Credit Gadgets

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We introduce a novel device that we call a *credit gadget*. A credit gadget is a theoretically cost-free portfolio of pure discount bonds that enables its user to assume a desired risk exposure to a particular region of the local credit spread surface, while at the same time preventing risk exposure to all other regions. A credit gadget may be used to hedge against particular forward absolute or relative credit spreads and forward credit spread skews, or to target and speculate on these spreads and skews. A credit gadget is a powerful instrument in the tool kit of fixed-income traders.

We derive the credit gadget in continuous time, and present an analogous discrete time exercise. We also provide a number of numerical illustrations.

I. KEY DEFINITIONS

Envision a matrix of the yield spreads on various (but matched) maturity credit risk-free and credit-risky pure discount securities that have no embedded options and are otherwise identical, most notably with the same liquidity and tax treatment. Such a matrix would reflect pure measures of currently prevailing absolute and relative credit spreads.

For example, suppose there are three pure discount Treasury bonds with maturities of one, two, and three years, with yields denoted by T1, T2, and T3. Similarly, let A1, A2, and A3 be the yields on A-rated pure discount bonds with maturities of one, two, and three years, and B1, B2, and B3 yields on B-rated bonds. The comparable-maturity bonds are identical, except for credit quality.[1]

The matrix of absolute and relative pure credit spreads is shown in Exhibit 1.

Using standard techniques such as piecewise cubic splines, one can depict this matrix in three-dimensional space, and thereby generate an entire three-dimensional surface of currently prevailing absolute and relative credit spreads.[2] We'll call this a credit spread surface. It is a graphical representation of a set of credit spreads for a range of credit ratings and expirations. Thus the credit spread surface is analogous to an implied volatility surface as in the options pricing literature.[3]

It is of course possible to use absolute and relative credit spreads to generate forward credit spreads and forward credit skews. For example, the pair (A1-T1) and (A2-T2) can be used to quantify a one-year credit spread between the A-rated bond and the Treasury that is expected to prevail one year hence. This is a forward absolute credit spread. A series of these spreads would represent a term structure of forward absolute credit spreads.

The pair (B2-A2) and (B3-A3) can be used to quantify a one-year credit spread expected to prevail between the B-rated and A-rated bonds two years hence. This is an example of a forward relative credit spread. A series of these spreads would represent a term structure of forward relative credit spreads.

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EXHIBIT 1
Bond Credit Spreads

1-Year	Maturity 2-Year	3-Year
(A1 – T1)	(A2 – T2)	(A3 – T3)
(B1 – T1)	(B2 – T2)	(B3 – T3)
(B1 – A1)	(B2 – A2)	(B3 – A3)

By combining the relevant forward absolute and relative credit spreads, one can in turn obtain a forward credit spread skew. Forward credit spreads and forward credit spread skews are completely analogous to the concepts of forward volatility and forward volatility skews that are well known in the options markets.[4]

More generally, just as an implied volatility surface can occasion a local volatility surface in the case of options, a credit spread surface can be used to generate what we shall call a *local credit spread surface*. A local credit spread surface is merely a graphical representation, in three-dimensional space, of forward absolute and relative credit spread term structures and forward credit spread skews. That is, a local credit spread surface represents the set of forward credit spreads for a range of credit ratings and future time periods.

A credit gadget is a theoretically cost-free portfolio of pure discount bonds that lets its user assume a desired risk exposure to a particular region of the local credit spread surface, while preventing risk exposure to all other regions. Thus a credit gadget may be used to hedge against particular forward absolute or relative credit spreads and forward credit spread skews. Also, a credit gadget can be used to target and speculate on these spreads and skews.[5]

II. DERIVATION AND PROPERTIES IN CONTINUOUS TIME

Let $B_T(\cdot)$ be the price of a T-period, zero-coupon credit risk-free bond (at time $\cdot$). So:

$$B_T(t)/B_{T-\Delta T}(t) = \exp(-f_T \Delta T) \quad (1)$$

where $f_T(t)$ is the forward rate between times $T - \Delta T$ and T. Assuming continuously compounded interest rates, we have:

$$f_T = [\ln(B_{T-\Delta T}[t]) - \ln(B_T[t])]/\Delta T \quad (2)$$

Similarly, let $B_T{}^*(\cdot)$ be the price of a T-period zero-coupon credit risky bond (at time $\cdot$). So:

$$B_T{}^*(t)/B_{T-\Delta T}(t) = \exp(-f_T{}^*\Delta T) \quad (3)$$

where $f_T{}^*(t)$ is the forward rate between times $T - \Delta T$ and T, and thus:

$$f_T{}^*(t) = [\ln(B_{T-\Delta T}{}^*[t]) - \ln(B_T{}^*[t])]/\Delta T \quad (4)$$

A forward absolute credit spread for the risky bond is equal to:

$$f_T{}^*(t) - f_T(t) \quad (5)$$

or equivalently:

$$\{[\ln(B_{T-\Delta T}{}^*[t]) - \ln(B_T{}^*[t])] - [\ln(B_{T-\Delta T}[t]) - \ln(B_T[t])]\}/\Delta T \quad (6)$$

A series of these forward absolute credit spreads represents a forward absolute credit spread term structure.

Let $f_T{}^{*B}$ represent the forward credit spread on a credit-risky Bond B whose price is given by $B_T{}^{*B}(\cdot)$. Let $f_T{}^{*A}$ represent the forward credit spread on a credit-risky Bond A whose price is given by $B_T{}^{*A}(\cdot)$. Bond B exhibits more credit risk than Bond A. Then a forward relative credit spread is represented by:

$$f_T{}^{*B} - f_T{}^{*A} \quad (7)$$

Equivalently, a forward relative credit spread is given by:

$$\{[\ln(B_{T-\Delta T}{}^{*B}[t]) - \ln(B_T{}^{*B}[t])] - [\ln(B_{T-\Delta T}{}^{*A}[t]) - \ln(B_T{}^{*A}[t])\}/\Delta T \quad (8)$$

A series of these forward relative credit spreads represents a forward relative credit spread term structure. Combining forward absolute and relative credit spreads, like those in Equations (5) and (7), produces a forward credit spread skew.

We have said that a credit gadget is a theoretically cost-free portfolio of pure discount bonds that has sensitivity only to a particular region of the local credit spread surface. To construct an infinitesimal credit gadget, Λ_T, we assume security positions as follows:

- A long position in B_T^*, a zero-coupon risky bond of maturity T whose value at time t is $B_T^*(t)$; and
- A short position of $\exp(-f_T^*(0)\Delta T)$ units of a zero-coupon risky bond $B_{T-\Delta T}^*$ of maturity T - ΔT whose value at time t is $B_{T-\Delta T}^*(t)$, where $f_T^*(0)$ is the forward rate (on the risky bond) between T – ΔT and T at the initial time t = 0; and
- A long position in B_T, a zero-coupon risk-free bond of maturity T whose value at time t is $B_T(t)$; and
- A short position of $\exp(-f_T(0)\Delta T)$ units of a zero-coupon risk-free bond $B_{T-\Delta T}$ of maturity T – ΔT whose value at time t is $B_{T-\Delta T}(t)$, where $f_T(0)$ is the forward rate on the risk-free bond between T – ΔT and T at the initial time t = 0.

The value of the infinitesimal credit gadget Λ_T at time T is given by:

$$\Lambda_T(t) = B_T^* - \exp[-f_T^*(0)\Delta T]B_{T-\Delta T}^*(t) + B_T - \exp[-f_T(0)\Delta T]B_{T-\Delta T}(t) \tag{9}$$

Now combine Equation (9) with Equations (1) and (3) to obtain the value of the credit gadget in terms of the infinitesimal forward absolute credit spread at time t:

$$\Lambda_T(t) = B_{T-\Delta T}^*(t)[\exp(-f_T^*(t)\Delta T) - \exp(-f_T^*(0)\Delta T)] + B_{T-\Delta T}(t)[\exp(-f_T(t)\Delta T) - \exp(-f_T(0)\Delta T)] \tag{10}$$

The initial value (at time 0) of this credit gadget is zero, and its value is sensitive only to the particular infinitesimal forward absolute credit spread $f_T^*(t) - f_T(t)$. If this spread does not change over time, the value of the credit gadget remains intact at zero. Otherwise, the value of the gadget will change accordingly; that is, as given by Equation (10). In this respect, a credit gadget is akin to a forward or futures contract that would be traded on a forward credit spread, i.e., a type of credit derivative.[6]

Equation (10) represents our principal result. A similar approach occasions an infinitesimal credit gadget whose initial value is zero and that is sensitive only to a particular infinitesimal forward relative credit spread, $f_T^{*B}(t) - f_T^{*A}(t)$:[7]

$$\Lambda_T(t) = B_{T-\Delta T}^{*B}(t)[\exp(-f_T^{*B}(t)\Delta T) - \exp(-f_T^{*B}(0)\Delta T)] + B_{T-\Delta T}^{*A}(t)[\exp(-f_T^{*A}(t)\Delta T) - \exp(-f_T^{*A}(0)\Delta T)] \tag{11}$$

From Equation (10) we may obtain a measure of the sensitivity of the credit gadget to changes in the infinitesimal forward absolute credit spread:

$$\begin{aligned}\partial\Lambda_T(t) &= -\Delta\partial f_T^*\exp(-f_T^*(t)\Delta T)B_{T-\Delta T}(T)\\ &= -\Delta T\partial f_T^* B_T^*(T)\end{aligned} \tag{12}$$

$$\begin{aligned}\partial\Lambda_T(t) &= -\Delta\partial f_T\exp(-f_T(t)\Delta T)B_{T-\Delta T}(t)\\ &= -\Delta\partial f_T B_T(T)\end{aligned} \tag{13}$$

and

$$\partial\Lambda_T(t) = -\Delta T\partial f_T^* B_T^*(T) + \Delta T\partial f_T B_T(T) \tag{14}$$

Equation (14) quantifies the sensitivity of the value of the credit gadget to an infinitesimal change in the forward absolute credit spread. It can be used to establish hedges or targeted risk exposures to the desired forward absolute credit spread and is therefore another important result. An analogous approach represents the sensitivity of the value of a credit gadget to an infinitesimal change in the relevant forward relative credit spread:[8]

$$\partial\Lambda_T(t) = -\Delta T\partial f_T^{*B} B_T^{*B}(T) + \Delta T\partial f_T^{*A} B_T^{*A}(T) \tag{15}$$

The intuition behind Equations (12) through (14) is quite straightforward. From (12), one obtains the sensitivity of the credit gadget with respect to a change in the relevant forward rate for the credit-risky asset. The change in this rate could be occasioned by either a change in the credit risk-free interest rate component of the risky asset's forward rate or a change in its credit spread component.

From (13), however, one isolates the sensitivity of the credit gadget with respect to a change in the credit risk-free interest rate component. After all, the credit risk-free asset's forward rate has no credit spread component, and its interest rate component is the same as that of the credit-risky asset due to the identical maturity of each asset. By controlling for the effects of changes in forward interest rates, Equation (14) correctly captures the sensitivity of the credit gadget to a change in the relevant forward credit spread per se.

Similar intuition holds for Equation (15) and for applications regarding forward credit spread skews. For instance, in (15) the effect of a change in the forward interest rate on the credit gadget is accommodated because the effect is equally reflected by the changes in the forward rates for both risky assets. Thus (15) correctly captures the sensitivity of the credit gadget to a change in the relative forward credit spread per se.

Generally speaking, Equations (14) and (15) facilitate the desired exposure to any particular region of the local credit spread surface, as we will illustrate. For instance, to hedge against a change in a particular forward relative credit spread, set $\partial\Lambda_T(t)$ equal to zero in Equation (15), and solve to obtain a portfolio long ∂f_T^{*A} of $B_T^{*A}(T)$ and short ∂f_T^{*B} of $B_T^{*B}(T)$. To hedge against a change in a particular forward absolute credit spread, set $\partial\Lambda_T(t)$ equal to zero in Equation (14), and solve to obtain a portfolio long ∂f_T of $B_T(T)$ and short ∂f_T^* of $B_T^*(T)$.

III. DISCRETE TIME DERIVATION AND PROPERTIES

We focus on a finite credit gadget that is sensitive to changes in forward absolute credit spreads. The discrete time derivation and properties of credit gadgets sensitive to changes in forward relative credit spreads or forward credit spread skews are analogous.

We seek a credit gadget that is sensitive only to changes in a particular forward credit spread between two finite time marks, T_2 and T_1 (where $T_2 < T_1$). Let $B_{T2}(0)$ be the price of a credit risk-free pure discount security with maturity T_2. Similarly, let $B_{T1}(0)$ be the price of a credit risk-free pure discount security with maturity T_1. Also, let $B_{T2}^*(0)$ and $B_{T1}^*(0)$ be the prices of credit-risky pure discount securities with maturities of T_2 and T_1. Then:

$$B_{T1}(0)/B_{T2}(0) = \exp[-f_{T1,T2}(0)(T_1 - T_2)] = \phi_{T1,T2} \quad (16)$$

and

$$B_{T1}^*(0)/B_{T2}^*(0) = \exp[-f_{T1,T2}^*(0)(T_1 - T_2)] = \phi_{T1,T2}^* \quad (17)$$

The variables $\phi_{T1,T2}$ and $\phi_{T1,T2}^*$ are the current forward discount factors, for the credit risk-free and credit-risky bonds, respectively, from time T_1 to time T_2. They are the discounted values at time T_2 of one dollar paid at time T_1, using forward rates obtained from the current ($t = 0$) credit risk-free (in the case of $\phi_{T1,T2}$) and credit-risky (in the case of $\phi_{T1,T2}^*$) yield curve to conduct the discounting.[9]

To construct a finite credit gadget, $\Lambda_{T1,T2}$, we assume a portfolio analogous to that occasioning an infinitesimal credit gadget, that is, a portfolio analogous to that described just before Equation (9). We have:

$$\Lambda_{T1,T2} = (B_{T1}^* - \phi_{T1,T2}^* B_{T2}^*) + (B_{T1} - \phi_{T1,T2} B_{T2}) \quad (18)$$

Equation (18) represents the discrete-time analogue of Equation (9).

The value of a finite credit gadget $\Lambda_{T1,T2}$ at time t is given by:

$$\begin{aligned}\Lambda_{T1,T2}(t) = {} & B_{T1}^*(t) - \exp[-f_{T1,T2}^*(0)(T_1 - T_2)]B_{T2}^*(t) + \\ & B_{T1}(t) - \exp[-f_{T1,T2}(0)(T_1 - T_2)]B_{T2}(t) \quad (19)\end{aligned}$$

Equivalently:

$$\begin{aligned}\Lambda_{T1,T2}(t) = {} & \{\exp[-f_{T1,T2}^*(t)(T_1 - T_2)] - \\ & \exp[-f_{T1,T2}^*(0)(T_1 - T_2)]\}B_{T2}^*(t) + \\ & \{\exp[-f_{T1,T2}(t)(T_1 - T_2)] - \\ & \exp[-f_{T1,T2}(0)(T_1 - T_2)]\}B_{T2}(t) \quad (20)\end{aligned}$$

Equation (20) shows that the value of a finite credit gadget is sensitive only to the difference in the forward rates for the credit-risky and comparable-maturity credit risk-free instruments corresponding to the interval between times T_2 and T_1; that is, $f_{T1,T2}^*$ and $f_{T1,T2}$. So, as expected, the value of a finite credit gadget is sensitive only to (here) the forward absolute credit spread that applies for a finite time period. Equation (20) represents the discrete-time analogue of Equation (10).

It is noteworthy that the ϕ-coefficients that define the finite credit gadget satisfy a compounding relation, e.g., $\phi_{T1,T3} = \phi_{T1,T2}\phi_{T2,T3}$. Thus, as anticipated, we can also view a finite credit gadget $\Lambda_{T1,T2}$ as a weighted combination of infinitesimal credit gadgets Λ_t.

Finally, the sensitivity of the finite credit gadget with respect to a small change in the forward absolute credit spread is given by:

$$\begin{aligned}\partial\Lambda_{T1,T2}(t) = {} & -(T_1 - T_2)\partial f_{T1,T2}^* B_{T1}^*(t) + \\ & (T_1 - T_2)\partial f_{T1,T2} B_{T1}(t) \quad (21)\end{aligned}$$

Equation (21) represents the discrete-time analogue of Equation (14).[10]

IV. NUMERICAL ILLUSTRATIONS

We illustrate several of the properties of credit gadgets, focusing on discrete-time credit gadgets that are sensitive to forward absolute and relative credit spreads.

Consider the matrix of yields, prices, and credit spreads in Exhibit 2. All securities are \$1 par value pure discount bonds with semiannually compounded yields.

Begin by focusing on the data in boldface. From Equations (16) through (19), we have:

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EXHIBIT 2

Bond Yields, Prices, and Credit Spreads

	0.5-Year Maturity			1.0-Year Maturity			1.5-Year Maturity		
Bond	**Yield**	**Price**	**Spread**	**Yield**	**Price**	**Spread**	**Yield**	**Price**	**Spread**
Treasury	**4.0%**	**0.980392**	**NA**	**4.2%**	**0.959287**	**NA**	*4.3%*	*0.938177*	*NA*
A-Rated	**4.5%**	**0.977995**	**0.5%**	**4.8%**	**0.953674**	**0.6%**	5.0%	0.928599	0.7%
B-Rated	5.0%	0.975610	1.0%	5.6%	0.946267	1.4%	5.7%	0.919152	1.4%

$$0.959287/0.980392 = \exp[-(0.043524)(0.5)] = 0.978473 = \phi_{1.0,0.5}$$

$$0.953674/0.977995 = \exp[-(0.050365)(0.5)] = 0.975132 = \phi_{1.0,0.5}^{*}$$

$$\Lambda_{1.0,0.5} = [(0.953674 - (0.975132)(0.977995)] + [(0.959287) - (0.978473)(0.980392)] = 0$$

and

$$\Lambda_{1.0,0.5} = \{\exp[-(0.050365)(0.5)] - \exp[-(0.050365)(0.5)]\}(0.977995) + \{\exp[-(0.043524)(0.5)] - \exp[-(0.043524)(0.5)]\}(0.980392) = 0$$

Here we have confirmed that the value of the (discrete-time) credit gadget that is sensitive to the forward absolute credit spread between the A-rated and Treasury bonds for the six-month period beginning six months hence is indeed zero. We have also quantified the relevant propagators for the credit risk-free and credit-risky assets ($\phi_{1.0,0.5} = 0.978473$ and $\phi_{1.0,0.5}^{*} = 0.975132$), as well as the relevant forward rates for the two assets ($f_{1.0,0.5} = 0.043524$ and $f_{1.0,0.5}^{*} = 0.050365$).

Now focus on the data in italics in Exhibit 2 for the 1.5-year Treasury bond. We can use these additional data and Equation (16) to compute the values of the propagators $\phi_{1.5,0.5}$ and $\phi_{1.5,1.0}$:

$$0.938177/0.980392 = 0.956941 = \phi_{1.5,0.5}$$

$$0.938177/0.959287 = 0.977994 = \phi_{1.5,1.0}$$

This permits us to confirm the compounding relation for the propagators. Specifically:

$$\phi_{1.5,0.5} = 0.956941 = \phi_{1.5,1.0}\phi_{1.0,0.5} = (0.977994)(0.978473)$$

Now focus again on the data in boldface, and how they relate to Equation (21). Recall that (21) speaks to the sensitivity of the discrete credit gadget with respect to a change in the relevant forward absolute credit spread. To quantify this sensitivity, we must permit a small change in the forward absolute credit spread. To do so, we allow the forward rate for the Treasury to increase by 1 basis point; that is, we let $f_{1.0,0.5}$ go from 0.043524 to 0.043624. At the same time, we let the forward rate for the A-rated bond increase by 2 basis points; that is, we let $f_{1.0,0.5}^{*}$ go from 0.050365 to 0.050565.

The logic behind this approach is simple. Permitting the forward rate on the credit-risky asset to change by 1 basis point will not necessarily capture the sensitivity of the credit gadget to a small change in the relevant forward absolute credit spread. This is because the basis point change could be occasioned by a change in the forward interest rate per se. By permitting the forward rates on the equivalent-maturity Treasury and risky bonds to change by 1 basis point and 2 basis points, respectively, we effectively capture the sensitivity of the credit gadget to a single basis point change in the forward absolute credit spread per se.

Indeed, we coin a term for this new measure—the value of a basis point change in the forward absolute credit spread, or VBP-FACS.

In the illustration, we have, from Equation (21):

$$\partial\Lambda_{1.0,0.5} = \text{VBP-FACS} = -(0.5)(0.0002)(0.953674) + (0.5)(0.0001)(0.959287) = -0.000047$$

It is of course sensible that the VBP-FACS is negative. Reexamining the portfolio formed to occasion Equation (9) (in continuous time) and its analogue, Equation (18) (in discrete time), it is apparent that the investor is essentially long a forward-starting credit-risky bond and short a forward-starting credit risk-free bond. Thus an increase in the forward absolute credit spread will occasion a loss in portfolio value.[11]

Next consider the data underlined in Exhibit 2; that is, the data for the B- and A-rated 1.5-year bonds. We can use these additional data along with the discrete-time analogue of Equation (15) to compute the sensitivity of a credit gadget with respect to a change in a forward relative credit spread (here involving the B- and A-rated bonds for the six-month period beginning one year hence).

Allowing the forward rate on the B-rated bond to increase by 2 basis points while that on the A-rated bond increases by just 1 basis point, we have:

$$\begin{aligned}\partial\Lambda_{1.5,1.0} &= \text{VPB-FRCS}\\ &= -(0.5)(0.0002)(0.919152) + (0.5)(0.0001)(0.928599)\\ &= -0.000045\end{aligned}$$

Again, the sensitivity of the credit gadget is inversely related to the widening of the relative credit spread between the B- and A-rated bonds. This result obtains because the portfolio that was formed to occasion this expression essentially entails a long position in a forward-starting B-rated bond and a short position in a forward-starting A-rated bond. The same result would have obtained had we allowed the forward rate on the B-rated bond to increase by 1 basis point while keeping the forward rate on the A-rated bond the same. This combination also reflects erosion of the credit quality of the B-rated bond vis-à-vis the A-rated counterpart.

Finally, we turn to the proactive use of credit gadgets to either hedge current or achieve desired forward credit spread and skew exposures. For this exercise, look again at the data in boldface in Exhibit 2, and assume that an investor thinks the forward absolute credit spread is too narrow. The investor wants to wager that the forward credit spread between the A-rated and Treasury bonds for the six-month period beginning six months hence will widen.

More specifically, given the risk tolerance of the investor, the investor wants a credit gadget with sensitivity to this spread of approximately –9.4; that is, $\partial\Lambda_{1.0,0.5} \cong -9.4$. This can be achieved by forming a credit gadget as follows:

- Long \$200,000 face value of the 1.0-year A-rated bond (cost = \$190,735).
- Short \$195,026 face value of the 0.5-year A-rated bond (proceeds = \$190,735).
- Long \$200,000 face value of the 1.0-year Treasury bond (cost = \$191,857).
- Short \$195,695 face value of the 0.5-year Treasury bond (proceeds = \$191,857).

This credit gadget will have zero initial cost and the desired sensitivity to the particular region of the local credit spread surface, $\partial\Lambda_{1.0,0.5} = -9.48$.

V. SUMMARY

We have introduced a device dubbed a credit gadget: a theoretically cost-free portfolio of pure discount credit risk-free and credit-risky bonds that permits an investor to hedge or otherwise assume the desired exposure to any particular region of the local credit spread surface. We think the credit gadget represents an important new tool for hedging or speculating on forward absolute and relative credit spreads and forward credit spread skews.

Future research might expand on the concepts we have introduced, including tackling the convexity bias occasioned by large changes in forward credit spreads and skews, and examining the effectiveness of credit gadgets using empirical data.

APPENDIX

Alternative Derivation

Credit gadgets may also be derived using the diffusion equations satisfied by traded pure discount bonds.[12] Our presentation here is limited to credit gadgets that are sensitive only to forward absolute credit spreads. Other credit gadgets may be constructed in an analogous fashion.

The credit risk-free and credit-risky pure discount bond prices satisfy the forward differential equations:

$$[(\partial/\partial T) + f_T(t)]B_T(t) = 0 \tag{A-1}$$

$$[(\partial/\partial T) + f_T^*(t)]B_T^*(t) = 0 \tag{A-2}$$

The bond prices $B_T(t)$ and $B_T^*(t)$ prevailing at time t for T-maturity bonds satisfy the boundary conditions $B_T(T) = 1$ and $B_T^*(T) = 1$, respectively. The solutions of Equations (A-1) and (A-2) are, respectively:

$$B_T(t) = \exp[-\textstyle\int_t^T f_u(t)du] \tag{A-3}$$

and

$$B_T^*(t) = \exp[-\textstyle\int_t^T f_u^*(t)du] \tag{A-4}$$

If $\phi_{T,T'}(t)$ denotes the propagator of $[\cdot]$ in Equation (A-1), then $\phi_{T,T}(t) = 1$ for all T and $t < T$, and that for all times $t < T' < T$:

$$[(\partial/\partial T) + f_t(t)]\phi_{T,T'}(t) = 0 \tag{A-5}$$

The solution of (A-1) can also be expressed as follows:

$$B_T(t) = \phi_{T,T'}(t)B_{T'}(t) \tag{A-6}$$

By an analogous argument, the solution of (A-2) can also be expressed as follows:

$$B_T^*(t) = \phi_{T,T'}^*(t)B_{T'}^*(t) \tag{A-7}$$

where $\phi_{T,T'}^*(t)$ is the propagator of the operator $[\cdot]$ in Equation (A-2). Note that since $B_t(t) = 1$ and $B_t^*(t) = 1$, then $B_T(t) = \phi_{T,t}(t)$ and $B_T^*(t) = \phi_{T,t}^*(t)$.

Now construct a portfolio $\Lambda_{T,T'}$ consisting of:

- Long one T-maturity credit-risky pure discount bond B_T^*.
- Short $\phi_{T,T'}^*(t = 0)$ of T′-maturity credit-risky pure discount bonds $B_{T'}^*$.
- Long one T-maturity credit risk-free pure discount bond B_T.
- Short $\phi_{T,T'}(t = 0)$ of T′-maturity credit risk-free pure discount bonds $B_{T'}$.

This portfolio is a credit gadget associated with the time interval between T′ and T:

$$\Lambda_{T,T'} = [B_T^* - \phi_{T,T'}(0)B_{T'}^*] + [B_T - \phi_{T,T'}(0)B_{T'}] \tag{A-8}$$

Equation (A-8) is completely analogous to Equation (18). From Equations (A-6) and (A-7), the price of this gadget at time $t = 0$ is zero. And from Equation (A-8), this price will change only if the forward absolute credit spread associated with the interval between T and T′, $f_{T,T'}^* - f_{T,T'}$, changes. For $T' = T - \Delta T$ with small ΔT, an infinitesimal credit gadget is obtained; i.e., $\Lambda_T = \Lambda_{T,T-\Delta T}$.

Finally, a finite credit gadget may be constructed from infinitesimal credit gadgets, described as follows:

$$\Lambda_{T,T'} = \int_{T'}^{T}\phi_{T,u}\Lambda_u d_u \tag{A-9}$$

ENDNOTES

[1]Bonds of course are not all identical. There are well-known methods for stripping pure discount bonds out of coupon-bearing bonds, and for adjusting yields and therefore credit spreads for the effects of embedded options (option-adjusted spread analysis, or OAS). In addition, there are increasingly methods for addressing the differential liquidity and tax treatment of bonds and the consequent effects on credit spreads. Our purpose is not to belabor the issue of the "pureness" of credit spreads. Rather, we simply assume that the analyst is using as pure a credit spread matrix as possible.

[2]See Jarrow, Lando, and Turnbull [1997] for a model of the term structure of credit spreads. See also Altman [1989], Duffie and Singleton [1999], Finger [2000], Litterman and Iben [1991], Merton [1974], and Rodriguez [1988].

[3]See, among others, Derman and Kani [1994], Derman, Kani, and Chriss [1996], Derman, Kani and Kamal [1997], Dupire [1994], and Rubinstein [1994].

[4]For any maturity, one requires at least two forward absolute credit spreads, or, equivalently, one forward absolute credit spread and one forward relative credit spread, to obtain a forward credit skew. This is analogous to requiring forward volatility from at least two different strike price options in order to generate a forward volatility skew.

[5]We do not consider whether one can trade liquid pure discount credit risk-free and credit-risky bonds that have no embedded options, are tax-free, and the like. Rather, we simply assume that such securities exist or otherwise can be readily manufactured. For example, if a bond is coupon-bearing, has an embedded call option, and is taxable, then we assume that a tax-exempt institutional entity can purchase the bond, strip away and sell its future coupon stream, and purchase some interest rate derivative that has the economic effect of eliminating the embedded call on the bond.

[6]For more on credit derivatives, see Das [1998], Hull and White [2000, 2001], Kijima [1998], and Tavakoli [1998].

[7]A similar approach produces an infinitesimal credit gadget whose initial value is zero and sensitive only to a particular forward credit spread skew.

[8]Again, an analogous approach represents the sensitivity of the value of a credit gadget to an infinitesimal change in a forward credit spread skew.

[9]In general, the variable $\phi_{T,\tau}$ is the forward discount factor that satisfies the forward equation $[\partial/\partial T + f_T]\phi_{T,\tau} = 0$ for all $\tau < T$ and boundary condition $\phi_{T,t} = 1$. $\phi_{T,\tau}$ is the propagator (Green's function) for the backward diffusion in time effected by the operator $\partial/\partial T + f_T$. It also satisfies the backward equation $[\partial/\partial t - f_t]\phi_{T,t} = 0$ and thus is the propagator for the diffusion forward in time effected by the operator $\partial/\partial t - f_t$. See Derman, Kani, and Kamal [1997] and the appendix for more on this topic.

[10]Using an analogous approach, one can readily derive finite credit gadgets whose values are sensitive only to changes in forward relative credit spreads or forward credit spread skews.

[11]Permitting the forward rate on the Treasury (A-rated) bond to decline by 1 (2) basis points will of course result in a positive value (0.000047) for $\partial\Lambda_{1,0,0.5}$. In addition, permitting the forward rate on the A-rated bond to decline by 1 basis point while keeping the forward rate on the Treasury intact—an unambiguous narrowing of the forward credit spread—occasions the same positive value (0.000047) for $\partial\Lambda_{1,0,0.5}$.

[12]This derivation is analogous to the derivation of volatility gadgets found in Derman, Kani, and Kamal [1997].

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