Valuing Default Swaps Under Market and Credit Risk Correlation

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The market for default swaps, as measured by the notional amounts of contracts traded per year, has grown from about \$50 billion in 1998 to over \$400 billion in 2000 (see "Credit Risk" [2000]). This exponential growth has generated significant interest in the fair valuation of default swaps in both the academic and practitioner communities.

The literature investigating the valuation of default swaps gives the impression that simple models for pricing default swaps are available only when credit and market risk are statistically independent (see Hull and White [2000, 2001], Martin, Thompson, and Browne [2000], Wei [2001], and the survey article by Cheng [2001]). Indeed, it is commonly believed that models incorporating correlated market and credit risk are quite complex, requiring burdensome recursive numerical procedures.

For example, from Hull and White:

> Like most other approaches, ours assumes that default probabilities, interest rates, and recovery rates are mutually independent. Unfortunately, it does not seem to be possible to relax these assumptions without a considerably more complex model [2000, p. 30].

We provide on the contrary a simple analytic formula for the valuation of default swaps when market and credit risk are correlated. This formula is easy to understand and to compute. It is derived in the context of a reduced-form credit risk model where correlated defaults arise because a firm's default intensities depend on common macroeconomic factors (see Jarrow [2001]). The common macro factor we use is the spot rate of interest, assumed to follow an extended Vasicek model in the Heath, Jarrow, and Morton [1992] framework.

We illustrate the numerical implementation of this model by deducing the default probability parameters implicit in the term structure of default swap quotes for 22 different companies over the time period August 21–October 31, 2000. The data used for this investigation come from Enron's web site. The 22 different firms are chosen to stratify various industry groupings: financial, food and beverages, petroleum, airlines, utilities, department stores, and technology.

For comparison purposes, the standard model with statistically independent market and credit risk (a special case of our model) is also calibrated to these market data. One can also easily calibrate our simple model to exactly match the observed default swap quote term structure in the case of correlated market and credit risk.

I. MODEL STRUCTURE

The reduced-form credit risk model in Jarrow [2001] is the basis for valuing default swaps. Trading can take place any time during the interval $[0, \bar{T}]$. Traded are default-free zero-coupon bonds and risky (defaultable) zero-coupon bonds of all maturities. Markets are assumed to be complete and frictionless, with no arbitrage opportunities.

Let $p(t, T)$ represent the time t price of a default-free dollar paid at time T where $0 \le t \le T \le \bar{T}$. The instantaneous forward rate at time t for date T is defined by $f(t, T) = -\partial log\, p(t, T)/\partial T$. The spot rate of interest is given by $r(t) = f(t, t)$.

Consider a firm issuing risky debt. Let $v(t, T)$ represent the time t price of a promised dollar to be paid by this firm at time T where $0 \le t \le T \le \bar{T}$. The debt is risky because if the firm defaults before time T, the promised dollar may not be paid. Let τ denote the first time that this firm defaults ($\tau > \bar{T}$ is possible if the firm does not default). The default time is a random variable.

We let

$$N(t) = 1_{\{\tau \le t\}} = \begin{cases} 1 & \text{if } \tau \le t \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

denote the point process indicating whether or not default has occurred prior to time t. It is assumed that the point process has a random intensity $\lambda(t)$, where $\lambda(t)\Delta$ gives the approximate probability of default for this firm over the time interval $[t, t + \Delta]$. The intensity process is defined under the risk-neutral probability.

If default occurs, we let the zero-coupon bond receive a *fractional recovery* of $\delta(\tau)v(\tau -, T)$ dollars, where $0 \le \delta(\tau)$, and $\tau -$ represents an instant before default. In this recovery rate structure, the debt is worth only a fraction of its pre-default value.

Under the assumption of no arbitrage and complete markets, standard arbitrage pricing theory implies that there is a unique equivalent probability Q such that the present values of the zero-coupon bonds are computed by discounting at the spot rate of interest and then taking an expectation with respect to Q.[1]

That is:

$$p(t,T) = E_t\left(e^{-\int_t^T r(u)du}\right) \tag{2}$$

and

$$v(t,T:\delta) = E_t\Big(\delta(\tau)v(\tau-,T)e^{-\int_t^\tau r(u)du} 1_{(t<\tau\le T)} + 1\, e^{-\int_t^T r(u)du} 1_{(T<\tau)}\Big) \tag{3}$$

where $E_t(.)$ is the conditional expectation with respect to Q at time t. The risky debt value is composed of two parts. The first part is the present value of the promised payment in default. The second part is the present value of the promised payment if default does not occur.

Credit derivatives are priced using the approach of Lando [1998], assuming a perfectly liquid market. We assume that the point process is modeled as a Cox process with an intensity function $\lambda(t, X_t)$ where $\{X_t: t \in [0, T]\}$ is a vector stochastic process representing the state variables underlying the evolution of the economy. These macroeconomic state variables induce the correlation between market and default risk. For example, if X_t quantifies market risk, and $\lambda(t, X_t)$ increases as X_t increases, then as market risk increases, the likelihood of the firm defaulting increases as well.

The cash flows from most credit derivatives can be of three different types:

- The first is a random payment Y_T at time T, but only if there is no default prior to time T.[2]
- The second is a random payment rate of $y_t dt$ at time t for the period $[0, T]$, but only if there is no default prior to the time the payment is received.
- The third is a random payment that occurs only at default of Ψ_τ, and is zero otherwise. This payment is made only if default occurs during the time period $[0, T]$.

Under appropriate integration conditions, the present values of these credit risky cash flows are (proofs are in the appendix):

$$\begin{aligned} V_t 1_{\{\tau>t\}} &= E_t\left(Y_T 1_{\{\tau>T\}} e^{-\int_t^T r(u)du}\right) \\ &= E_t\left(Y_T e^{-\int_t^T [r(u)+\lambda(u)]du}\right) \end{aligned} \tag{4}$$

$$V_t 1_{\{\tau>t\}} = E_t\left(\int_t^T \gamma_s 1_{\{\tau>s\}} e^{-\int_t^s r(u)du} ds\right)$$

$$= E_t\left(\int_t^T \gamma_s e^{-\int_t^s [r(u)+\lambda(u)]du} ds\right) \tag{5}$$

$$V_t 1_{\{\tau>t\}} = E_t\left(1_{\{t<\tau\le T\}} \Psi_\tau e^{-\int_t^\tau r(u)du}\right)$$

$$= E_t\left(\int_t^T \Psi_s \lambda(s) e^{-\int_t^s [r(u)+\lambda(u)]du} ds\right) \tag{6}$$

In each valuation, the second expectation is with respect to the state variables underlying the stochastic structure of the economy [including the spot rate $r(t)$]. Randomness attributable to the bankruptcy process is removed through the computation of an iterated expectation. What remains is the valuation of a random cash flow based on the state variables using a risk-adjusted discount rate. The risk adjustment is the expected loss per unit time that equals the intensity $\lambda(\mu)$ (with a zero recovery rate).

II. BINARY DEFAULT SWAP

We specialize the general structure to value a binary default swap (sometimes called a credit swap); see Rooney [1998]. In this credit swap there are two counterparties, the protection seller and the protection buyer. We assume that both counterparties are default-free. This assumption can easily be relaxed as discussed in Jarrow and Turnbull [1995].

The swap lasts for a fixed period of time $[0, T]$, the swap's maturity. The protection seller receives a fixed payment flow of c_T dollars per unit time from the protection buyer unless there is default on a reference credit [a particular firm with intensity $\lambda(t)$]. Default is called a *credit event*. If default occurs, the protection seller pays the protection buyer \$1 on the default date τ. If there is no default, the protection seller makes no payment to the protection buyer. The swap terminates either after the default event occurs or at the maturity date of the swap.

In traded default swaps, the payments occur at fixed intervals. If default occurs between payment dates, the accrued portion of the payment due at the next payment date is due at the time of default. The continuous payment structure provides a reasonable approximation of this condition.

The value of the swap to the protection seller is:

$$V_t 1_{\{t<\tau\}} = E_t\left(\int_t^T c_T 1_{\{\tau>s\}} e^{-\int_t^s r(u)du} ds\right) - E_t\left(1_{\{t<\tau\le T\}} e^{-\int_t^\tau r(s)ds}\right) \tag{7}$$

Using Equations (5) and (6), we obtain:

$$V_t 1_{\{t<\tau\}} = E_t\left(\int_t^T c_T e^{-\int_t^s [r(u)+\lambda(u)]du} ds\right) - E_t\left(\int_t^T \lambda(s) e^{-\int_t^s [r(u)+\lambda(u)]du} ds\right) \tag{8}$$

We can recognize this as the value of a risky coupon bond of maturity T paying a continuous cash flow of c_T dollars per unit time with a zero recovery rate less the cost of a dollar default insurance on the firm; that is:

$$V_t 1_{\{t<\tau\}} = c_T \int_t^T v(t,s:0)ds - E_t\left(\int_t^T \lambda(s) e^{-\int_t^s [r(u)+\lambda(u)]du} ds\right) \tag{9}$$

This simple formula applies when both market risk and credit risk are correlated. Whether or not we obtain a simple analytic formula for the price of a default swap depends on the evaluation of the second term on the right-hand-side of Equation (9).

In standard default swaps, the cash payment c_T, called the default swap rate, is determined at time 0 such that $V_0 = 0$; that is:

$$c_T = \frac{E_0\left(\int_0^T \lambda(s) e^{-\int_0^s [r(u)+\lambda(u)]du} ds\right)}{\int_0^T v(0,s:0)ds} \tag{10}$$

III. EMPIRICAL SPECIFICATION

To obtain a simple but realistic empirical formulation of the model, we use a special case of Jarrow [2001], where the economy is Markovian in a single state variable—the spot rate of interest. Although Jarrow includes the cumulative excess return on an equity market index, Janosi, Jarrow, and Yildirim [2000] find that this adds no additional explanatory power in the pricing of corporate debt.

For the spot rate of interest, we use a single-factor model with deterministic volatilities, sometimes called the extended Vasicek model (spot rate evolution):

$$dr(t) = a\left[\bar{r}(t) - r(t)\right]dt + \sigma_r dW(t) \tag{11}$$

where $\alpha \neq 0$, $\sigma_r > 0$ are constants, $\overline{r}(t)$ is a deterministic function of t chosen to match the initial zero-coupon bond price curve, and $W(t)$ is a standard Brownian motion under Q initialized at $W(0) = 0$.[3] The evolution of the spot rate is given under the risk-neutral probability Q.

We assume that the intensity function is almost linear with the spot rate of interest.

$$\lambda(t) = max[\lambda_0(t) + \lambda_1 r(t), 0] \tag{12}$$

where $\lambda_0(t) \geq 0$ is a deterministic function of time, and λ_1 is a constant. In this formulation, the (pseudo) probability of default per unit time is assumed to be the maximum of a linear function of the spot rate $r(t)$ and zero. The maximum operator is necessary to keep the intensity function non-negative. For analytic convenience, we drop the maximum operator in the empirical implementation. This linear approximation implies that negative default rates ($\lambda(t) < 0$) are possible.

Finally, to price risky debt, we assume that the recovery rate is a constant. In fact, in the valuation of default swaps, this constant recovery rate assumption is unnecessary. It is included only to provide Equation (14) for risky debt prices when the recovery rate is non-zero:

$$\delta(t) = \delta$$

where δ is a constant.

Given this structure, Jarrow [2001] obtains closed-form solutions for default-free and risky zero-coupon bond prices:

$$p(t,T) = e^{-\mu_1(t,T)+\sigma_1^2(t,T)/2} \tag{13}$$

and

$$v(t,T:\delta) = p(t,T)e^{-\int_t^T \lambda_0(s)(1-\delta)ds - \lambda_1(1-\delta)\mu_1(t,T) + \left(2\lambda_1(1-\delta)+\lambda_1{}^2(1-\delta)^2\right)\sigma_1^2(t,T)/2} \tag{14}$$

where no default has occurred at or prior to time t:

$$\mu_1(t,T) = \int_t^T f(t,u)du + \int_t^T b(u,T)^2 du \big/ 2,$$

$$\sigma_1^2(t,T) = \int_t^T b(u,T)^2 du$$

and

$$b(u,t) = \sigma_r\left(1 - e^{-a(t-u)}\right)\Big/ a$$

Substitution of the linear intensity into the default swap's valuation yields Equation (15). The proof is in the appendix.

$$V_t 1_{\{\tau>t\}} = c_T \int_t^T v(t,s:0)ds - \int_t^T [\lambda_0(s) + \lambda_1[\mu_0(t,s) - (1+\lambda_1)\sigma_{01}(t,s)]]v(t,\ s:0)ds \tag{15}$$

where

$$\mu_0(t,s) = f(t,s) + b(t,s)^2 / 2$$

and

$$\sigma_{01}(t,s) = b(t,s)^2 / 2$$

This is the desired analytic expression for the fair value of a default swap, easily computed using only knowledge of the default-free term structure evolution and the current valuation of the firm's risky debt prices. If risky debt prices are not available, the risky debt prices can be estimated using the analytic formula given by Equation (14).

Finally, the default swap rate is determined by solving Equation (15):

$$c_T = \frac{\int_0^T [\lambda_0(s) + \lambda_1[\mu_0(0,s) - (1+\lambda_1)\sigma_{01}(0,s)]]v(0,s{:}0)ds}{\int_0^T v(0,s{:}0)ds} \tag{16}$$

IV. DESCRIPTION OF THE DATA

Data from Enron's web site cover August 21–October 31, 2000. We select 22 different firms to represent various industry groupings: financial, food and beverages, petroleum, airlines, utilities, department stores, and technology. The 22 firms are listed in Exhibit 1. For parameter estimation of the spot rate process, daily U.S. Treasury bond, note, and bill prices were downloaded from Bloomberg for the same time period.

V. ESTIMATION OF SPOT RATE PROCESS PARAMETERS

To implement estimation of the default and liquidity discount parameters, we first need to estimate the parameters for the state variable processes ($r(t)$). The inputs to the spot rate process evolution are the forward rate curves ($f(t, T)$) and the spot rate parameters (a, σ_r).

We use a two-step procedure for estimation of the forward rate curves. First, for a given time t, the discount bond prices [$p(t, T)$ for various T] are estimated by solving a minimization problem:

EXHIBIT 1

Default Swap Quotes—8/21/00-10/31/00 (basis points)

		Avg c_1	Avg c_2	Avg c_3	Avg c_4	Avg c_5
American Airlines Inc.	AMR1	149.6944	167.0556	175.0278	183.9444	196.2222
Archer-Daniels-Midland Co.	ADM	32.4722	38.2222	43.5000	47.5556	53.5278
CP & L Energy Inc.	CPL	51.3056	57.4167	60.5278	64.0833	68.3889
Chase Manhattan Corp.	CMB	30.3889	35.3889	37.2778	39.0000	41.8333
Coca-Cola Enterprises	CCE	35.3056	40.4167	43.5833	45.0556	49.2778
Dow Chemical Company	DOW	23.5000	29.3333	35.6111	40.6667	44.6389
Delta Air Lines Inc.	DAL	130.1944	138.7222	146.4722	151.2222	161.3333
Eastman Kodak Co.	EK	26.7222	29.1667	31.5833	33.6944	36.6667
First Union Corp.	FTU	32.8333	36.6667	38.7222	41.0278	43.8056
J.P. Morgan & Company	JPM	19.8889	22.8056	26.3056	27.9444	31.0000
K Mart Corp.	KM	353.9167	369.3611	458.2778	471.0833	485.9167
Lyondell Chemical Company	LYO	349.6111	349.6389	367.5000	374.6667	387.7222
Merrill Lynch & Co.	MER	27.0278	30.0556	40.0833	43.6389	47.9722
Phillips Petroleum Co.	P	58.5278	67.0556	72.6667	75.1944	79.5000
Ralston-Ralston Purina Group	RAL	61.4444	69.6667	73.9167	76.4167	81.1111
Sears, Roebuck and Co.	S	60.5833	64.7222	71.4722	73.6111	77.0556
Southwest Airlines	LUV	34.8611	40.5278	42.7778	44.5278	46.3333
TXU Corporation	TXU	83.3611	89.0278	95.1389	103.0278	104.8611
Union Oil Co. of California	UCL1	59.9444	67.9167	72.3889	74.8889	79.5000
Wal-Mart Stores Inc.	WMT	13.5000	16.2778	17.2778	18.1389	20.4444
Xerox Corp.	XRX	161.2778	169.5278	214.5833	189.8333	227.8056
Texas Instruments Inc.	TXN	40.8889	44.3333	50.2778	52.6944	56.1389

Choose $(p(t,T)$ for all relevant $T \leq max\{T_i : i \in I_t\}$

$$\text{to minimize } \sum_{i \in I_t} \left[B_i(t,T_i) - B_i(t,T_i)^{bid} \right]^2 \tag{17}$$

where I_t is an index set including the various U.S. Treasury bonds, notes, and bills available at time t; $B_i(t, T_i)$ is the model price [Equation (4)] for the i-th bond with maturity T_i as a function of $[p(t, T)]$; and $B_i(t, T_i)^{bid}$ is the market bid price for the i-th bond with maturity T_i. The discount bond price maturity dates T coincide with the maturities of the Treasury bills, and the coupon payments and principal repayment dates are the same as for the Treasury notes and bonds.

Second, we fit a continuous forward rate curve to the estimated zero-coupon bond prices $[p(t, T)]$ for all $T \leq max\{T_i : i \in I\}$). We use the maximum-smoothness forward rate curve as developed by Adams and van Deventer [1994] and refined by Janosi and Jarrow [2000]. Briefly, we choose the unique piecewise fourth-degree polynomial with the left and right end points left "dangling" that minimizes

$$\int_t^{max\{T_i : i \in I_t\}} \left| \partial^2 f(t,s) / \partial s^2 \right| ds$$

For the spot rate parameter (a, σ_r) estimation, the procedure follows that used in Janosi, Jarrow, and Yildirim [2000]. The procedure is based on an explicit formula for the variance of the default-free zero-coupon bond prices derived using Equation (7); see Heath, Jarrow, and Morton [1992].

For $\Delta = 1/365$ (a day), the expression is:

$$\begin{aligned} & var_t[log(P(t+\Delta,T)/P(t,T)) - r(t)\Delta] \\ & = \left(\sigma_r^2 \left(e^{-a(T-t)} - 1 \right)^2 / a^2 \right) \Delta \end{aligned} \tag{18}$$

First we fix a time to maturity $T - t \in \{3$ months, 6 months, 1 year, 5 years, 10 years, the longest time to maturity of an outstanding Treasury bond closest to 30 years$\}$. Then, we compute the sample variance, denoted ν_T, using the smoothed forward rate curves generated over the sample period.

To estimate the parameters (σ_r, a) we run the non-linear regression:

$$\nu_T = \left(\sigma_r^2 \left(e^{-a(T-t)} - 1 \right)^2 / a^2 \right) \Delta + e_T \tag{19}$$

across the bond's time to maturities $T - t \in \{¼, ½, 1, 5, 10$, longest time to maturity closest to $30\}$, where e_T is the error term.

The parameter estimates (and their standard errors) for this non-linear regression are: $\sigma_r = 0.00593$ (0.0021) and $a = 0.03450$ (0.0003).

VI. DEFAULT PARAMETER ESTIMATION

To illustrate the use of Equation (16), we deduce the default parameters consistent with the given term structure of default swap quotes obtained from Enron's web page. The default swap rates obtained are for contracts of maturities one, two, three, four and five years. This gives five observations per day to estimate the default parameters in Equation (16). The default parameters to estimate are the deterministic function $\lambda_0(t)$ and the constant λ_r.

Exhibit 1 provides the averages of the default swap quotes for the various companies over the observation period. Notice that for all companies the average swap quotation increases with the maturity of the swap. For example, the American Airlines quotes are 149.6944 basis points for the one-year swap and 196.2222 bp for the five-year swap.

Two formulations of our analytic expression are estimated. Model 1, the standard model, has default risk independent of interest rate risk. This is the model in Martin, Thompson, and Browne [2001].[4] The second model, Model 2, is a simple two-parameter version of Equation (16) that incorporates correlated interest rate and default risk. We choose the simplest form of the general model to provide a base case for subsequent empirical investigation into the model's validity.

In Model 1, the standard model, $\lambda_0(t)$ is piecewise-constant and $\lambda_1 = 0$ where:

$$\lambda_0(t)=\begin{cases}\lambda_0^1 & for \quad t\in[0,1)\\ \lambda_0^2 & for \quad t\in[1,2)\\ \lambda_0^3 & for \quad t\in[2,3)\\ \lambda_0^4 & for \quad t\in[3,4)\\ \lambda_0^5 & for \quad t\in[4,5)\end{cases}$$

In this model, the function $\lambda_0(t)$ is calibrated to exactly match the term structure of default swap quotes. Because $\lambda_1 = 0$, default risk and interest rate risk are uncorrelated. Under this structure, Equation (16) becomes:

$$c_k=\frac{\sum_{j=1}^{k}\lambda_0^j\int_{j-1}^{j}e^{-\lambda_0^j s}p(0,s)ds}{\sum_{j=1}^{k}\int_{j-1}^{j}e^{-\lambda_0^j s}p(0,s)ds}\quad \text{for } k=1,\ 2,\ \ldots,\ 5$$

We can invert this system for $\lambda_0(t)$. This model gives an exact fit to the term structure of default swap quotes.

In Model 2, for correlated defaults, $\lambda_0(t) = \lambda_1$ and λ_1 are constants:

$$c_T=\frac{\int_0^T[\lambda_0+\lambda_1[\mu_0(0,s)-(1+\lambda_1)\sigma_{01}(0,s)]]v(0,s{:}0)ds}{\int_0^T v(0,s{:}0)ds}$$

This is also a special case of our model because the intercept of the intensity process, a deterministic function, is restricted to be a constant in order to provide a base case for subsequent investigation.

We can invert this valuation formula for both λ_0 and λ_1. Since this is only a two-parameter model, there will be errors in matching the term structure of default swap quotes. We choose the parameters to minimize the sum of squared errors between the theoretical and the market quotes.

Exhibit 2 shows the average default intensity parameters for Model 1 over the sample period, measured in basis points. These parameter values exactly match the term structure of default swap quotes.

EXHIBIT 2

Model 1 Average Default Intensity Parameters (bp)

		Avg λ_0^1	Avg λ_0^2	Avg λ_0^3	Avg λ_0^4	Avg λ_0^5	N
American Airlines Inc.	AMR1	149.6944	25.8551	25.9323	25.5596	25.7771	36
Archer-Daniels-Midland Co.	ADM	32.4722	24.2311	23.8048	24.3386	23.7516	36
CP & L Energy Inc.	CPL	51.3056	25.9018	26.2306	25.9217	25.2191	36
Chase Manhattan Corp.	CMB	30.3889	24.3501	23.1122	23.5853	23.6264	36
Coca-Cola Enterprises	CCE	35.3056	24.0027	24.9629	24.8912	24.9134	36
Delta Air Lines Inc.	DAL	130.1944	26.0913	25.9644	26.5291	25.9795	36
Dow Chemical Company	DOW	23.5000	21.7930	14.4593	14.4606	14.3817	36
Eastman Kodak Co.	EK	26.7222	19.3607	16.7585	17.6705	17.7911	36
First Union Corp.	FTU	32.8333	23.7851	24.5558	24.7381	24.3545	36
J.P. Morgan & Company	JPM	19.8889	20.9335	5.6304	5.9821	6.3262	36
K Mart Corp.	KM	353.9167	25.9111	26.4591	26.7432	26.2066	36
Lyondell Chemical Company	LYO	349.6111	19.4016	19.2960	19.2386	23.6301	36
Merrill Lynch & Co.	MER	27.0278	20.4044	21.4851	21.3277	20.9874	36
Phillips Petroleum Co.	P	58.5278	26.3940	25.1884	25.2548	25.4656	36
Ralston-Ralston Purina Group	RAL	61.4444	25.9633	25.7631	25.4154	25.3198	36
Sears, Roebuck and Co.	S	60.5833	20.8942	25.1632	25.5317	25.1548	36
Southwest Airlines	LUV	34.8611	24.6734	24.3807	24.7166	24.5128	36
TXU Corporation	TXU	83.3611	19.9234	21.5444	25.9118	23.9295	36
Union Oil Co. of California	UCL1	59.9444	26.3321	25.1084	26.0188	25.7945	36
Wal-Mart Stores Inc.	WMT	13.5000	13.8871	6.9627	6.9274	7.9734	36
Xerox Corp.	XRX	161.2778	25.3679	26.4067	25.9497	26.2770	36
Texas Instruments Inc.	TXN	40.8889	24.1582	25.5791	24.3188	24.5635	36

EXHIBIT 3

Model 2 Average Default Intensity Parameters (bp)

	Avg λ_0	Avg $StdErr_0$	Avg λ_1	Avg $StdErr_1$	SSE	N
American Airlines Inc.	173.6869	7.0377	11.0057	0.4488	22309.6399	36
Archer-Daniels-Midland Co.	42.8829	3.4159	2.7034	0.2179	4445.6399	36
CP & L Energy Inc.	60.1086	2.6531	3.6905	0.1696	3038.6800	36
Chase Manhattan Corp.	36.6415	1.7476	2.1326	0.1118	1571.4400	36
Coca-Cola Enterprises	42.5605	2.1703	2.6177	0.1389	2123.6799	36
Delta Air Lines Inc.	145.0000	4.8327	9.2301	0.3082	9815.5600	36
Dow Chemical Company	34.6153	3.4997	2.1120	0.2228	5401.7199	36
Eastman Kodak Co.	31.4411	1.6846	1.9607	0.1080	1130.6400	36
First Union Corp.	38.4624	1.7125	2.3262	0.1095	1250.4000	36
J.P. Morgan & Company	25.4682	1.7772	1.8859	0.1138	1247.8400	36
K Mart Corp.	426.0039	24.9005	26.9134	1.5758	250994.0401	36
Lyondell Chemical Company	364.3603	7.8434	23.0640	0.4992	23250.6800	36
Merrill Lynch & Co.	37.6026	3.6423	2.3920	0.2329	4427.4799	36
Phillips Petroleum Co.	70.3085	3.3042	4.3899	0.2111	5338.9999	36
Ralston-Ralston Purina Group	72.2258	3.0590	4.4654	0.1955	4628.8799	36
Sears, Roebuck and Co.	69.2179	2.8086	4.2433	0.1790	3715.8799	36
Southwest Airlines	41.6399	1.8105	2.5913	0.1157	1906.9599	36
TXU Corporation	94.6842	4.9341	6.2449	0.3151	10842.6399	36
Union Oil Co. of California	70.6460	3.0475	4.4091	0.1948	4535.6399	36
Wal-Mart Stores Inc.	17.0430	1.0653	1.3275	0.0680	577.7998	36
Xerox Corp.	191.8220	12.5238	12.2826	0.8001	64820.0399	36
Texas Instruments Inc.	48.6580	2.5356	3.2647	0.1618	2769.0399	36

EXHIBIT 4

American Airlines Parameter Estimates Across Time

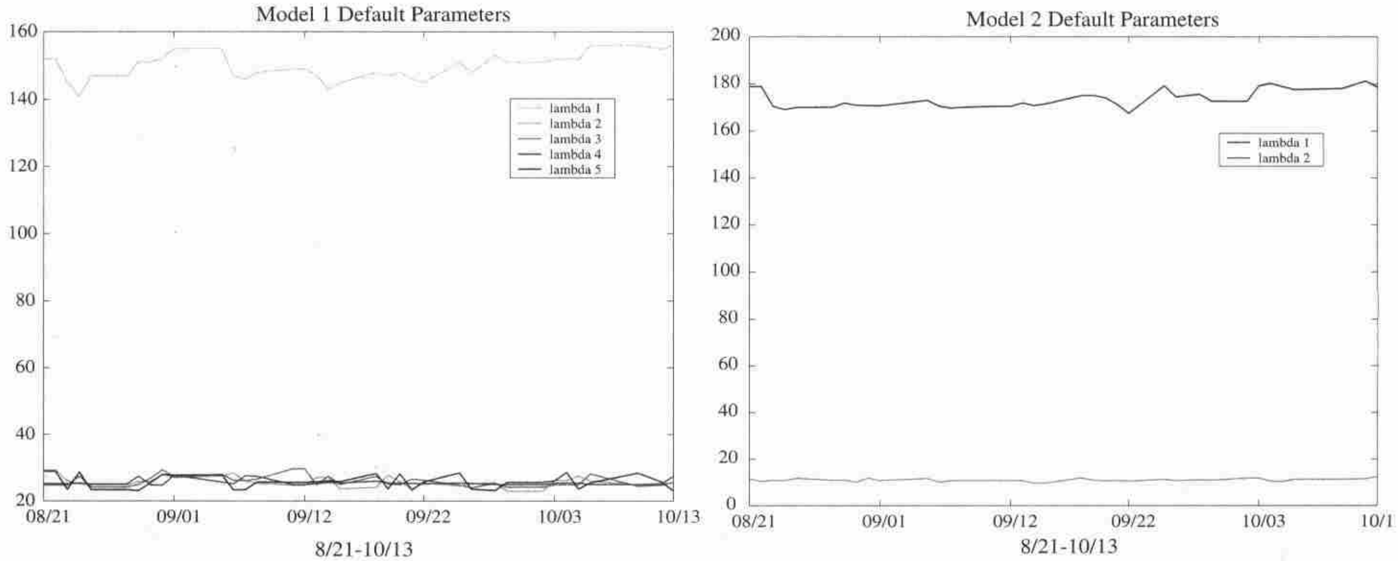
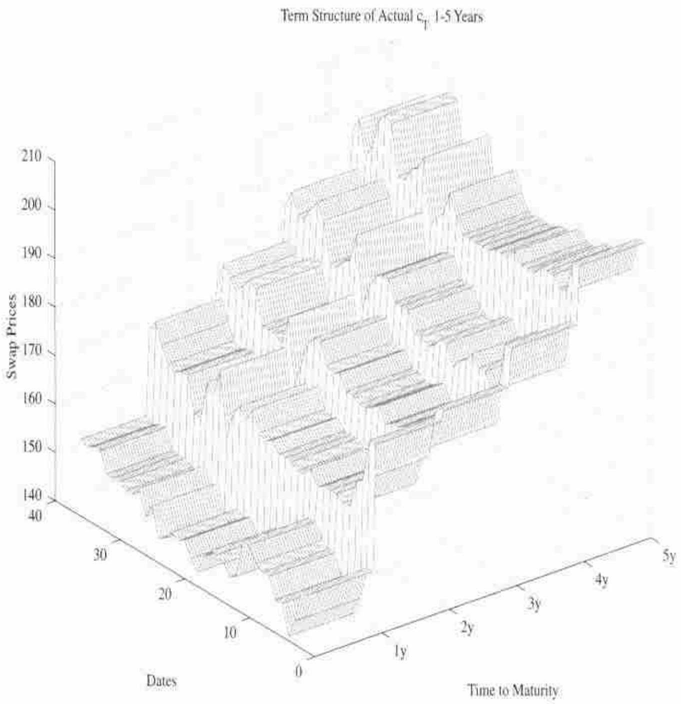


EXHIBIT 5
American Airlines Default Swap Quotes and Model 2 Default Swap Rates



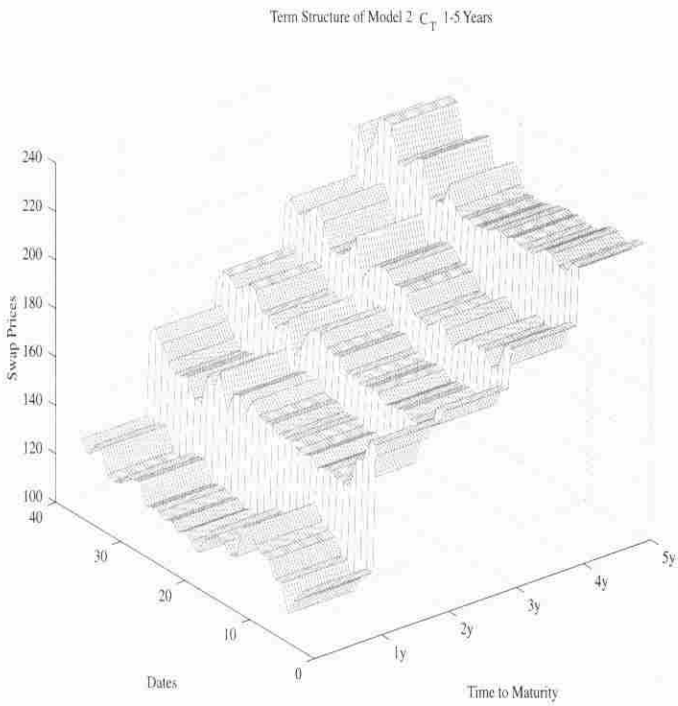


Exhibit 3 provides the average default intensity parameters for Model 2 over the sample period, measured in basis points. As noted, for all firms λ_1 is positive, indicating that as interest rates rise, the likelihood of default increases as well. The sign of the interest rate coefficient in these intensity functions is consistent with simple economic intuition.

Exhibit 4 graphs the parameter estimates for American Airlines for both Models 1 and 2 over the time period. Exhibit 5 shows the actual swap rate quotes for American Airlines over the period and the computed swap rate curves for Model 2. The simple two-parameter model is able to match the shape of the default swap term structure reasonably well. This is interesting, given that it is only a two-parameter model.

Exhibit 6 presents the average pricing errors and the standard errors for Model 2. Note that the errors tend to be negative at the short end (years 1 and 2) of the default swap term structure and positive at the long end (years 4 and 5). The best matching occurs for the three-year quote. The average pricing errors are −13.93, −7.75, 3.24, 5.83, and 12.61.

Although there is pricing error present in the estimated parameters in Exhibit 3, it is important to note that these errors can be eliminated completely. Indeed, by calibrating the intercept function of the intensity $\lambda_0(t)$ as in the standard model, one can exactly match the default swap term structure using a model with correlated market and credit risk.

VII. SUMMARY

We have provided a simple analytic formula for valuing default swaps with correlated market and credit risk. We illustrate its implementation by deducing the default probability parameters implicit in default swap quotes for 22 companies over a ten-week period. This simple analytic formula can be calibrated to exactly match the observed default swap term structure.

EXHIBIT 6
Model 2 Errors

	AvgErr c_1	AvgErr c_2	AvgErr c_3	AvgErr c_4	AvgErr c_5
American Airlines Inc.	−24.6944	−7.3333	0.6389	9.5556	21.8333
	3.1905	2.6952	1.3622	1.7463	1.9744
Archer-Daniels-Midland Co.	−10.5833	−4.8333	0.4444	4.5000	10.4722
	3.4367	3.4311	3.6392	3.6144	4.5645
CP & L Energy Inc.	−9.0389	−2.9278	0.1833	3.7389	8.0444
	1.6684	1.5684	1.5332	1.4003	1.9943
Chase Manhattan Corp.	−6.3889	−1.3889	0.5000	2.2222	5.0556
	1.7071	0.8879	1.0749	0.7646	1.3366
Coca-Cola Enterprises	−7.4222	−2.3111	0.8556	2.3278	6.5500
	2.0033	1.3027	2.0821	1.6938	1.9581
Delta Air Lines Inc.	−15.3944	−6.8667	0.8833	5.6333	15.7444
	6.0568	2.2111	1.8721	3.0971	4.3759
Dow Chemical Company	−11.2500	−5.4167	0.8611	5.9167	9.8889
	4.9149	4.1537	2.7036	2.8619	6.6045
Eastman Kodak Co.	−4.8444	−2.4000	0.0167	2.1278	5.1000
	2.8574	1.5314	0.8423	1.3758	2.4858
First Union Corp.	−5.7778	−1.9444	0.1111	2.4167	5.1944
	1.1786	1.1044	0.9555	0.8133	1.4307
J.P. Morgan & Company	−5.7000	−2.7833	0.7167	2.3556	5.4111
	1.4948	1.4653	1.0303	0.8157	1.6227
K Mart Corp.	−73.7944	−58.3500	30.5667	43.3722	58.2056
	39.6238	37.2448	24.0359	28.8007	31.0385
Lyondell Chemical Company	−16.2167	−16.1889	1.6722	8.8389	21.8944
	19.8447	7.1202	4.9350	5.5078	6.0216
Merrill Lynch & Co.	−10.7278	−7.7000	2.3278	5.8833	10.2167
	2.8506	2.5704	2.0564	1.5583	1.7776
Phillips Petroleum Co.	−12.0611	−3.5333	2.0778	4.6056	8.9111
	1.7077	1.7709	1.5472	2.2309	1.7737
Ralston-Ralston Purina	−11.0667	−2.8444	1.4056	3.9056	8.6000
	2.5067	1.8640	1.7660	1.7563	1.7882
Sears, Roebuck and Co.	−8.9056	−4.7667	1.9833	4.1222	7.5667
	4.9591	4.1491	3.3563	2.9039	2.8451
Southwest Airlines	−6.9444	−1.2778	0.9722	2.7222	4.5278
	2.2094	0.7518	1.2367	1.1782	1.3808
TXU Corporation	−11.7222	−6.0556	0.0556	7.9444	9.7778
	12.9790	8.4247	9.9607	7.1668	10.9853
Union Oil Co. of California	−10.9833	−3.0111	1.4611	3.9611	8.5722
	2.3472	1.9201	1.7500	1.4965	1.6823
Wal-Mart Stores Inc.	−3.6278	−0.8500	0.1500	1.0111	3.3167
	1.7239	0.8314	0.9941	0.8280	0.8911
Xerox Corp.	−31.3278	−23.0778	21.9778	−2.7722	35.2000
	29.0264	29.8426	43.4229	32.5142	44.3323
Texas Instruments Inc.	−7.9778	−4.5333	1.4111	3.8278	7.2722
	3.6949	4.0469	2.3890	2.4271	2.9460
Averages	−13.9295	−7.7452	3.2396	5.8280	12.6071
	6.9083	5.4949	5.2066	4.8433	6.1732

Average standard errors are under each point estimate.

APPENDIX
Proofs and Derivations

Facts about the Spot Rate *r(t)*

From Equation (7), we define

$$X_0 \equiv r(s) = f(t,s) + b(t,s)^2/2 + \int_t^s \rho(v,s)dW(v)$$

where $\rho(v, s) = \sigma_r e^{-a(s-v)}$

and

$$b(t,s) = \int_t^s \rho(t,v)dv = \begin{cases} \sigma_r(1-e^{-a(s-t)})/a & for \quad a \neq 0 \\ \sigma_r(s-t) & for \quad a = 0 \end{cases}$$

A direct computation gives:

$$\mu_0(t,s) \equiv E_t(r(s)) = f(t,s) + b(t,s)^2/2 \qquad \text{(A-1)}$$

and

$$\sigma_0^2(t,s) \equiv Var_t(r(s)) = \int_t^s \rho(v,s)^2 dv \qquad \text{(A-2)}$$

Define

$$X_1 \equiv \int_t^T r(s)ds = \int_t^T f(t,s)ds + \int_t^T b(t,s)^2 ds/2 + \int_t^T\int_t^s \rho(v,s)dW(v)ds$$

Changing the order of integration and a direct computation yields

$$\int_t^T b(t,s)^2 ds/2 = \int_t^T b(v,T)^2 dv/2$$

and

$$\int_t^T\int_t^s \rho(v,s)dW(v)ds = \int_t^T b(v,T)dW(v)$$

Substitution gives

$$\int_t^T r(s)ds = \int_t^T f(t,s)ds + \int_t^T b(v,T)^2 dv/2 + \int_t^T b(v,T)dW(v)$$

A direct computation gives

$$\mu_1(t,T) \equiv E_t(\int_t^T r(s)ds) = \int_t^T f(t,s)ds + \int_t^T b(v,T)^2 dv/2 \qquad \text{(A-3)}$$

and

$$\sigma_1^2(t,T) \equiv Var_t(\int_t^T r(s)ds) = \int_t^T b(v,T)^2 dv \qquad \text{(A-4)}$$

Also:

$$\sigma_{01}(t,s) \equiv E_t(r(s)\int_t^s r(u)du) = \int_t^s \rho(v,s)b(v,s)dv = b(t,s)^2/2 \qquad \text{(A-5)}$$

Finally, from Heath, Jarrow, and Morton [1992, p. 88]:

$$p(s,T) = p(t,T)e^{\int_t^s r(u)du - (1/2)\int_t^s b(u,T)^2 du - \int_t^s b(u,T)dW(u)} \qquad \text{(A-6)}$$

It can be shown that

$$cov_t(\int_t^T r(u)du, \int_t^s r(u)du) = \sigma_1^2(t,s) \quad for \quad T > s \qquad \text{(A-7)}$$

$$cov_t(\int_t^s b(u,T)dW(u), \int_t^s r(u)du) = \int_t^s b(u,T)b(u,s)du \qquad \text{(A-8)}$$

Facts about Normal Distributions

Given that (x, y) is bivariate-normal, we have (see Hogg and Craig [1970, p. 114]):

$$E_t\left[e^{Ax+By}\right] = e^{\mu_x A+\mu_y B+\left[\sigma_x^2 A^2+2\sigma_{xy}AB+\sigma_y^2 B^2\right]/2} \qquad \text{(A-9)}$$

where

$$\mu_x \equiv E_t(x), \mu_y \equiv E_t(y), \sigma_x^2 \equiv Var_t(x), \sigma_y^2 \equiv Var_t(y)$$

and $\sigma_{xy} \equiv Cov_t(x, y)$.

Also:

$$\frac{\partial E_t\left[e^{Ax+By}\right]}{\partial A} = E_t\left[xe^{Ax+By}\right] = e^{\mu_x A+\mu_y B+\left[\sigma_x^2 A^2+2\sigma_{xy}AB+\sigma_y^2 B^2\right]/2}[\mu_x + \sigma_x^2 A + \sigma_{xy}B] \qquad \text{(A-10)}$$

Proof of General Valuation Formulas

Let $E_t(\cdot)$ denote the conditional expectation with respect to the information set generated by $\{X_S, N_S\}$ for $s \leq t$ where $\{X_S\}$ is a vector stochastic process representing the state variables. Let $E_t\{\cdot| X_T\}$ denote the conditional expectation with respect to the information set generated by $\{X_S, N_S\}$ for $s \leq t$ and conditioned on the information set $\{X_S\}$ for $s \leq T$ as well.

Given the information set $\{X_S\}$ for all $s \leq T$, a Cox process N_t behaves like a Poisson process with intensity $\lambda(t, X_t)$. Thus:

$$Q_t(\tau > T|X_T) = E_t(1_{\{\tau>T\}}|X_T) = e^{-\int_t^T \lambda(s,X_s)ds} \qquad \text{(A-11)}$$

And:

$$Q_t(\tau > T) = E_t(E_t(1_{\{\tau>T\}}|X_T)) = E_t(e^{-\int_t^T \lambda(s,X_s)ds}) \qquad \text{(A-12)}$$

Hence, the probability of default prior to time T is:

$$Q_t(\tau \leq T) = 1 - Q_t(\tau > T) = 1 - E_t(e^{-\int_t^T \lambda(s,X_s)ds}) \qquad \text{(A-13)}$$

The conditional density function is:

$$Q_t(\tau \in (s, s+ds]|X_T) = dQ_t(\tau \leq s|X_T)/ds = \lambda(s, X_s)e^{-\int_t^s \lambda(u,X_u)du} \qquad \text{(A-14)}$$

Derivation of Equation (4)

Using the facts about Cox processes, we have that:

$$\begin{aligned}
E_t(Y_T 1_{\{\tau>T\}} e^{-\int_t^T r(u)du}) &= E_t(E_t(Y_T 1_{\{\tau>T\}} e^{-\int_t^T r(u)du}|X_T)) \\
&= E_t(Y_T e^{-\int_t^T r(u)du} E_t(1_{\{\tau>T\}}|X_T)) \\
&= E_t(Y_T e^{-\int_t^T r(u)du}[Q_t(\tau > T|X_T) + 0Q_t(t < \tau \leq T|X_T)] \\
&= E_t(Y_T e^{-\int_t^T r(u)du} e^{-\int_t^T \lambda(u)du})
\end{aligned}$$

Derivation of Equation (5)

Using the facts about Cox processes, we have that:

$$\begin{aligned}
E_t(\int_t^T y_s 1_{\{\tau>s\}} e^{-\int_t^s r(u)du} ds) &= E_t(E_t(\int_t^T y_s 1_{\{\tau>s\}} e^{-\int_t^s r(u)du} ds|X_T)) \\
&= E_t(\int_t^T y_s e^{-\int_t^s r(u)du} E_t(1_{\{\tau>s\}}|X_T)ds) \\
&= E_t(\int_t^T y_s e^{-\int_t^s r(u)du}[Q_t(\tau > s|X_T) + 0Q_t(t < \tau \leq s|X_T)]ds) \\
&= E_t(\int_t^T y_s e^{-\int_t^s r(u)du} e^{-\int_t^s \lambda(u)du} ds)
\end{aligned}$$

Derivation of Equation (6)

Using the facts about Cox processes, we have that:

$$\begin{aligned}
E_t(1_{\{t<\tau\leq T\}} \Psi_\tau e^{-\int_t^\tau r(u)du}) &= E_t(E_t(\int_t^T 1_{\{\tau=s\}} \Psi_s e^{-\int_t^s r(u)du}|X_T)ds) \\
&= E_t(\int_t^T \Psi_s e^{-\int_t^s r(u)du} E_t(1_{\{\tau=s\}}|X_T)ds) \\
&= E_t(\int_t^T \Psi_s e^{-\int_t^s r(u)du} Q_t(\tau \in (s, s+ds]|X_T)ds) \\
&= E_t(\int_t^T \Psi_s e^{-\int_t^s r(u)du} \lambda(s)e^{-\int_t^s \lambda(u)du} ds)
\end{aligned}$$

Derivation of Equation (15)

Substitution of the linear intensity into:

$$\begin{aligned}
I_t(T)1_{\{t<\tau\}} &= E_t\left(1_{\{t<\tau\leq T\}} \cdot e^{-\int_t^\tau r(u)du}\right) \\
&= E_t\left(\int_t^T \lambda(s)e^{-\int_t^s [r(u)+\lambda(u)]du} ds\right)
\end{aligned}$$

gives:

$$I_t(T)1_{\{t<\tau\}} = E_t\left(\int_t^T [\lambda_0(s) + \lambda_1 r(s)]e^{-\int_t^s [r(u)+\lambda_0(s)+\lambda_1 r(u)]du} ds\right)$$

Algebra and interchanging the order of integration and expectation gives:

$$I_t(T)1_{\{t<\tau\}} = \left(\int_t^T \lambda_0(s)E_t(e^{-\int_t^s [r(u)+\lambda_0(s)+\lambda_1 r(u)]du})ds\right) +$$

$$\left(\lambda_1 \int_t^T E_t(r(s)e^{-\int_t^s \lambda_0(u)du-(1+\lambda_1)\int_t^s r(u)du})ds\right)$$

Using Equation (14) for $v(t, s: 0)$, we get

$$\left(\int_t^T \lambda_0(s)v(t,s:\delta=0)ds\right) +$$

$$\left(\lambda_1 \int_t^T e^{-\int_t^s \lambda_0(u)du} E_t(r(s)e^{-(1+\lambda_1)\int_t^s r(u)du})ds\right)$$

Using Equation (A-9) with

$$x \equiv r(s) = X_0, \ y \equiv \int_t r(u)du$$

$$= X_1, \ \mathrm{A} \equiv 0, \text{ and } B \equiv -(1+\lambda_1)$$

and expressions (A-1)-(A-5) gives:

$$= \int_t^T \lambda_0(s)v(t,s:0)ds +$$

$$\lambda_1 \int_t^T e^{-\int_t^s \lambda_0(u)du-(1+\lambda_1)\mu_1(t,s)+[1+2\lambda_1+\lambda_1^2]\sigma_1^2(t,s)/2}$$

$$[\mu_0(t,s)-(1+\lambda_1)\sigma_{01}(t,s)]ds$$

Using Equation (14) gives the result.

ENDNOTES

[1]See Jarrow and Turnbull [1995]. No-arbitrage guarantees the existence but not the uniqueness of the probability Q. Without any additional hypotheses on the economy, the uniqueness of Q is equivalent to markets being complete. See Battig and Jarrow [1999].

[2]The random variables Y_T and y_t and the stochastic process Ψ_t are measurable with respect to the information set generated by $\{X_S: 0 \leq s < T\}$.

[3]In particular:

$$\bar{r}(t) = f(0,t) + \left(\partial f(0,t)/\partial t + \sigma_r^2\left(1-e^{-2at}\right)\Big/ 2a\right)\Big/ a \ \text{ for a } \neq 0$$

[4]Martin, Thompson, and Browne [2001] investigate a procedure for fitting a smoothed intensity function to market data. Although this smoothing procedure could also be used here, we do not focus on this aspect of the estimation.

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