Pricing Corporate Bonds with Rating-Based Covenants

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Prominent examples of corporate problems have refocused attention on the impact of debt covenants associated with ratings changes on firm value and debt prices. Standard & Poor's notes that many companies have clauses in their finance and trading agreements that would force them to restructure or pay back billions of dollars if their ratings fall.

For example, on May 15, 2002, S&P stated:

> Tyco International Ltd. and Vivendi Universal are among companies with ratings-based triggers in financial agreements that make them most vulnerable to a cash crunch.[1]

A prominent analyst said:

> S&P and Moody's Investors Service are under pressure to improve their scrutiny of borrowers after failing to spot the collapse of Enron last year. Enron's bankruptcy was fueled by the presence of triggers, many of which were undisclosed or buried in agreements.[2]

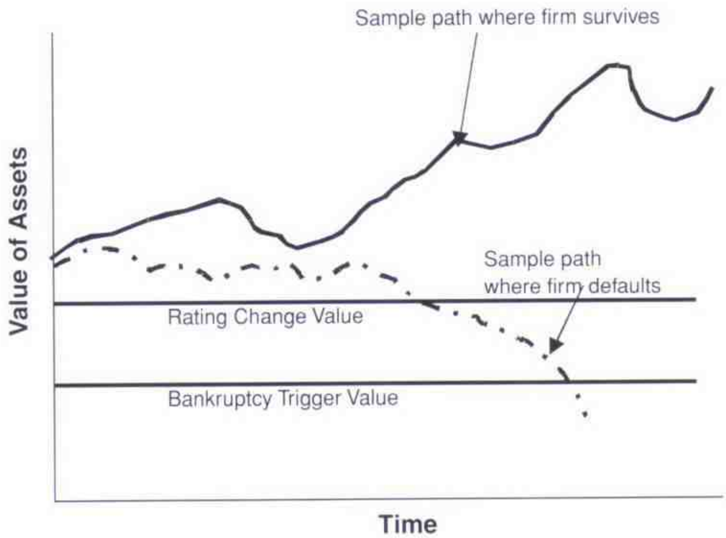
My purpose is to obtain the price of corporate debt after accounting for the presence of a rating-based trigger. In two seminal articles on optimal capital structure, Leland [1994] and Leland and Toft [1996] characterize the optimal capital structure and obtain the price of debt in the presence of credit risk. These articles permit closed-form solutions for debt values and equity values when firms can choose the amount as well as the maturity of debt, and when bankruptcy is determined endogenously. The research, however, does not explicitly consider other dimensions of debt choice such as covenants or debt priority.

I adapt the analysis of Leland and Toft [1996] to include rating-based covenants, assuming exogenous bankruptcy and credit rating triggers. The model has practical use for the pricing, analysis, and hedging of such bonds or their associated contingent claims.

I draw upon a growing literature on the pricing of corporate bonds, and use a structural approach to obtain the price of a corporate bond in the presence of a rating-based covenant. I model firm assets as a diffusion process whose volatility and drift parameters change when there is a change in the firm's credit rating. A credit rating change is triggered when the value of the firm's assets falls below a prespecified level (exogenous), while bankruptcy is triggered at a lower value level. Given these assumptions, debt prices are obtained as the risk-neutral expected present value of coupon cash flows and the face value or the recovery amount (in the case of bankruptcy). The analytical formula is easily implemented using numerical techniques.

The model provides interesting insights into the behavior of corporate bonds in the presence of covenants. After a ratings downgrade, there are two factors that impact a bond price.

EXHIBIT 1
Sample Asset Value Paths



Solid line: Asset values remain above bankruptcy trigger level and rating change level.
Dashed line: Asset values with a rating change and then default.

First, reorganization reduces payments to equityholders and results in a transfer of wealth to bondholders, because it reduces the probability of financial distress. On the other hand, downgrades cause increased interest expense. The sum total determines the net outflows and consequently the drift function for the evolution of firm value. In most cases, one would expect the increased interest expense to more than offset the reduced dividend payments. Second, an increase in the volatility of firm assets is likely to increase the probability of financial distress.

The net of these two factors determines the impact on debt values. The model does not account for the impact of a possible liquidity crunch on downgrades in an explicit manner, but instead assumes that the increased volatility of asset values accounts for that factor.

I. LITERATURE REVIEW

There are two approaches to the pricing of corporate bonds: a structural approach, and a reduced-form approach. The structural approach to modeling of credit risk assumes that a firm defaults when the value of its assets falls below a certain prescribed level (bankruptcy triggering value).

Exhibit 1 illustrates two sample paths. The solid line shows the path where the firm's assets grow and remain above the bankruptcy trigger value. The dashed line shows a path where the asset values decline and cross the bankruptcy trigger value (the firm defaults).

To calculate the chance that a sample path will cross the trigger value, a structural model of default requires us to characterize several aspects of the firm, and most structural models differ in how they model each aspect.

- Value of the firm's assets: Market value of a firm's assets at a given time.
- Asset risk: Volatility of the firm's assets that gives us the future dispersion of asset values.
- Leverage: Amount and maturity structure of liabilities.
- Recovery: Amount that is recovered in case of default.

The structural approach is based on the relationship between the market value of a firm's assets and its total obligations. A firm defaults on its debt if firm value falls below the nominal value of the outstanding debt. Equity is therefore considered a call option on the market value of the firm's assets with a strike price equal to the face value of the firm's debt. Given the value of a firm's equity, the face value of debt, and the value of a firm's assets, we can determine the probability of default using a contingent claims approach, as illustrated below.

Structural models of default typically assume that a firm value follows a diffusion process. For example, consider the classic approach of Merton [1974]. The value of a firm V_t is modeled as a geometric Brownian motion of the form:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dz \tag{1}$$

where μ is the instantaneous expected rate of return on the firm's assets per unit of time, σ is the instantaneous standard deviation of the firm's assets, and dz is s standard Wiener process.

If the firm is financed through zero-coupon debt with face value P maturing at time T, at maturity the bondholders receive the amount:

$$\min(V_T, P) = P - \max(P - V_T, 0) \tag{2}$$

which is the payoff of a risk-free zero-coupon bond minus a put option. At any time, the value of debt is equal to the firm value minus the value of equity. Equity is therefore considered a call option on the market value of the firm's assets with a strike price equal to the book value of the firm's debt.

The market value of debt $d(t, T)$ can be written as:

$$d(t,T) = V_t - C(V_t, P, T, \sigma) \tag{3}$$

where $C(V_t, P, T, \sigma)$ is a call option on the firm's assets that may be evaluated using the Black and Scholes [1973] formula. Given the firm characteristics and debt values, the implicit probability of default can be determined.

The structural framework has been extended in four primary ways:

- The asset value process in Equation (1) may include more realistic specifications.
- Recovery rates are non-zero and stochastic.
- The process for the risk-free rate is stochastic.
- The nature of the default barrier is exogenous or endogenous to the model.

The traditional formulation of the asset price process has been in the form of a diffusion. This implies that a firm cannot default unexpectedly or if it is not in current financial distress. Therefore, its probability of defaulting in the short term is zero, and short-term credit spreads should be zero. This implication is strongly rejected by the data.

To tackle this issue, Zhou [1997] proposes a jump-diffusion to model spread changes. Jumps create significant discontinuities that allow for a positive spread in short-dated debt. Recovery rates are allowed to be fixed or stochastic. A typical assumption is that a fixed fraction of the face value is recovered at bond maturity.

Because default risk parameters are implied from corporate bond prices, an assumption about interest rates is also an important input. The original formulation of debt prices assumes constant interest rates. Shimko, Tejima, and Van Deventer [1993] use a stochastic interest rate process that is correlated with asset prices. Other authors who use a two-factor process to deduce default rates include Kim, Ramaswamy, and Sundaresan [1993], Longstaff and Schwartz [1995], and Bryis and de Varenne [1997].

Finally, the assumption of a fixed default barrier that is equal to the face value of the debt is modified to make it more realistic. Leland [1994] endogenizes the value of assets that triggers bankruptcy (the idea is later extended by Leland and Toft [1996]).

Because the structural approach to bond pricing explicitly links default to the value of the firm's assets, it is rooted in economic fundamentals. Despite challenges to implementation of these models, organizations such as KMV and Moody's have conducted empirical studies to validate them with some success.

The reduced-form approach treats default instead as an unpredictable event involving a sudden loss in market value. This approach to modeling default is adopted by Madan and Unal [1994], Jarrow and Turnbull [1995], Jarrow, Lando, and Turnbull [1997], and Duffie and Singleton [1999]. These authors do not consider the relation between default and firm value in an explicit manner.

Under the reduced-form approach, the value of a zero-coupon bond with maturity T (under the risk-neutral measure Q) is given by:

$$d(0,T) = E^Q\left[\exp\left(-\int_0^T r_t dt\right)\left(1_{\{\tau > T\}} + \rho 1_{\{\tau < T\}}\right)\right] \tag{4}$$

where E is the expectations operator, r_t is the instantaneous risk-free rate, 1 is the indicator function, τ is the default time, and ρ is the recovery amount in the event of default.

Jarrow and Turnbull [1995] present one of the first models in this class that assumes a constant recovery rate (ρ) and an exponentially distributed default time (τ). They also assume that the processes driving default, interest rates, and recovery are mutually independent. The primary weakness of their model is a constant default intensity for differing horizons.

In response to this drawback, Jarrow, Lando, and Turnbull [1997] present a more sophisticated approach that models default as a continuous-time Markov chain with certain states characterizing default. A firm stays in a certain state for a time lag that is exponentially distributed. Jarrow, Lando, and Turnbull use data from Standard & Poor's transition matrices and recovery rates from Moody's to estimate the prices of corporate bonds. Madan and Unal [1994] provide another example of the stochastic intensity of default with random recovery rates.

In a significant contribution, Duffie and Singleton [1999] show that a claim on a defaultable bond may be priced as if it were default-free by replacing the usual short-term interest rate process r with the default-adjusted short rate process $R = r + hL$, where h is the hazard rate (the arrival intensity of a Poisson process whose first jump occurs at default), and $L = (1 - \rho)$ is the loss at default.

For a more detailed overview of reduced-form models of default, see Lando [1997]. Their primary drawback is the absence of an explicit economic relationship between default and firm value.

II. VALUATION FRAMEWORK

As in the structural models of finance, I assume that the firm has productive assets whose value V_t follows a continuous diffusion process with constant proportional volatility. The parameters of the firm's value process change after there is a shift in its credit rating (at a trigger value V^*). I assume that once a rating is changed, the parameter values remain constant for the foreseeable future:

$$\frac{dV_t}{V_t} = \left[\mu_1 - \delta_1\right]dt + \sigma_1 dz \tag{5-A}$$

if $V_t > V^* \; \forall t \in [0, t]$, and

$$\frac{dV_t}{V_t} = \left[\mu_2 - \delta_2\right]dt + \sigma_2 dz \tag{5-B}$$

if $\exists s \in [0, t]$ subject to $V_s \leq V^*$ where $\mu_{i=1,2}$ is the total expected rate of return on asset value V_t, $\delta_{i=1,2}$ is the constant fraction of value paid out to securityholders, and dz is the increment of a standard Brownian motion.

The process continues without limit unless the firm value falls to a default triggering value V_B. I also assume there is a default-free asset that pays a continuous interest rate r. One drawback of the model is the assumption that following a rating change the value process does not revert to the original parameters for the foreseeable future.

Consider a bond issue with maturity T periods from the present that pays a constant coupon flow c and has principal p. Let ρ be the fraction of asset value V_B that debt of maturity T represents in the event of bankruptcy. Then, using risk-neutral valuation, the price of a bond is written as the sum of three components: the expected present value of all payments if:

1. Firm value remains above V^*;
2. Firm value drops below V^* but remains above V_B; and
3. Firm value crosses V_B.

I assume that $f(s; V_1, V_2)$ denotes the density of the first passage time s from firm level V_1 to firm value V_2 (similarly, $F(s; V_1, V_2)$is the cumulative distribution function of the first passage density from V_1 to V_2). When the drift rate is the risk-free rate minus the payout each period, the debt value can be written as the sum of the three components of price as:

$$\begin{aligned} d\left(V, V^*, V_B, T\right) = & \\ & \int_0^T e^{-rs} c\left[1 - F\left(s, V, V^*\right)\right]ds + e^{-rT} p\left[1 - F\left(T, V, V^*\right)\right] \\ & \int_0^T \int_0^{T-s} e^{-r(s+y)} c\left[1 - F\left(T - s, V^*, V_B\right)\right] f\left(s, V, V^*\right) dy\,ds + \\ & \int_0^T e^{-rT} p\left[1 - F\left(T - s, V^*, V_B\right)\right] f\left(s, V, V^*\right) ds \\ & \int_0^T \int_0^{T-s} e^{-r(s+y)} \rho V_B f\left(T - s, V^*, V_B\right) f\left(s, V^*, V\right) dy\,ds \end{aligned} \tag{6}$$

The second line in Equation (6) represents the discounted expected value of the coupon flow and principal when the firm value remains above the value V^*. The third and fourth lines are the present value of the payments and principal if the firm value drops below V^* but there is no bankruptcy, and the last line is the recovery amount in the event of bankruptcy. For our analysis, the formulation of the density for the first passage time $f(s; V_1, V_2)$ is obtained from Cox and Miller [1970], and expressions for the cumulative density $f(s; V_1, V_2)$ come from Leland and Toft [1996].

Integrating Equation (6) by parts and collecting terms, we get the main result for the price of risky debt as:

$$d\left(V,V^*,V_B,T\right)=\frac{c}{r}+e^{-rT}\left(p-\frac{c}{r}\right)\left[1-H(T)\right]+\left(p-\frac{c}{r}\right)\left[K(T)\right] \tag{7}$$

where

$$H(T)=\int_o^T F\left(T-y,V^*,V_B\right)f\left(y,V,V^*\right)dy \tag{8}$$

and

$$K(T)=\int_o^T e^{-ry}G\left(T-y,V^*,V_B\right)f\left(y,V,V^*\right)dy \tag{9}$$

Here the expressions for the cumulative densities and other terms are given by (where $N[.]$ denotes the cumulative normal density function):

$$F\left(t,V^*,V_B\right)=N\left[h_1(t)\right]+\left(\frac{V^*}{V_B}\right)^{-2a}N\left[h_2(t)\right]$$

$$G\left(t,V^*,V_B\right)=\left(\frac{V^*}{V_B}\right)^{-2a}N\left[h_1(t)\right]+\left(\frac{V^*}{V_B}\right)^{-2a}N\left[h_2(t)\right]$$

$$f\left(t,V,V^*\right)=\frac{c}{\sigma_1\sqrt{2\pi t^3}}Exp\left[-\frac{1}{2}\left(\frac{-c-\left(r-\delta_1-0.5\sigma_1^2\right)}{\sigma_1\sqrt{t}}\right)^2\right]$$

$$q_1(t)=\frac{\left(-b-z\sigma_2^{\;2}t\right)}{\sigma_2\sqrt{t}}\quad q_2(t)=\frac{\left(-b+z\sigma_2^{\;2}t\right)}{\sigma_2\sqrt{t}}\quad h_1(t)=\frac{\left(-b-a\sigma_2^{\;2}t\right)}{\sigma_2\sqrt{t}}$$

$$h_2(t)=\frac{\left(-b+a\sigma_2^{\;2}t\right)}{\sigma_2\sqrt{t}}\quad a=\frac{\left(r-\delta_2-0.5\sigma_2^{\;2}\right)}{\sigma_2^{\;2}}\quad z=\frac{\left(\left(a\sigma_2^{\;2}\right)^2+2r\sigma_2^{\;2}\right)^{0.5}}{\sigma_2^{\;2}}$$

$$b=\log\left(\frac{V^*}{V_B}\right)\quad c=\log\left(\frac{V}{V^*}\right)$$

Equation (7) is simple to implement using standard numerical techniques. Mathematica code, for example, computes the price of a bond in a few seconds. Note that in the absence of a rating-based trigger, the model reduces to the standard structural bond pricing equation.

III. ANALYSIS OF DEBT VALUES WITH RATING-BASED COVENANTS

In general, the presence of a covenant alters the flow of coupon payments to securityholders, and the increased volatility of firm assets increases the probability of financial distress. A covenant is beneficial at time of bond issue because it reduces the asset substitution problem.

I assume for our purposes that the asset dynamics account for these decisions by management.

Debt Values as a Function of Firm Value

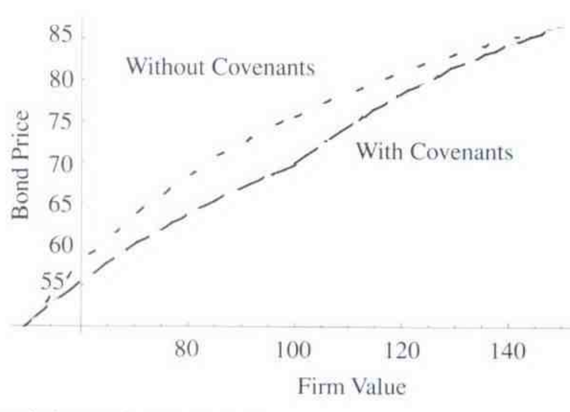
Exhibit 2 graphs the price of two bonds as a function of firm value. The dashed line represents the price of a bond when covenants are absent, and the solid line shows the impact of a covenant.

First, the true value of the bond is calculated assuming a covenant. Then the implied parameters for the model without covenants are computed. In other words, I compute the implied volatility of the asset process that makes the model bond price without covenants equal the true price.

In Exhibit 2, for example, the asset volatility is 30.0% prior to a rating change and 45.0% subsequent to a change.

EXHIBIT 2

Bond Prices versus Firm Value



Short dashes: Debt with covenants.

Long dashes: Debt without covenants.

Bond maturity: 5 years.

Risk-free rate: 7.5%.

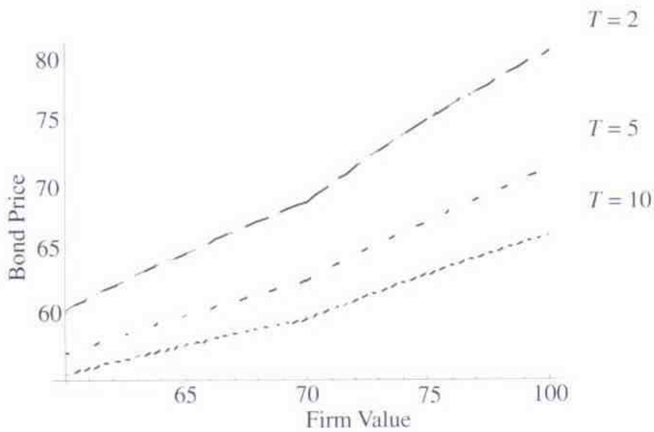
Payout rate: 7% before ratings change and 10% after ratings change.

Bankruptcy costs: 50%.

Corporate tax rate: 35%.

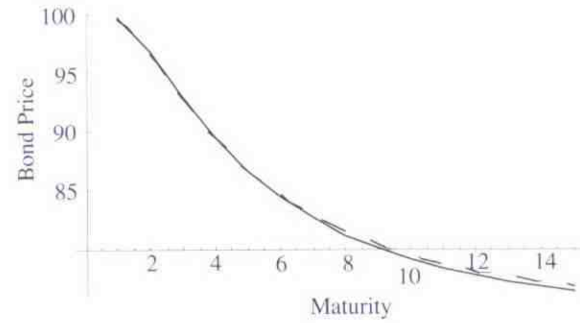
Initial value of firm's underlying assets $V = 150$, $V^* = 100$, and $V_B = 50$.

EXHIBIT 3
Bond Prices versus Firm Value—With Covenants



Bond maturity: 1 year, 5 years, 10 years.

EXHIBIT 4
Bond Prices versus Maturity



Solid line: Debt with covenants.
Dashed line: Debt without covenants.
Bond maturity: 5 years.

This gives a bond price of $86.50 per $100.00 face value. The corresponding implied volatility for a bond without rating change assumptions is 39.1%.

The graph shows that the analysis of bond prices, assuming no covenant, will overstate the impact of firm value on bond prices. Indeed the covenant-based price is lower than the price without covenants for all points in the graph. This is not to say that the covenant reduces bond prices, but rather that, conditional on the presence of a covenant, the model without covenants results in incorrect inferences insofar as it gives a higher price at each level of firm value.

Exhibit 3 illustrates the values of three bonds with different maturities as a function of firm value. Recall that we have assumed a change in credit rating causes two characteristics to change: the volatility of firm assets, and the outflows to equityholders (and an increased interest expense). As a result, the probability of bankruptcy is likely to increase.

There are kinks in the plots where the rating change occurs. The slopes change at this point due to the net effect of a change in cash flows to securityholders and an increase in the volatility of firm assets. As firm value declines, there is a greater likelihood of a ratings change. As a result, the price of a bond with covenants drops more quickly than for a bond with no covenants.

Debt Prices and Yields as a Function of Maturity

Exhibit 4 illustrates the value of a bond as a function of time to maturity when there is a covenant assumption (solid line) and when there is no such assumption (dashed line). As expected, debt prices decline with the maturity of the outstanding bonds.

Note that the prices of debt with covenants are lower than the prices of debt without covenants. This maturity impact is exacerbated as the maturity lengthens.

Bankruptcy Probabilities

Given the bond pricing equation and the value process, we can now compute the probability of going bankrupt. It is easily shown that the cumulative probability of going bankrupt in Equation (8) is given by:

$$H(t) = \int_0^t F\left(t - y, V^*, V_B\right) f\left(y, V, V^*\right) dy$$

where expressions for the functions $F(.)$ and $f(.)$ have been identified, and the drift function is the true drift rather than the risk-free rate.

Again the probability of going bankrupt is dependent on parameter values in both regimes. All else the same, if there is more asset volatility after a ratings change, the probability of bankruptcy may actually increase, propelling the firm toward insolvency. Our analysis takes a true bond price that accounts for such behavior after a ratings change, however, and asks how a model that does not assume the presence of a covenant will change the implied probability of default.

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Exhibit 5
Bankruptcy Probability versus Maturity

Maturity (Years)	With Covenants	Without Covenants
1	0.35	0.44
2	3.79	4.25
3	8.79	9.49
5	18.00	18.91
10	32.66	33.85
20	45.97	47.84

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Exhibit 6
Par Yields versus Maturity

Maturity (Years)	With Covenants	Without Covenants
1	7.83	7.89
2	9.27	9.37
3	10.28	10.34
5	11.15	11.14
10	11.52	11.44
20	11.45	11.35

Exhibit 5 presents the bankruptcy probabilities implied by two models, the first assuming the presence of covenants, and the second one without covenants. Note that the results for the model without covenants indicate a higher probability of bankruptcy, where the implied parameters for the misspecified model are obtained from a five-year bond.

Par Yields

Exhibit 6 gives the par yield of a bond as a function of time to maturity. First, the true value of a bond is calculated assuming the presence of a covenant. Then the parameters for a bond without covenants are computed so that the implied volatility of the assets makes the bond price equal the observed price. Finally, given the implied parameters, I compute the coupon rate that gives a bond price equal to the face value.

Par yields on both models are close, even though the model without covenants consistently gives higher par yields. Note that the model is able to generate the hump-shaped yield curve observed in the debt markets and documented in the literature.

IV. CONCLUDING COMMENTS

Widespread reports of corporate bond covenants triggered by a downward revision of credit ratings have attracted some attention. This article provides a simple framework for the valuation of corporate bonds whose covenants are triggered by a change in credit rating.

Analysis shows that a framework that assumes the presence of a covenant will produce more depreciation in bond prices as firm values decline than one that assumes no covenants. The results have important implications for evaluating values at risk, contingent claim prices, and related measures of bond price behavior.

ENDNOTES

The author is grateful to Mukarram Attari, Douglas Foster, and Palani-Rajan Kadapakkam.

[1]www.standardandpoors.com

[2]"Tyco Amongst Most Vulnerable." May 15, 2002, standardandpoors.com

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