THE JOURNAL OF FINANCE • VOL. LXIV, NO. 1 • FEBRUARY 2009

High-Water Marks: High Risk Appetites? Convex Compensation, Long Horizons, and Portfolio Choice

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ABSTRACT

We study the portfolio choice of hedge fund managers who are compensated by high-water mark contracts. We find that even risk-neutral managers do not place unbounded weights on risky assets, despite option-like contracts. Instead, they place a constant fraction of funds in a mean-variance efficient portfolio and the rest in the riskless asset, acting as would constant relative risk aversion (CRRA) investors. This result is a direct consequence of the in(de)finite horizon of the contract. We show that the risk-seeking incentives of option-like contracts rely on combining finite horizons and convex compensation schemes rather than on convexity alone.

A POPULAR INTUITION IN FINANCIAL ECONOMICS is that contracts with convex payoffs can potentially increase the incentive to take risks, that is, can induce risk-shifting.[1] According to this commonly held view, if a manager is rewarded for gains but not punished for losses, that manager will only gain from increasing underlying volatility.

In this paper, we ask the question: How are the incentives of the manager affected if the horizon of the compensation contract is infinite or indefinite?

Our interest in this question is both practical and theoretical. At a practical level, we want to understand the incentives provided by certain contracts called high-water mark contracts, which have proliferated with the growth of the hedge fund industry. As we show, such contracts can be thought of as a perpetually renewed call option on the value of the fund, and so, they present a particularly interesting type of option contract with an in(de)finite horizon. At the theoretical level, we want to understand whether the in(de)finite horizon of the contract will mitigate risk-shifting incentives.

Under the typical high-water mark contract, hedge fund managers receive a fraction of the increase in fund value in excess of the last recorded maximum—

*Panageas is from the Wharton School at the University of Pennsylvania and Westerfield is from the Marshall School at the University of Southern California. We are indebted to Domenico Cuoco for a very helpful discussion on a previous draft. We would also like to thank the associate editor; two anonymous referees; Andy Abel, Franklin Allen, Harry DeAngelo, Glenn Ellison, David Musto, Nick Souleles, and Anna Pavlova; and seminar participants at the MIT Theory Lunch, the Wharton Finance faculty lunch, and the Marshall School Center for Investment Studies Seminar for useful discussion and comments. Jianfeng Yu provided exceptional research assistance.

[1] This intuition has been challenged by Carpenter (2000) and Ross (2004). We discuss their works and their relation to ours in the literature review section of the introduction.

the high-water mark—provided that such an increase took place. Such performance fees are typical in the hedge fund industry, and they can lead to spectacular compensation for the hedge fund managers when the fund is successful.[2] They also seem to avoid the problem of biasing portfolios toward replicating an index, as is the case of relative performance evaluation.[3]

The high-water mark contract resembles an option in that the manager is paid when fund value increases above a given level. In keeping with option terminology, we refer to this level as the "strike price." When the fund gains value, the manager is paid and the strike price increases, but when the fund loses money, the strike price remains unchanged and the manager retains her implied option at the old strike price. The high-water mark contract should not be thought of as one option but as a sequence of options with a changing strike price. When the fund declines in value, the value of all the implied future options declines as well. Importantly, most hedge fund management contracts do not have a pre-specified termination date.

At a first pass, high-water mark contracts seem to be subject to standard risk-shifting problems. Because the manager can determine the fund's volatility—and increasing underlying asset volatility increases the value of an option—one might think that a manager would take an unboundedly large position in risky assets. We show that this need not be the case and provide examples to the contrary. To illustrate our point, we compute the optimal portfolio of a risk-neutral hedge fund manager who maximizes the expected discounted value of compensation fees. Surprisingly, we demonstrate that her portfolio coincides with the portfolio that would have been chosen by a constant relative risk aversion (CRRA) investor.

The intuition for our results is straightforward. A bolder portfolio today could help increase the probability of crossing the current high-water mark, but it will also increase the probability that the assets in the fund will be substantially lower next period, while the high-water mark will remain unchanged. In the latter case, the value of future options will decline because the assets in the fund will be lower for the same value of the high-water mark. Hence, future options will become more "out of the money." A hedge fund manager sees a trade-off between current and future payoffs, and she is led to the choice of an interior portfolio. Interestingly, this portfolio places a constant fraction of wealth in the risky asset and the rest in a mean-variance efficient portfolio of stocks. Hence,

[2] These fees can range from 15% to 50% of the recent increase and can earn billions of dollars. There have been several academic articles on the characteristics of hedge fund fees, such as Fung and Hsieh (1999) and Ang, Rhodes-Kropf, and Zhao (2008), as well as several articles in the *Wall Street Journal*, such as Zuckerman (2004). An example (reported in Goetzmann, Ingersoll, and Ross (2003)) is the Quantum Fund, which receives a high-water mark fee of 20% and a regular annual fee of 1%. For a (pre-fee) performance of 49% in 1995, this resulted in an estimated compensation of $393 million. In contrast, when the fund lost 1.5% in 1996 and did not reach its high-water mark, the compensation was just 1% on the net asset value or $54 million.

[3] For an analysis of the incentives to bias the portfolio in the direction of a particular index and take on additional risk, see also Chevalier and Ellison (1997), Carpenter (2000), Cuoco and Kaniel (2007), Basak, Shapiro and Tepla (2006), and Basak, Pavlova, and Shapiro (2007).

the risk-neutral hedge fund manager behaves exactly as a Merton-type investor with a specific CRRA coefficient that we determine explicitly.

Our analysis supports two broad conclusions. First, the commonly given risk-shifting intuition does not rely only on the implicit option contract per se, but on *the finiteness of the horizon* as well. In our model the manager is risk neutral, and yet she ends up choosing a bounded portfolio despite the option-like character of her compensation. The only disciplining force in our model is the presence of a continuation value function on an infinite horizon.

Second, absolute performance evaluation[4] seems to induce portfolio managers to invest in mean variance efficient portfolios as opposed to portfolios that contain a bias towards a specific index. This result has been confirmed empirically by studies such as Fung and Hsieh (1997). Somewhat surprisingly, the fractions invested in the risky and the riskless assets are constant over time in much the same way that they would be for an investor with CRRA.

We make several assumptions so as to sharpen our results. While they are not completely realistic, they serve either to make our results more surprising or to preserve comparability with the existing literature. First, we assume the manager is risk neutral and we place no restrictions on the manager's portfolio choice. This increases her incentives to risk-shift while giving her the ability to take extremely leveraged positions. We use these assumptions to illustrate circumstances in which one would think that incentives should have the most "bite," leading the manager to take unboundedly large risks.

Second, the fund has an effectively infinite horizon. It is liquidated either at a random time or when its value drops to zero, whichever comes first. This assumption is a key departure from most of the existing literature on portfolio choice by professional fund managers. A common thread behind all of our results is the disciplining effect of the infinite horizon.[5]

Third, and in line with the preexisting literature, we completely restrict the manager's ability to trade on her own account.[6] At a practical level this means that the manager's consumption stream and her compensation fees coincide and the manager is maximizing the (subjective) expected discounted sum of future incentive fees rather than their implied market value.

This paper is related to a number of strands in the literature. The paper closest to our study is Goetzmann, Ingersoll, and Ross (2003) (henceforth, GIR).

[4] We use the term absolute performance evaluation to mean that outcomes are not compared to an index. This contrasts with relative performance evaluation in which compensation is determined by evaluating performance relative to an index.

[5] Liquidating the fund when wealth drops to zero (and not at a higher threshold) is not realistic. However, endogenous termination—perhaps after several periods of bad performance—would provide its own disciplining effect. This assumption also provides a useful comparison to Goetzmann, Ingersoll, and Ross (2003, p. 1708), who demonstrate that in this case, a manager who is trying to maximize the implicit market value of fees would not try to maximize volatility.

[6] The usual choice in the literature is to assume that the compensation is paid out at some time T and the manager derives utility $U(W_T)$, where W_T is her compensation and U is some concave function. However, it is implicitly assumed that the manager cannot trade on a private account using these fees as collateral. For further discussion on this issue see Section III.A.

They estimate the implied market value of hedge fund management fees in a setting where the fund's value follows a simple log-normal process and the fund managers have no discretion over the choice of portfolio. However, they don't explicitly model the effects of hedge fund compensation schemes on portfolio choice. In their framework, maximizing the market value of the fund manager's compensation would lead to unbounded volatility, absent other constraints. Our analysis supports and extends their result. While a manager would indeed place an unbounded weight on risky assets if she tried to maximize the implied market value of fees, we show that maximization of expected discounted compensation can lead to different outcomes. The key difference is that in our paper the continuation value is sufficiently high to discipline the manager away from risk-shifting.

The basic intuition that convex compensation schemes will increase risk seeking has been challenged by both Carpenter (2000) and Ross (2004), who address this problem in finite horizon settings. In both cases, a risk-averse manager may not be made more risk seeking by an option-like contract. However, it appears that in both papers a risk-neutral manager would choose unbounded volatility. Our setup is different because we have an infinite horizon setting. In particular, we show that even a risk-neutral manager would not seek unbounded risks. This helps us demonstrate the new intuition produced by this paper: What mitigates risk seeking is the disciplining force of the continuation value on an infinite horizon.

This paper is also related to the literature on incentive fees for fund managers.[7] In this literature, it is typically assumed that managers receive management fees at the end of some finite period. In most of these papers there is also some form of benchmarking. By contrast, we examine high-water mark incentive schemes on an infinite horizon. Our results are different because in our model there is no explicit terminal time. Moreover, the "benchmarking" is not exogenous, but depends on the past performance of the fund.

On a methodological level, this paper shares many similarities to Browne (1997) on the maximization of the expected discounted reward from attaining a goal. In fact, the problem that we consider in this paper and the problem considered by Browne (1997) are identical in the region in which the high-water mark is not increasing. However, in our setup the reward from attaining the goal is not exogenous but endogenous, and it needs to be determined as part of the solution. We also use insights from Heinricher and Stockbridge (1991) to provide a new verification theorem that is appropriate for problems of the kind considered in this paper.

The structure of the paper is as follows: Section I presents the model. Section II presents a heuristic way of arriving at the solution, an exact proof, and a discussion of the results. Section III contains an extension to the risk-averse case, and Section IV contains an extension to multiple assets and a drift in the

[7] Recent papers in this literature include Carpenter (2000), Cuoco and Kaniel (2007), Basak, Shapiro, and Tepla (2006), Basak, Pavlova, and Shapiro (2007), Hodder and Jackwerth (2008), among many others.

high-water mark. Section V concludes. All proofs are contained in Appendix A and Appendix B.

I. The Model

We begin with a model in which the manager is risk neutral, there is one risky asset, there is no intermediate withdrawal of funds by investors, and the high-water mark is adjusted only when the value of the fund exceeds the previously recorded high-water mark. A number of possible extensions are considered in later sections.

A. Investment Opportunities

The hedge fund manager has two investment opportunities. The first opportunity is the money market, where she receives a fixed interest rate of $r > 0$. Formally, the price of an asset invested in the money market evolves according to

$$\frac{dP_{0,t}}{P_{0,t}} = r\,dt.$$

We place no limits on the positions that can be taken in the money market. In addition, the fund manager can invest in a risky security with a price per share that evolves as a geometric Brownian motion,

$$\frac{dP_{1,t}}{P_{1,t}} = \mu\,dt + \sigma\,dB_t,$$

where $\mu > r$ and $\sigma > 0$ are constants and B_t is a standard one-dimensional Brownian motion.[8]

We make no statement about the source of the hedge fund manager's investment opportunities. In particular, they could reflect the prevailing risk premia or simply the manager's subjective assessment of expected returns. We study only how the manager exploits investment opportunities, not how and why these investment opportunities exist.

B. Portfolio and Wealth Processes

The portfolio process π_t is the fraction of fund wealth (W_t) invested in the risky asset at time t. The remainder of fund wealth, $W_t(1 - \pi_t)$, is invested in the riskless asset. We allow short selling and both borrowing and lending at the riskless rate.[9]

[8] The process B_t is a one-dimensional Brownian motion on a complete probability space (Ω, F, P). We denote by $F = F_t$ the P-augmentation of the filtration generated by B_t.

[9] To have a well-defined problem, we also require that for $\theta_t = \pi_t W_t$, it is the case that

$$\int_0^T \theta_t^2\,dt < \infty \text{ for all } T > 0.$$

An important modification of the standard dynamic budget constraint is introduced by the presence of a high-water mark. This component is related to the running maximum process

$$H_t = \max\{W_s; s \in [0, t]\}.$$

Whenever $dH_t \neq 0$, meaning that the running maximum increases, fund wealth declines by $k\, dH_t$ to reflect the fraction of the increase that is given to the fund manager as compensation. Under these assumptions, the evolution of the wealth process is given by

$$dW_t = W_t \pi_t (\mu\, dt + \sigma\, dB_t) + W_t \left(1 - \pi_t\right) r\, dt - k\, dH_t. \tag{1}$$

C. Optimization Problem

The hedge fund manager is assumed to be maximizing the expected net present value of her fees. Mathematically, these are represented by an objective function of the form

$$\max_{\pi} E\left[\int_0^{\tau^0 \wedge \infty} e^{-(\beta+\lambda)t} k\, dH_t\right]. \tag{2}$$

The agent's discount factor, $\beta > 0$, is adjusted by the probability of fund termination. In order to perform some useful analysis later, we assume that the hedge fund is subject to exogenous random termination. This termination process is assumed to be Poisson with constant intensity $\lambda > 0$. At termination, the investor is assumed to withdraw all of her invested funds, the hedge fund management contract is terminated, and the manager's continuation value is zero. We also assume that the fund will be terminated the first time wealth drops to zero, that is, at τ^0 such that

$$\tau^0 = \inf\{t : W_t \leq 0\},$$

although this will never occur in finite time.

II. The Solution

A. A Heuristic Derivation

To solve the optimization problem (2) we proceed in the usual way: In a first step we construct a solution to a Hamilton–Jacobi–Bellman (HJB) partial differential equation subject to some heuristically derived boundary conditions, motivated by the optimization problem of equation (2). This helps us form a guess to the solution of the problem. In a second step, we prove a verification

This is a common assumption throughout the portfolio choice literature and is used to avoid doubling strategies (see Duffie (1996)).

theorem (in Appendix A). This theorem asserts that if a function satisfies certain conditions, it is the value function of the problem. In a third step we confirm that the solution that we guessed in the first step satisfies all the conditions of the verification theorem and hence is the value function.

We start in the usual fashion by rewriting the value function for the optimization problem (2) in time separable form as

$$e^{-(\beta+\lambda)t} V(W_t, H_t),$$

where

$$V(W_t, H_t) = \max_{\{\pi_s\}_{s=t}^{\infty}} E_t \left[\int_t^{\tau^0 \wedge \infty} e^{-(\beta+\lambda)(s-t)} k \, dH_s \right]. \tag{3}$$

We want to first study the problem in the region $\{W_t = H_t\}$. Whenever $W_t \geq H_t, H_t$ increases instantaneously by an amount dH_t to reflect the new high-water mark level. Comparing $V(W_t, H_t)$ before the event to its value after the event gives

$$V(W_t, H_t) = k \, dH_t + V(W_t - k \, dH_t, H_t + dH_t).$$

Taking a first-order Taylor expansion for "small" dH_t around $W_t = H_t$ gives[10]

$$V(H_t, H_t) = k \, dH_t + V(H_t, H_t) - V_W(H_t, H_t) k \, dH_t + V_H(H_t, H_t) \, dH_t.$$

Canceling $V(H_t, H_t)$ from both sides and collecting terms shows that the above equation can only hold if

$$k V_W = k + V_H, \tag{4}$$

whenever $W_t = H_t$.

We next study the problem on the set $\{W_t < H_t\}$. In this region $dH_t = 0$. Accordingly, we search for $V(W_t, H_t)$ that behaves as a discounted martingale and satisfies the standard HJB relation:

$$0 = -(\beta + \lambda)V + \max_{\pi_t} \left\{ V_W W_t (r + \pi_t(\mu - r)) + \frac{1}{2} V_{WW} W_t^2 \sigma^2 \pi_t^2 \right\}. \tag{5}$$

Observe that terms associated with dH_t do not appear in this relation because the running maximum does not increase in this region.

Assuming that there exists a solution to the maximization problem inside equation (5), we obtain

$$\pi_t^* = -\frac{V_W}{V_{WW} W_t} \frac{\mu - r}{\sigma^2}, \tag{6}$$

[10] We focus on "small" dH_t because the water mark is instantaneously reset when $W_t = H_t$.

which is the standard Merton-type solution. After substituting this optimal π_t^* into (5) and simplifying, we see that

$$0 = -(\beta + \lambda)V + rV_W W_t - \frac{1}{2}\frac{(V_W)^2}{V_{WW}}\frac{(\mu - r)^2}{\sigma^2}.$$

A solution of the above nonlinear ordinary differential equation is

$$V(W_t, H_t) = K(H_t)W_t^{\eta},$$

where $K(H_t)$ is a function that depends on H_t only, and the constant η solves

$$\eta^2 r - \eta(r + \beta + \lambda + \omega) + (\beta + \lambda) = 0 \tag{7}$$

with ω defined as

$$\omega = \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}. \tag{8}$$

Remember that $\mu > r$, and so, $\omega > 0$ by assumption.

Equation (7) admits two solutions:

$$\eta^{\pm} = \frac{r + \beta + \lambda + \omega \pm \sqrt{(r + \beta + \lambda + \omega)^2 - 4r(\beta + \lambda)}}{2r}. \tag{9}$$

We can rewrite the term inside the square root as

$$(\beta + \lambda + \omega - r)^2 + 4r\omega > 0, \tag{10}$$

which shows that both roots of (9) are real. The roots are also both positive since $\eta^+ > 0$ and $\eta^+\eta^- = \frac{\beta+\lambda}{r} > 0$. Additionally, $\eta^- \in (0, 1)$ and $\eta^+ > 1$, which can be seen by evaluating (7) and observing that the expression is positive for $\eta = 0$ and negative for $\eta = 1$.

We can ignore the root $\eta^+ > 1$ since this root would not be associated with a solution to (5) and just focus on $\eta^- \in (0, 1)$, so that the solution to (5) takes the form[11]

$$V(W_t, H_t) = K(H_t)W_t^{\eta^-}. \tag{11}$$

To simplify notation, from now on we shall write η instead of η^- and we will implicitly refer always to the root of the polynomial in (9) associated with η^-.

Finally, we must find $K(H_t)$, so that the boundary condition (4) is satisfied. Taking derivatives and evaluating at $W_t = H_t$, we have

$$k\eta K(H_t)H_t^{\eta-1} = k + K'(H_t)H_t^{\eta}.$$

[11] It is easy to verify that $K(H_t)W_t^{\eta^+}$ does not satisfy equation (5) since the maximum inside (5) cannot be attained.

This leads to the ordinary differential equation

$$K'(H_t) = k\left(\eta\frac{K(H_t)}{H_t} - \frac{1}{H_t^{\eta}}\right), \tag{12}$$

with the (general) solution

$$K(H_t) = \left[\frac{k}{\eta(1+k)-1}H^{1-(1+k)\eta} + C\right]H^{k\eta} \tag{13}$$

for some arbitrary constant C. To determine the constant C, we assume, hypothetically, that the fund will be terminated whenever $H_t = \bar{H}$ for a sufficiently high $\bar{H}$. This implies the boundary condition

$$V(W_t, \bar{H}) = 0,$$

which leads us to set the constant in (13) equal to

$$C = -\frac{k\bar{H}^{(1-\eta(1+k))}}{\eta(1+k)-1}. \tag{14}$$

To obtain the solution to the infinite horizon problem, let $\bar{H} \to \infty$ and assume that

$$\eta(1+k) > 1. \tag{15}$$

Then the constant in equation (13) converges to zero and our guess for the value function becomes[12]

$$V(W_t, H_t) = \frac{kH_t}{\eta(1+k)-1}\left(\frac{W_t}{H_t}\right)^{\eta}. \tag{16}$$

As we have already remarked, equation (16) is only a guess at the solution. The following proposition, proven in Appendix A, verifies that this conjecture corresponds to the actual value function.

PROPOSITION 1: *Assume that condition (15) holds and that $\omega > 0$. Then the value function is given by (16) and the optimal portfolio by (6).*

In deriving this value function, the key assumptions that we make on the parameters are equation (15) and $\omega > 0$. Both conditions are easy to interpret economically. Condition (15) is an inherently technical condition and says that discounting of future payoffs has to be sufficiently strong to ensure that the value function of the hedge fund manager remains finite. The condition $\omega > 0$ implies that the manager perceives that there exist some investment opportunities in the market that will yield more than r. We discuss this latter condition in Section II.D.

[12] The conjecture for the value function (16) also satisfies the properties

$$\lim_{H_t\to\infty} V_H(W_t, H_t) = 0, \lim_{W_t\to 0} V(W_t, H_t) = 0.$$

Combining (6) and (16), we obtain the optimal portfolio

$$\pi_t^* = \frac{1}{1-\eta}\frac{\mu - r}{\sigma^2}. \tag{17}$$

As already mentioned, this portfolio places a constant fraction in the risky assets independent of any state variable of the problem. In this sense, a risk-neutral hedge fund manager behaves as if she were a Merton-type investor with CRRA[13] of $1-\eta$. This effective risk aversion and its magnitude are endogenous rather than assumed.

In the next sections we develop the intuition behind equation (17) by identifying the key assumptions of the analysis and performing comparative statics exercises.

B. The Horizon Effect

As we argue in the introduction to the paper, the infinite horizon is key for our results. The goal of this section is to show that managers with finite horizons will opt for unbounded volatility as they approach the termination time T.

To preserve tractability, we examine a sequence of stochastic termination times. We choose this sequence so that the stochastic termination time (τ_n) will be equal to T with probability approaching one as $n \to \infty$. The method that we employ has been used effectively in the pricing of American put options on finite horizons by Carr (1998), and we refer the reader to that paper for details.

To start, let $N_{n,t}$ be a sequence of standard Poisson processes with interarrival intensity $\lambda_n = \frac{n}{T}$. Now define the termination time τ_n as

$$\tau_n = \inf\{t : N_{n,t} = n\}.$$

This means that $N_{n,t}$ measures the number of times that a Poisson "counter" with arrival intensity λ_n has increased, and the fund is terminated when $N_{n,t} = n$. Note that as n increases both the interarrival intensity λ_n and the number of Poisson jumps needed to trigger termination increase in proportion to n. Hence, the expected termination time is equal to T for all n. However, as n increases the variance of the termination time converges to zero. As Carr (1998) shows,

[13] Our portfolio is similar to that in Browne (1997). Browne (1997) considers the optimization problem of maximizing the expected discounted gain from attaining a target level of wealth and finds the same portfolio as in (17). Interestingly, since the value function takes (endogenously) the form of a power function in fund wealth, the portfolio of the manager is identical to the portfolio that would be chosen by a Merton investor with risk aversion $1-\eta$. Our similarity to Browne (1997) is no coincidence: Our hedge fund manager understands that her compensation increases whenever she attains the previously recorded high-water mark and tries to maximize the expected discounted gain from attaining that threshold. As a result, the two problems coincide whenever $W_t < H_t$. This correspondence explains the somewhat surprising fact that the optimal portfolio does not depend on k. In contrast to Browne (1997), however, the reward from attaining the target in our problem is endogenous because it depends on the last recorded high-water mark. Moreover, it is reset each time the water mark is reached. Browne (1997) by contrast assumes an exogenous reward.

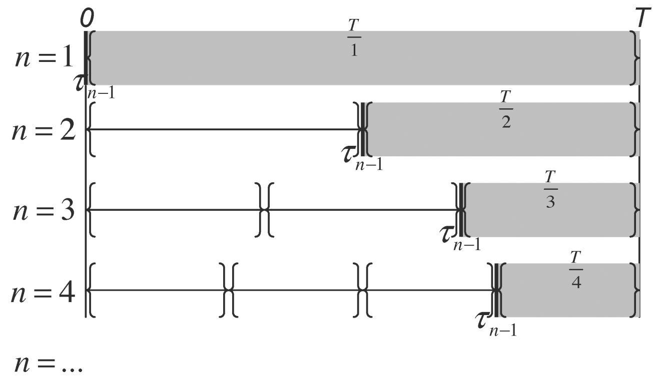

Figure 1. Poisson termination times. This figure illustrates the construction of the limit in Section II.B. The curly brackets indicate the expected arrival time of Poisson increments, given by $\frac{T}{n} = \frac{1}{\lambda_{(n)}}$, for different values of n. By construction the expected time of termination $E(\tau_n)$ is equal to T for all $n = 1, 2 \ldots \infty$. The shaded areas indicate the last "period" before termination.

the termination time τ_n has an Erlang distribution

$$\Pr(\tau_n \in dt) = \frac{(\lambda)^n}{(n-1)!} t^{n-1} e^{-\lambda t}\, dt,$$

where $\lambda = \lambda_n = \frac{n}{T}$. Moreover, $\Pr(\tau_n \in dt)$ converges to a Dirac delta function centered at T.[14]

Let us now fix a number n and consider the portfolio that the manager chooses when $N_{n,t} = n - 1$, that is, at the next-to-last "period", τ_{n-1}. Figure 1 provides a graphical illustration of this construction for different values of n. At that time, the arrival of termination is exponential with arrival intensity $\lambda_n = \frac{n}{T}$. Because this is the same setup we used in the previous section, we can re-use the formulas of that section to show that the portfolio of the manager will be given by (17) with η determined by (9):

$$\eta = \frac{r + \beta + \lambda_n + \omega - \sqrt{(r + \beta + \lambda_n + \omega)^2 - 4r(\beta + \lambda_n)}}{2r}.$$

As n approaches ∞, both τ_n and τ_{n-1} will be arbitrarily close to T. So, if we want to study portfolio choice as the manager approaches the termination time T, it suffices to examine the portfolio that she will choose at the (random) time τ_{n-1}.

LEMMA 1: *Let $\pi_{(n-1)}$ be the portfolio that a manager chooses at time τ_{n-1}. Then*

$$\lim_{n\to\infty} \pi_{(n-1)} = \infty.$$

[14] Intuitively, by splitting the random termination into n i.i.d. subperiods, we can control the randomness associated with the time of arrival of each subperiod, while keeping the mean arrival rate fixed. With a simple exponential distribution this is impossible because the mean and the variance are controlled by the same parameter. However, an Erlang distribution allows one to "control" its mean and its variance separately.

The intuition for this result comes from the fact that as the termination time approaches, the manager becomes effectively more impatient. In the infinite horizon setting, the manager behaves as if she were effectively risk averse, since her future prospects for fees are hurt whenever the level of wealth under management declines. By contrast, when the horizon is finite and near, there is no difference to the manager if she ends up moderately or extremely below the high-water mark.

C. An Alternative Derivation

Section II.A derived a solution to the manager's problem that was based on standard dynamic programming techniques. It is also possible to derive the same solution using only martingale theory. The advantage of this approach is that it helps illustrate in a direct and straightforward way why the value function is concave in W_t and the optimal portfolio policy is interior.

To this end, consider any admissible portfolio policy π_s and let τ denote the next time that W_t reaches the high-water mark H_t, assuming that the portfolio manager follows the policy π_s:

$$\tau = \inf\{s \geq t : W_s = H_t\}.$$

Letting $\Psi^{\{\pi_s\}}$ denote the payoff associated with the portfolio policy π_s, Bellman's principle of optimality implies that

$$\Psi^{\{\pi_s\}} \leq E_t\left\{e^{-(\beta+\lambda)(\tau-t)}V\left(W_\tau, H_t\right)\right\} = E_t\left\{e^{-(\beta+\lambda)(\tau-t)}V(H_t, H_t)\right\}. \tag{18}$$

Next, define $\xi = \sigma\eta$, where η is given by η^- in (9). Since $\eta \in (0, 1)$, $\xi < \sigma$. Define also the process M_s as

$$M_s = e^{-\xi^2 \int_t^s \frac{\pi_u^2}{2} du + \xi \int_t^s \pi_u \, dB_u} \text{ for } s > t. \tag{19}$$

The process M_s is a nonnegative local martingale and hence a supermartingale, so that the supermartingale stopping theorem implies

$$M_t = 1 \geq E_t\left(e^{-\xi^2 \int_t^{\tau\wedge T} \frac{\pi_u^2}{2} du + \xi \int_t^{\tau\wedge T} \pi_u \, dB_u}\right) \tag{20}$$

for any $T > t$.

Applying Ito's Lemma to (1) and noting that the high-water mark does not change between t and $\tau \wedge T$ leads to

$$\log\left(W_{\tau\wedge T}\right) = \log\left(W_t\right) + \int_t^{\tau\wedge T} \left[r + \pi_u\left(\mu - r\right) - \pi_u^2\frac{\sigma^2}{2}\right] du + \sigma \int_t^{\tau\wedge T} \pi_u \, dB_u, \tag{21}$$

which can be rearranged to yield

$$\int_t^{\tau\wedge T} \pi_u \, dB_u = \frac{1}{\sigma}\left(\log\left(W_{\tau\wedge T}\right) - \log\left(W_t\right) - \int_t^{\tau\wedge T}\left[r + \pi_u\left(\mu - r\right) - \pi_u^2\frac{\sigma^2}{2}\right] du\right). \tag{22}$$

Using (22) and substituting into (20) yields

$$\begin{aligned} 1 &\geq E_t\left[e^{-\left(\frac{\xi}{\sigma}\right)^2 \int_t^{\tau\wedge T} \frac{\pi_u^2}{2}\sigma^2 \, du + \frac{\xi}{\sigma}\left(\log(W_{\tau\wedge T}) - \log(W_t) - \int_t^{\tau\wedge T}\left(r+\pi_u(\mu-r)-\pi_u^2\frac{\sigma^2}{2}\right) du\right)}\right] \\ &= E_t\left[\left(\frac{W_{\tau\wedge T}}{W_t}\right)^{\eta} e^{\int_t^{\tau\wedge T}(-\eta^2+\eta)\frac{\pi_u^2}{2}\sigma^2 - \eta[r+\pi_u(\mu-r)]\, du}\right]. \end{aligned} \tag{23}$$

Since $\eta < 1$, the expression $(-\eta^2 + \eta)\frac{\sigma^2}{2}\pi_u^2 - \eta\left[r + \pi_u\left(\mu - r\right)\right]$ is convex (indeed quadratic) in π_u and hence

$$(-\eta^2 + \eta)\frac{\pi_u^2}{2}\sigma^2 - \eta\left[r + \pi_u\left(\mu - r\right)\right] \geq \min_{\pi_u}\left\{(-\eta^2 + \eta)\frac{\pi_u^2}{2}\sigma^2 - \eta[r + \pi_u(\mu - r)]\right\}. \tag{24}$$

Solving the minimization problem on the right-hand side of (24) leads to

$$\pi^* = \frac{1}{1-\eta}\frac{\mu - r}{\sigma^2}. \tag{25}$$

Substituting (25) into (24) implies

$$\begin{aligned} (-\eta^2 + \eta)\tfrac{\pi_u^2}{2}\sigma^2 - \eta[r + \pi_u(\mu - r)] &\geq (-\eta^2 + \eta)\tfrac{(\pi^*)^2}{2}\sigma^2 - \eta[r + \pi^*(\mu - r)], \\ &= -(\beta + \lambda), \end{aligned} \tag{26}$$

where the last equality follows from (7). Combining (26) with (23) yields

$$1 \geq E_t\left[\left(\frac{W_{\tau\wedge T}}{W_t}\right)^{\eta} e^{-(\beta+\lambda)(\tau\wedge T - t)}\right].$$

Letting $T \to \infty$ and using the dominated convergence theorem leads to[15]

$$1 \geq E_t\left[\left(\frac{W_\tau}{W_t}\right)^{\eta} e^{-(\beta+\lambda)(\tau - t)}\right] = \left(\frac{H_t}{W_t}\right)^{\eta} E_t\left[e^{-(\beta+\lambda)(\tau-t)}\right]. \tag{27}$$

Multiplying both sides of (27) by $\left(\frac{W_t}{H_t}\right)^{\eta} V\left(H_t, H_t\right)$ implies that

$$\left(\frac{W_t}{H_t}\right)^{\eta} V\left(H_t, H_t\right) \geq E_t\left[e^{-(\beta+\lambda)(\tau-t)}\right] V\left(H_t, H_t\right) \geq \Psi^{\{\pi_s\}}, \tag{28}$$

where the last inequality follows from (18). Equation (28) asserts that the expression $(\frac{W_t}{H_t})^{\eta} V\left(H_t, H_t\right)$ provides an upper bound to all admissible payoffs.

[15] Notice that for any random realization $(\frac{W_{\tau\wedge T}}{W_t})^{\eta} e^{-(\beta+\lambda)(\tau\wedge T - t)} \leq (\frac{H_t}{W_t})^{\eta}$.

Furthermore, if the manager uses the admissible portfolio policy (25), then all inequalities can be replaced with equalities. Then Bellman's principle of optimality implies that the value function can be written as

$$V(W_t, H_t) = \left(\frac{W_t}{H_t}\right)^{\eta} V(H_t, H_t). \tag{29}$$

Since $\eta < 1$, equation (29) implies that the value function is concave in W_t, and so, the portfolio policy is interior and given by (25). To gain some further intuition into this result, it is easiest to relate it to the analysis in Goetzmann, Ingersoll, and Ross (2003). This will help us illustrate in a simple way the key assumptions that drive the interior nature of the portfolio.

D. The Relation to Goetzmann, Ingersoll, and Ross (2003)

In this section, we show how our results compare to Goetzmann, Ingersoll, and Ross (2003). In particular, we re-derive the optimal policy by using the valuation techniques of GIR. This alternative derivation will help us illustrate most clearly the significance of our assumption that $\mu > r$.

To begin, we concentrate on policies that invest an exogenous constant fraction of wealth in the risky asset: $\pi_t = \pi$ for all t. Since the optimal policy belongs to this class, we will be able to recover our earlier results.

As in Section II.A, in the region $\{W_t < H_t\}$ the manager's payoff for any fixed portfolio π satisfies

$$0 = -(\beta + \lambda)V + V_W W_t[r + \pi(\mu - r)] + \frac{1}{2} V_{WW} \sigma^2 \pi^2 W_t^2.$$

This ODE has a general solution of the form

$$V(W_t, H_t) = K_1(H_t) W_t^{\alpha^+} + K_2(H_t) W_t^{\alpha^-},$$

where $K_1(H_t)$ and $K_2(H_t)$ represent arbitrary functions that can depend at most on H_t. The constants α^+ and α^- solve the quadratic equation

$$-(\beta + \lambda) + \alpha\left(r + \pi(\mu - r) - \frac{1}{2}\sigma^2\pi^2\right) + \frac{1}{2}\sigma^2\pi^2\alpha^2 = 0. \tag{30}$$

Define the mean growth rate of log-wealth as

$$l(\pi) = \left(r + \pi(\mu - r) - \frac{1}{2}\sigma^2\pi^2\right), \tag{31}$$

so that the solution to (30) can be written as

$$\alpha^{\pm}(\pi) = \frac{-l(\pi) \pm \sqrt{l(\pi)^2 + 2(\beta + \lambda)\sigma^2\pi^2}}{\sigma^2\pi^2}.$$

We will study the roots $\alpha^{\pm}$ as functions of π. Because α^{-} will be negative and we wish to satisfy the boundary condition

$$\lim_{W_t \to 0} V(W_t, H_t) = 0,$$

we will drop the negative root by setting $K_2(H_t) = 0$. Since we are focusing on the positive root α^{+}, we shall use the letter α instead of α^{+}. From this point on, the value function can be obtained by repeating the same sequence of steps as in equations 11–(16) in Section II.A. We arrive at an expression for V that takes the portfolio policy π as given:

$$V(W_t, H_t; \pi) = \frac{k H_t}{\alpha(\pi)(1+k) - 1} \left(\frac{W_t}{H_t} \right)^{\alpha(\pi)}. \tag{32}$$

This replicates the setup and the results in GIR with two important exceptions: μ can be different from r, and the log growth rate of the fund and its volatility depends on the choice of π. To continue, we can maximize the value function (32) over π.

PROPOSITION 2: *Assume that Condition (15) holds and that $\mu > r$. Then $V(W_t, H_t; \pi)$ is maximized over π independent of (W_t, H_t) when*

$$\pi = \pi^* = \frac{1}{1-\eta} \frac{\mu - r}{\sigma^2}. \tag{33}$$

A key assumption of the above proposition is that $\mu > r$. Clearly, this is identical to the assumption of a nonzero Sharpe ratio ($\omega > 0$) that we made in the previous section. This assumption is substantive. If we let $\mu \to r$, then the portfolio of the manager becomes unbounded.

LEMMA 2: *Assume that $\beta + \lambda > r$ and $\mu > r$. Then*

$$\begin{aligned} \lim_{\mu \to r} \eta &= 1, \\ \lim_{\mu \to r} \pi^* &= \infty. \end{aligned}$$

This is similar to the high-water mark valuation result shown in GIR. They value the manager's fees under the risk-neutral probability measure where, by construction, $\beta = r = \mu$ and hence $\beta + \lambda > r$. Therefore, the desired portfolio has infinite volatility. (Strictly speaking, the maximization problem is not well defined in this case.)

In the context of portfolio choice, however, these results may seem surprising. How is it that moving the Sharpe ratio towards zero leads to extremely bold portfolios? The resolution of the puzzle can be obtained by remembering that the manager's investment opportunities affect her continuation value, and thus, they affect the relative importance of short- and long-run objectives.

To make this basic intuition more explicit, we start by considering expression (32). Note that π affects this expression only through $\alpha(\pi)$. One can show that V is decreasing in α, so the manager will choose π so as to *minimize* $\alpha(\pi)$.[16]

If $\mu = r$, then V represents the net present value of the manager's fees. This implies that there is an upper bound to $V(W_t, H_t; \pi)$, namely, W_t. Indeed, the whole analysis in GIR revolves around the share of fund wealth that the manager appropriates due to the high-water mark contract. Clearly, this share cannot be above one.

Next, observe that the bound $V(W_t, H_t; \pi) \leq W_t$ implies that $\alpha(\pi) \geq 1$. To see this, note that $\alpha(\pi) = 1$ implies $V(W_t, H_t; \pi) = W_t$ and remember that V is *decreasing* in $\alpha(\pi)$. As a result, it cannot be that there exists a π for which $\alpha(\pi)$ is below one because if there were, we would have $V(W_t, H_t; \pi) > W_t$. As a result, it must be that $\alpha(\pi) \geq 1$, and so, the manager's value function is convex in W_t. Thus, the manager will want to take unbounded portfolio positions.

The key difference between the $\mu = r$ case and the $\mu > r$ case is that when $\mu > r$, the bound $V \leq W_t$ is no longer valid. Intuitively, this results from the fact that V is no longer the "fair value" of a payoff for which the value cannot exceed W. Instead V is the expectation of the discounted fees *under the natural measure*, and hence there is no reason for V to be less than W_t. In fact, an examination of (32) shows that V can exceed W_t. Thus, $\alpha(\pi)$ can attain values lower than one, and this has the effect of introducing concavity into the problem and making the optimal portfolio finite.

This reasoning explains what is so special about the $\mu = r$ case: When $\mu = r$, the continuation value function of the problem isn't sufficient to make the manager fear the long-run costs associated with increasing volatility in the short run.

We develop this intuition further in the next section.

E. Comparative Statics

Given the simple form of the optimal portfolio and the value function, one can obtain a number of comparative statics results. We isolate a few that seem to be central in understanding both the properties of the solution and the main intuition.

LEMMA 3: *If (15) holds, $\omega > 0$, and $\beta + \lambda > r$, then*

$$\frac{\partial \pi_t}{\partial \beta} = \frac{\partial \pi_t}{\partial \lambda} > 0.$$

If, in addition, $\beta + \lambda > r + \omega$, then

$$\frac{\partial \pi_t}{\partial \mu} < 0.$$

[16] To see this, differentiate $\frac{kH_t}{\alpha(1+k)-1}$ with respect to α, which gives $-\frac{k(1+k)H_t}{(\alpha(1+k)-1)^2} < 0$, and also differentiate $(\frac{W_t}{H_t})^\alpha$ with respect to α to obtain $(\frac{W_t}{H_t})^\alpha(\log(W) - \log(H)) \leq 0$.

These comparative statics are an extension of Lemmas 1 and 2, and they reinforce our findings in the previous two sections. The parameters β and λ control how short-termist a manager is: The more a manager discounts the future, the bolder is her optimal portfolio. The reasoning in Lemma 1 is that there was no continuation value to the problem as the exogenous termination time became near. Here we show more generally that managers who discount the future at higher rates will exhibit stronger risk-shifting behavior.

The result $\frac{\partial \pi_t}{\partial \mu} < 0$ is an extension of Lemma 2, and it also seems surprising at first sight. In most Merton-type setups one would expect the opposite, namely, that a higher μ will lead to bolder portfolios. The resolution of the puzzle is that in our setup the "effective" risk aversion of the manager is not exogenous, but instead it is endogenous and depends on the relative importance of the present payoffs compared with future payoffs.

Recall that the optimal portfolio is given by

$$\pi = \frac{1}{1-\eta}\frac{\mu - r}{\sigma^2}.$$

When $\mu - r$ increases, the second fraction clearly increases. Hence, if η were to remain constant, the portfolio would increase, as in any standard Merton setup. We shall refer to this as the "direct" effect of an increase in μ. However, one also expects that when μ increases, the value function will also increase; in our model, μ enters the value function through the parameter η, which is equal to one minus the coefficient of "effective" risk aversion. An increase in μ decreases η, and so, increasing μ also increases both the value function and the coefficient of effective risk aversion. Our assumption $\beta + \lambda > r + \omega$ guarantees that the indirect effect of increased effective risk aversion is stronger than the direct effect of μ on portfolio choice.[17]

We conclude this section with two features of the value function that appear odd at first: $V_H > 0$ and $V_k < 0$. We start with a discussion of the first result: One would expect the opposite, that is, that increases in the water mark would reduce the value function of the manager. It turns out that this unusual property is the result of (1) the risk-neutrality assumption and (2) the reduction in the fund growth rate that results from payments to the manager.

To understand why an increase in the high-water mark actually ends up increasing the expected discounted fees, evaluate the value function (16) at $W_t = H_t$. One obtains

$$V(H_t, H_t) = \frac{kH_t}{\eta(1+k) - 1},$$

which is linear in H_t. This equation means that the expected gain to a manager that has just reached her high-water mark is proportional to the value of funds under management. We call this the "scale effect": At the point at which the

[17] This result is anticipated by Lemma 2, where we show that the optimal portfolio will diverge to infinity when ω is close to zero. Hence, we should expect the portfolio to actually decline (for ω close to zero) as μ (and hence ω) increases.

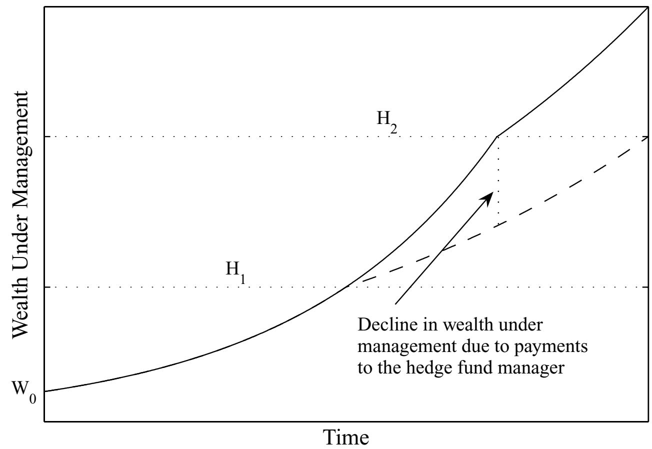

Figure 2. Reduction in the growth of fund wealth W_t due to payments to the hedge fund manager.

manager begins to collect her fees, she would prefer the fund to be as large as possible. However, the scale effect needs to be balanced against the "waiting effect" that is captured by the term $(\frac{W_t}{H_t})^\eta$. An increase in H_t for a fixed W_t means that it will take more time to hit the high-water mark. The fact that $\eta < 1$ implies that the scale effect will dominate the waiting effect—the manager is "patient."

Figure 2 depicts an alternative interpretation of the $V_H > 0$ result. We fix a wealth of W_0 and examine how fast fund wealth W_t can reach H_2. We compare two cases: in the first the current high-water mark is H_1, and in the second it is at H_2. For any given portfolio policy, the growth of the fund value between W_0 and H_1 will be unaffected by whether the water mark is at H_1 or H_2. However, once H_1 is reached, there is an important difference due to fund wealth being partially diverted to management fees. If the high-water mark is at H_2, then the elapsed time between H_1 and H_2 will be less (solid line) than in the case when the high-water mark is H_1 (dashed line). This is because if the high-water mark is H_1, then fund growth will be lower between H_1 and H_2 by the amount of the payments provided to the manager. As a result, a manager can attain a higher fund value in shorter time when she faces an increased high-water mark. The tradeoff for the manager is between payments now and increased wealth under management and correspondingly higher payments later.

The fact that $V_k < 0$ is due to a similar reasoning. Assuming that k is large enough to satisfy condition (15), then direct differentiation yields

$$V_k = \frac{(\eta - 1)H_t}{(\eta(1+k) - 1)^2}\left(\frac{W_t}{H_t}\right)^\eta < 0.$$

A higher k will increase the payouts to the manager, but it will also reduce the growth rate of W_t, as money gets diverted to the manager. If a manager puts heavy emphasis on the future, then the latter effect will dominate.

The purpose of the next section is to demonstrate how both of these counterintuitive results are linked with the extreme assumption of risk neutrality (which keeps the marginal utility of a dollar constant). With the more empirically relevant assumption of risk aversion neither of the two results will hold.

III. Risk Aversion

In this section we extend our analysis to cases in which the manager is risk averse. Our main goals are to show that our portfolio results are robust to different preferences and that some of the counterintuitive results obtained with risk neutrality can be changed in a risk-averse environment.

A. Exclusion from Trading in Private Accounts

Extending the model to the risk averse case introduces some technical complications. To understand these complications and the approach we take to resolve them, we start with a discussion of the maintained assumption of excluding trading in private accounts.

We have assumed throughout that the manager cannot trade in private accounts; she just consumes the incoming fees. This assumption is key in order to avoid unbounded volatility choices by the manager. To see why, consider what would happen if instead we assume that the manager is allowed to trade in an unconstrained manner in any type of security, so that she faces a dynamically complete market. It is then a standard result that the manager's consumption choices are constrained by a single intertemporal budget constraint (Karatzas and Shreve (1998), Chapters 1 and 4),

$$E^Q \int_0^\infty e^{-rs} dC_s \leq W_0 + E^Q \int_0^\infty e^{-(r+\lambda)s} k\, dH_s, \tag{34}$$

where Q is the *unique* risk neutral probability measure (equivalent martingale measure) and C_s is the cumulative consumption process of the manager. Intuitively, the above formula states that the fund manager's total wealth is given by the sum of her personal financial wealth (W_0) and the net present value of her fees $E^Q \int_0^\infty e^{-rs} k\, dH_s$.

Letting U be any function that maps consumption processes into levels of utility for the manager, and letting π_s be the portfolio allocation that the manager chooses for the fund, we can express the manager's joint consumption/fund asset allocation problem as

$$\max_{\{C_s, \pi_s\}_{s=0}^\infty} U(C_s),$$

subject to (34). Suppose now that we fix any two portfolio policies π_s^1 and π_s^2 and denote the resulting high-water mark processes as $H_s^{(\pi_s^1)}$, $H_s^{(\pi_s^2)}$, respectively.

Assume furthermore that

$$E^Q \int_0^\infty e^{-(r+\lambda)s} k \, dH_s^{(\pi_s^1)} > E^Q \int_0^\infty e^{-(r+\lambda)s} k \, dH_s^{(\pi_s^2)}.$$

The manager will prefer the policy π_s^1 because this policy will relax her budget constraint (34) more than policy π_s^2. More generally the manager will try to use her choice of π_s in order to maximize the net present value of her fees, since this is the way in which she can maximize her total wealth and her feasible consumption choices.

Intuitively, in a complete market there is a "separation" between choosing optimal consumption processes and choosing an optimal portfolio strategy for the fund. The manager should in a first step choose a portfolio strategy that maximizes the net present value of the fund and in a second step add the resulting present value to her financial wealth and determine her optimal consumption process.

This highlights the importance of excluding trades in personal accounts. Unless we restrict the manager's ability to "replicate" the fees that she is receiving in a private account, she would always try to maximize the net present value of these fees. As we have shown in Lemma 2, this would lead to unbounded volatility choices.

Accordingly, we restrict the manager's ability to trade, creating an incomplete markets setting. In an incomplete market, (34) is not even uniquely defined because there will typically be multiple equivalent martingale measures. Hence, the separation between the manager's consumption and asset allocation decisions for the fund no longer holds.[18]

B. Introducing Risk Aversion

The above discussion highlights the importance of restricting the manager's ability to trade in private accounts. As in the risk-neutral case, we will accordingly continue to assume for simplicity that the manager is completely excluded from financial markets.

However, maintaining this assumption, we run into a technical issue by introducing risk aversion: The hedge fund manager is receiving her compensation in a lumpy fashion. (Mathematically speaking, the high-water mark increases on a measure zero set of time.) Accordingly, if the fund manager cannot access financial markets, then she will face a severe problem smoothing her consumption stream.

A simple and intuitive way to overcome this problem is to use a variant of the utilities proposed by Hindy and Huang (1992, 1993) and Hindy, Huang, and Kreps (1992). According to these utilities consumption goods exhibit some degree of durability. Therefore, it is not the flow of consumption that matters

[18] In an incomplete market the "relevant" equivalent martingale measure Q in the set of equivalent martingale measures will depend on the form of the income process and the consumption decisions of the manager.

for the utility of the agent, rather the stock of accumulated goods. Since the hedge fund manager is excluded from trading in private accounts, she is using her fees $k\,dH_t$ to increase her *stock* of (durable) goods. Accordingly, her total stock of goods Z_t evolves as[19]

$$dZ_t = k\,dH_t. \tag{35}$$

The manager derives utility from the stock of these goods so that her expected utility is given by

$$J = \max_{\pi_s} E\left[\int_0^\infty e^{-\beta s} U(Z_s)\,ds\right]. \tag{36}$$

To solve the problem in (36), we again apply integration by parts to obtain

$$J = \frac{U(Z_0)}{\beta} + \frac{1}{\beta}\max_{\pi_s} E\left[\int_0^\infty e^{-\beta s} U'(Z_s)\,dZ_s\right].$$

The first term on the right-hand side of the above expression captures the utility to the agent from pre-existing durables (before the fund was started). The second term is the term we are interested in. It captures the expected utility of future fees. We isolate this term and define

$$\begin{aligned} V\left(W_t, H_t\right) &= \max_{\pi_s} E\left[\int_0^\infty e^{-\beta s} U'\left(Z_s\right) dZ_s\right] \\ &= \max_{\pi_s} E\left[\int_0^\infty e^{-\beta s} U'(Z_0 + k\left(H_s - H_0\right))k\,dH_s\right], \end{aligned}$$

where the second line follows from (35). An interesting observation is that V reduces to the same objective we have been studying in Sections II to III when $U'(\cdot) = 1$, that is, when the manager is risk neutral. To obtain an analytic characterization going forward, we make a specific parametric assumption on the utility function, namely,

$$U\left(Z\right) = \frac{Z^{1-\gamma}}{1-\gamma}. \tag{37}$$

The objective has now the same familiar form as in Section II.A. Accordingly, we can follow the same steps and obtain the HJB equation

$$0 = -\beta V + \max_{\pi_t}\left\{V_W W_t(r + \pi_t(\mu - r)) + \frac{1}{2} V_{WW} W_t^2 \sigma^2 \pi_t^2\right\}, \tag{38}$$

which holds whenever $W_t < H_t$. A solution to (38) is given by

$$V(W_t, H_t) = F(H_t) W_t^{\eta}, \tag{39}$$

[19] Hindy and Huang (1993) assume that durables have a depreciation rate χ. We set $\chi = 0$ for simplicity. We can also solve special cases where $\chi \neq 0$ as we explain below.

where η is the same as in Section II. Accordingly,

$$\pi_t^* = \frac{1}{1-\eta}\frac{\mu - r}{\sigma^2}. \tag{40}$$

Notice that the portfolio allocation has not changed from our original model despite the presence of risk aversion. We explain this result in detail in the next section.

To study V at the boundary $\{W_t = H_t\}$, we can apply the same logic as in Section II.A to arrive at

$$V_W k = U'(Z_0 + k(H_t - H_0))k + V_H.$$

Using (39), we obtain the ODE

$$\eta k F(H_t) H_t^{\eta-1} = F'(H_t) H_t^{\eta} + (Z_0 + k(H_t - H_0))^{-\gamma} k, \tag{41}$$

and its solution is

$$F(H_t) = k^{1-\gamma}\left(\int_{H_t}^{\infty} x^{-\eta(1+k)}\left(x - H_0 + \frac{Z_0}{k}\right)^{-\gamma} dx\right) H_t^{\eta k}.$$

With that, we can verify that $V = F(H_t)W_t^{\eta}$ is indeed the value function under the additional assumption

$$\gamma + \eta(1+k) - 1 > 0, \tag{42}$$

by following the same steps as in Section II.A. This last condition plays the same role as assumption (15) in our risk-neutral derivation: It is a technical assumption that keeps the value function well defined and finite.

One new property of the value function is that if γ is large enough then $V_H < 0$, in line with common intuition. We attributed the earlier $V_H > 0$ result (in the risk-neutral case) to a scaling effect that dominates the waiting effect. With risk aversion, however, the marginal value to additional consumption declines asymptotically, and so, we would expect the scaling effect to be less important for more risk-averse managers. Indeed,

LEMMA 4: *Assume that $kH_0 > Z_0$ and that $\gamma > 1 - \eta$.*[20] *Then $V_H < 0$.*

Another counterintuitive property of the value function under risk neutrality is that $V_k < 0$. With risk aversion, this property will no longer be necessarily true. In particular we have the following result:

LEMMA 5: *Assume that $kH_0 > Z_0$ and that*

$$\sqrt{\eta k} < \frac{\eta}{2}(1+k) < 1. \tag{43}$$

Then, there exists an open set of γ such that $F_k > 0$ and hence $V_k > 0$.

[20] This condition is sufficient but not necessary for the results.

C. Remarks on the Optimal Portfolio

We now return to the surprising result of equation (40). How is it that risk aversion does not enter the optimal portfolio? To fully understand the source of this result, it is most useful to take an *arbitrary* t such that $W_t < H_t$ and denote as τ the first time that $W_\tau = H_t$:

$$\tau = \inf\{s \geq t : W_s = H_t\}.$$

Then, Bellman's principle of optimality applied to (36) implies that the agent's value function between time t and τ can be rewritten as

$$J(W_t, H_t) = \sup_{\{\pi_s\}_{t \leq s \leq \tau}} \left[E_t \left(\int_t^\tau e^{-\beta(s-t)} U(Z_0 + k(H_s - H_0))\, ds + e^{-\beta(\tau - t)} J(H_\tau, H_\tau) \right) \right].$$

This equation states that the manager's payoff can be decomposed into her expected utility between t and τ and the (expected discounted) value function after time τ.

Since $H_s = H_t$ for all s between t and τ, and H_t is a continuous function of time, we can rewrite the above expression as

$$\begin{aligned}
J(W_t, H_t) &= \sup_{\{\pi_s\}_{t \leq s \leq \tau}} \left[E_t \left(U(Z_0 + k(H_t - H_0)) \int_t^\tau e^{-\beta(s-t)} ds + e^{-\beta(\tau - t)} J(H_t, H_t) \right) \right] \\
&= \sup_{\{\pi_s\}_{t \leq s \leq \tau}} \left[E_t \left(U(Z_0 + k(H_t - H_0)) \frac{1 - e^{-\beta(\tau - t)}}{\beta} + e^{-\beta(\tau - t)} J(H_t, H_t) \right) \right] \\
&= \frac{U(Z_0 + k(H_t - H_0))}{\beta} \\
&\quad + \left(J(H_t, H_t) - \frac{U(Z_0 + k(H_t - H_0))}{\beta} \right) \sup_{\{\pi_s\}_{t \leq s \leq \tau}} \left[E_t e^{-\beta(\tau - t)} \right].
\end{aligned}$$

The last equation shows that the value function is equal to the utility associated with existing durables plus the product of the reward to attaining the high-water mark and the discounted value of doing so. The size of the reward from attaining the water mark depends only on H_t, which is a fixed number between t and τ. Accordingly, when the manager varies the portfolio, she is only varying the time it takes to obtain that reward. Thus, the portfolio process only enters the expression

$$\sup_{\pi_s} \left[E_t e^{-\beta(\tau - t)} \right]. \tag{44}$$

Importantly, the shape of the utility function doesn't enter equation (44), and this is why risk aversion does not enter the optimal portfolio. Intuitively, between t and τ there are no variations of the utility index $U(Z_0 + k(H_s - H_0))$, and hence the shape of U is irrelevant for *portfolio choice*.

Of key importance in the above reasoning is the assumption that the level of durables Z_t does not change between t and τ. It is possible to show that this is no longer true when durables are depreciating at a rate $\chi \neq 0$. In this case, higher risk aversion will generally lead to more conservative portfolio choice.[21]

IV. Extensions

The extension of the model to the multiple asset case is straightforward. Assume that there are n risky assets, all of which follow a log-normal process

$$\frac{dP_i}{P_i} = \mu_i\,dt + \sum_{j=1}^{n} \sigma_{ij}\,dB_t^{(j)},\ i = 1\ldots n,$$

where $dB_t^{(j)}$ denotes an n-dimensional Brownian motion. We continue to assume that the riskless asset evolves according to

$$\frac{dP_0}{P_0} = r\,dt.$$

The wealth evolution equation (for a given vector π) is given as

$$dW_t^{\pi} = W_t^{\pi}\left(\left[r + \sum_{i=1}^{n} \pi_i(\mu_i - r)\right]dt + \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i\sigma_{ij}\,dB_t^{(j)}\right) - k\,dH_t.$$

Denoting by σ the matrix σ_{ij} and by μ, π the respective vectors of means and portfolios, one can define the infinitesimal generator of any function $f(W, H)$ as[22,23]

$$\mathcal{A}f(W, H) = (\pi'(\mu - r\mathbf{1}_n) + r)Wf_W + \frac{1}{2}\pi'\sigma\sigma'\pi\, f_{WW}W^2.$$

Now it is straightforward to verify that all of the results of Section III hold with the constant ω redefined to be

$$\omega = \frac{1}{2}(\mu - r\mathbf{1}_n)'(\sigma\sigma')^{-1}(\mu - r\mathbf{1}_n).$$

The optimal portfolio is still the mean-variance efficient portfolio

$$\pi^* = \frac{1}{1-\eta}(\sigma\sigma')^{-1}(\mu - r\mathbf{1}_n).$$

This confirms our previous conclusion that the portfolio manager will choose a potentially leveraged but efficient mean variance portfolio that keeps a constant exposure to the risky assets.

It is equally straightforward to extend the model to allow for a more general adjustment to the high-water mark. In particular, a common stipulation is to allow the high-water mark to increase at the rate g. It turns out that we can

[21] We make the calculations for this case available upon request.
[22] We assume throughout the invertibility of $\sigma\sigma'$.
[23] See Oksendal (1998).

allow for this extension within our framework by appropriately adjusting both the interest rate and the discount rate of the manager.

To see this result, notice that allowing the high-water mark to increase at the rate g will change the Bellman equation (5) to

$$0 = V_H H g - (\beta + \lambda)V + \max_{\pi_t} \left\{ V_W W_t (r + \pi_t(\mu - r)) + \frac{1}{2} V_{WW} W_t^2 \sigma^2 \pi_t^2 \right\}. \quad (45)$$

The boundary condition (4) remains the same. We next conjecture a value function of the form

$$V(H_t, W_t) = C H_t^{1-\eta} W_t^{\eta} \quad (46)$$

for appropriate constants C and η. Plugging (46) into (45) leads to

$$\eta^2(r - g) - (r + \beta + \lambda - 2g + \omega)\eta + (\beta + \lambda - g) = 0.$$

Notice that this equation is identical to (7) with the exception that $\beta + \lambda - g$ replaces $\beta + \lambda$ and $r - g$ replaces r. Accordingly, if the high-water mark increases at the rate g, this is observationally equivalent to reducing both the discount rate of the manager and the interest rate by g and then repeating the analysis in Section II.A.

V. Conclusions

The purpose of this paper is to study the portfolio choice incentives that are implied by high-water mark fees for hedge fund managers. We show that under certain regularity conditions, even a risk-neutral hedge fund manager has an incentive to choose an interior mean-variance efficient portfolio. The surprising result is that the optimization problem of the hedge fund manager allows an interior solution despite risk neutrality and the fact that termination is enforced either when wealth reaches zero or at some stochastic time that is independent of asset market events.

Our analysis yields two conclusions.

First, this paper draws a distinction between the effects of *option-like compensation* and those of *finite horizons*. In our model, compensation has option-like features, yet the portfolio is constant and bounded. By contrast, were the hedge fund to terminate at some predetermined time, the manager's preferred portfolio would become unbounded as she approached termination. We believe that this intuition extends beyond the specifics of the option-like compensation contract considered here. It appears that one can mitigate risk-shifting by making sure that there is a tradeoff between increasing variance in the "current" period and some sort of "punishment" in terms of the options that will follow after the current period.

Second, high-water mark contracts seem to provide the right incentives for portfolio choice, in the sense that the portfolio chosen exhibits two-fund separation: A constant fraction is invested in the riskless asset and the rest in a mean-variance efficient portfolio (given the manager's information set). By contrast, most of the literature on portfolio choice in the presence of benchmarking demonstrates the presence of biases in the portfolio towards a particular index.

Our conclusions are robust across several extensions of the basic framework to multiple assets, risk-averse preferences, and changes in the way the high-water mark is calculated.

We conclude with a caveat. The purpose of this paper is not to provide an exhaustive analysis of the incentive effects of high-water mark contracts. We make some extreme assumptions in order to demonstrate most clearly the disciplining effect of a continuation value on an in(de)finite horizon. In terms of future research, we believe that there are at least two interesting directions: first, to relax the exclusion of private accounts and instead impose the constraint that the manager cannot borrow against the future net present value of her income and second, to allow for optimal early termination of the manager by the investor.

Appendix A: Proof of Proposition 1

We present a verification theorem that can be invoked to verify that the conjecture that we made about the value function in (16) is actually correct. In doing so, we focus on the setup of Section II. Extending the proof to the setup of Sections III and IV is straightforward. The proof proceeds in two steps. First, we prove the verification theorem assuming that the fund is terminated when the high-water mark reaches a level $\bar{H}$. This is done in Proposition 3. Then we let $\bar{H} \to \infty$ and use the monotone convergence theorem.

It will save some notation to give the proof in terms of $\theta = \pi W$, total stock holdings, instead of π.

PROPOSITION 3: *Let $\tau^0 = \inf\{t : W_t = 0\}$ and $\tau^H = \inf\{t : H_t = \bar{H}\}$. Suppose that there exists a function that is twice continuously differentiable in W_t and once continuously differentiable in H_t, such that*

$$0 = -(\beta + \lambda)V + \max_{\theta}\left\{V_W(\theta\mu + (W_t - \theta)r) + \frac{1}{2}V_{WW}\sigma^2\theta^2\right\} \tag{A1}$$

and that

$$kV_W = k + V_H, \tag{A2}$$

whenever $W_t = H_t$. In addition, assume that the contract is terminated when $H_t = \bar{H}$, so that $V(W_t, \bar{H}) = 0$ for all $W_t \geq 0$, $V(0, H_t) = 0$ for all H_t, and that $V(W_t, H_t) \geq 0$. Suppose also that $V_W > 0$ and $V_{WW} < 0$ and let

$$\theta^* = -\frac{V_W}{V_{WW}}\frac{(\mu - r)}{\sigma^2}. \tag{A3}$$

Assume finally that θ^* and $\theta^* V_W$ are bounded in $[0, \bar{H}]$. For any admissible policy θ_t, denote by H_t^θ and W_t^θ the resulting processes for W_t and H_t.

Then

$$V(W_0, H_0) \geq E\left[\int_0^{\tau^0 \wedge \tau^{\bar{H}}} e^{-(\beta+\lambda)s} k\, dH_s^{\theta}\right] \tag{A4}$$

for all admissible controls θ, and equality holds for the optimal control (A3).

REMARK 1: *It is straightforward to verify that the function*

$$V(W_t, H_t) = W_t^{\eta} H_t^{k\eta}\left(\frac{kH^{1-(1+k)\eta}}{\eta(1+k)-1} - \frac{k\bar{H}^{1-(1+k)\eta}}{\eta(1+k)-1}\right) \tag{A5}$$

with η given in (9) satisfies all the conditions of the above proposition.

Proof of Proposition 3: The proof proceeds in essentially the same way as every verification theorem (see Fleming and Soner (1993), Chapter III; Oksendal (1998), Chapters 10 and 11; Browne (1997)). Applying Ito's Lemma to

$$M(t, W_t, H_t) = e^{-(\beta+\lambda)t} V(W_t, H_t) + \int_0^t e^{-(\beta+\lambda)s} k\, dH_s, \tag{A6}$$

for a time $t < \tau^0 \wedge T \wedge \tau^{\bar{H}}$, one obtains

$$\begin{aligned} M(t, W_t, H_t) = M(0, W_0, H_0) &+ \int_0^t e^{-(\beta+\lambda)s}\left[\mathcal{A}V(W_s, H_s; \theta_s)\right] ds \\ &+ \int_0^t e^{-(\beta+\lambda)s} \sigma\theta_s V_W\, dB_s + \int_0^t e^{-(\beta+\lambda)s} \left(k - V_W k + V_H\right) dH_s, \end{aligned}$$

where

$$\mathcal{A}V(W_s, H_s; \theta_s) = \frac{1}{2}\sigma^2\theta_s^2 V_{WW} + \theta_s\left[(\mu - r) V_W\right] + rWV_W - (\beta+\lambda)V.$$

By condition (A2), one can rewrite this as

$$\begin{aligned} M(t, W_t, H_t) = M(0, W_0, H_0) &+ \int_0^t e^{-(\beta+\lambda)s}\left[\mathcal{A}V(W_s, H_s; \theta_s)\right] ds \\ &+ \int_0^t e^{-(\beta+\lambda)s} \sigma\theta_s V_W\, dB_s. \end{aligned} \tag{A7}$$

Since $V_{WW} < 0$, $\mathcal{A}V(W_s, H_s; \theta_s)$ is a concave quadratic expression in θ_s, which attains its maximum at

$$\theta^* = -\frac{V_W}{V_{WW}} \frac{(\mu - r)}{\sigma^2},$$

with corresponding maximal value

$$\mathcal{A}V(W_s, H_s; \theta_s^*) = rWV_W - \omega\frac{V_W^2}{V_{WW}} - (\beta+\lambda)V = 0,$$

where the second equality follows from (A1). This observation implies that the second term on the right-hand side of (A7) is less than or equal to zero for any portfolio policy. Accordingly,

$$\begin{aligned}\int_0^{\tau^0\wedge T\wedge\tau^{\bar{H}}} e^{-(\beta+\lambda)s}\sigma\theta_s V_W\,dB_s &= M\left(\tau^0\wedge T\wedge\tau^{\bar{H}}, W_{\tau^0\wedge T\wedge\tau^{\bar{H}}}, H_{\tau^0\wedge T\wedge\tau^{\bar{H}}}\right)\\ &\quad - M(0, W_0, H_0) - \int_0^{\tau^0\wedge T\wedge\tau^{\bar{H}}} e^{-(\beta+\lambda)s}\left[\mathcal{A}V(W_s, H_s; \theta_s)\right]ds\\ &\geq M\left(\tau^0\wedge T\wedge\tau^{\bar{H}}, W_{\tau^0\wedge T\wedge\tau^{\bar{H}}}, H_{\tau^0\wedge T\wedge\tau^{\bar{H}}}\right) - M(0, W_0, H_0).\end{aligned} \tag{A8}$$

Now, by virtue of the assumption $V \geq 0$ for all H_t and W_t, it follows that the left-hand side is a local martingale that is bounded below by $(-M(0, W_0, H_0))$ and is thus a supermartingale. Because of this we can take expectations in (A7) and claim that

$$\begin{aligned}E\left[M(\tau^0\wedge T\wedge\tau^{\bar{H}}, W_{\tau^0\wedge T\wedge\tau^{\bar{H}}}, H_{\tau^0\wedge T\wedge\tau^{\bar{H}}})\right] &\leq E\int_0^{\tau^0\wedge T\wedge\tau^{\bar{H}}} e^{-(\beta+\lambda)s}\left[\mathcal{A}V(W_s, H_s; \theta_s)\right]ds\\ &\quad + M(0, W_0, H_0)\\ &\leq E\int_0^{\tau^0\wedge T\wedge\tau^{\bar{H}}} e^{-(\beta+\lambda)s}\sup_{\theta_s}\left[\mathcal{A}V(W_s, H_s; \theta_s)\right]ds\\ &\quad + M(0, W_0, H_0)\\ &= M(0, W_0, H_0).\end{aligned}$$

Because θ^* and $\theta^* V_W$ are bounded in $[0, \bar{H}]$, we can replace inequalities with equalities for the optimal policy as the left-hand side in (A8) is a martingale. Accordingly, for any admissible policy

$$\begin{aligned}V(W_0, H_0) &= M(0, W_0, H_0)\\ &\geq E\left[e^{-(\beta+\lambda)(\tau^0\wedge T\wedge\tau^{\bar{H}})}V\left(W_{\tau^0\wedge T\wedge\tau^{\bar{H}}}, H_{\tau^0\wedge T\wedge\tau^{\bar{H}}}\right)\right.\\ &\qquad \left. + \int_0^{\tau^0\wedge T\wedge\tau^{\bar{H}}} e^{-(\beta+\lambda)s}k\,dH_s\right].\end{aligned}$$

Taking limits as $T \to \infty$ and using the fact that for any policy

$$\lim_{T\to\infty} E\left[e^{-(\beta+\lambda)(\tau^0\wedge T\wedge\tau^{\bar{H}})}V\left(W_{\tau^0\wedge T\wedge\tau^{\bar{H}}}, H_{\tau^0\wedge T\wedge\tau^{\bar{H}}}\right)\right] = 0,$$

the monotone convergence theorem implies

$$\begin{aligned}
&\lim_{T\to\infty} E\Big[e^{-(\beta+\lambda)(\tau^0\wedge T\wedge\tau^{\bar H})}V\big(W^{\theta^*}_{\tau^0\wedge T\wedge\tau^{\bar H}}, H^{\theta^*}_{\tau^0\wedge T\wedge\tau^{\bar H}}\big) + \int_0^{\tau^0\wedge T\wedge\tau^{\bar H}} e^{-(\beta+\lambda)s} k\, dH^{\theta^*}_s\Big]\\
&= E\Big[\int_0^{\tau^0\wedge\tau^{\bar H}} e^{-(\beta+\lambda)s} k\, dH^{\theta^*}_s\Big]\\
&\geq \lim_{T\to\infty} E\Big[e^{-(\beta+\lambda)(\tau^0\wedge T\wedge\tau^{\bar H})}V\big(W^{\theta}_{\tau^0\wedge T\wedge\tau^{\bar H}}, H^{\theta}_{\tau^0\wedge T\wedge\tau^{\bar H}}\big) + \int_0^{\tau^0\wedge T\wedge\tau^{\bar H}} e^{-(\beta+\lambda)s} k\, dH^{\theta}_s\Big]\\
&= E\Big[\int_0^{\tau^0\wedge T} e^{-(\beta+\lambda)s} k\, dH^{\theta}_s\Big].
\end{aligned}$$

This concludes the proof. Q.E.D.

To obtain a verification theorem for the infinite horizon problem one needs to observe that the expression in (A5) approaches our conjecture for the value function as $\bar H \to \infty$. The result that the solution to the infinite horizon problem is given by (16) follows by the monotone convergence theorem.

Appendix B: Other Proofs

Proof of Lemma 1: In light of (17), it suffices to show that $n \to \infty$ implies $\eta \to 1$. Remember that $0 \leq \eta \leq 1$ for any n, as shown in the discussion following formula (10) in Section II.A. From equation (7) and $\lambda_n = \frac{n}{T}$, we have that η solves

$$\eta^2 r - \Big(r + \beta + \frac{n}{T} + \omega\Big)\eta + \Big(\beta + \frac{n}{T}\Big) = 0.$$

Dividing by n and rearranging gives

$$\eta = \frac{\frac{1}{T} + \frac{\eta^2 r}{n} + \frac{\beta}{n}}{\frac{1}{T} + \frac{r+\beta+\omega}{n}}.$$

Because η is bounded between zero and one, so is η^2. Since T is constant, we obtain the limit

$$\lim_{n\to\infty} \eta = 1.$$

The fact that $\pi \to \infty$ is then a direct implication of (17). Q.E.D.

Proof of Lemma 2: Remember that the optimal portfolio is given as

$$\pi^* = \frac{1}{1-\eta}\frac{\mu - r}{\sigma^2},$$

and that η is described by

$$\eta^2 r - (r + \beta + \lambda + \omega)\eta + (\beta + \lambda) = 0.$$

Since $\mu \to r$ implies $\omega \to 0$, we have that $\eta \to 1$ by inspection. (Remember that the second quadratic root is disregarded.)

Now, $\mu \to r$ implies that both the numerator and denominator of the optimal portfolio policy go to zero, so, we apply L'Hospital's rule. This gives

$$\lim_{\mu \to r} \pi^* = \frac{1}{\sigma^2} \lim_{\mu \to r} \frac{1}{-\eta_\omega \omega_\mu}, \tag{B1}$$

where the subscripts represent derivatives. We apply the implicit function theorem to the equation describing η and we find that

$$\frac{d\eta}{d\omega} = -\frac{-\eta}{2\eta r - (r + \beta + \lambda + \omega)}.$$

When $\mu = r$, we have $\omega = 0$ and $\eta = 1$, so that

$$\lim_{\mu \to r} \frac{d\eta}{d\omega} = \frac{1}{r - (\beta + \lambda)}.$$

By assumption, we have $\beta + \lambda > r$, and so, equation (B1) becomes

$$\lim_{\mu \to r} \pi^* = \frac{1}{\sigma^2} \lim_{\mu \to r} \frac{1}{-\eta_\omega \omega_\mu} = \frac{\beta + \lambda - r}{\sigma^2} \lim_{\mu \to r} \frac{1}{\frac{\mu - r}{\sigma^2}} = \infty.$$

Q.E.D.

Proof of Proposition 2: The value function is

$$\frac{kH_t}{\alpha(\pi)(1+k) - 1} \times \left(\frac{W_t}{H_t}\right)^{\alpha(\pi)}.$$

Because of condition (15) and $\frac{W_t}{H_t} \leq 1$, both terms are separately maximized when $\alpha(\pi)$ is minimized. Hence, the value function will be maximized when $\alpha(\pi)$ is minimized. We will show that the minimum value of $\alpha(\pi)$ is achieved when equation (33) is met.

One can directly differentiate $\alpha(\pi)$ w.r.t. π, but this appears to be cumbersome. However, there is an easier way to proceed. First, we start with the observation that the π that minimizes $\alpha(\pi)$ must be positive as long as $\mu > r$. To see this, assume otherwise that the π that minimizes $\alpha(\pi)$ is negative, and consider $\alpha(-\pi)$ which is given by

$$\alpha(-\pi) = \frac{\sqrt{l(-\pi)^2 + 2(\beta + \lambda)\sigma^2\pi^2} - l(-\pi)}{\sigma^2\pi^2}$$

and also

$$l(-\pi) = \left(r + (-\pi)(\mu - r) - \frac{1}{2}\sigma^2\pi^2\right) > l(\pi),$$

since $\mu > r$. Note now that any expression of the form

$$\sqrt{l^2 + c} - l$$

is decreasing in l for any l and $c > 0$, which means that

$$\alpha(-\pi) < \alpha(\pi),$$

which is a contradiction. Hence, as long as $\mu > r$, we can constrain attention to the nonnegative axis to determine the value of π that minimizes $\alpha(\pi)$. The next step is to use the implicit function theorem on equation (30) to obtain

$$\frac{d\alpha}{d\pi} = -\frac{\alpha\left[(\mu - r) - \sigma^2\pi\right] + \sigma^2\pi\alpha^2}{\left(r + \pi(\mu - r) - \frac{1}{2}\sigma^2\pi^2\right) + \sigma^2\pi^2\alpha}. \tag{B2}$$

Setting the numerator equal to zero yields

$$\pi^* = \frac{1}{1 - \alpha(\pi^*)}\frac{(\mu - r)}{\sigma^2}.$$

We now show that $\alpha(\pi^*) = \eta$. Replace π^* in equation (30) and use the definition of ω from (8) to obtain

$$-(\beta + \lambda) + \alpha\left(r + \frac{1}{1 - \alpha}2\omega - \frac{\omega}{(1 - \alpha)^2}\right) + \frac{\omega\alpha^2}{(1 - \alpha)^2} = 0.$$

Rearranging yields

$$-\alpha^2 r + \alpha\,(r + \beta + \lambda + \omega) - (\beta + \lambda) = 0,$$

which is exactly equation (7), which defines η^- and η^+. Since we want $\pi^* > 0$, we can restrict attention to $\eta^- = \eta < 1$.

To check the second-order conditions, the uniqueness of the local extremum implies we need only verify

$$\alpha(\pi^*) < \lim_{\pi \to 0} \alpha(\pi), \tag{B3}$$

$$\alpha(\pi^*) < \lim_{\pi \to \infty} \alpha(\pi). \tag{B4}$$

Since $\lim_{\pi \to \infty} \alpha(\pi) = 1$, equation (B4) holds because $\alpha(\pi^*) < 1$. Since $\lim_{\pi \to 0} \alpha(\pi) = \infty$, equation (B3) holds. Q.E.D.

Proof of Lemma 3: We prove the first result only for β. The result for λ follows similarly. Remember that $0 \leq \eta \leq 1$. Using the implicit function theorem on

$$\eta^2 r - (r + \beta + \lambda + \omega)\eta + (\beta + \lambda) = 0 \tag{B5}$$

yields

$$\eta_\beta = \frac{\eta\,(1-\eta)}{(\beta+\lambda)-\eta^2 r} > 0,$$

where the inequality follows from $\eta \in (0,1)$ and the assumption $r < \beta+\lambda$. This means that

$$\frac{\partial\left(\frac{1}{1-\eta}\right)}{\partial\beta} > 0,$$

and $\frac{\partial}{\partial\beta}\pi^* > 0$ follows from (17). Identical steps lead to $\frac{\partial}{\partial\lambda}\pi^* > 0$.

We now examine $\frac{\partial}{\partial\mu}\pi^* > 0$. Differentiating (17) yields

$$\frac{\partial\pi}{\partial\mu} = \frac{1}{\sigma^2}\left(\frac{1-\eta+\eta_\omega\omega_\mu\,(\mu-r)}{(1-\eta)^2}\right). \tag{B6}$$

In the proof of Lemma 2, we obtain

$$\frac{d\eta}{d\omega} = \frac{\eta}{2\eta r-(r+\beta+\lambda+\omega)},$$

and from the definition of ω in (8), we obtain

$$\omega_\mu\,(\mu-r) = 2\omega.$$

Using these in (B6), we have

$$\operatorname{sgn}\left(\frac{\partial\pi}{\partial\mu}\right) = \operatorname{sgn}\left[1-\eta+\frac{2\eta\omega}{2\eta r-(r+\beta+\lambda+\omega)}\right].$$

Using (7) and rearranging gives

$$1-\eta+\frac{2\eta\omega}{2\eta r-(r+\beta+\lambda+\omega)} = \frac{(\beta+\lambda-r-\omega)(1-\eta)}{2\eta r-(r+\beta+\lambda+\omega)},$$

which is negative when $\beta+\lambda > r+\omega$. Q.E.D.

Proof of Lemma 4: Since

$$\operatorname{sgn}(V_H) = \operatorname{sgn}\left(F'(H_t)\right),$$

it suffices to show that $F'(H_t) < 0$. The derivative $F'(H_t)$ is given by

$$\begin{aligned}F'(H_t) &= k^{1-\gamma}\left[-H_t^{-(\gamma+\eta)}\left(1-\frac{H_0-\frac{Z_0}{k}}{H_t}\right)^{-\gamma}+\eta k H_t^{\eta k-1}\int_{H_t}^{\infty}x^{-\eta(1+k)-\gamma}\left(1-\frac{H_0-\frac{Z_0}{k}}{x}\right)^{-\gamma}dx\right]\\ &= k^{1-\gamma}H_t^{-(\gamma+\eta)}\left(1-\frac{H_0-\frac{Z_0}{k}}{H_t}\right)^{-\gamma}\left[\eta k H_t^{\eta(1+k)+\gamma-1}\int_{H_t}^{\infty}x^{-\eta(1+k)-\gamma}\left(\frac{1-\frac{H_0-\frac{Z_0}{k}}{x}}{1-\frac{H_0-\frac{Z_0}{k}}{H_t}}\right)^{-\gamma}dx-1\right].\end{aligned}$$

Since $0 < H_0 - \frac{Z_0}{k} \leq H_t \leq x$, it follows that

$$\frac{H_0 - \frac{Z_0}{k}}{x} \leq \frac{H_0 - \frac{Z_0}{k}}{H_t},$$

so that

$$\left(\frac{1 - \frac{H_0 - \frac{Z_0}{k}}{x}}{1 - \frac{H_0 - \frac{Z_0}{k}}{H_t}} \right)^{-\gamma} \leq 1.$$

As a result,

$$\begin{aligned}
F'(H_t) &\leq k^{1-\gamma} H_t^{-(\gamma+\eta)} \left(1 - \frac{H_0 - \frac{Z_0}{k}}{H_t} \right)^{-\gamma} \left[\eta k H_t^{\eta(1+k)+\gamma-1} \int_{H_t}^{\infty} x^{-\eta(1+k)-\gamma} dx - 1 \right] \\
&= k^{1-\gamma} H_t^{-(\gamma+\eta)} \left(1 - \frac{H_0 - \frac{Z_0}{k}}{H_t} \right)^{-\gamma} \left[\frac{\eta k}{\eta(1+k)+\gamma-1} - 1 \right] \\
&= k^{1-\gamma} H_t^{-(\gamma+\eta)} \left(1 - \frac{H_0 - \frac{Z_0}{k}}{H_t} \right)^{-\gamma} \left[\frac{1-\gamma-\eta}{\eta(1+k)+\gamma-1} \right] \\
&< 0
\end{aligned}$$

and therefore $V_H < 0$. Q.E.D.

Proof of Lemma 5: Since

$$\operatorname{sgn}(V_k) = \operatorname{sgn}(F_k),$$

it suffices to check that $F_k \geq 0$. In this proof we shall illustrate the existence of values of $\gamma \in (0, 1)$, such that $F_k \geq 0$, whenever the conditions of the lemma hold.

The derivative F_k is given by

$$\begin{aligned}
F_k = (1-\gamma)k^{-\gamma} &\left(\int_{H_t}^{\infty} x^{-\eta(1+k)} \left(x - H_0 + \frac{Z_0}{k} \right)^{-\gamma} dx \right) H_t^{\eta k} \\
&- \eta k^{1-\gamma} \left(\int_{H_t}^{\infty} \left[\log(x) - \log(H_t) \right] x^{-\eta(1+k)} \left(x - H_0 + \frac{Z_0}{k} \right)^{-\gamma} dx \right) H_t^{\eta k} \\
&+ \gamma Z_0 k^{-\gamma-1} \left(\int_{H_t}^{\infty} x^{-\eta(1+k)} \left(x - H_0 + \frac{Z_0}{k} \right)^{-\gamma-1} dx \right) H_t^{\eta k}.
\end{aligned}$$

Since the last term is clearly nonnegative, we can combine the first two terms in one expression and obtain

$$F_k \geq \frac{H_t^{\eta k}}{k^{\gamma}} \left(\int_{H_t}^{\infty} [(1-\gamma) - \eta k(\log(x) - \log(H_t))] \left(1 - \frac{H_0 - \frac{Z_0}{k}}{x} \right)^{-\gamma} x^{-\eta(1+k)-\gamma} dx \right).$$

Examining the terms inside the integral we observe that $1-\gamma>0$ and that $-\eta k(\log(x)-\log(H_t))<0$. Moreover, it is clear that $1-\gamma-\eta k(\log(x)-\log(H_t))>0$ if $x=H_t$ and that $1-\gamma-\eta k(\log(x)-\log(H_t))<0$ as $x\to\infty$. Hence, the term $(1-\gamma)-\eta k(\log(x)-\log(H_t))$ will be positive close to the lower integration limit and will become negative for large enough x. Let $\bar{x}$ denote the (unique) value of x where $(1-\gamma)-\eta k(\log(x)-\log(H_t))=0$.

Given the assumption $H_0>\frac{Z_0}{k}$, it is straightforward to show that $(1-\frac{H_0-\frac{Z_0}{k}}{x})^{-\gamma}$ is a declining function of x. This implies that

$$F_k \geq$$

$$\frac{H_t^{\eta k}}{k^{\gamma}}\left(1-\frac{H_0-\frac{Z_0}{k}}{\bar{x}}\right)^{-\gamma}\left(\int_{H_t}^{\infty}\left[(1-\gamma)-\eta k(\log(x)-\log(H_t))\right]\frac{\left(1-\frac{H_0-\frac{Z_0}{k}}{x}\right)^{-\gamma}}{\left(1-\frac{H_0-\frac{Z_0}{k}}{\bar{x}}\right)^{-\gamma}}x^{-\eta(1+k)-\gamma}\,dx\right)$$

$$>\frac{H_t^{\eta k}}{k^{\gamma}}\left(1-\frac{H_0-\frac{Z_0}{k}}{\bar{x}}\right)^{-\gamma}\left(\int_{H_t}^{\infty}\left[(1-\gamma)-\eta k(\log(x)-\log(H_t))\right]x^{-\eta(1+k)-\gamma}\,dx\right),$$

since the declining function

$$\frac{\left(1-\frac{H_0-\frac{Z_0}{k}}{x}\right)^{-\gamma}}{\left(1-\frac{H_0-\frac{Z_0}{k}}{\bar{x}}\right)^{-\gamma}}$$

will "weight" by a factor more than one the positive parts of

$$[(1-\gamma)-\eta k(\log(x)-\log(H_t))]$$

and by a factor less than one the negative parts of it. Hence, we are left with showing that

$$\int_{H_t}^{\infty}[(1-\gamma)-\eta k(\log(x)-\log(H_t))]x^{-\eta(1+k)-\gamma}\,dx>0.$$

This integral can be rewritten as

$$[(1-\gamma)+\eta k\log(H_t)]\int_{H_t}^{\infty}x^{-\eta(1+k)-\gamma}\,dx-\eta k\int_{H_t}^{\infty}\log(x)x^{-\eta(1+k)-\gamma}\,dx,$$

which, after using integration by parts on the second term, is equal to

$$-\frac{(1-\gamma)+\eta k\log(H_t)}{1-\eta(1+k)-\gamma}H_t^{1-\eta(1+k)-\gamma}+$$

$$+\eta k\left[\log(H_t)\frac{H_t^{1-\eta(1+k)-\gamma}}{1-\eta(1+k)-\gamma}\right]+\frac{\eta k}{1-\eta(1+k)-\gamma}\int_{H_t}^{\infty}x^{-\eta(1+k)-\gamma}\,dx.$$

Computing the integral associated with the third term and simplifying, we can reduce this expression to

$$\left(-\frac{1-\gamma}{1-\eta(1+k)-\gamma}-\frac{\eta k}{\left(1-\eta(1+k)-\gamma\right)^2}\right)H_t^{1-\eta(1+k)-\gamma}.$$

In order to establish the claim of the lemma it suffices to show that

$$-\frac{1-\gamma}{1-\eta(1+k)-\gamma}-\frac{\eta k}{\left(1-\eta(1+k)-\gamma\right)^2}>0$$

or

$$\frac{\left(1-\gamma\right)\left(\gamma+\eta(1+k)-1\right)-\eta k}{\left(1-\eta(1+k)-\gamma\right)^2}>0.$$

The denominator is positive. The sign of the numerator

$$\left(1-\gamma\right)\left(\gamma+\eta(1+k)-1\right)-\eta k \tag{B7}$$

will depend on the parameters of the problem. However, we can show that there will always exist values of γ that will make the numerator positive as long as condition (43) holds. To see this, maximize (B7) over γ to obtain the optimal γ^*:

$$\gamma^*=1-\frac{\eta}{2}(1+k).$$

Given assumption (43), γ^* is between zero and one. Substituting γ^* into (B7) gives

$$\frac{\eta^2(1+k)^2}{4}-\eta k,$$

which is larger than zero by assumption (43). Q.E.D.

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