A Note on Common Interest Rate Risk Measures

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To control the interest rate risk of a portfolio, a trading strategy, or a dealer position, it is necessary to be able to quantify this risk. Several studies of the historical returns on U.S. Treasury STRIPS using principal components analysis have found there are three factors that explain almost all the variance of returns: 1) changes in the overall level of interest rates; 2) changes in the shape of the yield curve (i.e., steepening and flattening; and 3) changes in the curvature of the yield curve.[1]

Changes in the first factor explain about 90% of the historical returns on U.S. Treasury STRIPS, while the second factor explains from 7% to 8%. Similar results are reported for historical returns on various types of collateralized mortgage obligations by Schumacher, Dektar, and Fabozzi [1994].

The most popular measure used to estimate interest rate risk attributable to a level change in interest rates is duration. This measure assumes a parallel shift in the level of interest rates. While there are several measures of duration, the appropriate measure to use is *effective duration*, a measure that takes into account the effect of any embedded options on the value of a security due to a parallel change in interest rate. Most definitions of effective duration use a change in the level of the spot curve.[2]

Measures beyond effective duration that measure exposure to changes in the level, slope, and curvature of the yield curve have been proposed by Kuberek [1998], Willner [1996], and Nelson and Carleton [1987]. More finely tuned measures that focus only on measuring exposure to changes in the shape of either the yield curve or spot curves are partial and key rate durations.

Partial durations, first introduced by Reitano [1990, 1991], measure the sensitivity of a bond's price or portfolio's value to changes in maturity sections of the par yield curve or the on-the-run Treasury yield curve.[3] Key rate durations (KRD), introduced by Ho [1992], are similar to partial durations except that the underlying curve manipulated is the spot curve. KRD measures the sensitivity of a bond's price or a portfolio's value to changes in maturity sections of the spot curve.

Both measures are easily converted into level, slope, and curvature durations by performing straightforward linear transformations. Of course, the resulting level, slope, and curvature durations are with respect to changes in different reference curves.[4]

The appropriate measure should depend on the particular application considered. Traders often prefer the partial-duration framework because it is based on a yield curve and is easier to understand when placing trades based on changes in yield. In asset liability management, however, where the liabilities are most often (and appropriately) evaluated using the spot curve, the key rate duration approach is the more appropriate approach.

It is not that one measure is correct or incorrect. They simply measure very different sensitivities. Unfortunately, the two measures are often used interchangeably in the literature and in practice, but they are very different. We aim to shed some light on the important differences between key rate durations and partial durations so that users can better apply them to both trading and risk management strategies.

I. COMPUTING KEY RATE AND PARTIAL DURATIONS

We first define key rates along the interest rate curve (spot curve for key rate durations and par curve for partial durations) for: 1, 2, 3, 5, 10, 20, and 30-year maturities. To compute a key rate duration (KRD), we shift the appropriate key rate up and down by ΔY. For the upward shift, the key rate is shifted up by ΔY, and the rates between the adjacent key rates shift up in a linearly declining (increasing for a downward shift) manner so that the shift at the two adjacent key rates is zero. For U.S. Treasuries we price the security for each spot curve change.[5]

For each key rate j, we compute the KRD as follows:

$$KRD_j = \frac{1}{P_0}\frac{P_j(-) - P_j(+)}{2\Delta Y} \qquad (1)$$

where P_0 is the current price of the security; $P_j(-)$ is the price of the security following the corresponding downward shift at key rate j; $P_j(+)$ is the price of the security following the corresponding upward shift at key rate j; and ΔY is the shift in the key rate expressed in percentage terms. Note that for j below and above the adjacent key rates ($j-1$ and $j+1$) Equation (1) will be equal to zero as rates outside this range are not changed, so the numerator in Equation (1) is always zero.

Partial durations (PD) are computed in a similar manner except that the curve shifted is the par curve. Following the par curve shift, the corresponding spot curve change is determined in order to compute the new security price. The PD is then computed as in Equation (1), with ΔY the shift in the par curve and the prices computed from the corresponding spot curve changes.

To illustrate the difference between the two approaches, we use the spot curve as of the middle of February 2003 as an example. Exhibit 1 presents the spot curve and the corresponding par curve. The spot curve is related to the par curve as follows:

$$SR_t = 2\left[DF_t^{-\frac{1}{2t}} - 1\right]$$

where

$$DF_t = \begin{cases} \left(1 + \frac{PY_t}{2}\right)^{-1} & \text{for } t = 0.5 \\ \dfrac{1 - \frac{PY_t}{2}\sum_{i=1}^{2t-1} DF_{i/2}}{1 + \frac{PY_t}{2}} & \text{for } t > 0.5 \end{cases}$$

EXHIBIT 1
Spot and Par Yield Curves

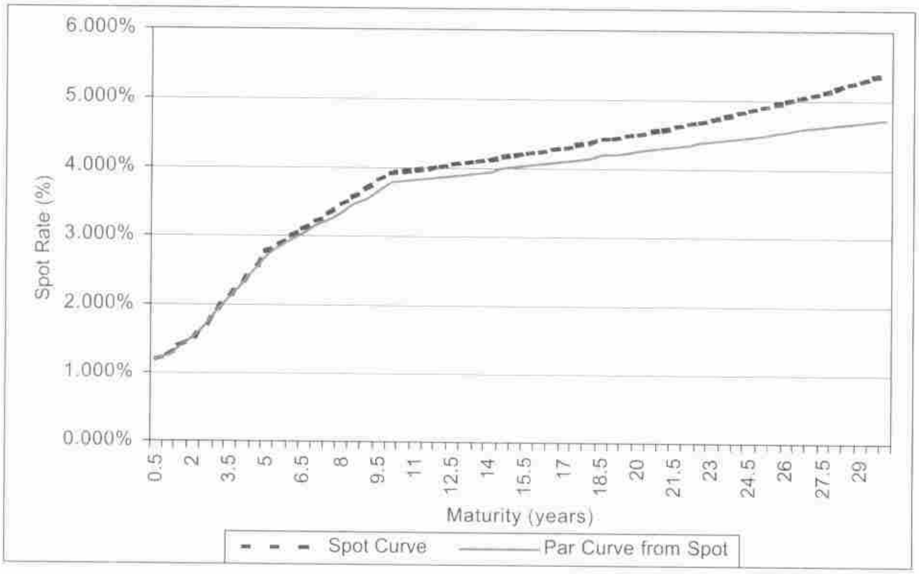
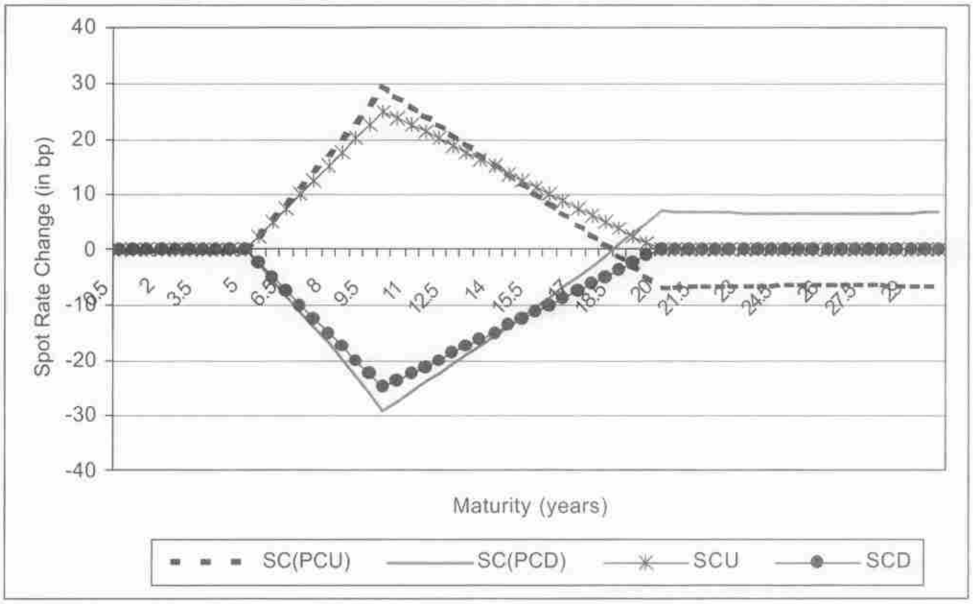


EXHIBIT 2

Different Spot Rate Curve Changes Resulting From KRD and PD Approaches



where PY_t is the par yield, and SR_t is the spot rate corresponding to a maturity of t years.

Let's consider a simple example. Suppose the one-year par yield $PY_1 = 2.040\%$, and $SR_1 = 2.040\%$, while the five-year par yield and spot rate are $PY_5 = 2.381\%$ and $SR_5 = 2.390\%$, respectively. If both the one-year and the five-year par yields are increased by 25 basis points, the spot rates become $SR_{1,up} = 2.291\%$ and $SR_{5,up} = 2.652\%$. The corresponding changes in the spot rate are not the same as the par yield changes, and they also differ across the maturity spectrum ($\Delta SR_1 = 0.251\%$ and $\Delta SR_5 = 0.262\%$).

To illustrate these effects in the PD/KRD framework, we use the rate for a ten-year maturity as a key rate. Exhibit 2 shows the spot curve changes resulting from each method. SCU and SCD are the spot curve changes from an up and down move, respectively, for the computation of the ten-year KRD. Note that the changes in rates at and beyond the adjacent key rates are zero.

These graphs also illustrate the change in par yields used to compute the partial duration. SC(PCU) and SC(PCD) are the changes in the spot curve from the up and down changes to the par curve, respectively. Recall that the spot curve is the pricing curve, so Exhibit 2 illustrates the significant differences between the two approaches. To compute the ten-year duration measures we add the changes in Exhibit 2 to the original spot curve in Exhibit 1, compute the prices, and then use Equation (1).

Exhibit 3 presents a comparison of the two approaches for some selected zero-coupon securities (i.e., STRIPS) and two coupon securities. The exhibit shows PDs and KRDs at each of the maturity points (1, 2, 3, 5, 10, 20, and 30 years) for a range of the two security types. The coupon securities are on-the-run 10- and 30-year U.S. Treasury securities.[6] The ΔY used for all computations is 25 basis points.

The risk profiles for the different securities differ significantly. All PDs for zero-coupon bonds at maturity points shorter than the actual maturity of the security are negative, while the corresponding KRDs are zero. The PDs that correspond to the actual maturity of the zero-coupon bonds are considerably higher than the corresponding KRDs. For the coupon bonds, the differences between PDs and KRDs are less pronounced. PDs are never negative, and PDs at the actual bond maturities do not exceed the corresponding KRDs by as much.

Differences between PDs and KRDs are due to the spot curve reshaping caused by the PD approach as shown in Exhibit 2. Particularly interesting is the PD profile for the long-dated zeros as compared to the KRD profile.

Exhibit 4 presents the PDs and the KRDs for the 30-year zero-coupon bond where the differences are most pronounced. It illustrates the extreme differences between the risk measures.

The PD profile suggests that an increase in the middle of the par curve results in an increase in the value

EXHIBIT 3
Partial Durations and Key Rate Durations for Selected Securities

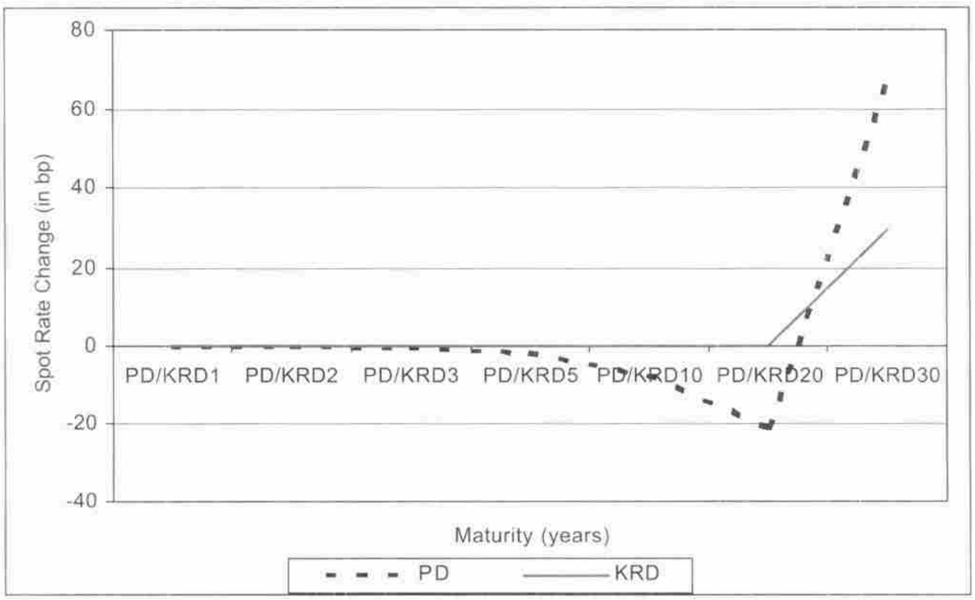
Partial Durations

	Maturity	PD1	PD2	PD3	PD5	PD10	PD20	PD30	Sum
Zero-Coupon	5	-0.025	-0.075	-0.145	5.252	0.000	0.000	0.000	5.007
	10	-0.038	-0.113	-0.217	-0.938	11.562	0.000	0.000	10.255
	20	-0.047	-0.142	-0.272	-1.176	-5.372	27.995	0.000	20.982
	25	-0.055	-0.166	-0.319	-1.375	-6.282	12.193	23.325	27.322
	30	-0.068	-0.205	-0.394	-1.698	-7.760	-20.919	66.394	35.351
Coupon	10	0.005	0.014	0.027	0.891	7.093	0.000	0.000	8.030
	30	0.002	0.006	0.012	0.052	0.237	3.551	12.022	15.882

Key Rate Durations

	Maturity	KRD1	KRD2	KRD3	KRD5	KRD10	KRD20	KRD30	Sum
Zero-Coupon	5	0.000	0.000	0.000	4.931	0.000	0.000	0.000	4.931
	10	0.000	0.000	0.000	0.000	9.808	0.000	0.000	9.808
	20	0.000	0.000	0.000	0.000	0.000	19.567	0.000	19.567
	25	0.000	0.000	0.000	0.000	0.000	12.203	12.203	24.406
	30	0.000	0.000	0.000	0.000	0.000	0.000	29.238	29.238
Coupon	10	0.035	0.104	0.190	1.318	6.097	0.000	0.000	7.744
	30	0.041	0.121	0.221	0.816	2.520	4.822	5.812	14.351

EXHIBIT 4
KRD and PD Comparison for 30-Year Zero-Coupon Bond



of the 30-year security. This result is due to the large negative partial durations at 10- and 20-year maturities caused by the spot curve reshaping illustrated in Exhibit 2. The rate inversions of the spot curve as a result of shifting the par curve produce these results. The KRD profile is perhaps more intuitive since its interpretation is that the only rate that affects the price is the 30-year rate.

This is not a matter of one measure being correct or incorrect. Each is equally justifiable, depending on the application. Each simply quantifies a very different interest rate sensitivity. Many users of these measures seem to be unaware of this characteristic.[7]

In Exhibit 4, the differences between the two measures seem muted for shorter-maturity securities with coupons. Even for the coupon securities, however, the spot rate change profiles differ significantly. At first glance, the PDs seem to overweight the influence of cash flows at maturity and underweight the interim cash flows.

The PD profiles suggest that to hedge against adverse yield moves the long-dated zero-coupon securities would be hedged by increasing the PD exposure to the intermediate portion of the curve. In other words, due to the spot/par curve relationship, we would have to buy securities to reduce our long interest rate risk. Although this seems paradoxical, it is perfectly consistent with the characteristics exhibited in Exhibit 2. As a result of the negative PDs, in order to reduce our interest rate risk at the center of the curve, we may actually have to increase the effective duration of our overall position.

As long as the PDs at each point across the maturity spectrum are zero, though, the overall risk associated with changes in both curves is also zero. Since both KRDs and PDs ultimately measure the risk associated with changes in the spot curve when we eliminate either exposure, the overall position will be protected against adverse changes in the spot curve.

The PD approach works well if the strategy is to eliminate risk exposure along the entire maturity spectrum. If the strategy requires altering exposure to different portions of the curve, the question arises as to which curve is appropriate. Do we want to modify exposure to changes in forward rates, spot rates, or yields? If it is to spot or forward rates, then PDs must be applied with care.

For example, in Exhibit 2 we illustrate the net changes to the spot curve from shifting the par curve. Notice the oscillatory effect this has on changes to the spot curve [SC(PCU) and SC(PCD)]. This geometric result may have consequences for the corresponding forward rates since they are very sensitive to changes in the slope of the spot curve. Moreover, if the desire is to increase exposure to portions of the spot curve, using PDs is more difficult for the same reasons.

To put it simply, the user has to be aware of the interrelationships between the par and spot curves before implementing a trading strategy. Using KRDs to increase exposure to forward and spot curves is more straightforward since it involves direct shifts in the spot curve.

The added layer of complexity incorporated into the PD computation and interpretation becomes more complicated in the case of bonds with embedded options. Many embedded options are highly sensitive to changes in the forward curves, so using the par curve as the reference point may produce some peculiar results.

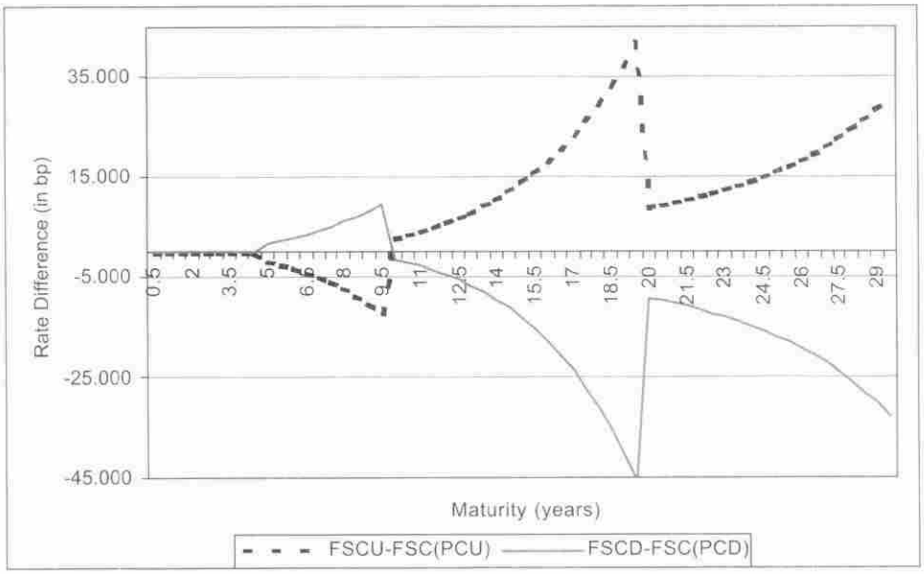
Exhibit 5 illustrates the differences in the implied forward rates corresponding to Exhibits 1 and 2. FSCU is the implied six-month forward curve that corresponds to the spot curve that results from adding SCU in Exhibit 2 to the spot curve in Exhibit 1. FSC(PCU) is the implied six-month forward curve that results from adding PCU to the par curve. Hence, FSCU corresponds to a direct spot curve change used to compute KRDs and FSC(PCU) corresponds to an indirect spot curve change caused by a change in the par curve used to compute PDs. FSCD and FSC(PCD) are defined similarly for downward shifts in the spot/par curve.

Exhibit 5 illustrates differences in the impact of both kinds of rate changes on the implied forward curve. The plot should make it clear that any trading strategy based on forward rates must be implemented with care even for securities without embedded options. We intentionally keep the analysis simple by using securities without embedded options as we are trying to illustrate only the significant differences in the approaches.[8]

Exhibit 6 presents the traditional duration measures for each of our selected securities. Duration and modified duration are shown before the effective duration measures using the par curve [Effective Duration (PC)] and the spot curve [Effective Duration (SC)] as a starting point for the computations.[9]

The effective duration values resulting from the spot curve are extremely close to the modified duration numbers for all securities except the long-dated coupon bond. This result is due to the relationship between the spot curve and the yield to maturity of coupon bonds when we have an upward-sloping spot curve. If the spot curve were flat, there would be no difference between the two measures. Parallel shifts in the spot curve, however, do not result in exactly the same changes in the yield to maturity for long-dated coupon-paying securities.

EXHIBIT 5
Differences in Applied Forward Rates between KRD and PD Approaches



Differences in implied forward rates corresponding to Exhibits 1 and 2. FSCU is the implied six-month forward curve that corresponds to the spot curve that results from adding SCU in Exhibit 2 to the spot curve in Exhibit 1. FSC(PCU) is the implied six-month forward curve that results from adding PCU to the par curve. FSCD and FSC(PCD) are defined similarly for downward shifts in the spot/par curve.

EXHIBIT 6
Traditional Duration Measures for Selected Securities

	Maturity	Duration	Modified Duration	Effective Duration (PC)	Effective Duration (SC)
Zero-Coupon	5	5.000	4.931	5.018	4.931
	10	10.000	9.807	10.173	9.808
	20	20.000	19.558	21.014	19.567
	25	25.000	24.402	27.363	24.418
	30	30.000	29.211	35.415	29.238
Coupon	10	0.000	7.786	8.029	7.730
	30	0.000	15.154	15.890	14.339

The effective durations of the zero-coupon bonds from shifting the par curve are higher than the maturity of the securities. This is due to the effects illustrated in Exhibit 2. When the par curve is shifted by ΔY, the corresponding change in the spot curve is greater than ΔY, resulting in very high effective durations.

A comparison of the last column of Exhibit 3 to the results in Exhibit 6 indicates that the sum of the PDs is very close to the effective durations from shifting the par curve, and the sum of the KRDs is very close to the effective durations from shifting the spot curve. Any differences are due to rounding.

II. LEVEL, SLOPE, AND CURVATURE DURATIONS

Following the approach outlined by Kuberek [1998], we compute the level duration (which is the same as the effective duration, ED), the slope duration (SD), and the curvature duration (CD) using both the spot curve and the par curve.[10] The results are presented in Exhibit 7. The level duration resulting from par curve shifts is significantly higher than the level duration from spot curve shifts, while the curvature durations are significantly lower and even negative for the longer-maturity securities.

EXHIBIT 7

Level, Slope, and Curvature Durations

Par Curve Changes

	Maturity	Effective Duration	Slope Duration	Curvature Duration
Zero-Coupon	5	5.017	2.611	4.834
	10	10.272	8.183	9.570
	20	21.014	21.346	6.030
	25	27.362	28.942	0.269
	30	35.415	38.362	-7.441
Coupon	10	8.028	5.920	7.617
	30	15.889	15.497	3.512

Spot Curve Changes

	Maturity	Effective Duration	Slope Duration	Curvature Duration
Zero-Coupon	5	4.931	2.517	4.687
	10	9.808	7.457	9.127
	20	19.566	18.442	8.724
	25	24.417	23.730	6.660
	30	29.238	28.835	4.683
Coupon	10	7.730	5.462	7.284
	30	14.338	12.858	6.315

These results have significant implications for those who use this framework of risk assessment. These duration measures are really just linear combinations of the KRD and PD methods, so the same caveats apply as above. Any hedging or trading strategy must be carefully implemented, whether using the PD or the KRD approach.

III. AN EXAMPLE

A straightforward example illustrates how misusing the duration measures can adversely affect the outcome of a strategy. Suppose we have a liability consisting of a series of four $10 million payments at 5, 10, 20, and 30 years. The liability is valued using the U.S. Treasury spot curve as suggested in Ryan and Fabozzi [2002]. The cash flows correspond to the maturities of our zero-coupon securities outlined in Exhibit 3, so they have the same sensitivities to changes in the spot curve as quantified by the key rate durations at the same maturity points. Our objective is to immunize the liability series.

Further assume that we use partial durations to implement the immunization strategy. That is, we misapply the measures and match the key rate durations by using partial durations. Exhibit 8 shows the original spot curve and the steepened spot curve used for the example.

We implement the strategy under the current spot curve and then evaluate it under the steepened curve scenario. Exhibit 9 presents the results. The fourth row gives the difference in the value of the liabilities following the steepening of the spot curve.

For properly immunized liabilities, we expect the change in the assets to be equal to the change in the liabilities. Under "Asset Changes," we present the change in assets using each measure. Here we consider only the first-order effects. The "Immunization Results" section computes the net change between assets and liabilities. Notice that the PD approach is not nearly as effective as the KRD approach.[11]

The last row of Exhibit 9 presents the actual PDs that would need to be applied in order to have equally effective immunization strategies.

The example is offered for illustration purposes only. The reverse is also valid. If we were to misapply the KRD approach within a yield curve strategy, the results would be equally problematical.

IV. CONCLUSION

We have shown that using the par curve to measure interest rate sensitivity is very different from using the spot curve. We do not suggest that one is correct and the other is not. Instead we simply want to highlight the sig-

EXHIBIT 8
Spot Curves for Asset/Liability Example

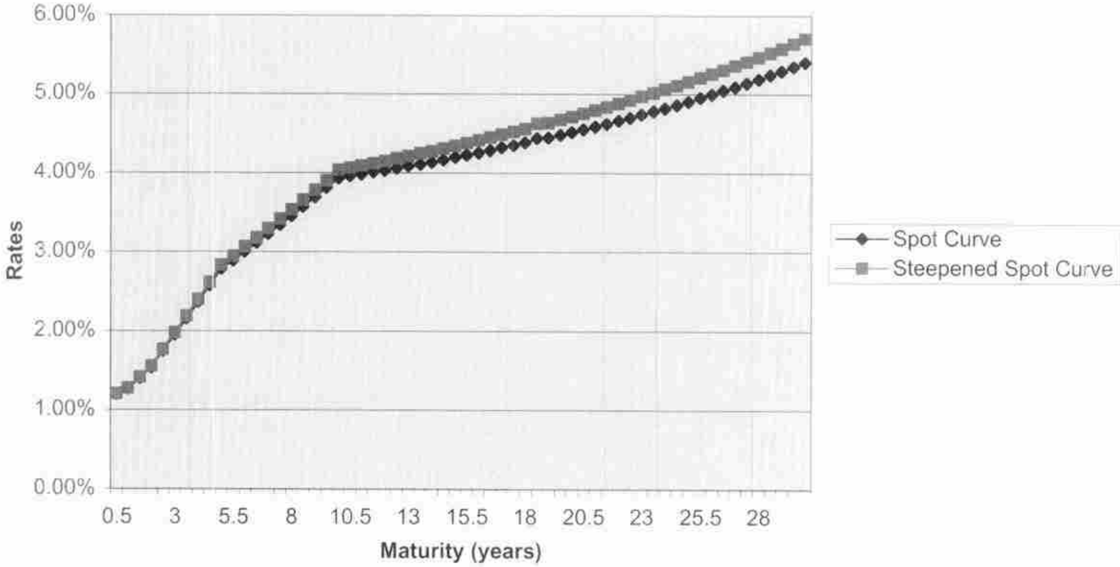


EXHIBIT 9
Asset Liability Example

	L5	L10	L20	L30
Liability	$10,000,000.00	$10,000,000.00	$10,000,000.00	$10,000,000.00
Present Value (L: Original Spot)	$8,708,466.50	$6,772,742.57	$4,093,626.59	$2,020,782.91
Present Value (L: Steepened Spot)	$8,687,023.43	$6,706,662.73	$3,936,663.92	$1,851,351.50
Change in Liability from Steepening	($21,443.08)	($66,079.85)	($156,962.67)	($169,431.40)
KRDs	4.93	9.81	19.57	29.24
Par Yield Change from Steepening Spot Curve	0.048%	0.089%	0.159%	0.193%
Spot Rate Change from Steepening Spot Curve	0.050%	0.100%	0.200%	0.300%
Asset Changes				
ΔValue from PD & Corresponding Par Curve	($20,617.28)	($59,420.79)	($127,416.24)	($114,313.64)
ΔValue from KRD & Spot Curve	($21,466.37)	($66,440.60)	($160,224.54)	($177,263.08)
Immunization Results				
Net Differences from PD based Immunization	$825.80	$6,659.06	$29,546.43	$55,117.76
Net Differences from KRD based Immunization	($23.29)	($360.76)	($3,261.88)	($7,831.68)
Actual PDs Required to Match KRDs	5.128	10.969	24.609	45.342

nificant differences that result, and how the interpretation and implementation of the results vary significantly. Users of these measures should ensure they understand the underlying relationships before they apply any strategy.

Users of the partial duration methodology are explicitly assuming that the par curve is the driving interest rate curve in the market, and that all strategies should be based on par yield changes, not spot rate changes. Users of the key rate duration methodology base their strategies directly on the spot curve. Some may argue that the spot curve is more theoretically sound, given that it is the pricing curve for fixed-income securities. If it is the expected change in price that we base our strategies on, then the pricing curve should be more appropriate to use. An equally cogent argument is that the on-the-run yields are the driving force behind the fixed-income market, so the par curve should be the appropriate curve as the securities will be priced close to par.

Which of the two measures to use will depend on the application under consideration. Users must understand the dynamics of each, and ensure that they properly interpret the measures prior to implementing a trading or risk management strategy.

ENDNOTES

The authors thank Thomas Ho and Robert Reitano for helpful comments.

[1]The first studies are Jones [1991] and Litterman and Scheinkman [1991].

[2]For a discussion of the difference between effective duration and a measure called modified duration, see Fabozzi, Buetow, and Johnson [2002].

[3]Partial duration is also discussed by Waldman [1992].

[4]We are taking some liberties with terminology. Both approaches are often referred to as key rate duration or partial duration. We refer throughout to the Ho [1992] approach as the key rate duration and the par curve approach as partial duration to distinguish the two approaches.

[5]There are some callable Treasury issues in the market. The Treasury stopped issuing callable bonds in the early 1980s. We ignore these issues in our discussion.

[6]The coupon securities are the U.S. Treasury note with a coupon of 4-3/8% and a maturity of August 15, 2012, and the U.S. Treasury bond with a coupon of 5-3/8% and a maturity of February 15, 2031. We will refer to these as the 10- and 30-year bonds (although their exact maturities are 9.5 and 28 years).

[7]We can confirm that a wide range of users of these measures in the fixed-income industry do not differentiate between PD and KRD.

[8]Securities with embedded options are properly evaluated using a term structure model. Most term structure models use the spot curve as an input to establish the no-arbitrage relationship. Exhibit 5 highlights the different spot curves to use in a term structure model, depending on the approach used. The obvious result of this is that the sensitivity measures would also be very different.

[9]We use both reference curves to compute the effective duration in order to continue to highlight the differences that result between the two approaches. As we note earlier, most sources suggest using the spot curve directly for the computation of effective duration. As result, the user should be particularly careful in comparing effective duration to partial duration. If the effective duration is computed using the spot curve, it cannot be compared directly to the partial durations without making the appropriate adjustments we outline.

[10]As before we use a ΔY of 25 basis points for the computations.

[11]The non-zero differences under the KRD approach are due to the convexity of the liabilities (ignored in computation of the change in asset values).

REFERENCES

Fabozzi, F., G. Buetow, and R. Johnson. "Measuring Interest Rate Risk." In F.J. Fabozzi, ed., *The Handbook of Fixed Income Securities*, 6th Ed. New York, NY: McGraw-Hill, 2001.

Ho, T. "Key Rate Durations: Measures of Interest Rate Risks." *The Journal of Fixed Income*, Vol. 2, No. 2 (1992), pp. 29-44.

Jones, F. "Yield Curve Strategies." *The Journal of Fixed Income*, (1991) pp. 33-41.

Kuberek, Robert C. "Term Structure Factor Models." Chapter 23 in F.J. Fabozzi, ed., *Handbook of Portfolio Management*. Hoboken, NJ: John Wiley & Sons, 1998.

Litterman, R., and J. Scheinkman. "Common Factors Affecting Bond Returns." *The Journal of Fixed Income*, Vol. 1, No. 1 (1991), pp. 54-61.

Nelson, C., and W. Carleton. "A Parsimonious Modeling of Yield Curves." *Journal of Business*, 60 (1987), pp. 473-489.

Reitano, R. "Multivariate Duration Analysis." *Transactions of the Society of Actuaries*, XLIII (1991), pp. 335-391.

——. "Non-Parallel Yield Curve Shifts and Duration Leverage." *The Journal of Portfolio Management*, Vol. 16, No. 4 (1990), pp. 62-67.

Ryan, R., and F. Fabozzi. "Rethinking Pension Liability and Asset Allocation." *The Journal of Portfolio Management*, Vol. 28, No. 4 (2002), pp. 7-15.

Schumacher, M.P., D.C. Dektar, and F.J. Fabozzi. "Yield Curve Risk of CMO Bonds." Chapter 15 in Frank J. Fabozzi, ed., *CMO Portfolio Management*. New Hope, PA: Frank J. Fabozzi Associates, 1994.

Waldman, M. "Beyond Duration: Risk Dimensions of Mortgage Securities." *The Journal of Fixed Income*, Vol. 2, No. 3 (1992), pp. 5-15.

Willner, R. "A New Tool for Portfolio Managers: Level, Slope, and Curvature Durations." *The Journal of Fixed Income*, Vol. 6, No. 1 (1996), pp. 48-59.