

Risk in Dynamic Arbitrage: The Price Effects of Convergence Trading

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ABSTRACT

I develop an equilibrium model of convergence trading and its impact on asset prices. Arbitrageurs optimally decide how to allocate their limited capital over time. Their activity reduces price discrepancies, but their activity also generates losses with positive probability, even if the trading opportunity is fundamentally riskless. Moreover, prices of identical assets can diverge even if the constraints faced by arbitrageurs are not binding. Occasionally, total losses are large, making arbitrageurs' returns negatively skewed, consistent with the empirical evidence. The model also predicts comovement of arbitrageurs' expected returns and market liquidity.

MANY HEDGE FUNDS AND OTHER FINANCIAL institutions attempt to exploit the relative mispricing of assets. However, from time to time, the prices of the assets diverge, forcing these institutions (whom I will refer to as convergence traders or arbitrageurs) to unwind some of their positions at spectacular losses. The near-collapse of the Long-Term Capital Management (LTCM) hedge fund in 1998 is frequently cited as an example of this phenomenon.¹ To what extent can these losses be attributed to the actions of arbitrageurs as opposed to unforeseen shocks? Why do other institutions with liquid capital not eliminate the abnormal returns around these events? In this paper, I develop a theoretical model to address these questions. I show that such losses can occur in the absence of any shock. I also show that even if the constraints faced by

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¹For detailed analysis of the LTCM crisis, see Edwards (1999), Loewenstein (2000), and MacKenzie (2003). For similar episodes on derivative markets see Mitchell, Pedersen, and Pulvino (2007), Berndt et al. (2004), Gabaix, Krishnamurthy, and Vigneron (2007), and Froot and O'Connell (1999). See also Coval and Stafford (2007), Greenwood (2005), and Bollen and Whaley (2004) for related phenomena in equity markets and option markets.

arbitrageurs are not binding, it still might be in their best interest not to leap into the fray and trade away an arbitrage opportunity. Indeed, in the equilibrium of the model, an arbitrage opportunity remains, because it might get better tomorrow.

Specifically, I present an analytically tractable equilibrium model of convergence trading. I consider the problem of arbitrageurs facing a dynamic arbitrage opportunity. They can take opposite positions on two assets with identical cash flows but temporarily different prices. Each of the two assets is traded in a local market. Initially, there is a gap between the prices of the assets because local traders' demand curves differ. In each instance, however the difference across local markets disappears with positive probability. I label this interval of asymmetric local demand a *window* of arbitrage opportunity. The arbitrage is fundamentally riskless, because, in the absence of arbitrageurs, the gap remains constant until the random time at which it disappears. Arbitrageurs have limited capital, and to take a position they have to be able to collateralize their potential losses. If their trades did not affect prices, the development of the gap would provide a one-sided bet, as prices could only converge. However, by trading, they endogenously determine the size of the gap for as long as the window remains open. If the aggregate position of arbitrageurs decreases, prices diverge.

The main observations of the paper are that prices can diverge even if the constraint that arbitrageurs face is not binding, and arbitrageurs can suffer losses in the absence of any shock. The idea is that as arbitrageurs optimally allocate their capital across time, the expected payoff to an arbitrage trade at any point in time must be the same. This implies that there are two types of dynamic equilibria. Corresponding to the textbook concept of arbitrage-free markets, the gap can be zero at each time instant so the expected payoff of a trade is always zero. More interestingly, it is also possible that the gap is positive at each point in time so long as the asymmetry in demand curves is present. As arbitrageurs must be indifferent to investing a unit of capital early or investing it later, the expected payoff of investing at any point in time must be the same. This is possible only if prices diverge further with positive probability at each point in time. Thus, if arbitrageurs decide to save some capital for later, it increases the chance that they will miss out on the current arbitrage opportunity, but the potential gain is also larger because the gap might increase further. Therefore, even if the gap is large, arbitrageurs will save some capital for later, since prices might diverge further. I also show that the unique equilibrium that is robust to the introduction of arbitrarily small trading costs belongs to this second group.

The model makes predictions about how the intensity of arbitrageurs' competition—as reflected in their aggregate level of capital—affects the characteristics of the arbitrage opportunity. I show that competition reduces the profitability of the arbitrage opportunity in a particular way. Not only does the expected level of the gap become smaller, but the half-life of the gap gets longer. As more arbitrageurs enter the market, the expected size of the future gap diminishes at a slower rate. This relationship is monotone as long as the

level of arbitrage capital is not sufficient to fully integrate the two markets. Intuitively, greater competition among arbitrageurs must lower the available profit in the market. If just the level of the gap decreased, arbitrageurs could still leverage up and the expected profit would remain large. Thus, in equilibrium, potential price divergence also has to increase, as this tightens the collateralizing constraint and reduces arbitrageurs' expected profit. As a consequence, the arbitrage opportunity is transformed into a speculative bet, where the probability-weighted gains exceed the probability-weighted losses less and less as competition increases among arbitrageurs.

The model makes several empirical predictions. First, it predicts that if market liquidity is measured by the price effect of a transitory demand shock (in the spirit of Pastor and Stambaugh (2003) and Acharya and Pedersen (2005)), then the average hedge fund will suffer the largest losses when market liquidity drops the most. In this sense, the average hedge fund should have positive exposure to liquidity shocks, that is, a positive liquidity beta. Second, the model predicts a left-skewed distribution of hedge funds' returns, which is consistent with the empirical observations. By the logic of the equilibrium, a large price divergence and the correspondingly large hedge fund losses will arise in small probability states. In large probability states, the average hedge fund realizes a positive return. Third, the model implies that the half-life of the Sharpe ratio (the normalized version of the modeled gap) is monotonically related to the popularity of a given arbitrage trade, that is, the aggregate capital of hedge funds specializing in a given trade. Thus, as the competition among hedge funds drives the expected return of different strategies to the same level, the half-life of the Sharpe ratio following a liquidity shock should also be driven to the same level across various markets.

The model presented here is closely related to theoretical models of endogenous liquidity provision (e.g., Grossman and Miller (1988) and Huang and Wang (2006)). However, these models concentrate on the decision of liquidity providers, who must pay an exogenous cost to enter the market. Furthermore, these models focus on the provision of liquidity in a single period. By contrast, in my setup arbitrageurs have to decide *when* to provide liquidity. The driving force of the mechanism in my model is that the equilibrium dynamics of the gap endogenously imply the opportunity cost of providing a unit of capital at a given point in time. If the gap increases, arbitrageurs investing early not only lose their invested unit of capital, they also miss a better opportunity to invest later on.

The model belongs to the literature on the general equilibrium analysis of risky arbitrage (e.g., Gromb and Vayanos (2002), Zigrand (2004), Xiong (2001), Kyle and Xiong (2001), Basak and Croitoru (2000)). A large part of this literature focuses on potential losses in convergence trading. The common element in these models is that they describe mechanisms that force arbitrageurs to liquidate part of their positions after an initial adverse shock in prices, which creates further adverse price movements and further liquidations. In contrast, my mechanism is not based on the amplification of an exogenous shock. Arbitrageurs require higher returns in longer windows independent of the

magnitude of their past losses. The higher returns and the corresponding divergence in prices reflect the higher marginal value of liquid capital as future opportunities are getting more attractive.

The partial equilibrium model of Liu and Longstaff (2004) also illustrates how arbitrageurs with capital constraints might suffer significant losses and might not invest fully in the arbitrage opportunity. However, in their model this outcome is the result of an exogenously defined price gap process, while my focus is on the determination of the price process.

This paper proceeds as follows. Section I presents the structure of the model. Section II derives the unique robust equilibrium. Section III discusses the results, and Section IV analyzes the robustness of the equilibrium. Section V further discusses the related literature. Finally, Section VI concludes.

I. A Simple Model of Risky Arbitrage

Two assets have identical cash flows and are traded in separate markets. A unit mass of risk-neutral arbitrageurs can trade in both markets. The representative arbitrageur shorts $x(t)$ shares of the expensive asset and buys $x(t)$ units of the cheap asset. Since my focus is on arbitrageurs, I simply specify that local traders provide a static demand curve for the price difference or gap, $g(t)$, across the two markets. That is,

$$g(t) = f(\bar{x}(t)), \quad (1)$$

in the random time interval $[0, \bar{t}]$, where $\bar{x}(t)$ is the aggregate activity of arbitrageurs in each of these two markets in $t \in [0, \bar{t}]$. The inverse demand function, $f(\cdot)$, is continuous and monotonically decreasing in $\bar{x}(t)$. In the interval $[0, \bar{t}]$, demand curves in the two separate markets differ, so $g^* \equiv f(0) > 0$. A finite long-short position, $\bar{x}^{\max} \equiv f^{-1}(0)$, would eliminate the price difference. At $\bar{t}$ the difference in local demand curves disappears, so the inverse demand curve for the gap, $f(\cdot)$, collapses to $f^0(\cdot)$ with $f^0(0) = 0$. Time $\bar{t}$ is distributed exponentially with a constant hazard rate of δ , so $\bar{t} \leq t$ with probability $e^{-\delta t}$. I use the term “window of arbitrage opportunity” to refer to the interval $[0, \bar{t}]$, and say that the window is open in $t \in [0, \bar{t}]$ and closes at $\bar{t}$.

The representative arbitrageur starts her activity with $v(0) = v_0$ capital. After time 0, arbitrageurs do not get additional capital, and no new arbitrageurs arrive in the market. Arbitrageurs are required to have positive mark-to-market capital at all times. Given a path for the price gap, $g(t)$, they solve the problem

$$J(v(0)) = \max_{x(t)} \int_0^\infty \delta e^{-\delta t} (g(t)x(t) + v(t)) dt, \quad (2)$$

$$\begin{aligned} s.t. \quad v(t) &= v_0 - \int_{g(0)}^{g(t)} x(u) dg(u) \\ 0 &\leq v(t). \end{aligned} \quad (3)$$

The maximand shows that if the window closes at t , the arbitrageur gains $g(t)x(t)$ profit on her current holdings, and she realizes her cumulated profit or loss, $v(t)$. The final payoff in this event is weighted by the corresponding value of the density function, $\delta e^{-\delta t}$. The first constraint shows the dynamics of the arbitrageur's capital level. At each point in time, the capital level is adjusted for the current gains or losses, $x(t)dg(t)$. Inequality (3) is the capital constraint. It shows that arbitrageurs are not allowed to take positions that could make them go bankrupt in any state of the world, given that their liabilities are marked to market. I guess, and later verify, that in equilibrium $g(t)$ is continuous and continuously differentiable in t .

Note that if arbitrageurs did not intervene, the gap would be constant and positive at g^* until the random time $\tilde{t}$ and would collapse to zero thereafter. Prices could only converge, providing a one-sided bet to the first arbitrageur with access to both local markets. In this sense, arbitrageurs face an arbitrage opportunity that is *fundamentally* riskless. However, if arbitrageurs' aggregate position, $\bar{x}(t)$, decreases in equilibrium, the gap will increase, leading to capital losses for arbitrageurs. Thus, the arbitrage opportunity might become risky in equilibrium.

As problem (2)–(3) shows, the focus of the analysis is the price effect of arbitrageurs' activity when they face an arbitrage opportunity of random length. The structure of the opportunity captures the intuition that the prices of similar assets traded by different groups of traders can temporarily differ if arbitrageurs do not eliminate the price gap. The exact source of the arbitrage opportunity is immaterial for the purpose of the model. It can be an asymmetric shock to local traders' risk aversion or their income stream or any other type of demand shock. We can also think of the window as the dynamic version of a liquidity event as introduced in Grossman and Miller (1988), which results from the asynchronous arrival of traders with matching demand.

In the next section, I focus on the symmetric equilibria of the model in which each arbitrageur follows the aggregate strategy and individual capital dynamics follow the dynamics of aggregate capital, that is, $x(t) = \bar{x}(t)$, $v(t) = \bar{v}(t)$, and $v_0 = \bar{v}_0$. I discuss asymmetric equilibria in Section IV. In all symmetric equilibria, the equilibrium position, $\bar{x}(t)$, must solve (2)–(3) given the gap, $g(t)$. The equilibrium also requires that the market clears, that is, that the demand curve, equation (1), is also satisfied. I discuss the role and significance of the model's assumptions in Section IV.

II. The Robust Equilibrium

In this section I present the unique robust equilibrium of the model. I proceed in two parts. First, I show that, in general, the model has a continuum of equilibria. Second, I select the unique robust equilibrium by introducing a perturbation of diminishingly small trading costs. I discuss the implications of this robust equilibrium in Section III.

I solve for the equilibria in four steps. I provide the intuition behind these steps here and relegate the details to the Appendix. First, the maximum

principle (see the Appendix for details) implies the first-order condition

$$\delta \frac{g(t)}{\frac{dg(t)}{dt}} = J'(v(t)), \quad (4)$$

together with the envelope condition

$$\frac{dJ'(v(t))}{dt} = J'(v(t))\delta - \delta, \quad (5)$$

where $J'(v(t))$ is the marginal value function differentiated in $v(t)$. The two conditions reflect the main concern of arbitrageurs: how to allocate their limited capital across time. At each point in time, they have to decide what proportion of their capital to commit to the arbitrage now and what proportion to save for later. The danger of saving capital for later is that the window might close today, in which case the arbitrageur would miss out on the opportunity. The danger of committing the capital today is that if the window gets wider, the arbitrageur will have less capital available for investing when doing so is more profitable. Similar to any other Euler equation, if (4) holds, the optimizing agent is indifferent to investing a unit today or saving it for later. Because of the risk-neutrality of arbitrageurs, this condition is independent of the quantities invested.

Second, the general solutions of the linear differential equations (4) and (5) are given by

$$J'(v(t)) = 1 + \frac{g_0}{g_\infty - g_0} e^{\delta t} \quad (6)$$

$$g(t) = \frac{g_\infty g_0}{g_\infty e^{-\delta t} + g_0(1 - e^{-\delta t})}, \quad (7)$$

where g_0 and g_∞ are the (yet undefined) starting point and limit of the $g(t)$ path, respectively. Thus, to make arbitrageurs indifferent to how they will allocate their capital across time, the gap path must be determined by (7) with a given g_0 and g_∞ .

Third, the market-clearing condition, (1), determines the path of aggregate positions $\bar{x}(t)$ for any $g(t)$ path given by (7).

The last step is to pick a g_∞ and find a corresponding g_0 so that the implied aggregate positions $\bar{x}(t)$ and the conditional gap path $g(t)$ are consistent with the capital constraint (3) for each t . Toward this end, observe that if the capital of the representative arbitrageur, $v(t)$, is zero for any given t' , then $x(t) = 0$ and $v(t) = 0$ for all $t > t'$, because arbitrageurs need capital to take positions. Thus, if a finite t' existed for which all arbitrageurs' capital were zero, the gap, $g(t)$, would be constant at g^* for all $t > t'$, providing a riskless arbitrage opportunity for anyone who would rather save a unit of capital until t' . However, this is inconsistent with the requirement that arbitrageurs must be indifferent as to when to invest. (This is why there is no g_0 and g_∞ , which would imply $g(t) = g^*$ by equation (7) for all $t > t'$.) Consequently, in a symmetric equilibrium the

budget constraint is not binding for any finite t , that is, $\bar{v}(t) > 0$ for all finite t . On the other hand, if the expected return on capital at time 0 is positive, that is, if

$$J'(v(0)) = \frac{g_\infty}{g_\infty - g_0} > 0, \quad (8)$$

then the capital constraint must bind in the limit as otherwise arbitrageurs would be motivated to increase their positions at some point in time. This implies that in the symmetric equilibrium,

$$\lim_{t \rightarrow \infty} \bar{v}(t) = \bar{v}_0 - \int_{g_0}^{g_\infty} \bar{x}(t) dg(t) = 0. \quad (9)$$

Any pair of g_0 and g_∞ that solves (9) with $g(t)$ given by (7) and $\bar{x}(t)$ given by (1) determines an equilibrium.

These four steps pin down all equilibria. All equilibria but one have very similar properties. The only equilibrium that is different is the *efficient market equilibrium*, in which the gap is constant at the zero level, that is, $g(t) = 0$ for all t . This equilibrium is implied by the choice of $g_\infty = 0$. Thus, from (6), in this equilibrium the expected return on capital, $J'(v(t))$, is zero at any point in time, so arbitrageurs are indifferent between different timing strategies. Further, at each point in time the position of the representative arbitrageur is $\bar{x}(t) = \bar{x}^{\max}$. Because arbitrageurs do not have to face potential losses, in this case no capital is needed to collateralize their positions. Arbitrageurs could increase their positions as $\lim_{t \rightarrow \infty} \bar{v}(t) = \bar{v}_0 > 0$, but there is no point in doing so.

In each of the other equilibria, the gap path converges to a positive $g_\infty \in (0, g^*]$, starting at a positive level $g_0 < g_\infty$, so the expected return on capital, $J'(v(t))$, is positive at any t . The level of capital of any of the arbitrageurs, $\bar{v}(t)$, is positive at any finite time t . However, increasing the investment level at some time intervals without decreasing it in others is not possible, because the capital constraint is binding in the limit, that is, (9) holds. The indifference condition ensures that all arbitrageurs weakly prefer the equilibrium strategy over cutting back positions at some points in time in order to increase positions at others. Consistent with (7), in each of these equilibria the conditional gap path, $g(t)$, is monotonically increasing.

The efficient market equilibrium is the only equilibrium if and only if $\bar{v}_0$ is sufficiently large. Intuitively, if the aggregate level of arbitrageurs' capital is sufficiently large, they can integrate the two local markets and the law of one price applies. Otherwise, both types of equilibria exist. The next proposition summarizes the properties of all symmetric equilibria.

PROPOSITION 1: *There is a critical value $\bar{v}^{\max}$, such that for any $\bar{v}_0 \in (0, \bar{v}^{\max})$, the model has a continuum of symmetric equilibria characterized by the conditional gap path, $g(t)$, and the conditional investment path, $\bar{x}(t)$. In any of these equilibria, either $g(t) = 0$ and $\bar{x}(t) = \bar{x}^{\max}$ for all t , or $g(t)$ is monotonically increasing,*

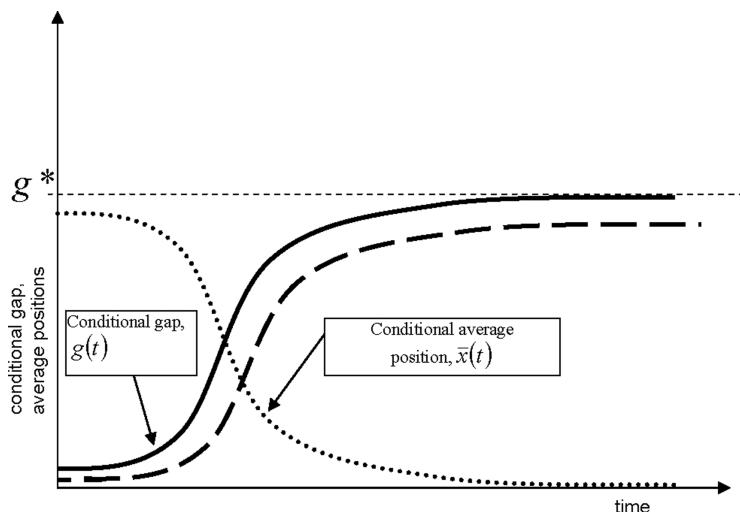


Figure 1. The conditional gap path and the conditional trading position in equilibrium. Increasing curves show the qualitative features of the conditional gap paths, $g(t)$, in two possible equilibria. The solid curve represents the equilibrium in which $\lim_{t \rightarrow \infty} g(t) = g^*$, while the dashed curve shows another equilibrium in which $\lim_{t \rightarrow \infty} g(t) = g_\infty < g^*$. The decreasing curve represents the conditional average position path, $\bar{x}(t)$, corresponding to the first conditional gap path.

$\bar{x}(t)$ is monotonically decreasing, and $g(t)$ is given in the form of

$$g(t) = \frac{g_\infty g_0}{g_\infty e^{-\delta t} + g_0(1 - e^{-\delta t})},$$

where $g_\infty \in (0, g^*]$ and the capital constraint is not binding for any finite t , that is, $\bar{v}(t) > 0$, but it is binding in the limit $\lim_{t \rightarrow \infty} \bar{v}(t) = 0$.

Proof: The proof is in the Appendix.

Figure 1 shows the qualitative properties of two of the equilibrium gap paths together with the corresponding paths of the conditional positions, $\bar{x}(t)$. Remember that the path $g(t)$ shows the conditional size of the gap given that $t \in [0, \bar{t}]$. In reality, only the beginning of the paths will be observed. For example, if the window closed at $\bar{t} = 1$, we would observe the increasing path from $g(0)$ to $g(1)$ and then the gap would jump back to zero at time 1. Hence, the increasing pattern implies that as long as the window survives, the gap must increase and each arbitrageur must suffer losses of $\bar{x}(t)dg(t) > 0$. These losses are consistent with the fact that $\bar{v}(t) > 0$ for any finite t , that is, arbitrageurs' capital constraint is not binding in any finite t , because the increasing pattern of $g(t)$ is not a consequence of the liquidation induced by past losses. Rather, it is a consequence of the required indifference along the path. The gap has to increase to provide sufficiently high returns to those who wait. The larger returns implied for these arbitrageurs compensates them for the risk of missing out on the opportunity to invest today.

It might seem counterintuitive that no equilibria exist where the gap remains constant at a positive level. Because arbitrageurs have limited capital, they might be expected to invest all of their capital in the arbitrage, which would push the gap down to a positive level. Arbitrageurs would then hold the same position until the window closed. This would keep the gap at this positive level as long as the window were open. The reason this does not happen lies in the endogenous nature of the capital constraint. Arbitrageurs are not constrained in terms of the size of their positions but in terms of their capital levels, which they use only to collateralize their potential losses. If there are no potential losses, they are not constrained at all. So if the gap were constant and positive in a given interval, arbitrageurs could always invest more at the beginning of the interval. This would push the level of the gap down at the beginning of the interval, in line with an increasing conditional gap path.

Although the efficient market equilibrium seems intuitive, this—like every other equilibrium but one—does not survive a simple equilibrium selection criterion. For the rest of this section only, I introduce a small perturbation to the model. I assume that there is a small positive cost, m , associated with short-selling the gap. If the arbitrageur holds a position $x(t)$ between t and $t + dt$, she pays $mx(t)dt$ as a trading cost. This is the carry cost of the position. I assume that $m < \delta g^*$. Otherwise, arbitrageurs would not invest at all. I also assume that $x(t)$ must be continuous in a non-zero measure set containing t for all t .

As I show in the next proposition, with a positive carry cost, there is a unique equilibrium for any initial capital, $\bar{v}_0$. In this equilibrium the conditional gap path $g(t)$ is monotonically increasing, reaches the theoretical maximum g^* in a finite point T , and remains at this level as long as the window is open. Importantly, as m diminishes, T increases without bound, and the equilibrium converges to the equilibrium described in Proposition 1, with $g_\infty = g^*$. This is why I call the equilibrium in which $g_\infty = g^*$ the *robust equilibrium* of the system.

PROPOSITION 2: *When $m > 0$, there is a unique symmetric equilibrium. In this equilibrium there is a $T > 0$, such that $g(t)$ is strictly monotonically increasing in t , and $g(t) \leq g^*$ for all $t \in [0, T]$ and $g(t) = g^*$ for all $t > T$.*

If $\bar{v}_0 \in (0, \bar{v}^{\max})$, as $m \rightarrow 0$, the equilibrium of the perturbed system with $m > 0$ converges to the equilibrium described in Proposition 1, where

$$g(t) = \frac{g^* g_0}{g^* e^{-\delta t} + g_0(1 - e^{-\delta t})}. \quad (10)$$

Proof: The proof is in the Appendix.

Intuitively, the main reason an arbitrarily small exogenous holding cost eliminates all but one equilibrium lies in the capital constraint. In all equilibria but the one in which the conditional gap path converges to g^* , arbitrageurs commit to investing at a nondiminishing level for an arbitrarily long period. This requires them to keep betting on the convergence of the price gap even if prices

have been diverging for a very long time. Even if the cost of investing is very small, sustaining such a strategy for a very long time will be extremely costly. An arbitrageur with limited capital cannot commit to such a strategy. Formally,

$$\lim_{t \rightarrow \infty} \bar{v}(t) = \lim_{t \rightarrow \infty} \left(\bar{v}_0 - \int_{g_0}^{g^*} \bar{x}(t) dg(t) - m \int_0^\infty \bar{x}(t) dt \right)$$

can converge only if $\lim_{t \rightarrow \infty} \bar{x}(t) = 0$. This implies that the conditional gap path must converge to g^* .

In the next section I discuss the implications of the robust equilibrium.

III. Main Implications

The robust equilibrium of the model illustrates the two main observations of this paper. First, prices can diverge, providing increasing abnormal returns to active arbitrageurs, even in times when none of the arbitrageurs face binding capital constraints. In the robust equilibrium, both the gap, $g(t)$, and the expected return on capital in period t , $J'(v(t))$, are increasing in t , and $\bar{v}(t) > 0$ for all t . Second, arbitrageurs can suffer losses even in the absence of any shock. In the robust equilibrium arbitrageurs lose capital at each point in time in the interval $[0, \bar{t}]$ as a result of arbitrageurs' actions only. It is apparent that if arbitrageurs did not trade, the window would be a riskless arbitrage opportunity.

The first result sheds new light on the existing evidence on slow arbitrage capital. Recently, it has been observed in several derivatives markets (e.g., Mitchell et al. (2007), Berndt et al. (2004), Gabaix et al. (2007), and Froot and O'Connell (1999))² that if an unexpected shock dislocates prices, unconstrained funds do not provide enough liquidity to eliminate the abnormal returns, that is, arbitrage capital is slow to move in, and the prices might move further away from their fundamental values. Furthermore, the suggested inefficiency seems to survive for several months after these events; that is, the half-life of the premium is surprisingly long. The model suggests that slow capital and the survival of high returns are not necessarily signs of the high cost of entry for new arbitrageurs. Even those arbitrageurs who are already present in the particular market and able to invest with no explicit cost will not invest up to their capital limit. The equilibrium possibility of a widening gap implies that arbitrage trading creates its own opportunity cost. Investing a unit today will lead to capital losses exactly at those states when investing would be the most profitable.

To highlight the implications of the second main result, I provide further intuition on how arbitrageurs' actions transform the riskless arbitrage opportunity

² Examples include the convertible arbitrage market in 2005–2006, the credit default swaps in 2002, the mortgage-backed securities market in late 1993, and the catastrophe reinsurance market in the early 1990s. See also Coval and Stafford (2007), Greenwood (2005), and Bollen and Whaley (2004) for related phenomena in equity markets and options markets. The observed violations of the law of one price are also related (see Froot and Dabora (1999) and Lamont and Thaler (2003)).

into a risky bet. First, I discuss the effect of arbitrageurs' aggregate capital on equilibrium price dynamics. I interpret this variable as a proxy for three related characteristics: the intensity of the competition of arbitrageurs in a particular market segment, the proportion of arbitrageurs that knows about the particular arbitrage opportunity, and—borrowing the term from Brunnermeier and Pedersen (2007)—the aggregate funding liquidity in the market. Second, I analyze the model's implications for the interaction of the gap asset's illiquidity and the distribution of hedge fund returns. The illiquidity of an asset is the price effect of a transitory shock of a given size. Most empirical measures of illiquidity are close to this concept (see Amihud, Mendelson, and Pedersen (2005) for a survey). As a window of arbitrage opportunity is the model's equivalent of a transitory shock, illiquidity of the gap asset is measured by $g(\tilde{t})$, the peak of the price effect of the shock. While funding liquidity, $\bar{v}_0$, is a primitive of the model, illiquidity of the gap asset and hedge fund returns are endogenously determined. I will spell out the testable implications of the model along the way.

A. Funding Liquidity and Prices

To see how the aggregate level of arbitrage capital affects the dynamics of the gap, I analyze three related measures. As a first measure, it is instructive to look at the effect of increasing competition on the expected return on capital. As (8) makes apparent, the expected return on capital, $J'(v(0))$, is independent of the individual level of capital, $v(0)$, for given prices. However, in equilibrium, it does depend on the aggregate level of capital, $\bar{v}_0$, through g_0 .

Alternatively, we can focus directly on the effect of increasing the level of arbitrage capital on the dynamics of the gap. From equation (10), the expected future value of the gap given the size of the gap in any $t \in [0, \tilde{t}]$ is

$$E(\tilde{g}(t+u) | g(t)) = e^{-\delta u} g(t+u) = g(t) \left(1 - \frac{g(t)(1 - e^{-\delta u})}{g(t)(1 - e^{-\delta u}) + g^* e^{-\delta u}} \right), \quad (11)$$

where $\tilde{g}(t)$ is the unconditional gap, which is $g(t)$ if $t \in [0, \tilde{t}]$ and zero otherwise. As $g(t) \in (0, g^*]$, the gap is always expected to decrease. Using expression (11), I define the second and third measures. One is the expected change in the gap during a given interval, u , which is

$$\|E(\tilde{g}(t+u) | g(t)) - g(t)\|.$$

The expected change in the gap is increasing in $g(t)$. The other measure is the speed at which the gap converges at a given time $t \in [0, \tilde{t}]$. The speed of convergence can be measured by the half-life of the gap, h^t , defined implicitly by

$$E(\tilde{g}(t+h^t) | g(t)) \equiv \frac{1}{2} g(t).$$

Using (11),

$$h^t(g(t)) \equiv \frac{1}{\delta} \ln \left(\frac{g^*}{g(t)} + 1 \right). \quad (12)$$

As $g(t)$ depends on $\bar{v}_0$ through g_0 , both measures also depend on $\bar{v}_0$.

These three measures—the expected return on capital, the expected change in the gap, and the speed of convergence—are closely related to one another in this model. An arbitrage opportunity provides a higher expected return if the gap converges faster, which implies that the expected change must be larger in a given interval.

The following proposition shows that if the competition is more fierce, the expected return on capital, the expected change in the gap, and the speed of convergence all decrease. Furthermore, as the level of capital increases toward its theoretical maximum, $\bar{v}^{\max}$, the gap approaches a martingale process and the half-life increases without bound. Arbitrageurs' competition transforms arbitrage opportunities into speculative bets.

PROPOSITION 3: *In the robust equilibrium:*

- (i) *for any fixed $t \in [0, \bar{t}]$ and $u > t$, the gap, the expected return on capital, and the expected change in the gap are all decreasing in the level of aggregate capital, that is,*

$$\frac{\partial g(t)}{\partial \bar{v}_0}, \frac{\partial J'(v(t))}{\partial \bar{v}_0}, \frac{\partial \|g(t) - E(\tilde{g}(t+u) | g(t))\|}{\partial \bar{v}_0} < 0,$$

- (ii) *the half-life of the gap, $h^t(g_0)$, is increasing in $\bar{v}_0$, and*
 (iii) *as $\bar{v}_0 \rightarrow \bar{v}^{\max}$, $g(t)$ approaches a martingale, that is*

$$\lim_{\bar{v}_0 \rightarrow \bar{v}^{\max}} E(\tilde{g}(t+u) | \tilde{g}(t) = g(t)) = g(t),$$

and the half-life gets arbitrarily long

$$\lim_{\bar{v}_0 \rightarrow \bar{v}^{\max}} h^t(g(t)) = \infty.$$

Proof: It is clear from the proof of Proposition 1 that $\frac{\partial g_0}{\partial \bar{v}_0} < 0$ and $\lim_{\bar{v}_0 \rightarrow \bar{v}^{\max}} g_0 = 0$. All statements of Proposition 3 are straightforward consequences of this fact and equations (11), (10), (12), and (6). Q.E.D.

Proposition 3 not only shows that the marginal return on aggregate arbitrage capital is decreasing; it also demonstrates the exact way in which the marginal return has to diminish. Although the size of the gap decreases as $\bar{v}_0$ increases, this would not be enough to reduce the arbitrageurs' expected profit. The reason is that arbitrageurs' collateralization constraint is endogenous. If potential losses did not increase, arbitrageurs could increase their leverage and their return on a unit of arbitrage capital. In equilibrium, not only do potential gains decrease as an arbitrage trade gets more popular, but potential losses rise on

account of the potential widening of the gap. This increases the half-life of the gap.

This logic implies the first testable implication of the model. At least some convergence trading opportunities must be widely known in the hedge fund industry. Thus, the allocation of the aggregate capital of those hedge funds specializing in each of these trades has to drive expected profits to the same level in the long term. In this model, this implies that at a liquidity event the Sharpe ratio (the normalized version of the gap) should be expected to shrink with the same speed: the half-life should be approximately the same in any of these episodes across markets.

B. Illiquidity of Assets and Hedge Fund Returns

To obtain a better sense of the implied distribution of hedge fund returns, let us keep the aggregate level of capital, $\bar{v}_0$, fixed and have a look at the return distribution of the representative hedge fund that invests in several subsequent windows of arbitrage opportunity. If the window closes at time $\bar{t}$, in the robust equilibrium the representative arbitrageur's realized gross return is

$$\bar{r}(\bar{t}) \equiv \frac{\bar{x}(\bar{t})g(\bar{t}) + \bar{v}(\bar{t})}{\bar{v}_0}.$$

First, note that the model implies the skewed distribution of returns that has been observed in the data. In particular, in the robust equilibrium, the distribution of the arbitrageur's total return is skewed toward the left. The reason is that $\bar{r}(\bar{t})$ is a decreasing function of the length of the window, $\bar{t}$. Intuitively, the representative arbitrageur loses capital as the window gets longer. As the arbitrageur's position must decrease as the gap increases, the arbitrageur liquidates a part of her portfolio as the gap widens and suffers losses on the liquidated units. The arbitrageur makes a net gain if the window closes instantly as $\bar{r}(0) > 1$, but if the window is long enough, most of the arbitrageur's capital will be lost. Since long windows occur with small probability, the representative arbitrageur will make a positive net return with large probability and large losses with small probability during each window.

The last testable implication of my model that is closely related to the recent empirical literature on the connection between liquidity risk and expected returns (e.g., Pastor and Stambaugh (2003), Acharya and Pedersen (2005)) is that hedge funds should have negative exposure to aggregate illiquidity shocks, that is, hedge fund returns should have a positive liquidity beta. With a fixed aggregate level of capital, the price effect of a transitory shock—the measured illiquidity of the asset—depends only on the persistence of the shock, that is, the length of the window. In longer windows, arbitrageurs reduce the level of liquidity they provide, so the price effect will be larger. However, these are also the windows that result in the largest losses for the representative arbitrageur. Thus, the representative hedge fund suffers the largest losses when illiquidity of the asset increases the most. If there is any positive correlation between

the persistence of transitory shocks across markets, then the largest losses of hedge funds will tend to correspond to the largest hikes in aggregate illiquidity, which validates the positive liquidity beta.³ The next proposition shows the two formal results discussed in this subsection.

PROPOSITION 4: *The realized return of arbitrageurs conditional on a window of length $\tilde{t}$, $\bar{r}(\tilde{t})$, is monotonically decreasing in $\tilde{t}$. Consequently,*

- (i) *the distribution of $\bar{r}(\tilde{t})$ is skewed toward the left,*
- (ii) *the covariance of the illiquidity of the gap, $g(\tilde{t})$, and arbitrageurs' return, $\bar{r}(\tilde{t})$, is positive, that is,*

$$\text{Cov}(\bar{r}(\tilde{t}), g(\tilde{t})) > 0.$$

Proof: The first statement is the direct consequence of the exponential distribution of $\tilde{t}$, and the fact that $r(\tilde{t})$ is decreasing:

$$\frac{\partial \bar{r}(\tilde{t})}{\partial \tilde{t}} = \frac{\partial (\bar{x}(\tilde{t})g(\tilde{t}) + \bar{v}(\tilde{t}))}{\partial \tilde{t}} = \frac{\partial \bar{v}(\tilde{t})}{\partial \tilde{t}} + \frac{\partial \bar{x}(\tilde{t})}{\partial \tilde{t}} g(\tilde{t}) + \frac{\partial g(\tilde{t})}{\partial \tilde{t}} \bar{x}(\tilde{t}) = \frac{\partial \bar{x}(\tilde{t})}{\partial \tilde{t}} g(\tilde{t}) < 0.$$

The second statement is implied by the fact that $g(\tilde{t})$ and $\bar{r}(\tilde{t})$ are monotonic functions of the same random variable (see McAfee (2002) for the general result). Q.E.D.

Finally, the next proposition shows the intuitive relationship between funding liquidity and the illiquidity of assets: If the aggregate capital of arbitrageurs, $\bar{v}_0$, is smaller, arbitrageurs tend to provide less capital, so the gap asset tends to be less liquid in the first-order stochastic dominance sense.

PROPOSITION 5: *Let $\Omega(g(\tilde{t}) | \bar{v}_0)$ be the conditional cumulative density of $g(\tilde{t})$ for a given level of aggregate capital of arbitrageurs, $\bar{v}_0$. Then*

$$\frac{\partial \Omega(g(\tilde{t}) | \bar{v}_0)}{\partial \bar{v}_0} > 0.$$

Proof: The statement is a simple consequence of the exponential distribution of $\tilde{t}$, the fact that $g(t)$ is increasing in t , and the result in Proposition 3 that $\frac{\partial g(t)}{\partial \bar{v}_0} < 0$. Q.E.D.

IV. Robustness

The main idea of the model is that if hedge funds with limited capital have to decide *when* to provide liquidity, in equilibrium the expected payoff of the

³ An indication that hedge funds might have a positive exposure to liquidity risk is implied by Bondarenko (2004), who shows that hedge fund returns are significantly exposed to variance risk as measured by the changes of implied volatility of traded options. It is well known that there is a strong correlation between volatility risk and liquidity risk measures (see Pastor and Stambaugh (2003)).

arbitrage trade has to be the same at every point in time. This payoff might always be zero, in which case there is no arbitrage opportunity in the market, or it can be positive, in which case price divergence must be possible at every point in time, even though the aggregate capital constraint does not bind. This implies that arbitrageurs who follow their individually optimal strategies create losses endogenously even in the absence of any shocks. Their competition reduces the predictability of relative price movements, transforming the arbitrage opportunity into a speculative bet. I expect that these results are robust to a wide range of setups, but I have to emphasize a few critical points.

The duration of the window of arbitrage opportunity is uncertain in this model and, in particular, it can be arbitrarily long. An example of a departure from this assumption is Gromb and Vayanos (2002). They assume a window with a fixed length, that is, the gap disappears after an exogenously fixed interval. They show, in contrast to my result, that the gap path typically decreases.

Another critical assumption is that arbitrageurs take both prices and the probability of convergence as given. Zigrand (2004) presents a model where there is imperfect competition among arbitrageurs, while Carlin, Lobo, and Vishwanathan (2007) study the occasional breakdown in hedge funds' cooperative behavior of providing liquidity to each other. This element of strategic interaction is missing from my framework.

It is also important to see that the main mechanism behind the results is driven by the hedging demand of arbitrageurs. In equilibrium, as the window remains open, future investment opportunities improve. Because the relative risk aversion of this model's arbitrageurs is less than one, as future investment opportunities improve they need higher compensation for investing in the gap asset.⁴ This is why prices must diverge. In contrast, the demand of the agents of Xiong (2001) and Kyle and Xiong (2001) is independent of future returns, as they assume a logarithmic utility function. Thus, even if the assumption of risk neutrality per se is not crucial for the intuition behind my results (although it simplifies the derivation substantially), it is important that arbitrageurs are not too risk averse.

The assumption that the aggregate capital level of arbitrageurs is limited is also critical. However, it might not be necessary to assume that there is absolutely no capital inflow into the market during the window of arbitrage opportunity. The effect of relaxing this assumption depends on how we think about the supply of arbitrage capital.

One view is that, as time goes by, more arbitrageurs learn about the opportunity, so more capital enters the market. This effect might be reinforced by the increasing marginal return on additional liquid capital as the window gets longer. It turns out that if the newly entering capital is a deterministic function of the price gap and its level is not too large, the equilibrium remains unchanged. To see this, observe that if $x^i(t)$ is the individual position of arbitrageur $i \in [0, 1]$

⁴ See Merton (1992) and Campbell and Viceira (2002) for further discussions on the relationship between risk aversion and hedging demand.

in time $t \in [0, \bar{t}]$, v_0^i is her initial capital, and

$$\int_0^1 x^i(t) di = \bar{x}(t), \int_0^1 v_0^i di = \bar{v}_0, v_0^i - \int_{g_0}^{g(t)} x^i(t) dg(t) \geq 0,$$

for all t , then it is an asymmetric equilibrium resulting in the same aggregate positions $\bar{x}(t)$ and the same gap path $g(t)$ as the unique symmetric robust equilibrium. The reason is that in the robust equilibrium arbitrageurs are indifferent as to when to invest. The gap, $g(t)$, and the dynamics of the aggregate level of capital, $\bar{v}(t)$, depend only on the dynamics of the aggregate positions, so as long as the aggregate positions do not change, and individual positions are consistent with individual capital constraints, individual positions can be arbitrary. For example, one can divide arbitrageurs into two groups: incumbents who have positive positions from time 0 on, and new entrants who take positive positions only if the gap is sufficiently large. Thus, there are asymmetric equilibria in which new capital flows into the market if the abnormal return is sufficiently high, but the dynamics of $g(t)$ remain the same.

Another argument is related to the agency view of the arbitrage sector. Hedge funds get their capital from investors who delegate their portfolio decisions in the hope that hedge funds know of and have access to better opportunities. However, these investors do not have exact information on the abilities and opportunities of hedge funds. Hence, investors use arbitrageurs' past performance as a signal about their abilities. If this effect is strong, there might even be a capital outflow from the market when the gap increases, as this is when arbitrageurs lose money.⁵ This would strengthen the dynamics described in the paper.

A related point is that I do not consider the possibility that arbitrageurs, after taking positions at the beginning of the window, could reveal their information to others to attract new entrants and induce faster convergence. This possibility seems to be unrealistic for two reasons. First, market participants might not give credit to such announcements, because even convergence traders with no information on the speed of convergence are motivated to announce that they are sure of a fast convergence. A good example is the effect of the fax sent by John Merriwether, the founding partner of the Long-Term Capital Management hedge fund, just before the fund's near-collapse:

... the opportunity set in these trades at this time is believed to be among the best that LTCM has ever seen. But, as we have seen, good convergence trades can diverge further. In August, many of them diverged at a speed and to an extent that had not been seen before. LTCM thus believes that it is prudent and opportunistic to increase the level of the Fund's capital

⁵ Alternatively, it is possible that arbitrageurs themselves learn about the probability of convergence during the window. As the window gets longer, arbitrageurs might conclude that the probability that the prices will converge is smaller than they thought. This would result in similar withdrawal of capital and even larger endogenous losses.

to take full advantage of this unusually attractive environment. (Quoted by MacKenzie (2003), p. 365)

The fax was sent to the fund's investors to invite new investment, but it rapidly leaked out to the public. Given that LTCM had just suffered huge losses, investors reacted by pulling capital out of the fund, and other hedge funds reportedly began to trade against LTCM. Second, hedge funds might be reluctant to share information about a newly discovered trading opportunity if there is a chance that future liquidity shocks will provide recurrent opportunities in the same market. Even if information disclosure increased their profit in the current window, more arbitrageurs exploring the given opportunity would reduce the profitability of the trade in future windows.

V. Related Literature

It is interesting to contrast the presence of endogenous losses with the intuition of other models of limits to arbitrage—for example, Shleifer and Vishny (1997), Xiong (2001), and Gromb and Vayanos (2002). The common element of all these models is that they present mechanisms that amplify exogenous shocks: Arbitrageurs lose capital because some initial loss makes them liquidate part of their positions, which widens the gap and results in further losses.⁶ I present a very different mechanism. In my model, the only exogenous shock is the opening of a window of arbitrage opportunity. Arbitrageurs initially reduce the impact of this shock. Prices might diverge later, even though there are no further shocks and the representative arbitrageur's capital constraint does not bind in equilibrium. The reason behind the price divergence is not the effect of past losses. In fact, for the equilibrium prices, at any point in time arbitrageurs are indifferent to how to allocate their capital in the future regardless of their past losses. The possible price divergence and the corresponding higher expected returns reflect the higher marginal value of liquid capital as future opportunities are becoming more attractive.

The idea that financially constrained agents require a premium for sacrificing future investment possibilities has been pointed out before in different contexts. The idea gained popularity in the literature on corporate risk management (see Froot, Scharfstein, and Stein (1993) and Holmström and Tirole (2001)) and in investment theory (see Dixit and Pindyck (1994)). All of the work in these fields emphasizes that assets should be more valuable if they provide free cash flow in states when investment possibilities are better. This point is also made in Gromb and Vayanos (2002) in relation to the fact that arbitrageurs might not invest fully in the arbitrage, expecting better opportunities in the future.

⁶ Apart from the literature on limited arbitrage, there is also a related literature that concentrates on endogenous risk as a result of amplification due to financial constraints (see Danielsson and Shin (2002), Danielsson et al. (2004), He and Krishnamurthy (2007), Morris and Shin (2004), Bernardo and Welch (2004)). In a related paper, Brunnermeier and Pedersen (2005) show that predatory trading of nondistressed traders can also amplify exogenous liquidity shocks.

Regarding this point, the novelty of my paper is the exploration of this effect on the dynamics of prices.

My focus on the timing of arbitrage trades connects my work to Abreu and Brunnermeier (2002, 2003). They analyze a model in which the development of the gap between a price of an asset and its fundamental value is exogenously given, and informational asymmetries cause a coordination problem in strategies over the optimal time to exit the market. In contrast, in my model information is symmetric, arbitrageurs want to be in the market when others are not (i.e., there is strategic substitution instead of complementarity), and my focus is on the endogenous determination of the price gap.

Finally, Duffie, Garleanu, and Pedersen (2002) also focus on the endogenous dynamics of an arbitrage opportunity that disappears in a random amount of time. However, their mechanism is very different. They consider arbitrageurs with differences in opinion who can trade with each other only occasionally. The equilibrium price deviates from the frictionless case because buyers of the asset take into account the expected lending fees they can collect from potential future shorters. The dynamics of prices are driven by the endogenous change of lending fees and expected short interest.

VI. Conclusion

Why capital does not materialize and eliminate price discrepancies is one of the outstanding puzzles in finance. I present a model that shows arbitrageurs might not eliminate the abnormal returns corresponding to diverging prices even if they do not face binding constraints. Indeed, in the unique robust equilibrium, prices can diverge even if all arbitrageurs have positive levels of capital at all finite future points in time. The arbitrage opportunity remains, because arbitrage opportunities get better in the future. The equilibrium also illustrates that arbitrageurs can suffer large losses even in the absence of unforeseen shocks. In the unique robust equilibrium, arbitrageurs lose capital with positive probability as a result of their individually optimal strategies only.

I introduce a novel, analytically tractable equilibrium model of dynamic arbitrage. In my model, arbitrage opportunities arise because of temporary pressure on the local demand curves of two very similar assets that are traded in different markets. The temporary demand pressure is present for an uncertain, arbitrarily long time span but disappears in finite time with probability one. Risk-neutral arbitrageurs can take positions in both local markets, and they have to decide how to allocate their limited capital across uncertain future arbitrage opportunities. This allocation—together with the uncertain duration of the local demand pressure—determines the future distribution of the price gap between the two assets. Hence, the individually optimal intertemporal allocation of capital and the distribution of future arbitrage opportunities are determined simultaneously in equilibrium.

The simplicity of the framework presented provides potential for its application to a wide variety of problems related to limited arbitrage. In future work

I plan to consider the applicability of this model to a multi-asset setup for the purpose of analyzing contagion across markets and the effects of flight-to-quality and flight-to-liquidity in times of market depression. I believe that this framework can be used to shed more light on these issues.

Appendix

Proof of Proposition 1: The maximum principle implies that if $x^*(t)$ is a solution of

$$J(v(t)) = \max_{x(t)} \int_0^\infty e^{-\delta t} U(x(t), v(t)) dt$$

$$s.t.$$

$$\frac{dv(t)}{dt} = G(c(t), k(t)),$$

then there exists $\lambda(t)$ such that

$$\frac{dU(x^*, k)}{dx} = -\lambda \frac{dG(x^*, v)}{dv}$$

$$\frac{d\lambda}{dt} = \lambda\delta - \left(\frac{dU(x^*, v)}{dv} + \lambda \frac{dG(x^*, v)}{dv} \right)$$

$$\frac{dJ(v(t))}{dv(t)} = \lambda.$$

Problem (2)–(3) and equations (4) and (5) are the corresponding expressions with substitution,

$$\delta(x(t)g(t) + v(t)) = U(x(t), v(t))$$

$$x(t) \frac{dg(t)}{dt} = G(x(t), v(t)).$$

See Obstfeld (1992) for the necessary regularity conditions on $U(\cdot)$ and $G(\cdot)$ and an economist-friendly presentation of the proof.

Let us check whether corner solutions of problem (2)–(3) are possible. In a corner solution, there is a time T when the aggregate budget constraint binds and arbitrageurs cannot take positions. However, if $\bar{v}(T) = 0$, $\bar{v}(T + \tau) = 0$ for all $\tau \geq 0$, because arbitrageurs cannot realize gains without taking positions. Hence, if such a point in time existed, $\bar{x}(T + \tau) = 0$ and $g(T + \tau) = g^*$ for $\tau \geq 0$ as in autarchy. But observe that the expected marginal profit in autarchy is infinity because arbitrageurs' positions are not constrained when $\frac{dg(t)}{dt} = 0$. This is a contradiction, as it would imply that arbitrageurs would not save capital for investing at a time providing infinite marginal profit. Thus, we have only interior solutions.

By standard methods, the general solution of the nonhomogenous linear differential equation of (5) is

$$J'(v(t)) = 1 + c_1 e^{\delta t},$$

where c_1 is a constant. Plugging in the general solution of (5) into (4) gives the homogenous linear differential equation

$$\frac{dg(t)}{g(t)} = \frac{\delta}{1 + c_1 e^{\delta t}},$$

with the general solution of

$$g(t) = c_2 \frac{1}{1 + c_1 e^{\delta t}} e^{\delta t}, \quad (\text{A1})$$

where c_2 is a constant. Thus, $g_\infty \equiv \lim_{t \rightarrow \infty} g(t) = \frac{c_2}{c_1}$ and $g_0 \equiv g(0) = \frac{c_2}{1+c_1}$, which gives (6) and (7).

There is evidently an equilibrium if $g(t) = 0$ for all t . This satisfies all the conditions with $g_\infty = g_0 = 0$, and the budget constraint in this case is

$$\int_{g_0}^{g_\infty} \bar{x}(t) dg(t) = 0 \leq \bar{v}_0. \quad (\text{A2})$$

To find other solutions, note first that from the construction of the system, $g(t)$ and $\frac{dg(t)}{dt}$ must have the same sign. Otherwise, the gap can move only in one direction and arbitrageurs have a safe profit opportunity. This would make them increase their positions without bound, so $g(t)$ would reach zero, as $\bar{x}^{\max}$ is finite. Thus, from

$$\frac{dg(t)}{dt} = \frac{g_\infty \delta g_0 e^{-t\delta} (g_\infty - g_0)}{(g_0(1 - e^{-\delta t}) + g_\infty e^{-\delta t})^2},$$

either $0 < g_0 < g_\infty$ and $\frac{dg(t)}{dt} > 0$ or $0 > g_0 > g_\infty$ and $\frac{dg(t)}{dt} < 0$. Let us suppose that there exists an $\bar{x}(t)$, which supports the path in the latter case. Because of the monotonicity of $f(\bar{x}(t))$, $\bar{x}(t) > 0$ for all t , and it is bounded from above as $\lim_{t \rightarrow \infty} \frac{dg(t)}{dt} = 0$. Thus, arbitrageurs would lose $g(t)$ units when the window closes and gain $\frac{dg(t)}{dt} < 0$ when the window remains open. Thus, as $t \rightarrow \infty$ their gains diminish, but their losses do not, so they cannot be indifferent as to when to invest. Analogous arguments rule out the possibility of $g_\infty > g^*$, where the aggregate positions are negative from the point when $g(t) > g^*$. Thus, the only possibility is that $g_\infty \in (0, g^*]$.

From (6) the expected profit is positive at each point t if the window is still open. Thus, the capital constraint has to be strict in the limit

$$\int_{g_0}^{g_\infty} \bar{x}(t) dg(t) = \bar{v}_0. \quad (\text{A3})$$

The last step is to find a g_0 as a function of $\bar{v}_0$ for a given g_∞ to satisfy the budget constraint (A3). The path $g(t)$ will give the path $x(t)$ by the market-clearing condition. First note that for a given g_∞ , if $g(t)$ is the path with $g(0) = g_0$ and $\{g^+(t)\}_{t \geq 0}$ is the path with a $g(0) = g_0^+$, where $g_0 < g_0^+$, then the two paths

differ only in their starting points, that is, there is a u such that $g^+(t) = g(t + u)$ for all t . To see this, observe that as $g(t)$ is continuous and monotonically increasing between g_0 and g_∞ , there must be a u such that $g(u) = g_0^+$. But then

$$\begin{aligned} g^+(t) &= \frac{g_\infty g_0^+}{g_\infty e^{-\delta t} + g_0^+(1 - e^{-\delta t})} = \frac{g_\infty \frac{g_\infty g_0}{g_\infty e^{-\delta u} + g_0(1 - e^{-\delta u})}}{g_\infty e^{-\delta t} + \frac{g_\infty g_0}{g_\infty e^{-\delta u} + g_0(1 - e^{-\delta u})}(1 - e^{-\delta t})} \\ &= g_0 \frac{g_\infty}{e^{-\delta(t+u)} g_\infty + g_0(1 - e^{-\delta(t+u)})} = g(u + t). \end{aligned}$$

This implies that

$$\int_0^\infty \bar{x}(t) dg(t) = \int_0^u \bar{x}(t) dg(t) + \int_0^\infty \bar{x}(t) dg^+(t) > \int_0^\infty \bar{x}(t) dg^+(t).$$

Thus, $\int_0^\infty \bar{x}(t) dg(t)$ is monotonically decreasing in g_0 for any fixed g_∞ . Consequently, if

$$\bar{v}^{\max}(g_\infty) = \lim_{g_0 \rightarrow 0} \int_0^\infty \bar{x}(t) dg(t),$$

for any $\bar{v}_0 \in (0, \bar{v}^{\max}(g_\infty))$ there exists a single g_0 such that

$$\int_0^\infty \bar{x}(t) dg(t) = \bar{v}_0.$$

In the main text, I use the notation of $\bar{v}^{\max} \equiv \bar{v}^{\max}(g^*)$. Q.E.D.

Proof of Proposition 2: Interior solutions of the dynamic system with $m > 0$ are given by

$$\dot{J}'(v(t)) = J'(v(t))\delta - \delta \quad (\text{A4})$$

$$\delta g(t) = J'(v(t)) \left(\frac{dg(t)}{dt} + m \right). \quad (\text{A5})$$

The general solution is

$$J'(v(t)) = 1 + c_1 e^{\delta t} \quad (\text{A6})$$

$$g(t) = \frac{m \left(\frac{1}{\delta} - e^{\delta t} c_1 t \right) + e^{\delta t} c_2}{1 + c_1 e^{\delta t}}. \quad (\text{A7})$$

It is clear that the budget constraint can only converge if $\lim_{t \rightarrow \infty} \bar{x}(t) = 0$, so $\lim_{t \rightarrow \infty} g(t) = g^*$. As $\lim_{t \rightarrow \infty} \frac{m(\frac{1}{\delta} - e^{\delta t} c_1 t) + e^{\delta t} c_2}{1 + c_1 e^{\delta t}}$ is not convergent, there cannot be an interior solution for all t . Hence, with a large enough T , arbitrageurs will lose all their capital and we must have a corner solution of $\bar{x}(T + \tau) = 0$ and

$g(T + \tau) = g^*$ for $\tau \geq 0$. As this implies that $J'(v(t)) = J'(v(T))$ for any $t > T$ and $\delta > 0$, arbitrageurs would choose to invest all of their capital at time T if they had any. From the capital constraint, one unit of capital collateralizes a position of the size $\frac{1}{\Delta m}$ for a Δ interval. Thus,

$$J'(v(T)) = \lim_{\Delta \rightarrow 0} (1 - e^{-\Delta \delta}) \left(\frac{1}{\Delta m} (g^* - \Delta m) + 1 \right) = \frac{\delta g^*}{m},$$

where I use the assumption that $x(t)$ must be continuous in a nonzero measure interval containing t . Then, (A6) implies $c_1 = \frac{\delta g^* - m}{e^{\delta T} m}$. From $g(T) = g^*$,

$$c_2 = e^{-\delta T} \left(g^* \delta \frac{g^*}{m} - m \frac{1}{\delta} + (\delta g^* - m)T \right).$$

Consequently, for all $0 \leq t < T$,

$$g(t) = \frac{m \frac{1}{\delta} (1 - e^{-\delta(T-t)}) + e^{-\delta(T-t)} \left(g^* \delta \frac{g^*}{m} + (\delta g^* - m)(T - t) \right)}{1 + \frac{\delta g^* - m}{m} e^{-\delta(T-t)}}.$$

Observe that

$$\frac{\partial g(t)}{\partial t} = \frac{(\delta g - m)e^{-\delta(T-t)} \frac{(\delta g - m)(1 - e^{-\delta(T-t)}) + \delta m(T - t)}{m}}{\left(1 + \frac{\delta g - m}{m} e^{-\delta(T-t)} \right)^2} > 0.$$

Similar to the proof of Proposition 1, we know that there is $x(t) \in [0, \bar{x}^{\max}]$, which satisfies the market-clearing condition

$$g(t) = f(\bar{x}(t))$$

for all t . The last step is to find t . From the definition of T , we know that

$$\int_0^T \bar{x}(t) \left(\frac{dg(t)}{dt} + m \right) dt = \bar{v}_0 \quad (\text{A8})$$

must hold. Since $g(t)$ depends on time only through $T - t$, the left-hand side of (A8) is monotonically increasing in T . As

$$\lim_{T \rightarrow 0} \int_0^T \bar{x}(t) \left(\frac{dg(t)}{dt} + m \right) dt = 0$$

and

$$\lim_{T \rightarrow \infty} \int_0^T \bar{x}(t) \left(\frac{dg(t)}{dt} + m \right) dt = \infty,$$

for any $\bar{v}_0 > 0$, there must be a unique T that satisfies (A8).

For the second part of the proposition, recall that the robust equilibrium is given by the general solution of the differential equations (4)–(5), (A1), and two boundary conditions. The first one is $\lim_{t \rightarrow \infty} g(t) = g^*$, which gives $c_2 = g^* c_1$. The second is the budget constraint (A3). In the proof of Proposition 1, I define g_0 in terms of c_1 and use (A3) to pin down g_0 . An equivalent step is to pin down c_1 by (A3). Observe also that the unique equilibrium when $m > 0$ can be given by the following steps. We know that for all $t \leq T$ the general solution is

$$g(t) = \frac{m \left(\frac{1}{\delta} - e^{\delta t} c_1^m t \right) + e^{\delta t} c_2^m}{1 + c_1^m e^{\delta t}}, \quad (\text{A9})$$

where I add subscript m to the constants c_1^m and c_2^m to distinguish them from the corresponding constants in the robust equilibrium. Instead of expressing c_1^m from the boundary condition

$$J'(v(T)) = 1 + c_1^m e^{\delta T} = g^* \frac{\delta}{m},$$

we can equivalently express T as

$$T = \frac{1}{\delta} \ln \frac{\delta g^* - m}{m c_1^m}.$$

Substituting T into (A9), the boundary condition $g(T) = g^*$ gives

$$c_2^m = c_1 \left(\frac{\delta (g^*)^2}{(\delta g^* - m)} + m \frac{1}{\delta} \ln \frac{\delta g^* - m}{m c_1^m} \right). \quad (\text{A10})$$

Finally, c_1^m is pinned down by the modified budget constraint after substituting (A10) into (A9):

$$\int_0^{\frac{1}{\delta} \ln \frac{\delta g - m}{m c_1^m}} x(t) \left(\frac{d g(t)}{d t} + m \right) = \bar{v}_0. \quad (\text{A11})$$

Observe that as $m \rightarrow 0$, the general solution (A7) converges to (A1), $\lim_{m \rightarrow 0} c_2^m = c_1^m g^*$, and $\lim_{m \rightarrow 0} T = \infty$; thus, the left-hand side of (A11) converges to the left-hand side of (A3). Consequently, $\lim_{m \rightarrow 0} c_1^m = c_1$ and the equilibrium with $m > 0$ must converge to the robust equilibrium. Q.E.D.

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