Pricing Defaultable Coupon Bonds Under a Jump-Diffusion Process

MARK C.W. WONG AND STEWART D. HODGES

MARK C.W. WONG
is a doctoral researcher at the Warwick Business School, University of Warwick, in the U.K.
Mark.Wong@warwick.ac.uk

STEWART D. HODGES
is professor of financial management and director of the Financial Options Research Centre at the Warwick Business School, University of Warwick, in the U.K.
forcsh@razor.wbs.warwick.ac.uk

The valuation of corporate debt is a central issue in theoretical and empirical work in corporate finance. The literature on pricing risky debt has evolved in two main directions: the structural approach and the reduced-form approach. As in Merton [1974], the structural approach takes the dynamics of the assets of the issuing firm as given, and prices corporate bonds as contingent claims on the assets. Assuming a constant interest rate economy, Merton's model provides an important insight into the determinants of risk structure, and shows how the default risk premium is affected by changes in the firm's business risk, debt maturity, and the prevailing interest rate.

Black and Cox [1976] and Geske [1977] provide generalizations that take into account the effects of coupon and bond indenture provisions. Shimko, Tejima, and Van Deventer [1993] discuss the application of stochastic interest rates in the valuation of corporate debt using Merton's [1974] framework.

Longstaff and Schwartz [1995] adapt the Black and Cox [1976] model for a more realistic setting. First, they allow interest rates to be stochastic, with dynamics as proposed by Vasicek [1977]. Second, they do not require the recovery rate to be equal to the boundary value upon first passage, but assume an exogenously given rate of face value K. Third, the default boundary is assumed to be K. This approach explicitly allows for deviations from absolute priority.

The introduction of bankruptcy costs and tax effects allows extension of the framework to a richer extent, taking into account issues in corporate finance. Examples that consider endogenous capital structure, liquidation policy, recapitalization, and reorganization of debt include Brennan and Schwartz [1984], Leland [1994, 1998], Anderson and Sundaresan [1996], Leland and Toft [1996], and Mella-Baral and Perraudin [1997]. These models allow for endogenous default, optimally determined by equityholders when asset levels fall to some low level. Anderson and Sundaresan [2000] conduct an empirical analysis of structural models of corporate bond yields. Their results suggest that recent modifications of the contingent claims models to allow for endogenous default barriers have improved the performance of the models in tracking observed yield spreads.

The structural approach to the valuation of risky debt has been criticized for not being able to generate sufficient credit spreads for short maturities of debt. In practice, even for short maturities, the market recognizes that some disaster may happen.

More recent literature has adopted an alternative approach that offers a high degree of tractability for credit-risky bonds. The reduced-form approach relates default time to an exogenously given hazard rate process, and derived formulas are calibrated to market data. This is illustrated by Jarrow and Turnbull [1995], Lando [1995], Jarrow, Lando, and Turnbull [1997], Duffee [1999], Duffie and Singleton [1999], and Madan and Unal [2000]. This approach provides a model that is close to the data, and it is always possible to fit some version of the model, although a fitted model may not perform well on out of-sample analysis.

An in-between approach has been suggested. Cathcart and El-Jahel [1999] propose a framework that combines structural and reduced-form approaches. It assumes a default event occurs in an expected or unexpected manner when the value of a signaling process reaches a certain lower barrier or at the first jump time of a hazard rate process. Although the model can generate strictly positive credit spreads for short maturities, the simple assumption that the firm goes bankrupt immediately when a jump in the asset value occurs for the first time needs empirical justification.

Zhou [1997] proposes a numerical model for pricing discount bonds in much the same spirit as in Longstaff and Schwartz [1995], when the underlying asset value follows a jump-diffusion process that is similar to the stock price process in Merton [1976]. Coupon bonds are not considered.

Instead of employing a jump process as a determinant of the default mechanism, Duffie and Lando [2001] study the implications of imperfect accounting information for modeling corporate bonds. They suppose that bond investors cannot directly observe the issuer's assets, and receive only periodic and imperfect accounting reports. As a consequence of the uncertainty in asset values, bounding short spreads away from zero can be obtained in their model.

There is a basic incompatibility in default mechanisms between the Duffie and Singleton [1999] model and the traditional models based on Merton. The reduced form of a structural model is usually taken to mean a version of the model in which endogenous variables are expressed as a function of predetermined variables only.[1] By this definition, the reduced-form model must be equivalent to the original structural one. This is not true of the Duffie and Singleton [1999] model, however.

In the Merton framework, where a firm's asset value follows a pure diffusion process, default can happen only expectedly. Given that default has not happened yet, there is a zero probability that the firm will become bankrupt at the next instant. The reduced-form approach typically assumes that default events are surprises. For example, Duffie and Singleton [1999] consider that, given a positive hazard rate process h_t, default occurs at a rate of h_t.

Neither the pure diffusion nor the reduced-form approach appears to comport completely with empirical evidence that default can happen in both an expected and an unexpected manner.[2] Motivated by this observation, we provide an alternative to the problem by modeling the firm's asset value as a jump-diffusion process. While a model based on jump-diffusion captures both expected and unexpected defaults, we show that it also exhibits the interesting properties of the leptokurtic feature that are empirically observable in asset returns.

We present a tractable discrete-time model for valuing typical coupon bonds under a jump-diffusion process in asset value. For simplicity, we assume that a default event can happen only on payment dates.[3] The firm goes into bankruptcy *expectedly* when the asset level hits a certain lower barrier through a continuous diffusion crossing, or *unexpectedly* when its value drops precipitously below the barrier.[4] Consistent with Geske [1977], Leland [1994, 1998], and Leland and Toft [1996], the default boundary is determined endogenously by requiring the value of equity to be at least the amount of the coupon just paid, in order to avoid bankruptcy. As investors in corporate bonds are subject to state and local taxes, we also consider the effects of tax premiums when jump risk is correlated with a market portfolio.

Our contributions are several. We show some significant implications of the jump process for the level and the term structure of credit spreads. For example, it is interesting to note that while the presence of jumps in asset values eliminates the undesirable qualitative feature of credit spreads going to zero at the short end, negative jumps can have significant and persistent effects on spread levels. The jump effects on spread levels are conspicuous for short maturities. For long maturities, credit spreads are indistinguishable from those generated by a pure diffusion model. When downward jumps are of higher variance, while the total variance of the firm's asset value remains the same, however, the effects on credit spreads become more persistent.

Other important factors include taxes and dividends. As suggested by Elton et al. [2001], taxes do have significant effects on the levels of credit spreads. A further contribution is that we provide characterizations of the default boundary, and show how the endogenous default mechanism is affected by the presence of jumps and taxes. Our results suggest that a reduction in federal tax rates may affect the earlier default of low-grade bonds.

We first review a continuous jump-diffusion model as presented by Zhou [1997]. We show a generalization of the model to incorporate systematic jump risk. To facilitate efficient computation, we adopt Amin's [1993] discrete-time approach to approximate the continuous models.

We then extend Geske's [1977] model to incorporate a jump component by the discrete-time approximation, as in Amin [1993], and price a defaultable coupon bond with general jump risk in total firm value. We investigate some properties of credit spreads, and show how term structures of spread levels under the jump-diffusion process differ from those generated by a pure diffusion approach. We present some noteworthy results about the effects of non-diversifiable jumps on debt pricing.

Finally, we discuss a more practical approach to pricing corporate bonds, by introducing state and federal taxes into the model. The effects of state taxes and a dividend payout rate on spread levels and bankruptcy mechanisms are also investigated.

I. THEORETICAL MODELS

We review two continuous-time models with different types of jump risk.

Continuous-Time Model with Non-Systematic Jump Risk

Zhou's [1997] model is in much the same spirit as Longstaff and Schwartz [1995]. The underlying process of the firm's asset value is modeled as a jump-diffusion process where the jump risk is assumed to be diversifiable. Such an approach is analogous to the modeling of the stock price process in Merton [1976].

Let V_t be the total market value of the assets of the firm at time t. Under an equivalent martingale measure, Zhou [1997] assumes that the dynamics of the firm's asset value process V_t follow the jump-diffusion process:

$$dV_t/V_t = (r - \lambda_J \overline{m})\ dt + \sigma\ dZ_t + m\ dJ, \qquad (1)$$

where

- r is the constant spot rate of interest based on continuous compounding;
- λ_J is the intensity of the Poisson jump process J: $P[dJ = 1] = \lambda_J\ dt$;
- m is the random percentage change in firm value if a Poisson jump occurs: $1 + m$ is lognormally distributed; $\log(1 + m) \sim \mathrm{N}[\gamma m - \frac{1}{2}\sigma_m^2, \sigma_m^2]$, $E(m) = \overline{m} = e^{\gamma m} - 1$;
- σ is the instantaneous volatility conditional on no jumps; and
- Z_t is a standard Brownian motion under the risk-neutral measure.

The process most often resembles geometric Brownian motion, but on average λ_J times per year, the price jumps discretely by a random amount. Thus the total change in asset value of the firm is posited to be made up of two types of changes. The first is the normal variations in value due to changes in the economic outlook, which cause marginal changes in the asset value. This component is modeled by standard geometric Brownian motion, Z_t, with constant variance per unit time.

The second type of change is the abnormal variations in asset value due to the arrival of important new information about the firm. This has more than a marginal effect on price, and is modeled by the jump process, Z, reflecting that such information arrives only at discrete times. Under the assumptions of Merton [1976] and Zhou [1997] that the two sources of uncertainties are independent of each other and the information is specific to the firm, the jump risk is uncorrelated with the market and is not priced in equilibrium.

Zhou [1997] also assumes the presence of a positive threshold value, K, for the firm at which financial distress occurs. If the firm value, V_t, falls to or below the threshold level, K, the firm defaults on all its obligations immediately, and some form of corporate restructuring takes place. The bondholder receives $1 - w(X_t)$ times the face value of the security at maturity T, where $X_t = V_t/K$ is the ratio of the firm value V_t to K. Under these assumptions, the bond price $B(X_0, 0)$, with a promised final payment of 1 at time 0, is given by

$$B(X_0, 0) = \exp(-rT)E[I_{V_T>K} + (1 - w(X_T))I_{V_T\leq K}].$$

Zhou [1997] proposes a numerical algorithm for computation of the bond price. He also considers that interest rates may follow a Vasicek [1977] process. Unlike Zhou, we deal with endogenous default mechanisms by employing Geske's [1977] idea. We extend Zhou's model in three ways: 1) we take coupons into account; 2) we assume that default barrier is determined endogenously; and 3) we consider the effect of systematic jumps.

Continuous-Time Model with General Jump Risk

The jump-diffusion processes as described by Merton [1976] and Zhou [1997] are perhaps the simplest type of models to include jumps in asset prices. The crucial assumption is that the jump risk is diversifiable and non-systematic. This assumption is questionable, as asset prices appear to be correlated with market movements.

In an empirical study of an economy where stock prices are assumed to follow a jump-diffusion process, Jarrow and Rosenfeld [1984] investigate the effect of assuming jumps to be diversifiable. Evidence has been found to show that the jump component of a stock's returns has a strong correlation with the market portfolio; that is, the market portfolio appears to include a jump component.

A similar conclusion is drawn by Kim, Oh, and Brooks [1994], who study 20 component stocks of the Major Market Index. They find that Poisson-type jumps observed from both the index and its component stocks constitute non-diversifiable risk. This implies that the standard assumption in option pricing as in Merton [1976] that these jumps are not priced may be invalid.

Relationships of common risk factors between returns on stocks and on bonds have been investigated in Fama and French [1993], and Elton et al. [2001]. Elton et al. [2001] use the Fama-French three-factor model [1993] to find that expected default accounts for a surprisingly small fraction of the premium in credit spreads of corporate bonds. They conclude that, while state taxes explain a substantial portion of the discrepancy, the remaining portion of the spread is closely related to factors commonly accepted as explaining risk premiums for common stocks. They show that a significant portion of the spread is compensation for systematic risk that is affected by the same influences of systematic risks in the stock market.

These results imply that it is hardly plausible to maintain Merton and Zhou's simplifying assumptions that jump risk is non-systematic and diversifiable. To incorporate systematic jump risk into Equation (1), we adopt a jump-diffusion process as in Bates [1991] as a generalization of the model.

First, we assume that under the objective measure the dynamics of the firm's asset value are as follows:

$$dV_t/V_t = (\mu - \lambda\bar{k} - \delta)\ dt + \sigma\ dB_t + k\ dq, \qquad (2)$$

where

- μ is the instantaneous cum-dividend expected return on the asset;
- k is the random percentage change in firm value if a Poisson jump occurs: $1 + k$ is lognormally distributed, $\log(1 + k) \sim N[\gamma - \frac{1}{2}\sigma^2, \sigma^2]$, $E(k) = \bar{k} = e^{\gamma} - 1$; λ is the intensity of the Poisson jump process q: $P[dq = 1] = \lambda\ dt$;
- δ is a constant payout rate as a fraction of firm value;
- σ is the instantaneous volatility conditional on no jumps; and
- B_t is a standard Brownian motion under the objective measure.

Second, several restrictions on utility preferences are imposed:

A1: Frictionless markets.

A2: Optimally invested wealth W_t follows a jump-diffusion:

$$dW_t/W_t = (\mu_w - \lambda\bar{k}_w - C/W)\ dt + \sigma_w\ dB_t' + k_w\ dq,$$

where μ_w is constant, and k_w is the percentage change in wealth when the Poisson jump happens. $1 + k_w$ is lognormally distributed, $\log(1 + k_w) \sim N[\gamma_w - \frac{1}{2}\sigma^2_{k_w}, \sigma^2_{k_w}]$; $E(k_w) = \bar{k}_w = e^{\gamma_w} - 1$, and $Cov[\log(1 + k), \log(1 + k_w)] = \sigma_{vw}$.

A3: The representative consumer has time-separable power utility:

$$E_\tau \int_\tau^\infty e^{-\rho t} U(C_\tau)\ dt,\ U(C) = (C^{1-R} - 1)/(1 - R).$$

Assuming that jump risk is systematic, all asset prices and wealth jump simultaneously, possibly by different amounts. As in Bates's model [1991] of systematic jump risk, we assume that under a risk-neutral measure, the jump-diffusion model in Equation (2) takes the form:

$$dV_t/V_t = (r - \lambda^*\bar{k}^* - \delta)\ dt + \sigma\ dB_t^* + k^*\ dq^*, \qquad (3)$$

where

$\lambda = \lambda\ E[1 + \Delta J_w/J_w] = \lambda\exp[-R\gamma_w + \frac{1}{2}R(1 + R)\sigma^2_{k_w}]$;

k^* is the random percentage change in firm value if a Poisson jump occurs: $1 + k$ is lognormally distributed, $\log(1 + k^*) \sim N[\gamma^* - \frac{1}{2}\sigma^2_k, \sigma^2_k]$, $E(k^*) = \bar{k}^* = e^{\gamma^*} - 1$, and $\gamma^* = \gamma - R\sigma_{vw}$; and

q^* is a Poisson counter with intensity λ^*.

It is noteworthy that Equation (3) is a generalization of Equation (1). It reduces to (1) when the dividend payout rate $\delta = 0$, and the coefficient of relative risk aversion $R = 0$, meaning that investors are risk-neutral. Furthermore, in the event that jump risk is firm-specific, $\gamma = \sigma_{k_w} = \sigma_{vw} = 0$, and the model reduces to the Merton model with $\overline{k}^* = \overline{k}$, $\overline{\lambda}^* = \lambda$.

A notable feature of the two processes is that they share the same modeling structure. The implicit nature of the relative risk aversion coefficient, R, adds to their structural similarity and makes pricing on the basis of Equation (3) convenient. For example, pricing European options using Equation (3) is straightforward (see Merton [1976]).

This is also the case for parameter estimation.[5] Parameters estimated from a pricing model based on the underlying risk-neutral jump-diffusion process in Equation (3) are of course those of the same process. Deducing the true parameters in Equation (2) requires additional assumptions about the coefficient of relative risk aversion and about the degree to which jumps in Equation (3) are related to jumps in wealth.[6]

II. DISCRETE-TIME APPROXIMATION

To implement the model, we need a discrete-time formulation that provides us with a computational tool for pricing derivative instruments. We apply a discrete-time approach to the solution of the process in Equation (3). As in Merton [1976] and Zhou [1997], the asset value of the firm under the risk- neutral measure is of the form:

$$V_t = V_0 \exp\left([r - \frac{1}{2}\sigma^2 - \lambda^* \overline{k}^* - \delta]t + \sigma B_t^* + \sum_{j=0}^{NJ} \log(1 + k_j^*)\right), \tag{4}$$

where NJ is the total number of Poisson jumps over time t, and the Poisson jump sizes $(1 + k_j)$ are independent and identically lognormally distributed random variables with parameters $N[\gamma^* - \frac{1}{2}\sigma_k^2, \sigma_k^2]$.

For computational convenience, we turn V_t into a logarithmic scale and define the drift of the logarithm of the asset value in Equation (4) by:

$$\alpha = r - \delta - \frac{1}{2}\sigma^2 - \lambda^* \overline{k}^*.$$

Equation (4) can be rewritten as:

$$X_t = \log(V_t/V_0) = \alpha t + \sigma B_t^* + \sum_{j=0}^{NJ} \log(1 + k_j^*).$$

To approximate X_t, we adopt a method Amin [1993] uses to discretize the process. The discrete-time formulation is based on the work of Cox, Ross, and Rubinstein [1979] as the starting point. Multivariate jumps are superimposed on the model to obtain the model with a limiting jump-diffusion process.

Let T be the maturity of a coupon bond. For a fixed positive integer, n, we divide the interval $[0, T]$ into n subintervals of width $h_n = T/n$. Let $i = 1, \ldots, n$. At each date, ih_n, the value of the approximate process, X_i, is shifted upward by αh_n relative to the grid at time $(i-1)h_n$. Therefore, the asset value at time ih_n and in state j relative to date 0 is given by $V_i = V_0 \exp(\alpha i h_n + j\sigma\sqrt{h_n})$. At any time $(i-1)h_n$ can move to any other points at time ih_n.

As discussed in Amin [1993], there are two types of movements as to the dynamics of changes in the asset values over time, namely, local and non-local change. The asset price undergoes either of the two different types of mutually exclusive price changes. In most periods, asset prices undergo only local changes. This is as in Cox, Ross, and Rubinstein [1979], where the asset price moves up or down by one step. This price change is due to the diffusion component of the continuous-time case. Given that a local change of asset prices has occurred, for a sufficiently large integer, n, the probabilities of an upward movement and of a downward movement are taken to be $p = \frac{1}{2}$.

If the change in asset price is in any other number of steps, a jump has occurred. This event has a low probability of occurring in any given period. In this case, determining the corresponding transition probabilities is more tedious. At each discrete date, we approximate the jump distribution on the entire real number line by breaking it into non-overlapping intervals of equal width. The entire probability mass over each of these intervals is assigned to the state included in one of these intervals.

Let the cumulative density function of the jump size be $F(.)$. For any state, l, not equal to -1, 0, or 1, the jump probability P_j is given by:

$$P_j = F(\alpha h_n + (l + \frac{1}{2})\sigma\sqrt{h_n}) - F(\alpha h_n + (l - \frac{1}{2})\sigma\sqrt{h_n})$$

When $l = 0$, we take:

$$P_j = F(\alpha h_n + (1 + \frac{1}{2})\sigma\sqrt{h_n}) - F(\alpha h_n - (1 + \frac{1}{2})\sigma\sqrt{h_n})$$

Finally, when $l = -1$ or $+1$:

$$P_j = 0$$

We have thus specified the risk-neutral measure for the discrete-time framework.[7]

Let $B_s(i)$ be the value of the risky bond at time ih_n and in state s. Then, for any state k, the bond price between two consecutive payment dates is given by the iterative formula:

$$B_k(i) = e^{-rh_n}\Big(\lambda^* h_n E_Y\Big[B_{k+y}(i+1)\Big] + \frac{1}{2}(1-\lambda^* h_n)\Big(B_{k-1}(i+1) + B_{k+1}(i+1)\Big)\Big).$$

The first term on the right-hand side represents a fraction of the bond price as a consequence of non-local changes in the asset prices. The second term is the expected value of two bond prices resulting from local movements of the asset value. Here, we assume that the probability of a jump in the discrete model at any time is equal to $\lambda^* h_n$. We also assume that h_n is so small that multiple jumps cannot occur within the same period.

At each coupon date $t = ih_n$, the bond price immediately before coupon payment is given by:

$$B_k(i^-) = Min\Big(V_t, coupon + B_k(i^+)\Big).$$

We assume Geske's [1977] condition that coupon payments are financed by issues of new equity. The firm goes bankrupt only when its stock value immediately after a coupon date is lower than the total coupon payment. Black and Cox [1976] argue that this will happen whenever the value of the equity, after payment is made, is less than the value of the payment. The argument is intuitive in that the firm will find no takers for its stock if investors know that the stock will become less valuable than the total value they need to contribute to the promised payment. This condition endogenously determines an asset level position of default barriers that is consistent with Leland [1994, 1998] and Leland and Toft [1996].

This condition is not completely necessary, however; other bankruptcy criteria can also be used in the model. For example, Kim, Ramaswamy, and Sundaresan [1993] assume a lower reorganization boundary for the firm's value, at which the total cash flow per unit time will be just sufficient to pay the contractual coupon. Furthermore, we also assume that bankruptcy costs are not a significant determinant of firm value.[8] With this discrete-time formulation, we can compute bond prices by dynamic programming using a backward recursion on the state space that we have developed.

III. PRICING WITH GENERAL JUMP RISK

We demonstrate the model using a numerical example. We consider a corporate bond with half-yearly coupon payments, $cK/2$, where K is the face value. In order to study the properties of credit spreads, we consider a base case environment in the risk-neutral world with parameters as follows: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, $c = 0.05$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, and $\sigma_k = 0.25$. As observation indicates that downward jumps are more likely to happen than upward jumps, we also assume $\gamma^* = -0.1$. The parameter k^* in Equation (3) becomes –9.5%, implying a negative average jump size in asset value.

Exhibit 1 shows the corresponding term structures of credit spreads for three jump-diffusion cases and a pure-diffusion case. In the jump-diffusion cases, the total variances under the risk-neutral measure are kept the same, and the contributions of jump components to total variance are 55% and 9%, respectively.[9] We assume that the misspecified pure-diffusion process has a constant variance equal to the total variance of the jump-diffusion case.

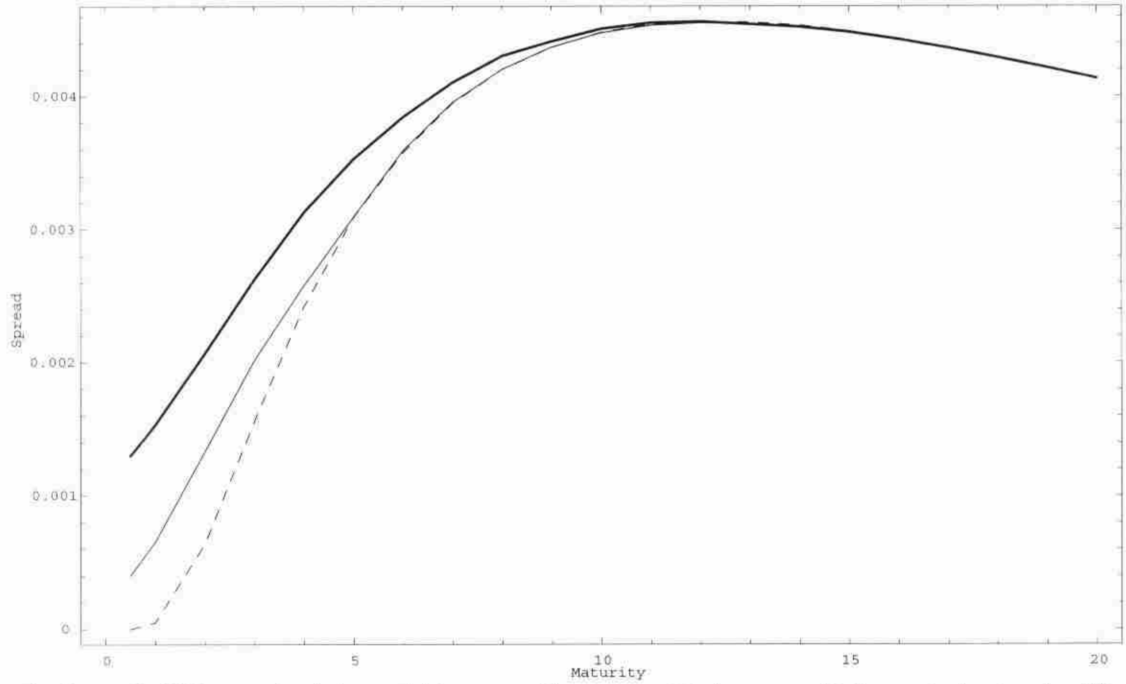
While the term structures are similar in shape for both jump-diffusion and pure-diffusion cases, the gap between credit spreads in the two cases narrows with maturities. In the presence of jumps, the possibility of sudden default raises the levels of credit spread only for short maturities.[10] This effect becomes more evident for bonds with shorter maturities when jumps occur more frequently and have higher variability.

As maturity lengthens, the differences in credit spreads narrow, implying that the effect of a jump becomes less important for bonds with long maturities. Similar results are shown in Exhibit 2, which plots the term structure of credit spreads for a different face value $K = 70$.

The rationale behind these results is as follows. Recall that the total variance per unit of time remains constant in all cases. When maturity is short, the diffusion volatility in the pure-diffusion model can cause only relatively small changes in the asset value. A single jump, however, can cause a relatively large change in the asset value.

EXHIBIT 1

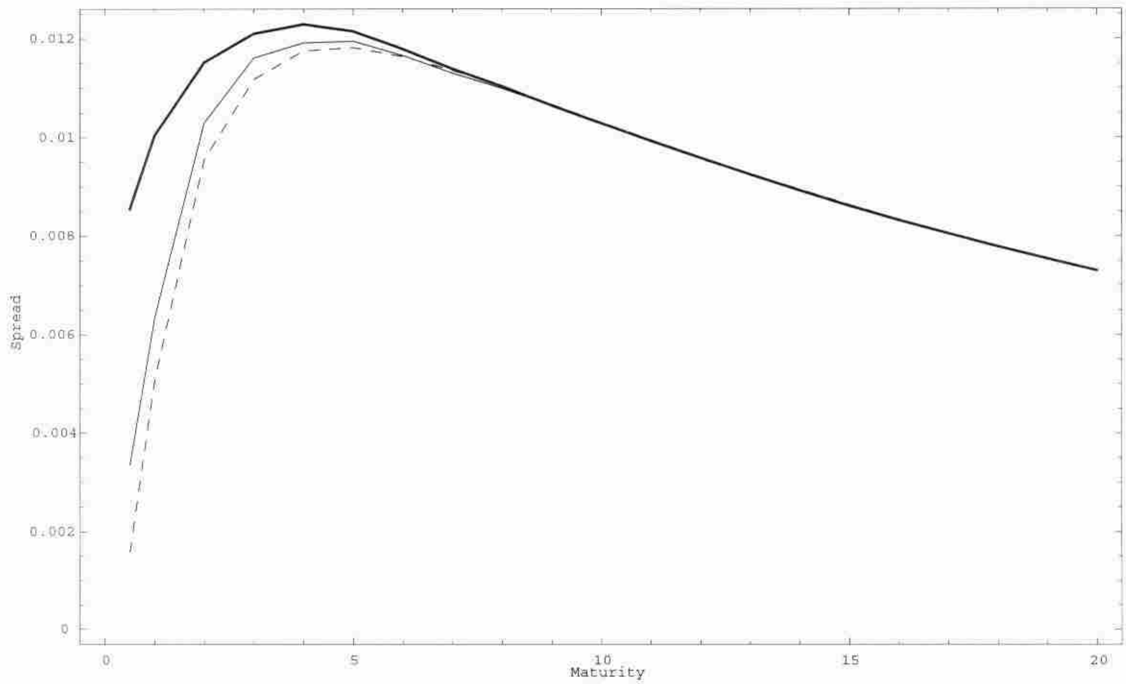
Term Structures of Credit Spreads Under Jump-Diffusion Process



Term structures of credit spreads of 5% coupon bond when underlying process follows a jump-diffusion process with the same total variance but different jump components: 1) $\lambda^* = 0.3$ (heavy solid line); 2) $\lambda^* = 0.05$ (lighter solid line). Term structure (dashed line) under a pure-diffusion process with a constant variance equal to the total variance of the jump-diffusion case shown for comparison. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, $c = 0.05$, $V_0 = 100$, $K = 50$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, and $\bar{k}^* = -9.5\%$. Total variance per unit of time remains constant in all cases.

EXHIBIT 2

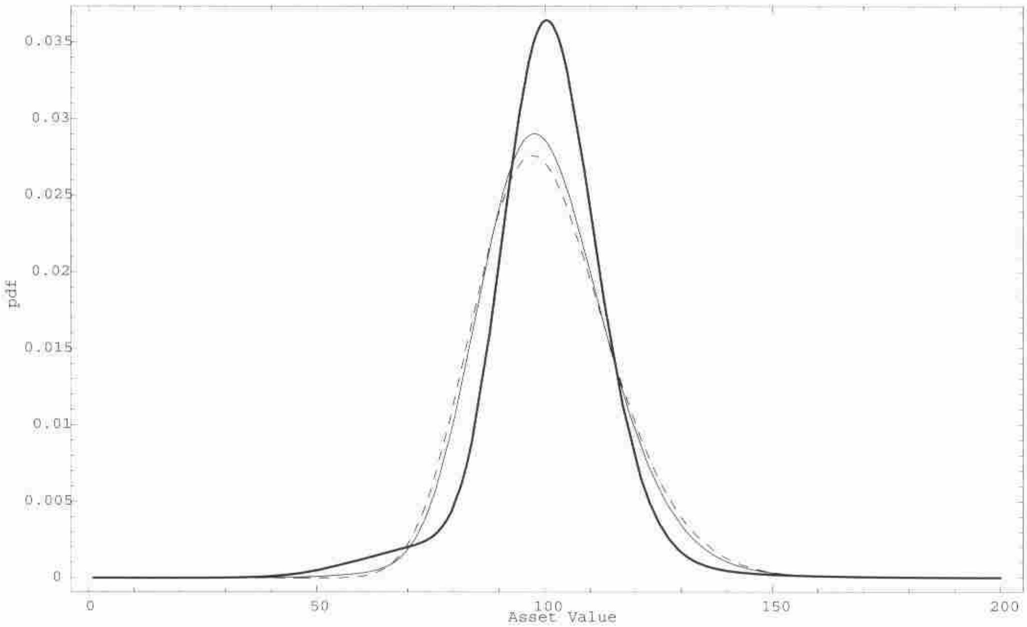
Term Structures of Credit Spreads Under Jump-Diffusion Process



Same conditions as in Exhibit 1, except for different face value, $K = 70$.

EXHIBIT 3

Risk-Neutral Probability Density Functions of Asset Value Under Jump-Diffusion Process



Three probability density functions of V_t under a jump-diffusion process with the same total variance but different jump components: 1) $\lambda^* = 0.3$ (heavy solid line); 2) $\lambda^* = 0.05$ (lighter solid line). Corresponding density function (dashed line) of asset value under a pure-diffusion process with a constant variance equal to the total variance of the jump-diffusion case is shown for comparison. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, $c = 0.05$, $V_0 = 100$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, and $\overline{k}^* = -9.5\%$. Total variance per unit of time remains constant in all cases.

Exhibit 3 shows the risk-neutral probability density functions for the jump-diffusion and pure-diffusion cases. It is evident that the functions are fat-tailed when jumps can happen.[11] Here, the time horizon is half a year. Numerical computations show that the differences in shape gradually disappear as the time horizon lengthens.

According to the central limit theorem, one insight into the nature of these results is that the horizon distribution of a jump-diffusion process converges to the pure-diffusion one as maturity lengthens. This shows that jumps bring about additional risk for short-term bonds only. Similar results are obtained when we increase the jump variance, σ_k, and reduce the jump frequency λ^*, while keeping its total contribution and total variance constant.

To plot the corresponding objective density functions of the firm's asset return, estimation of an instantaneous expected return on the asset in the real world is required, but this issue is beyond the scope of this article. Here, we simply assume that the asset return in the real world is characterized by $\mu = 15\%$, $R = 0.5$, $\gamma_w = \gamma$, and $\sigma_{vw} = \sigma_k^2 = \sigma_{kw}^2$.

Two points are revealed. First, the risk-neutral density function tends to give a higher estimate of default probability than the objective one, as a consequence of the lower instantaneous asset return in the risk-neutral world. Second, the close resemblance in shape to the corresponding risk-neutral density functions shows that the empirical property of fat tails in asset returns can also be observed in the real world.

Note that our comparison is based on the assumption of constant total variance, measured in the risk-neutral world. One point is of paramount importance in our credit spread analysis. In the case of non-systematic jumps, the total variances of the jump-diffusion process in the real and the risk-neutral world are the same. This implies that the theoretical levels of credit spread due to default risk can be estimated from the observable parameters in the real world. If the jump risk is systematic, however, estimation of spread levels becomes more complicated. This is particularly the case if the average jump sizes in the asset value process and market portfolio are negative; that is, γ, $\gamma_w < 0$.[12]

Under this assumption, it is easy to see that:

$$\lambda^* = \lambda \exp[-R\gamma_w + \frac{1}{2}R(1+R)\sigma_{k_w}^2] > \lambda,$$
$$\gamma^* = \gamma - R\sigma_{vw} < \gamma < 0.$$

The implication is that total variance measured in the risk-neutral world is higher than that observed in the real world. Jumps in the risk-neutral world tend to be more influential as they become more negative. This is also true in times of economic recession when investors become more averse to risk. In the presence of systematic jumps, the true spread levels cannot be approximated accurately by the theoretical levels based on the observable parameters without making specific assumptions about risk aversion and the market portfolio. In fact, without knowledge of risk aversion or the market portfolio, the jumps would not be priced correctly, so there is a tendency to underestimate the spread levels. Consistent with the findings in Elton et al. [2001], this result implies that without taking the systematic jump risk into account, Merton-type models tend to underestimate the credit spreads.

When we compare them with the empirical properties of credit spreads, it is evident that the spread levels generated in Exhibits 1 and 2 do not quite resemble those observed in the market. The empirical findings in Kim, Ramaswamy, and Sundaresan [1993] indicate that, over the 1926-1986 period, the yield spreads on high-grade corporates (AAA-rated) ranged from 15 to 215 basis points and averaged 77 basis points; and the yield spreads on BAA bonds ranged from 51 to 787 basis points and averaged 198 basis points.

IV. PRICING WITH GENERAL JUMP RISK, TAX, AND DIVIDEND EFFECTS

State taxes have been ignored in almost all modeling of defaultable bonds (see, for example, Jarrow, Lando, and Turnbull [1997] and Duffee [1999]). To improve spread level estimation, we introduce the two important factors of taxes and dividends. We then investigate their effects on spread levels and default mechanisms.

Tax effects are important because investors in corporate bonds are subject to state and local taxes on interest payments, while government bonds are not subject to these taxes. Thus, corporate bonds have to offer a higher pre-tax return to investors to compensate for tax expenses.

Elton et al. [2001] show that expected default accounts for a small fraction of the premium in corporate yields over Treasuries. State taxes explain a substantial portion of the difference. Taxes account for a significantly greater portion of the spreads than do expected losses. They find that for ten-year A-rated bonds, taxes accounted for 36.1% of the spreads, compared to the 17.8% accounted for by expected losses.

Since state taxes are deductible from income for federal tax purposes, the state tax rate is reduced by the federal tax rate. Hence, the effective tax rate, as a measure of marginal impact of state taxes, is of the form:

$$\tau = \tau_s(1 - \tau_g),$$

where τ_s is a state tax, and τ_g is the federal tax rate. We assume $\tau = 4.875\%$ by following the arguments in Elton et al. [2001] that we choose $\tau_s = 7.5\%$ as the midpoint of maximum marginal state taxes and $\tau_g = 35\%$ as the maximum federal tax rate. It is easy to modify the model to accommodate this tax factor.

We assume that default can happen only on coupon payment dates. There are two cases that affect the bond price. On each coupon payment date, if default does not happen, the actual bond value is the original bond value less the total amount of tax on the interest payment. If the bond defaults, the bond price becomes the residual asset value plus the tax refund due to a capital loss.[13] We take the after-tax coupon rate to be 5%.

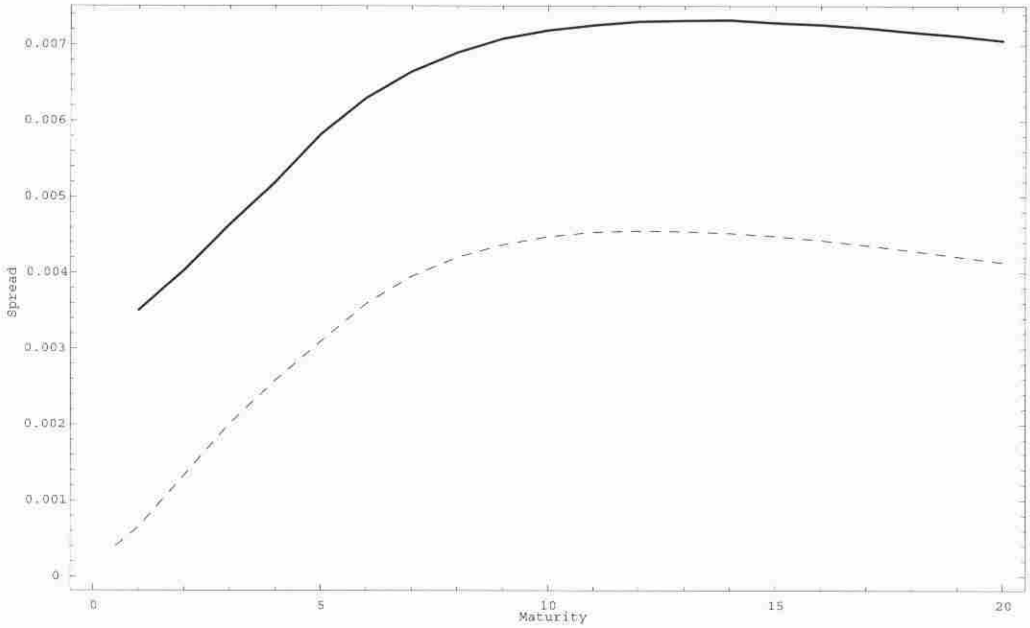
The tax effects on spread levels are shown in Exhibits 4 and 5. Exhibit 5 shows the term structures of the credit spread of two risky bonds with different face values and the spread component generated by pure tax effects.[14] Note that the pure tax level is the same in both cases.[15]

As expected, tax effects represent a considerable portion of total spread levels. The persistent difference in spread levels between the spread curves indicates that taxes are a more important influence on spreads. There are two important points worth noting.

First, pure tax effects appear to be proportionally more significant when the face value becomes smaller. This is because the spread levels remain the same while the fraction of premium due to default risk becomes smaller when $K = 50$. Second, the spread levels are closer to the empirical averages of yield spreads on high-grade and on medium-grade bonds as found in Kim, Ramaswamy, and Sundaresan [1993].[16] The model including tax effects is more able to produce realistic spread levels.

EXHIBIT 4

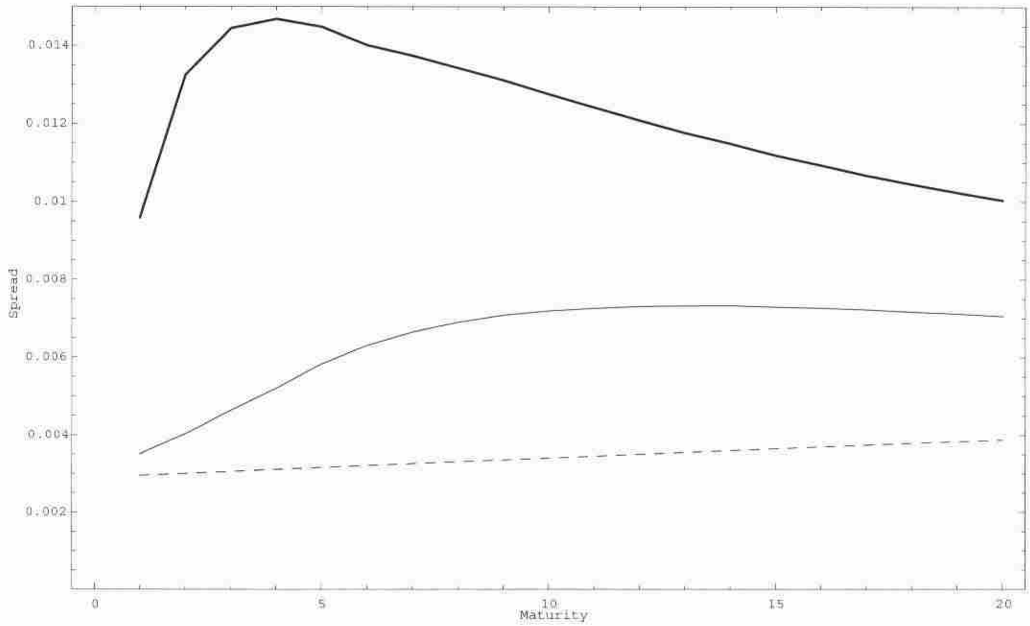
Term Structures of Credit Spreads Under Jump-Diffusion Process for Face Value K = 50 with Tax Effects



Term structures of credit spreads (solid line) of 5% (after-tax rate) coupon bond when underlying process follows a jump-diffusion process. Credit spread without taking tax effects into account is the dashed line. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, *after-tax* $c = 0.05$, $V_0 = 100$, $\lambda^* = 0.05$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, *and* $\overline{k}^* = -9.5\%$. *Total variance per unit of time remains constant in all cases.*

EXHIBIT 5

Term Structures of Credit Spreads Under Jump-Diffusion Process for Different Face Values of K = 70 and K = 50 with Tax Effects



Term structures of credit spreads of 5% (after-tax rate) coupon bonds with different face values: 1) $K = 70$ *(heavy solid line) and 2)* $K = 50$ *(lighter solid line), when underlying process follows a jump-diffusion process. Credit spread generated by pure tax effects is the dashed line. Parameter values:* $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, *after-tax* $c = 0.05$, $\tau = 4.875\%$, $V_0 = 100$, $\lambda^* = 0.05$, $\sigma_k = 0.25$. *Total variance per unit of time remains constant in all cases.*

The term structure of default barrier remains nearly stationary, even when there is a change in the average jump size in asset value $\bar{k}^*$ from −9.5% to 0%. The implication is that spread levels rise, as downward jumps accelerate the default mechanism by increasing the probability of bankruptcy, rather than raising the barrier to a higher level. The same result holds when different face values K are used.

We also look at the properties of spread levels and default barriers when there is a change in the effective tax rate. We assume that the distribution of state taxes remains unchanged. When we reduce the federal tax rate by 5 percentage points to 30%, there is an increase in the effective tax rate to $\tau = 5.25\%$. Numerical computations show that the spread level becomes higher as a result of the increase in τ. Moreover, Exhibit 6 indicates that the default barrier becomes higher.

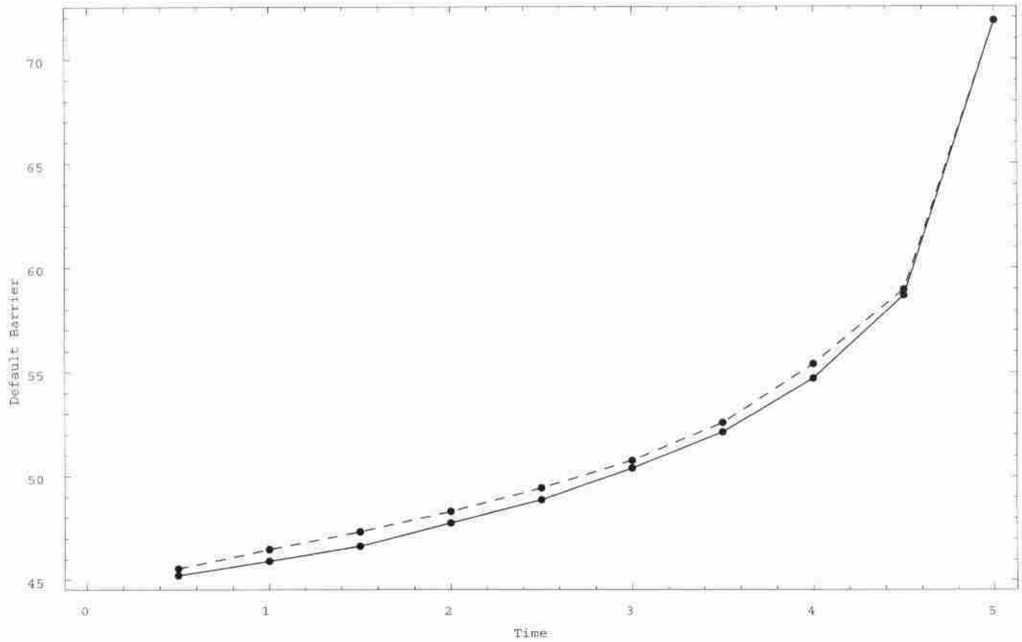
This can be explained as follows. In order to avoid bankruptcy, the asset value has to be maintained at such a level that the stock immediately after a coupon date is worth at least the coupon payment. The bankruptcy boundary is determined at the asset level where the stock price is equal to the coupon payment. If a default is imminent, the total debt value increases slightly because of the refund of capital loss in the event of bankruptcy. As a consequence, an increase in the effective tax rate raises the bankruptcy barrier. Furthermore, as the firm's value drops to a low level, the tax shelter for coupon payments will not be fully realized.[17]

While how firms consider taxes in making default decisions is a complicated issue, our result suggests that a change in the federal tax rate may be a factor in the earlier default of low-grade bonds.

Finally, the effect of dividends is shown in Exhibit 7, which plots the term structures of the credit spread under a jump-diffusion process for different values of dividend payout rate $\delta = 7\%$ and 5%. It is evident that a small change in dividend rates can have significant and persistent effects on spread levels.

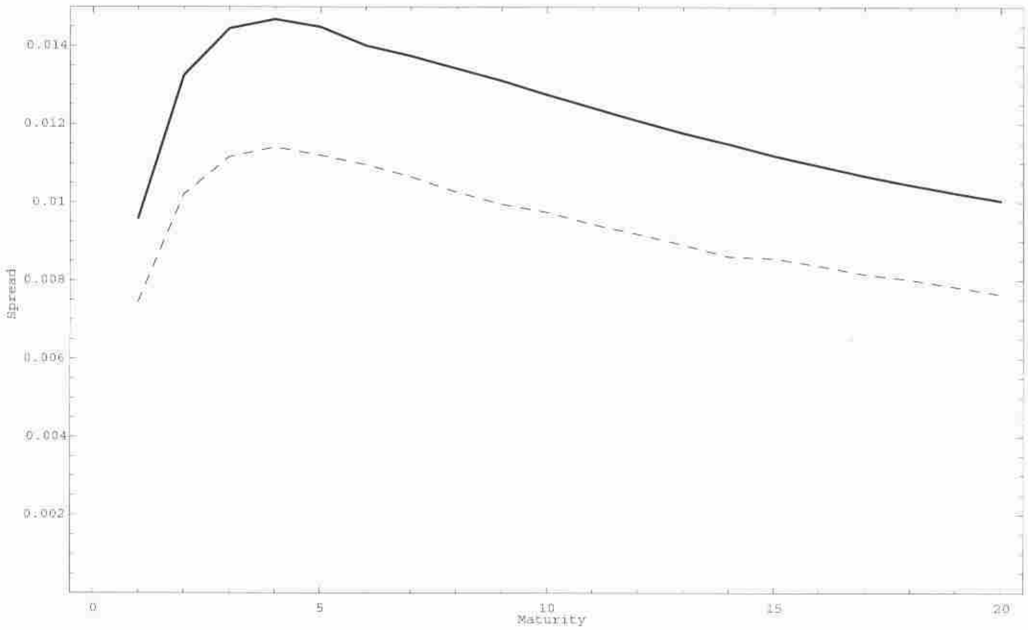
EXHIBIT 6

Default Boundaries Under Jump-Diffusion Process for Different Values of Tax Rate



Default boundaries of 5-year 5% (after-tax rate) coupon bonds with different values of τ: 1) $\tau = 4.875\%$ (solid line) and 2) $\tau = 5.25\%$ (dashed line), when underlying process follows jump-diffusion process. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, after-tax $c = 0.05$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, $\sigma_k = 0.25$.

EXHIBIT 7
Term Structures of Credit Spreads Under Jump-Diffusion Process for Different Values of Dividend Payout



Term structures of credit spreads of 5% (after-tax rate) coupon bonds with different values of delta: 1) $\delta = 7\%$ (solid line), and 2) $\delta = 5\%$ (dashed line), when underlying process follows jump-diffusion process. Parameter values: $r_0 = 0.08$, $\sigma = 0.2$, after-tax $c = 0.05$, $\tau = 4.875\%$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, $\sigma_k = 0.25$.

V. SUMMARY

We have compared the structural framework of bond pricing models under a jump-diffusion process and under a pure-diffusion process. We use a tractable discrete-time model for the valuation of defaultable coupon bonds when the underlying firm value process follows a jump-diffusion process. The method yields a framework that requires only simple mathematics.

The modeling of a firm's total asset value as a jump-diffusion process can provide a more realistic model of spread levels that, unlike diffusion-based models, do not go to zero for short maturities. This is because the jump-diffusion model enables us to generate a leptokurtic (fat-tailed) distribution for the firm's asset values. We have also found that negative jumps can have significant and persistent effects on spread levels.

As in Geske [1977], we assume that a default event can happen only on payment dates. Bankruptcy can happen *expectedly* when the asset level hits a certain lower barrier through a continuous diffusion crossing, or *unexpectedly* when its value drops precipitously below the barrier. Consistent with Leland [1994, 1998] and Leland and Toft [1996], the default boundary is determined endogenously by requiring the value of equity to be at least the amount of the coupon just paid, in order to avoid bankruptcy. Alternative bankruptcy criteria can also be used in the model. The model is flexible enough, that is, to accommodate a given lower reorganization boundary below which the firm will be unable to pay contractual coupons.

The effects of jumps on the levels of credit spreads can be significant and persistent over time. The effects on spread levels are conspicuous for short maturities. The higher the jump frequency and variability, the higher the short-term spreads. For long maturities, credit spreads are not distinguishable from those generated by a pure diffusion-model. When downward jumps are of higher volatility, however, the effects on credit spreads become more persistent. Furthermore, if the downward jumps are systematic, there is a tendency to underestimate spread levels. This result may partly explain why without taking the systematic jump risk into account, models after Merton tend to underestimate credit spreads.

Other important factors include taxes and dividends. State taxes have been ignored in almost all modeling of defaultable bonds. We have shown that taxes do have significant and persistent effects for bonds with long maturities. In fact, credit spreads increase with the effective tax rate on coupon payments. Tax effects appear to be the second-most important factor for spread levels, as documented in Elton, et al. [2001]. We have also found that while downward jumps in firm value increase the probability of default, the bankruptcy boundary does not seem to be affected.

We have also investigated the effects of state and federal taxes on default mechanisms. Assuming the distribution of state taxes remains unchanged, we have shown that a change in the federal tax rate may be a factor in the earlier default of low-grade bonds. Finally, we have found that dividend payout rates can have significant and persistent effects on spread levels.

With deployment of the additional factors of taxes and dividends, the jump-diffusion model has been shown to be more flexible than pure-diffusion models in fitting empirical spreads. It remains to be seen whether it is sufficiently flexible and sufficiently easy to fit to be useful in empirical work.

ENDNOTES

[1]For an econometric definition, see Koutsoyiannis [1973].

[2]See Delianedis and Geske [1998]. This empirical study indicates that risk-neutral default probabilities using the Merton [1974] and Geske [1977] diffusion models appear to incorporate significant information about credit migrations and default.

[3]The method is flexible enough to be modified and to allow bankruptcy events to happen in any between-payment period.

[4]Hence one would expect to see a marked increase in the volatility of bond returns and a sudden drop in equity prices.

[5]See Bates [1991].

[6]It is not necessary to know the true parameters in the process in Equation (2) as far as pricing issues are concerned.

[7]Amin [1993] shows that the discrete-time process converges weakly to the continuous-time process. This guarantees that the prices of European options computed from the discrete-time model will converge to their corresponding continuous-time values under fairly undemanding regularity conditions. For example, the option payoff must be uniformly integrable in h_n.

[8]See Kliger and Sarig [2000] for an empirical justification.

[9]The total variance is $\sigma^2 + \lambda^*(\sigma_k^2 + (\gamma^* - \frac{1}{2}\sigma_k^2)^2)$, where the first and second terms are due to the diffusion and jump components, respectively.

[10]Analytically, we can prove that there is a positive instantaneous default hazard at time 0, which is equal to λ^* times the probability of default in the case of a jump.

[11]The leptokurtic properties of fat tails, especially in the left tail, and a high peak that are evident in Exhibit 3 are normally observable in empirical densities of market indexes, such as the S&P 500 index.

[12]Downward jumps are more likely than upward jumps.

[13]When a bond defaults at time t, the amount $K - V_{t+}$ lost in default is a capital loss, and taxes $\tau(K - V_{t+})$ are recovered. See Elton et al. [2001].

[14]We assume the bonds are issued by two identical firms.

[15]Spread levels are computed when the firm's asset value is large relative to the amount of debt.

[16]The remaining discrepancies may be due to the illiquidity of corporate bonds.

[17]Under U.S. tax codes, to benefit fully from a tax shelter, the firm must have earnings before interest and taxes that are not lower than total coupon payments. When default is imminent, it is quite possible that profits will be less than the coupon payout and tax savings will not be fully realized. See Leland [1994].

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