A Two-Factor Term Structure Model under GARCH Volatility

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Extending the work of Vasicek [1977], Cox, Ingersoll, and Ross [1985] (henceforth CIR), and Heath, Jarrow, and Morton [1992] (henceforth HJM), researchers have developed many models of the term structure of interest rates that often involve pricing bonds or interest rate derivatives. One class of models often takes a short-term interest rate to be the variable that drives the term structure, and then prices bonds and bond options by deriving the appropriate risk-neutral or risk-adjusted dynamics of the short rate. The HJM approach, however, takes the entire yield curve (or, equivalently, the set of forward rates or bond prices) to be the state variable and derives prices of options on bonds. It also matches the current term structure by default as the term structure (or, equivalently, the set of forward rates) is itself the state variable.

It is well known that the dynamics of the term structure cannot be captured by one factor (see Litterman and Scheinkman ([1991]). In fact, Dybvig [1997] emphasizes a second factor related to the volatility of interest rates that may not have any major impact on the prices of the spot bonds, but may be very important for bond options. Andersen and Lund [1996] also find that a factor that helps explain the curvature of the yield curve is, in fact, closely related to the volatility of the short rate. Unlike the HJM approach, the short rate-based approaches can easily accommodate a second non-traded state variable such as volatility and maintain analytical as well as numerical tractability.

Fong and Vasicek [1991] develop a two-factor model in which one factor is the instantaneous variance of the short rate, and provide solutions for bond prices. Longstaff and Schwartz [1992] (henceforth LS) and Chen and Scott [1992] (henceforth CS) develop continuous-time two-factor models along the lines of CIR [1985] that can incorporate random volatility in the evolution of the short rate and offer analytical solutions for values of bonds and bond options.

One difficulty of the continuous-time models is that the volatility of the short rate is unobservable. Unobservable volatility implies that the spot volatility at time t that is needed to price bonds and bond options is not known at the time the price needs to be calculated. As a result, one would be constrained to use the spot volatility inferred from a previous period, which may not necessarily reflect the current information in the term structure. For example, one can calibrate the continuous-time stochastic volatility models to market data (observed at time $t-1$) on options and bonds, and therefore get an estimate of the volatility at $t-1$ that may very well be different from the volatility at time t.

Estimating these models on large data sets spanning multiple days also represents a computational burden. Each day's volatility must be estimated as a separate parameter; this considerably increases the number of parameters over which a criterion function such as squared pricing errors needs to be minimized. For example, if we have data for daily prices of bonds of various maturities for four weeks (20 days), we would have to estimate 20 extra parameters (other than the time-invariant parameters of the model).

We develop a discrete-time model of interest rates with analytical solutions for values of bonds and bond options in which the second state variable is a random volatility following a GARCH process, while the first state variable is the mean-reverting short rate, and the two state variables can be correlated. Besides bond and bond futures prices, the model generates analytical solutions for values of options on discount bonds and discount bond futures as well as other interest rate derivatives such as caps or floors.

The advantage of this model besides its analytical tractability is that the time-varying random volatility is easily filtered from the discrete observations of interest rates. Therefore, the spot volatility at time t that is needed to value bonds and bond options at that time is known or observable. Furthermore, observability of volatility implies that during model calibration, the number of parameters that needs to be estimated remains finite (equal to the number of time-invariant parameters of the model) instead of increasing proportionately with the number of days or time periods. (For expositional clarity, we often refer to the state variables as factors henceforth.)

Discrete-time volatility models based on the GARCH dynamics have been very popular in equity and currency markets due to their abilities to capture certain salient features of data such as volatility clustering and the fact that the time-varying volatility is observable (see Bollerslev, Chou, and Kroner [1992]). Brenner, Harjes, and Kroner [1996], Koedjik et al. [1994], and others have shown that GARCH processes can also capture the volatility dynamics of interest rates.

Until now, though, there were not explicit solutions for bond prices and prices of various interest rate derivatives under a GARCH process (to the best of our knowledge). Therefore one of our contributions lies in providing analytical solutions for bond prices and especially bond option prices in a GARCH framework, thus simplifying the computational burden of non-analytical solutions.

Calibrating our model to the yield curve (eight different maturities) for several two-week intervals in the period 1990-1996, we find that the two-factor version does not improve (statistically and economically) upon the nested one-factor model (which is a discrete-time version of the Vasicek model) in terms of pricing the cross-section of spot bonds. This occurs even though the one-factor is rejected in favor of the two-factor model in explaining the time series behavior of a chosen short rate (computed from the three month T-bill yields). This conclusion is robust to the length of the interval used to sample the term structure.

Given option prices (on discount bonds) from the two-factor model, however, the implied volatilities from the Black model exhibit a much more pronounced smirk or skew than the implied volatilities generated from the prices of the nested one-factor model. These option prices are generated not by conjectured parameter estimates, but by actual parameter estimates obtained though calibrating the models to the observed yield curve.

Thus our results show that the effects of random volatilities of interest rates are visible mostly in bond option prices rather than in bond prices.

I. MODEL

We assume that the short rate, r_t, which is the interest rate on a loan at time t that is to be repaid at time $t + 1$, follows the processes:

$$r_{t+1} = \mu_0 + \mu_1 r_t + \lambda h_{t+1} + \sqrt{h_{t+1}} z_{t+1} \tag{1}$$

$$h_{t+1} = \omega + \beta h_t + \alpha(z_t - \gamma\sqrt{h_t})^2 \tag{2}$$

where z_t is a standard normal. Consequently h_{t+1} is the variance of $r_{t+1} - r_t$, conditional on the information at time t. A non-zero γ allows correlation between the level of interest rates and the conditional variance in that $Cov_{t-1}[h(t+1), r_t] = 2\alpha\gamma h_t$ (Cov_{t-1} ( ) is the covariance conditional on the information at time $t-1$). The volatility of variance or volatility is controlled by the parameter, α.

In particular, note that h_{t+1} is known as of time t, given the history of r_t until t and an initial variance h_0 as follows:

$$h_{t+1} = \omega + \beta h_t + \alpha \frac{(r_t - \mu_0 - \mu_1 r_{t-1} - \lambda h_t - \gamma h_t)^2}{h_t} \tag{3}$$

In this model, we have a mean-reverting short rate with GARCH volatility. This is very similar to being the discrete-time counterpart of the Vasicek [1977] model, augmented with GARCH volatility. In fact, if we restrict

h_t to be constant, we have exactly the discrete-time counterpart of the continuous-time Vasicek model.[1]

It can be shown following Heston and Nandi [2000] that if r_t is taken to be the instantaneous short rate as in continuous-time models of the term structure, the laws of motion of r_t are:

$$dr_t = (\mu_0 + \mu_1^* r_t + \lambda v_t)\,dt + \sqrt{v_t}dW_t \qquad (4)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t \qquad (5)$$

where μ_1^*, θ, κ, and σ are related to the parameters of the discrete-time model (see Heston 1 and Nandi [1999] for the details).

It can be verified that the continuous-time model lends itself to an affine structure in that the logarithm of a bond yield is affine (i.e., linear plus a constant) in r_t and v_t as in Fong and Vasicek [1991]. As a result, one can work out analytical solutions for the continuous-time model also. We concentrate only on the discrete-time model, as parameter estimation and therefore model implementation are much easier this way. This is simply because, unlike in the discrete-time GARCH model, one cannot filter the spot variance, v_t, of the continuous-time model as a function of the path of r_t as r_t is observed only at discrete intervals of time.

Let the time t price of a discount bond that matures at T be $P(t, T)$. It can be shown that under the risk-neutral distribution (which is needed for pricing) z_t is replaced by z_t^q, so that $z_t^q = z_t - \eta\sqrt{h_{t+1}}$ is a standard normal under the risk-neutral distribution, where η is a constant (see Heston and Nandi [1999]).

In other words, the risk-neutral dynamics of r_t and h_t are given by

$$r_{t+1} = \mu_0 + \mu_1 r_t + \lambda^* h_{t+1} + \sqrt{h_{t+1}}z_{t+1}^q \qquad (6)$$

$$h_{t+1} = \omega + \beta h_t + \alpha(z_t^q - \gamma^*\sqrt{h_t})^2 \qquad (7)$$

where $\lambda^* = \lambda + \eta$, and $\gamma^* = \gamma - \eta$.

II. BOND PRICES

Consider the price of a zero-coupon bond at t that expires at $t + 2$, i.e., $P(t, t + 2)$. Since $P(t, t + 2)$ is the discounted expected value of a single-period bond:

$$\begin{aligned} P(t, t+2) &= \exp(-r_t)E_t^q(P(t+1, t+2)) \\ &= \exp(-r_t)_t E_t^q(\exp(-r_{t+1})) \\ &= \exp(-(\mu_0 + (\mu_1 + 1)r_t + \\ &\quad (\lambda^* + 0.5)h_{t+1})) \end{aligned} \qquad (8)$$

where $E_t^q()$ is the expectation (under the risk-neutral distribution) conditional on the information set at time t.

Thus the yield of the two-period bond is $\frac{1}{2}(\mu_0 + (\mu_1 + 1)r_t + (\lambda^* + 0.5)h_{t+1})$. Note that the bond yield is affine (i.e., linear plus a constant) in the state variables r_t and h_{t+1}. We can similarly calculate the yield on a three-period bond by using iterated conditional expectation and show that the three-period yield is also affine in the state variables. The affine nature of the bond yield in the state variables r_t and h_{t+1} suggests that we can write the price of a bond with $T - t$ periods to maturity in the exponential-affine form as:

$$P(t, T) = \exp(A(t, T) + B(t, T)r_t + C(t, T)h_{t+1}) \qquad (9)$$

as in CIR [1985], Heston [1990], Duffie and Kan [1996], and many others. As r_t and h_{t+1} are known as of time t, it remains to solve for the coefficients $A(t, T)$, $B(t, T)$, and $C(t, T)$ in terms of the model parameters. These coefficients can be solved recursively from a terminal or boundary condition. For the boundary condition, consider the price of a bond (that will mature at T) at $T - 2$:

$$\begin{aligned} P(T-2, T) = \exp(&A(T-2, T) + \\ &B(T-2, T)r_{T-2} + C(T-2, T)h_{T-1}) \end{aligned} \qquad (10)$$

But, following (8), it is also true that $P(T - 2, T) = \exp[-(\mu_0 + (\mu_1 + 1)r_{T-2} + (\lambda^* + 0.5)h_{T-1})]$. Equating the two expressions for $P(T - 2, T)$, and matching coefficients on the state variables and the constant, we get the boundary conditions:

$$A(T-2, T) = -\mu_0 \qquad (11)$$

$$B(T-2, T) = -(\mu_1 + 1) \qquad (12)$$

$$C(T-2, T) = -(\lambda^* + 0.5) \qquad (13)$$

Now we will derive the backward recursion formula for the coefficients $A(t, T)$, $B(t, T)$, and $C(t, T)$, given the boundary conditions. At each time period, the bond price is the discounted expected value of its next-period price. In other words:

$$
\begin{aligned}
P(t,T) &= \exp(-r_t)E_t^q\left[P(t+1,T)\right] \\
&= \exp(-r_t)E_t^q\left[\exp(A(t+1,T)+B(t+1,T)r_{t+1}\right. \\
&\quad + \left. C(t+1,T)h_{t+2})\right]
\end{aligned} \tag{14}
$$

Substituting the risk-neutralized dynamics for r_{t+1} and h_{t+1}, and doing the algebra as shown in the appendix, we get that the coefficients are:

$$
\begin{aligned}
A(t,T) &= A(t+1,T)+\mu_0 B(t+1,T)+\omega C(t+1,T) \\
&\quad - \frac{1}{2}\log(1-2\alpha C(t+1,T))
\end{aligned} \tag{15}
$$

$$
B(t,T) = \mu_1 B(t+1,T)-1 \tag{16}
$$

$$
\begin{aligned}
C(t,T) &= \lambda^* B(t+1,T)+\beta C(t+1,T)+ \\
\alpha C(t+1,T)&\left[\frac{\frac{B(t+1,T)^2}{2\alpha C(t+1,T)}-2\gamma^* B(t+1,T)+\gamma^{*2}}{1-2\alpha C(t+1,T)}\right]
\end{aligned} \tag{17}
$$

In other words, starting from the boundary conditions (11), (12), and (13), we have to perform a simple backward recursion using (15), (16), and (17) to find the coefficients $A(t, T)$, $B(t, T)$, and $C(t, T)$. Once we have these, $P(t, T)$ is given by (9).

III. BOND OPTIONS

Now we are ready to calculate the price of a European option on a discount bond. Let the time to expiration of the option be τ and that of the underlying bond be T, and $T > \tau$. Let the price of a call option (at time t) on the discount bond be $C_o(t, \tau, T)$. The payoff from the option at maturity is $C_o(\tau, \tau, T) = \max[P(\tau, T) - K, 0]$.

As in Merton [1973], instead of the money market account, we choose the τ-maturity bond as the numeraire. Deflated by the numeraire, all asset prices are martingales under the martingale measure, which we shall refer to as the forward measure (as in Jamshidian [1989], Brace, Gatarek, and Musiela [1997], and others). Hence, $\frac{C_o(t,\tau,T)}{P(t,\tau)} = E_t^F\left(\frac{C_o(\tau,\tau,T)}{P(\tau,\tau)}\right)$ where $E_t^F(\,)$ is the conditional expectation under the forward measure.

Noting that $P(\tau, \tau) = 1$, we have that

$$
\frac{C_o(t,\tau,T)}{P(t,\tau)} = E_t^F\left(\max[\exp(x)-K,0]\right) \tag{18}
$$

where $x \equiv A(\tau, T) + B(\tau, T)r_\tau + C(\tau, T)h_{\tau+1}$. Thus, in order to calculate the option price, we have to know the conditional density of x under the forward measure, which in turn is known if we know its corresponding characteristic function.

Let $f(\phi) = E_t^F[\exp(\phi x)]$ denote the conditional moment-generating function of x at time t under the forward measure. The function, $f(\phi)$, also depends on the state variables and the parameters of the model, although we suppress them for notational convenience. We shall guess the exponential-affine functional form for $f(\phi)$ as:

$$
f(\phi) = \exp(A_1(t;\phi,\tau)+B_1(t;\phi,\tau)r \tag{19}
$$

The coefficients $A_1(t; \phi, \tau)$, $B_1(t; \phi, \tau)$, and $C_1(t; \phi, \tau)$ can be obtained from a boundary condition using a recursive procedure. The boundary conditions are:[2]

$$
A_1(\tau; \phi, \tau) = \phi A(\tau, T) \tag{20}
$$

$$
B_1(\tau; \phi, \tau) = \phi B(\tau, T) \tag{21}
$$

$$
C_1(\tau; \phi, \tau) = \phi C(\tau, T) \tag{22}
$$

where $A(\tau, T)$, $B(\tau, T)$, and $C(\tau, T)$ are known from the recursions needed to calculate the price of the discount bond that expires at T, $P(t, T)$.

Let $A^*(t + 1; \phi, \tau) \equiv A_1(t + 1; \phi, \tau) + A(t + 1; \tau)$; $B^*(t + 1; \tau) \equiv B_1(t + 1; \phi, \tau) + B(t + 1; \tau)$; and $C^*(t + 1; \tau) \equiv C_1(t + 1; \phi, \tau) + C(t + 1; \tau)$, where we already have $A(t + 1; \tau)$, $B(t + 1; \tau)$, and $C(t + 1; \tau)$ from the recursion needed to compute the price of the discount bond that expires at τ, $P(t, \tau)$. The recursions required for calculating $f(\phi)$ from the boundary conditions are then given as below (note that these are very similar to the recursions for bond prices). The derivations of these recursions are similar to those for the recursions for computing the bond price, and can be found in Heston and Nandi [1999].

$$A_1(t;\phi,\tau) = -A(t;\tau) + A^*(t+1;\phi,\tau) + \mu_0 B^*(t+1;\phi,\tau) + \omega C^*(t+1;\phi,\tau) - \frac{1}{2}\log(1-2\alpha C^*(t+1;\phi,\tau)) \tag{23}$$

$$B_1(t;\phi,\tau) = -(B(t;\tau)+1) + \mu_1 B^*(t+1;\phi,\tau) \tag{24}$$

$$C_1(t;\phi,\tau) = -C(t;\tau) + \lambda^* B^*(t+1;\phi,\tau) + \beta C^*(t+1;\phi,\tau) + \alpha C^*(t+1;\phi,\tau) \times \left[\frac{\frac{B^*(t+1;\phi,\tau)^2}{2\alpha C^*(t+1;\phi,\tau)} - 2\gamma^* B^*(t+1;\phi,\tau) + \gamma^{*2}}{1-2\alpha C^*(t+1;\phi,\tau)}\right] \tag{25}$$

At time t, now that we have A_1 $(t;\ \phi,\ \tau)$, B_1 $(t;\ \phi,\ \tau)$, and C_1 $(t;\ \phi,\ \tau)$, we know the conditional moment-generating function of x at time t. If $f(\phi)$ is the moment-generating function, $f(i\phi)$ is the characteristic function. Inverting the characteristic function as in Heston and Nandi [2000], one gets the price of the call option to be

$$C_o(t;\tau,T) = \frac{1}{2}P(t,T) + P(t,\tau)\frac{1}{\pi}\int_0^\infty \times Re\left[\frac{e^{-i\phi\log(K)}f(i\phi+1)}{(i\phi)}\right]d\phi - KP(t,\tau)\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\log(K)}f(i\phi)}{(i\phi)}\right]d\phi\right) \tag{26}$$

The derivation of (26) is shown in Heston and Nandi [1999].

By a simple rearrangement, we can also write Equation (26) in the typical Black-Scholes format as C_o $(t;\ \tau,\ T)$ $= P(t,\ T)M_1\ () - KP(t,\ \tau)M_2\ ()$ where M_1 ( ) and M_2 ( ) are two probability distribution functions. The two univariate integrals converge very quickly and are very easy to integrate numerically. They can be combined and evaluated as a single univariate integral in fractions of a second using any good integration routine. We use the Romberg integration routine of Press et al. [1992] to generate option prices.

Although we do not have explicit representations for the prices of American options on puts, one could compute the early exercise premium from the continuous-time analogue of the nested one-factor model (the Vasicek model) along the lines of Carr, Jarrow, and Myneni [1992], Huang, Subrahmanyam, and Yu [1996], Ju [1998], and others by using as the numeraire the price of the discount bond that matures at the same time as the option. This early exercise value can be added to the European value to obtain an approximate American value.

Prices of other types of European interest rate options such as caps and floors can be calculated in the same way as above by noting that these are portfolios of options on discount bonds (see Hull [1999] for the analogies). Also, the value of an option on the average of an interest rate process can be computed as in Bakshi and Madan [2000] as we have an affine model. Options on coupon bonds cannot be evaluated through a straightforward analytical procedure, however, contrary to the one-factor model (see Jamshidian [1989]).

IV. OPTIONS ON BOND FUTURES

Now we show how to calculate the value of a European option on a discount bond future under our model. Let $F(t,\ T_1,\ T_2)$ denote the current futures price for a contract on a discount bond that expires at T_2; the futures contract expires at T_1. Suppose a call option is traded on the futures contract, and the option expires at τ. Let $\tau < T_1 < T_2$.

As with options on discount bonds, $f(\phi)$ is the moment-generating function of the logarithm of the futures price when the option matures and is given as follows:

$$f(\phi) = \exp(A_1(t;\phi,\tau) + B_1(t;\phi,\tau)r_t + C_1(t;\phi,\tau)h_{t+1}) \tag{27}$$

$f(\phi)$ can be calculated recursively exactly as with options on discount bonds, but with a slightly different boundary or terminal condition as given below:

$$A_1(\tau;\phi,\tau) = \phi A_f(\tau,T_1,T_2) \tag{28}$$

$$B_1(\tau;\phi,\tau) = \phi B_f(\tau,T_1,T_2) \tag{29}$$

$$C_1(\tau;\phi,\tau) = \phi C_f(\tau,T_1,T_2) \tag{30}$$

Note that the terminal coefficients, $A_f(\tau,\ T_1,\ T_2)$, $B_f(\tau,\ T_1,\ T_2)$, and $C_f(\tau,\ T_1,\ T_2)$, are known from the recursions needed to calculate the futures price.

The characteristic function corresponding to $f(\phi)$ is $f(i\phi)$. Given $f(i\phi)$, it can be shown that the price of the option, $C_o(t;\ \tau,\ T_1,\ T_2)$ is (see Heston and Nandi [1999] for the derivation):

$$C_o(t;\tau,T_1,T_2)$$

$$= P(t,\tau)\left(\frac{1}{2}F(t;T_1,T_2) + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\log(K)}f(i\phi+1)}{(i\phi)}\right]d\phi\right.$$

$$\left. - K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\log(K)}f(i\phi)}{(i\phi)}\right]d\phi\right)\right) \tag{31}$$

V. ESTIMATION

How well does the model fit the observed term structure? And can we account for strike price biases (such as the smile or smirk in implied volatilities in some interest rate derivatives markets) from a one-factor model such as the Black model after we have estimated the model parameters? This exploration will help illustrate the computational feasibility and efficiency of the model. Calibrating the model to the actual term structure should also generate some basic insights about the basic functionality of the model in the real world that may inspire more empirical research.

For the zero-coupon bond prices of various maturities, we use a modification of the Fisher, Nychka, and Zervos [1995] method of constructing zero-coupon yield curves constructed from the daily Center for Research in Security Prices bond file data due to Waggoner [1996](see Bliss 3 [1997] for the details).[3] The criterion function for parameter estimation is minimization of the sum of squared errors between model and market zero-coupon bond prices for a sample of two weeks in a period spanning 1990-1996.[4]

Specifically, we minimize the criterion function:

$$\min_{\mu_0,\mu_1,\omega,\alpha,\beta,\gamma,\lambda,\eta} \sum_{t=1}^{T}\sum_{i=1}^{N_t}(P_{i,t} - M_{i,t})^2 \tag{32}$$

where $P_{i,t}$ and $M_{i,t}$ are, respectively, the model and market prices for bond i on date t. For the short rate, we use the continuously compounded overnight rate implicit in the three month T-bill prices.

Note that in computing this criterion function, one needs to know h_{t+1}, which is the conditional variance of the r_{t+1}, at time t. For each t, h_{t+1} is known from the history of the short rate until time t, and need not be estimated as a parameter. In continuous-time stochastic volatility models, however, one would have to estimate the variance as a separate parameter, making the estimation procedure very cumbersome. Given our sample, we would have to estimate ten extra parameters (for the volatilities on ten separate trading days in two weeks) if we were using a continuous-time stochastic volatility model.

The parameter estimates obtained by fitting the model to market prices for two weeks starting January 3, 1994, are as follows: $\mu_0 = 2.13 \times 10^{-8}$, $\mu_1 = 0.999$, $\lambda = -3.36$, $\nu = 15.58$, $\omega = 1.44 \times 10^{-11}$, $\beta = 0.256$, $\alpha = 1.093 \times 10^{-11}$, and $\gamma = 12.68$ (note that the parameter estimates are based on the values of a daily (not annualized) yield). According to these parameter estimates, the long-run mean of the annualized volatility of the changes (not the levels) of the annualized short rate is around 0.041, and the half life of the volatility process is around 185 days.

Exhibit 1 shows the model yields for various maturities on January 3, 1994. Now, if we fit the nested one-factor model (the discrete-time version of the Vasicek model) to the same sample, we find that the average absolute pricing error is only a little higher than the two-factor model (around 2 basis points).[5]

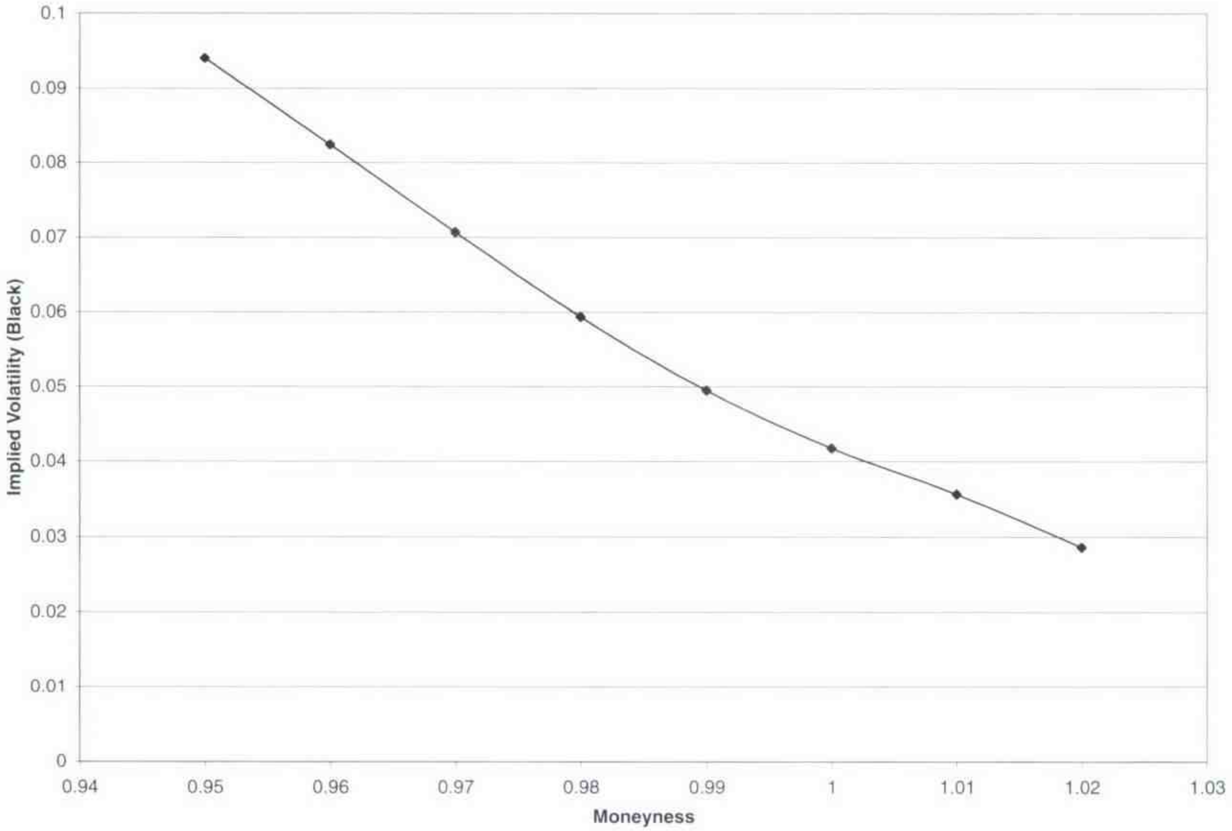
A statistical test such as a likelihood-ratio test testing the nested one-factor model against the two-factor model cannot reject the one-factor model (see Judge et al. [1988]). In other words, the two-factor model is only a trivial improvement if fitting the entire yield curve over a given period of time is the criterion function.[6]

In the options markets, volatility is expected to be of much more importance. Using the parameters estimated from the cross-section of bond prices (i.e., from the yield curve calibration), we generate prices of options on discount bonds of different strike prices under our two-factor model. Then, using the Black [1976] model, a widely used one-factor model for pricing these options in the marketplace (see Hull [1998]), we back out the implied volatility of different strike prices for an option (with 60

EXHIBIT 1
Market and Model Yields

Maturity (in days)	Market Yield	Model Yield
90	3.13	3.23
180	3.3	3.36
270	3.46	3.49
360	3.67	3.62
730	4.26	4.11
1095	4.65	4.53
1460	5.04	4.9
3650	6.09	5.95

EXHIBIT 2
Implied Volatility Smile



days to maturity) on a two-year zero-coupon bond.

Exhibit 2 shows that the implied volatilities are reduced as the strike price increases; the implied volatilities display a pronounced smirk or skew, a feature often observed in the bond options market, but the skew from the nested one-factor Gaussian model is much more flat.[7]

Thus our results suggest that volatility as a second factor is important primarily for bond options, rather than for spot bonds or bond futures.

VI. CONCLUSION

We have developed a discrete-time two-factor model of interest rates, in which the second factor is a time-varying volatility following a GARCH process that can be correlated with the level of the short rate. The model can be used to price spot bonds, bond futures, and bond options using easily computable analytical solutions. Unlike continuous-time two-factor models with random volatility, volatility in our model is observable on the basis of the history of interest rates. This gives our model a distinct computational advantage in terms of estimation and implementation over continuous-time stochastic volatility models of interest rates.

We find that the second factor (volatility) does not matter as much for valuing spot bonds as for valuing bond options. In fact, we cannot reject the nested one-factor version of the model for the entire cross-section of spot bonds on the basis of parameters estimated from the U.S. Treasury yield curve. Option values from the two- factor model produce a much more pronounced smirk or skew in implied volatilities (using the one-factor Black model) for bond options than for option values from the one-factor model. This suggests that the effects of random volatilities of interest rates would be visible mostly in bond option prices rather than in bond prices.

APPENDIX

Bond Valuation Equation

Here we show how to derive Equations (15), (16), and (17), the coefficients needed for bond pricing. Substituting the dynamics of r_{t+1} and h_{t+1} in (14):

$$\begin{aligned} &\exp(A(t,T) + B(t,T)r_t + C(t,T)h_{t+1}) \\ &\quad= \exp(-r_t + A(t+1,T) \\ &\quad+ B(t+1,T)(\mu_0 + \mu_1 r_t + \lambda^* h_{t+1}) \\ &\quad+ C(t+1,T)(\omega + \beta h_{t+1})). \end{aligned}$$

$$\begin{aligned} &E_t^q \Big[\exp(B(t+1,T)\sqrt{h_{t+1}} z_{t+1} \\ &+ \alpha C(t+1,T)(z_{t+1}^q - \gamma^* \sqrt{h_{t+1}})^2)\Big] \end{aligned}$$

Completing the square in the portion to which the expectation applies, using the fact that for a standard normal z, $E(a(z+b)^2) = \exp(-\frac{1}{2}\log(1-2a) + \frac{ab^2}{1-2a})$ and matching the coefficients on r_t, h_{t+1}, and the constant, we get (15), (16), and (17).

ENDNOTES

The views expressed here are those of the authors and not those of Fannie Mae.

[1]Note that we can easily drop λ from Equation (1) and still generate all our results. Similarly, we can include r in the right-hand side of Equation (2) to reflect that the level of the variance depends on the level of interest rates, and still get the analytical solutions.

[2]The boundary conditions can be derived as in Heston [1993] or Heston and Nandi [2000].

[3]We thank Daniel Waggoner for constructing the zero-coupon yield curves.

[4]The data for daily coupon bond prices that we have access to ended in 1996.

[5]The discount bond prices for the one-factor or the constant-volatility model are easily derived by noting that $z_t^* = z_t = \eta$, instead of $z_t^* = z_t - \eta\sqrt{h_{t+1}}$ as in the two-factor model.

[6]When we repeat this estimation exercise for other randomly chosen two-week periods between 1990 and 1996, our conclusions regarding the mispricing remain essentially unchanged, although the parameter estimates are somewhat different, depending on the period.

[7]It is possible that a different one-factor model such as that of CIR [1985] would also exhibit a smirk or smile because the Black formula is non-linear in volatility for out-of-the-money options.

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