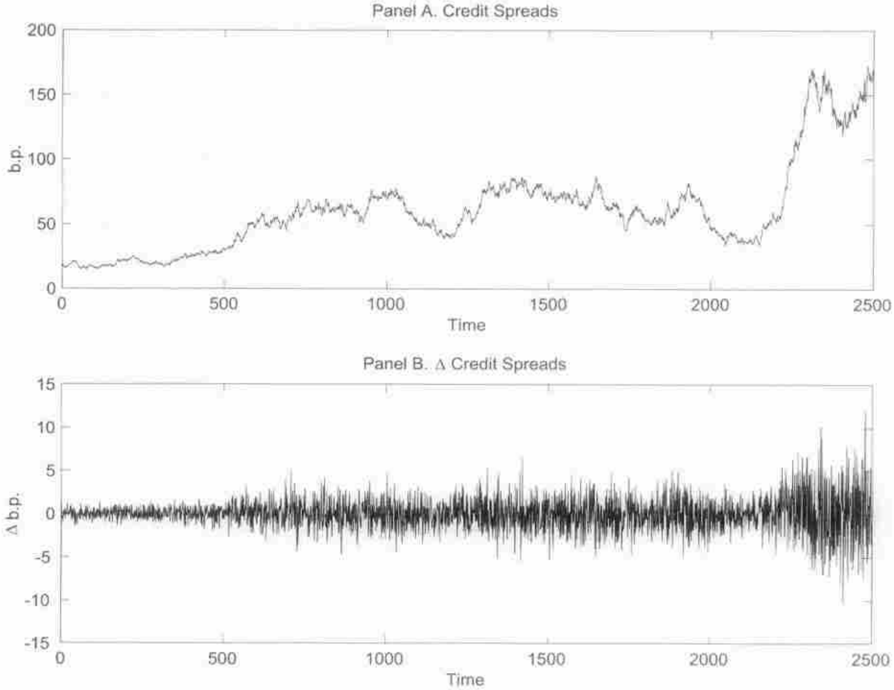
One bond market with sufficient liquidity and low credit quality is the Brady bond market. Although applying structural models to sovereigns is problematic on several

EXHIBIT 8
Change in Maximal Volatility Point: Asset Volatility



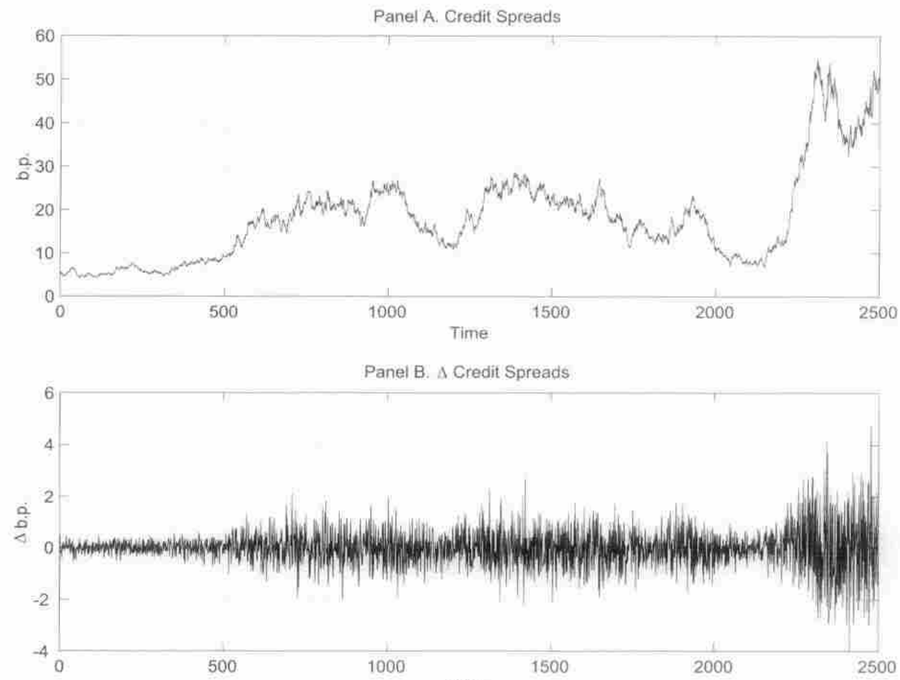
Movement in maximum volatility point as asset volatility increases. The face value of debt is 90; the risk-free rate is 7%; and time to maturity is 10 years.

EXHIBIT 9
Simulated Credit Spread: High Leverage



Simulated path for credit spreads (Panel A) and their differences (Panel B) for a highly levered firm under the Merton model of default. Asset volatility is 20%; the risk-free rate is 7%; and the starting value for firm value is 100. The face value of debt is 90 and initial maturity is 20 years.

EXHIBIT 10
Simulated Credit Spread: Low Leverage



Simulated path for credit spreads (Panel A) and their differences (Panel B) for a low leverage firm under the Merton model of default. Asset volatility is 20%; the risk-free rate is 7%; and the starting value for firm value is 100. The face value of debt is 55 and initial maturity is 20 years.

EXHIBIT 11

Descriptive Statistics for Merton Simulated Paths (bp)

	Mean	Std. Dev.	Skew.	Kurt.	Max Δ	Min Δ	AR(1)
High	.06	1.87	.20	6.29	11.92	−10.37	.000
Low	.02	.73	.21	6.78	4.75	−3.97	−.001

Both firms have asset volatility of 20% and starting firm value of 100; risk-free rates are 7%; and initial maturity of debt is 20 years. The face value of debt is 90 in the high-leverage case and 55 in the low-leverage case.

fronts, pricing models by Lehrbass [1999] and Wiggers [2002] use variants of the structural approach to value these bonds. I use the liquid market to merely provide an example of the dynamics in one market that are inconsistent with the implications of structural models. It is an open empirical question whether these dynamics are present in the short-term volatility of low-rated corporate credit.

The process that default probabilities (and credit spreads) follow under structural models of default is dramatically different from the patterns observed empirically in emerging market data. It is very difficult to generate spreads of the extent observed in these markets without astronomically high levels of asset volatility. This stems directly from the interpretation of the debt price as a put option; this option has very little value unless it is deep in the money.

Since under the risk-neutral measure (it is even worse under the true measure, as this drift must be greater than the risk-free rate) the firm gains value at the risk-free rate, very high levels of volatility are required to give large weight to the chance that the option goes deep in the money.

More problematic is the smoothness inherent in the trade-off implied by the process for dH, the probability of default in the Merton model. As I discuss above, default probability volatility is bounded above by the asset volatility of the firm.[14]

At high firm values, the put is deep out of the money and thus has a minuscule delta; at low firm values, the put delta is relatively large, but firm value is low. This trade-off fundamentally limits the ability of such models to generate large changes in default probabilities and, by extension, credit spreads. As a contrast, Exhibit 12 shows that changes in Mexican par default probabilities are both very low and very high, and there is strong time variation in the volatility.

The dynamics of default probabilities under structural models are very inconsistent with those observed in emerging markets. Emerging market default probabilities exhibit significant mean reversion, large (jumpy) changes, and high absolute values—none of which is theoretically consistent with structural default dynamics.

VII. CONCLUSION

While much attention has been paid to the level of credit spreads, very little research has focused on the volatility dynamics implied by common models of credit risk. I demonstrate that the structural framework for modeling credit risk implies that credit spread volatility is bounded from above by asset volatility and displays smoothness as a result of interaction between the put delta and the underlying firm value. Such smoothness is very inconsistent with data from emerging markets where high frequency data are available to test such models. Evidence in Tauren [1999] suggests that such models may also be inadequate descriptors of corporate credit spreads.

I derive a simple expression for the process followed by the default probability in structural models and show that volatility is decreasing in both time to maturity and the risk-free rate. Structural models also imply that maximal volatility is reached only at very low asset values, and well below the face value of the debt.

ENDNOTES

[1]The option pricing assumptions are essentially that markets are frictionless and complete; that assets (including the firm itself) trade continuously and are infinitely divisible; and that (less important for Merton's development) interest rates are non-stochastic. Other assumptions not made explicitly concern the transferability of assets and constant and completely determined property rights.

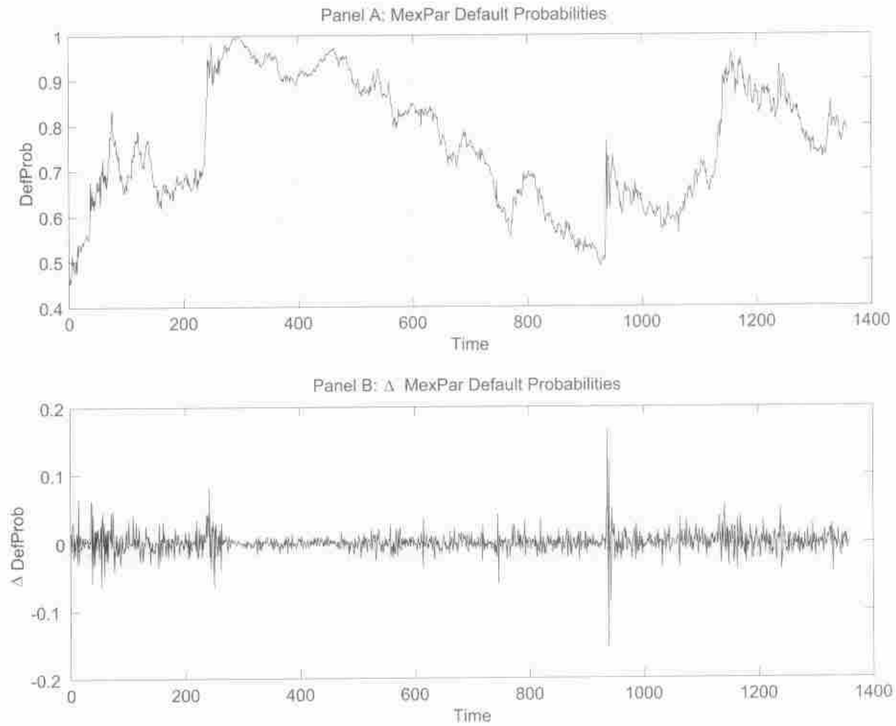
[2]For example, note the inverse Gaussian cumulative probability of default in Equation (15) of Leland and Toft [1996], and compare this with my Equation (9), which is from Merton [1974].

[3]I focus here on the discount bond case for notational simplicity. In this framework, coupons are handled by analogy with dividends in the option pricing literature.

[4]Equation (9) also directly determines the relationship between spreads and default probabilities in the discount case, since

EXHIBIT 12

Default Probabilities for Mexican Par



Cumulative default probabilities (Panel A) and their differences (Panel B) implied by the stripped static spreads on the Mexican par bond.

$1-\frac{e^{-(r+s)T}}{e^{-rT}}=p_{def}$ can be inverted to yield $p_{def}=-\frac{\ln(1-s)}{T}$.

[5]The complexity (and art) here often relates to the difficulty of choosing sufficient sets of homogeneous risky bonds and the statistical methodology used to fit smooth par curves for each risk rating or country. Once such curves are derived, continuous term structures of default probabilities are directly available by standard bootstrapping algorithms, and default swap prices can be computed easily.

[6]Note that in Merton [1974] the function Q is denoted P, which is reserved here for the price of a put option struck at B.

[7]Another equivalent interpretation is that the default probability in such a structural model is the put value discounted by the numeraire bank account, a result with some relevance for pricing derivatives on the default probability (such as default swaps). I am indebted to Hong Yan for pointing out the numeraire interpretation.

[8]Specifically, any derivative security must solve

$$\frac{\partial f}{dt}+rS\frac{\partial f}{dS}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{dS^2}=rf$$

for a generic underlying security S following geometric Brownian motion.

[9] This property of structural models (and, more important, the desire to avoid this property and have non-zero instantaneous default probability) is responsible for the popularity of hazard rate models and structural models that augment the diffusion process dV with a jump component or with incomplete information (as in Duffie and Lando [1999]).

[10] This correlation carries through to extensions of the model that allow for non-constant deterministic risk-free rates and stochastic interest rate models. To generate positive correlation, it is necessary for the firm value to be strongly negatively correlated with risk-free rates, due either to capital structure or industry cyclicality with business cycles.

[11]The volatility coefficient is negative, and reflects the negative correlation between value and default probability. Default probabilities become more volatile as the coefficient becomes more negative (declines). A more conventional representation would result by reflecting the innovations in the Brownian motion around the origin.

[12]Experiments with lower volatilities yield exceptionally low standard deviations and levels for spreads and default probabilities. The (more likely) trajectory where V drifts higher has default probability dynamics that are extremely dampened (since the Δ_p is far out of the money).

[13]T-statistics for the AR(1) coefficient are -0.04 for the low- and 0.80 for the high-leverage cases.

[14]Asset volatility in a sovereign context is a difficult con-

cept to interpret. If the underlying value process is that of the country itself, the volatility of gross national product would seem to be a reasonable proxy (albeit one measured with considerable error). Even for extraordinarily risky countries this quantity is likely to be very low, thus making the high volatility of their debt issues more puzzling and very unlikely to be described by such structural models.

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