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Cross-Asset Speculation in Stock Markets

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ABSTRACT

In practice, heterogeneously informed speculators combine private information about multiple stocks with information in prices, taking into account how their trades influence the inferences of other speculators via prices. We show how this speculation causes prices to be more correlated than asset fundamentals, raising price volatility. The covariance structure of asset fundamentals drives that of prices, while the covariance structure of liquidity trade drives that of order flows. We characterize how speculator profits vary with the distributions of information and liquidity trade across assets and speculators, and relate the cross-asset factor structure of order flows to that of returns.

STOCK MARKETS ARE CHARACTERIZED by many stocks and many traders. Speculators, especially institutional traders, often interact strategically in many stocks simultaneously. When asset values are correlated, private information about one stock may well provide a speculator information about other stocks that he will incorporate in his trading. For example, a speculator with private information about a new drug from Merck that affects the value of Glaxo Smith Kline may trade both stocks.

But strategic interactions are far richer than just this observation suggests. In practice, speculators also observe prices and use that information when trading. Concretely, traders may use information in the price of Glaxo to adjust trades of Merck. Speculators also understand that their trades convey information to others via their impacts on prices and that prices reflect information of other speculators that they, themselves, can use.

The contribution of this paper is to characterize equilibrium outcomes in this setting. To do so, we first solve for how speculators combine private information about different assets with the public information in prices to determine how much of each stock to trade. We then derive explicit analytical characterizations of the correlation structure of prices and order flows, determining both how are they driven by the correlation structure of asset fundamentals and liquidity trade, and how they are affected by the observability of prices; in addition, we quantify the links between the factor structures of order flows and

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prices. Finally, we document numerically how trading strategies and profits vary with the distributions of information and liquidity trade across assets and speculators.

The central studies of multi-asset stock markets are Admati (1985) and Caballe and Krishnan (1994). Each of these papers simplifies the strategic setting along one dimension to obtain answers about other dimensions. Admati develops a multi-asset, noisy rational expectations model with a continuum of agents. Her model has the feature that agents condition trades on both their private signals and stock prices. However, the continuum of agents precludes individual strategic behaviour—the informationally small agents ignore the price impact of their trades and do not have to account for how their trades affect information release. But, in practice, informed agents for a given stock are few in number. Those few speculators are typically institutional traders who understand that their significant trading has price impacts that they should and do anticipate and internalize. To realistically model trading, it is important to account for such strategic informed trade.

Caballe and Krishnan (1994) develop a static multi-asset formulation of the competitive dealership model of Kyle (1985) and Admati and Pfleiderer (1988): Speculators internalize the price impacts of their trades, but do not see prices. Hence, as in Admati (1985), speculators need not worry about the information their trades convey to other speculators. But, in practice, speculators use information in stock prices when trading, and they worry about information dissemination via prices.

Our model melds these two models, creating a strategic analogue of Admati's noisy rational expectations equilibrium (REE) model, along the lines of Kyle (1989).[1] As in Admati, risk-neutral speculators combine private information about various stocks with the information in prices to determine how much of each stock to trade. As in Caballe and Krishnan, speculators internalize how trades influence prices. In dynamic stock markets, speculators use information in existing prices when determining trades, and strategically account for how, via prices, their trades influence the inferences of other speculators, and hence the trades of other speculators. Combining observable prices with strategic speculators lets us capture this key feature of real world markets in a tractable static setting. In particular, a speculator in our model must account for how, via prices, his trades influence those of other speculators.

We first derive the form of equilibrium strategies. We prove that strategies have a forecast error structure: From their direct trades on private signals,

[1] There is a large literature on the Kyle (1985, 1989) models of speculation with one asset in one market; see the review by Biais, Glosten, and Spatt (2005). More closely related is the literature on splitting orders for the same asset across multiple markets, initiated by Chowhdry and Nanda (1991) and Dennert (1993), and extended by Bernhardt and Hughson (1997) and Biais, Martimort, and Rochet (2002) to allow for separated, strategic market makers who internalize how their pricing influences the division of orders by traders across markets. Other papers consider parameterized multi-asset models in which market makers only see their own order flow and look at cross-autocorrelations in returns (see Chan (1993), Bernhardt and Mahani (2007), or Chordia and Swaminathan (2004)).

traders subtract the projections onto net order flows, so that a speculator's net trades correspond to the errors in market maker forecasts of his direct trades. The economics underlying this result is that a speculator's relevant private information consists of the differences between what his signals are and what market makers perceive those signals to be.

We then prove that there is a unique linear equlibrium. Establishing the existence of an equilibrium is a formidable task due to the matrix structure inherent in multi-asset settings. With observable prices and nonstrategic traders, Admati (1985) sets out (p. 633) the technical difficulties that explain why the methods used by Hellwig (1980) to prove existence for a single asset do not extend. Indeed, Admati does not prove existence. Strategic, informationally large speculators present further challenges: We must resolve the forecasting-the-forecasts-of-others issues that arise in this multi-asset strategic setting when speculators use private information to filter the information in prices (see Townsend (1985), Pearlman and Sargent (2005), or Malinova and Smith (2003)).

To prove existence, we set up an iterative best-response mapping. Given an initial conjecture about trading strategies—a conjectured matrix of trading intensities—we compute the associated pricing, and solve for how speculators should filter prices. We then compute best responses by speculators to these conjectures and iterate, using these best responses as the next conjectures. While best responses are not well behaved, their inner product is, and for this recursion we derive conditions under which this mapping is a contraction, whose fixed point corresponds to the equilibrium. This proof is novel and unrelated to standard contraction mapping constructions in growth theory.[2]

We go beyond this fundamental analysis to provide new analytical and quantitative characterizations of the economic structure of both our environment and that of Caballe and Krishnan. By contrasting equilibrium outcomes when speculators can see prices with those when they cannot, we show how the informational structure of the market matters.

(1) We derive analytically the correlation structures of order flows and prices. We prove that in our observable price framework, these correlation structures are dual in the following sense: The correlation structure of prices is driven only by the correlation structure of asset value fundamentals, while the correlation structure of order flows is driven only by the correlation structure of liquidity trade. The economic logic for these results is also dual: Were the correlation structure of liquidity trade to affect that of prices, then prices would be systematically driven by processes other than value, and risk-neutral speculators would exploit such systematic mispricing; and were the asset value correlation structure to affect that of order flows, then market makers would unravel it, lowering speculator profits.

[2] Bernhardt, Taub, and Seiler (2007) analyze a dynamic stationary model with a single stock and provide a related contraction mapping argument on a function space.

(2) We prove that in integrated markets, prices are *more* correlated than the underlying fundamentals, especially when traders see prices. This is because in integrated markets, market makers exploit the information in cross-asset order flows. The resulting high cross-asset correlations in prices raise price volatility.

(3) Outcomes are very different when speculators cannot see prices. Most starkly, with unobservable prices, symmetrically situated speculators, and uncorrelated liquidity trade, speculators do *not* trade on cross-asset information. For example, a speculator with a signal about Merck's value that is correlated with Glaxo's value only uses that signal to trade Merck. It is not that the information structure is irrelevant, but the opposite: Speculators trade only on direct signals *because* order flow for Merck also affects the price of Glaxo. In contrast, when speculators see prices, they use Merck signals to trade Glaxo because they can subtract from their trades on signals the projections onto prices, reducing the information that market makers can extract from each order flow, thereby mitigating the total price impacts of their trades.

(4) We derive how the factor structure of order flow is related to that of prices. Hasbrouck and Seppi (2001) apply a principal component decomposition of cross-asset order flows and returns, and find that commonality in order flows explains roughly two-thirds of the commonality in returns. We carry out a similar decomposition in our model. We find that a moderate cross-asset correlation in asset value fundamentals plus a slight positive correlation in liquidity trade across assets can quantitatively match their empirical finding.[3] The primitive parameters that we identify—the correlations in information and liquidity trade—thus provide a theoretical underpinning to Hasbrouck and Seppi's purely empirical analysis.

We conclude by providing the first quantitative characterizations of equilibrium outcomes in multi-asset strategic settings for both symmetric and asymmetric environments.

(1) We find that observable prices drive expected speculator profits down sharply, especially when there are more speculators. Underlying this result is the information exchange through prices, which induces speculators to trade more aggressively. In particular, each speculator internalizes the fact that as he increases an order, other speculators see the price impacts and reduce their orders.

(2) We find that the impact of the division of information between speculators is very different from the impact of the division of information or liquidity trade across assets. Total speculator profits are very sensitive to the division of information between speculators, falling sharply when information is more evenly divided and prices are observable. In stark contrast, divisions of information or liquidity trade across *assets* have

[3] Interestingly, this qualitative result also holds when prices are unobservable.

only second-order effects on total speculator profits. However, this profit irrelevance result *masks* the complicated ways in which asymmetries across assets interact with the market structure to influence *how* speculators trade, and how equilibrium prices weight order flows in each market. For example, speculators may trade against their cross-asset information in a relatively illiquid market.

We next set out the model and analyze a speculator's optimization problem. In Section II, we then specialize to a symmetric setting to obtain analytical results. Section III offers numerical characterizations, showing how asymmetries affect outcomes. A conclusion follows in Section IV. All proofs are given in the Appendix.

I. The Model

In our multi-asset model of speculative trade in a stock market, speculators are strategic and internalize how their trades affect prices, and hence the inferences and trades of other speculators. N risk-neutral informed speculators and exogenous noise traders trade claims to M assets. Prices are set by risk-neutral, competitive, uninformed market makers.

We denote the vector of asset values by $v' = (v_1, \ldots, v_M)$. We consider a very general formulation in which asset values are linear functions of K underlying fundamentals, $e' = (e_1, \ldots, e_K)$. These K fundamentals are jointly normally distributed with means that we normalize to zero and an arbitrary variance–covariance matrix, Σ_e. The value of asset j is given by

$$v_j = v_{j1}e_1 + v_{j2}e_2 + \cdots + v_{jK}e_K.$$

We can write the vector of asset values as $v = Ve$, where V is the $M \times K$ matrix,

$$\begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1K} \\ v_{21} & v_{22} & \cdots & v_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ v_{M1} & v_{M2} & \cdots & v_{MK} \end{pmatrix}.$$

We allow for the possibility that each speculator has access to many sources of private information about asset values. Specifically, we let speculator i see a vector of signals about the value fundamentals, $s^i = A^i e$, where A^i is an $L_i \times K$ matrix,

$$\begin{pmatrix} s^i_1 \\ s^i_2 \\ \vdots \\ s^i_{L_i} \end{pmatrix} = \begin{pmatrix} A^i_{11} & A^i_{12} & \cdots & A^i_{1K} \\ A^i_{21} & A^i_{22} & \cdots & A^i_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ A^i_{L_i1} & A^i_{L_i2} & \cdots & A^i_{L_iK} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_K \end{pmatrix}.$$

Example: One speculator sees an asset's value. There are M assets, $N = M$ speculators, and $K = M$ fundamentals. The value of asset j is $v_j = e_j$, that is, V is the $M \times M$ identity matrix. Only speculator i sees $v_i = e_i$, that is, $s^i_i = e_i$ and $s^i_j = 0, j \neq i$. The A^i matrix capturing speculator i's information is an $M \times M$ matrix, which has only zero entries save for A^i_{ii}, which is one.

Example: Each speculator receives one signal for each asset. There are M assets, N speculators, and $K = MN$ fundamentals. The value of asset j is $v_j = e_{(j-1)N+1} + \cdots + e_{(j-1)N+i} + \cdots + e_{jN}$. Thus, V is an $M \times MN$ matrix, where the j^{th} row of V captures the value of asset j, so it has $v_{j,(j-1)N+i} = 1$ for $i = 1, \ldots, N$, and zeros elsewhere. Speculator i sees $s^i_j = e_{(j-1)N+i}$ for asset $j = 1, \ldots M$. Then A^i is an $M \times MN$ matrix, where the j^{th} row of the A^i matrix captures speculator i's information about asset j: It has $A^i_{j,(j-1)N+i} = 1$, and zeros elsewhere.

In addition to his signals s^i, speculator i knows the prices at which orders for each asset will be executed. That is, our economy is a noisy rational expectations economy in which agents are strategic and internalize how their trades affect prices and the information content of prices. It is in this sense that our model extends and melds the models of Admati (1985) and Caballe and Krishnan (1994), allowing us to capture key aspects of a dynamic market in a simpler static setting.

Let x^i_j be speculator i's order for asset j, let $x^{i'} = (x^i_1, x^i_2, \ldots, x^i_M)$ be the vector of his orders, and let $X' = (X_1, X_2, \ldots, X_M)' = (\sum_{i=1}^N x^i_1, \sum_{i=1}^N x^i_2, \ldots, \sum_{i=1}^N x^i_M)$ be the vector of net order flows across assets from speculators. In addition to trade from speculators, there is exogenous liquidity trade of asset j of u_j. We let $u' = (u_1, u_2, \ldots, u_M)$ be the vector of liquidity trades. Liquidity trade is distributed independently of the value fundamentals, and hence of speculator signals. We allow for a general correlation structure in liquidity trade across assets, assuming only that liquidity trade is jointly normally distributed with zero mean and variance–covariance matrix Σ_u.

Markets are cleared by competitive market makers who use the trading activity throughout the entire stock market when setting prices. In particular, a market maker observes the net order flow for his asset, and the prices of other assets. Equivalently, with linear pricing, market makers see the vector of net order flows for all assets, $(X+u)' = (X_1 + u_1, X_2 + u_2, \ldots X_m + u_m)$. Market makers set the price of each asset equal to its expected value given the vector of net order flows, that is, $p_j = E[v_j|(X+u)']$: Competition drives market maker expected profits from filling each order down to zero. At the period's end, assets are liquidated and trading profits are realized.

We focus on equilibria in which speculators adopt trading strategies that are linear functions of both their private signals and net order flows, and market makers set prices that are linear functions of net order flows. Specifically, we conjecture that speculator i's order for asset j takes the form

$$\begin{aligned} x^i_j &= b^i_{j1}s^i_1 + b^i_{j2}s^i_2 + \cdots + b^i_{jL_i}s^i_{jL_i} + B^i_{j1}(X_1 + u_1) + B^i_{j2}(X_2 + u_2) \\ &\quad + \cdots + B^i_{jM}(X_M + u_M). \end{aligned}$$

Hence, the vector of speculator i's orders is $x^i = b^i s^i + B^i(X + u)$, that is,

$$\begin{pmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_M^i \end{pmatrix} = \begin{pmatrix} b_{11}^i & b_{12}^i & \cdots & b_{1L_i}^i \\ b_{21}^i & b_{22}^i & \cdots & b_{2L_i}^i \\ \vdots & \vdots & \ddots & \vdots \\ b_{M1}^i & b_{M2}^i & \cdots & b_{ML_i}^i \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{L_i} \end{pmatrix} + \begin{pmatrix} B_{11}^i & B_{12}^i & \cdots & B_{1M}^i \\ B_{21}^i & B_{22}^i & \cdots & B_{2M}^i \\ \vdots & \vdots & \ddots & \vdots \\ B_{M1}^i & B_{M2}^i & \cdots & B_{MM}^i \end{pmatrix} \begin{pmatrix} X_1 + u_1 \\ X_2 + u_2 \\ \vdots \\ X_M + u_M \end{pmatrix}.$$

Thus, b^i represents i's trading intensities on his signals, and B^i represents i's trading intensities on the net order flows that he sees. The conjecture that pricing is a linear function of net order flows implies that $p = \lambda(X + u)$, where the pricing matrix λ is an $M \times M$ invertible matrix that aggregates information from the order flows of all assets to determine prices, that is,

$$\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_M \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1M} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{M1} & \lambda_{M2} & \cdots & \lambda_{MM} \end{pmatrix} \begin{pmatrix} X_1 + u_1 \\ X_2 + u_2 \\ \vdots \\ X_M + u_M \end{pmatrix}.$$

The extent to which a market maker uses information from other stocks to price a particular stock, and not just a stock's own order flow, is captured by the nondiagonal elements of λ. Because the pricing matrix λ is invertible, observing prices is equivalent to observing net order flows.

Equilibrium. In a linear equilibrium, speculators maximize expected trading profits given correct conjectures, net order flows are consistent with speculator optimization, and market makers earn zero expected profits from filling each order given the vector of net order flows.

Speculator's Optimization Problem. Speculator i seeks to maximize expected trading profits given his signals s^i and the net order flows $X + u$, solving

$$\max_{x^i} E[(v - p)' x^i | s^i, X + u] = \max_{x^i} E\left[\left(v - \lambda\left(\sum_{k=1}^{N} x^k + u\right)\right)' x^i | s^i, X + u\right]. \quad (1)$$

To solve for speculator i's best response to the conjectured linear trading strategies of the other speculators and the linear pricing, we first rewrite the vector of net order flows from i's perspective,

$$X + u = x^i + \sum_{k \neq i} (b^k s^k + B^k (X + u)) + u. \tag{2}$$

Defining $q^i \equiv (I - \sum_{k \neq i} B^k)^{-1}$, we solve equation (2) for

$$X + u = q^i \left(x^i + \sum_{k \neq i} b^k s^k + u \right). \tag{3}$$

Speculator i understands and internalizes how his orders affect the trades of other speculators through price via q^i. Substituting for $X + u$ into speculator i's objective yields

$$\max_{x^i} E \left[\left(v - \lambda q^i \left(x^i + \sum_{k \neq i} b^k s^k + u \right) \right)' x^i | s^i, X + u \right]. \tag{4}$$

The associated first-order condition with respect to x^i is

$$0 = E[(v - \lambda(X + u))' - q^{i'} \lambda' x^i | s^i, X + u]. \tag{5}$$

Proposition 1 describes the forecast error structure of a speculator's orders.

PROPOSITION 1: *Speculator i's vector of orders, x^i, is a linear function of the forecast errors of his trades on his private signals.*

Proposition 1 reflects that a speculator's relevant private information consists of the differences between what his signals s^i are, and what the market maker perceives them to be—that is, the errors in the market maker's forecasts of his signals, $s^i - E[s^i | X + u]$. Accordingly, the vector of speculator i's net orders, x^i, is a linear function of these forecast errors. That is, B^i is a matrix of projection coefficients of trading intensities on private signals onto net order flows: From their direct trades on private signals, traders subtract the projections of these trades onto net order flows, so that a speculator's net trades correspond to the market maker's forecast error of his trades on his signals.

Substituting this forecast error structure, we drop the conditioning on $X + u$ from the speculator's first-order conditions and express the first-order conditions solely in terms of the fundamentals. To do so, we first solve for net order flows in terms of trading intensities on fundamentals:

$$X + u = \sum_{k=1}^{N} b^k s^k + \sum_{k=1}^{N} B^k (X + u) + u$$
$$\Rightarrow X + u = \left(I - \sum_{k=1}^{N} B^k \right)^{-1} \left(\sum_{k=1}^{N} b^k s^k + u \right).$$

We then substitute this solution for $X + u$ into speculator i's trade on net order flow,

$$B^i(X+u) = B^i\left(I - \sum_{k=1}^{N} B^k\right)^{-1}\left(\sum_{k=1}^{N} b^k s^k + u\right),$$

and defining $\gamma^i \equiv B^i(I - \sum_k B^k)^{-1}$, we write speculator i's vector of trades as

$$x^i = b^i s^i + \gamma^i\left(\sum_{k=1}^{N} b^k s^k + u\right). \tag{6}$$

We next add across traders, and defining $\Gamma \equiv I + \sum_k \gamma^k$,we write the vector of net order flows as

$$X + u = \left(I + \sum_{k=1}^{N}\gamma^k\right)\left(\sum_{k=1}^{N} b^k s^k + u\right) = \Gamma\left(\sum_{k=1}^{N} b^k s^k + u\right). \tag{7}$$

Lemma 1 exploits this structure to write i's first-order conditions solely as functions of γ^i and Γ.

LEMMA 1: *The first-order condition characterizing speculator i's orders is given by*

$$0 = E\left[(I + \gamma^{i'})v - (I + \gamma^{i'})\lambda\Gamma\sum_{k=1}^{N} b^k s^k - \Gamma'\lambda'\left(b^i s^i + \gamma^i\sum_{k=1}^{N} b^k s^k\right)\middle| s^i\right]. \tag{8}$$

The crucial point to note in this first-order condition is that the "same" b^i sets the expectation to zero for *each* s^i. As a result, we can integrate out over s^i: The b^i that maximizes expected trading profits given any s^i also maximizes expected profits unconditionally. The unconditional maximization problem is easier to solve because it exploits the fact that the same b^i works for every s^i.

A. The Unconditional Problem

Informed speculator i's trading intensities maximize unconditional expected trading profits, solving

$$\max_{b^i, B^i} E\left[\left(Ve - \lambda\Gamma\left(\sum_{k=1}^{N} b^k A^k e + u\right)\right)'\left(b^i A^i e + \gamma^i\left(\sum_{k=1}^{N} b^k A^k e + u\right)\right)\right],$$

where we recall that $v = Ve$ and $s^i = A^i e$. Turning to the market maker's objective, competition drives expected market maker profit to zero. This implies that pricing minimizes the variance of the forecast error between prices and values—prices equal the projection of the vector of equity values onto net order flows. Hence, pricing solves the following least-squares prediction problem:

$$\min_{\lambda} E\left[\left(Ve - \lambda\Gamma\left(\sum_{k=1}^{N} b^k A^k e + u\right)\right)'\left(Ve - \lambda\Gamma\left(\sum_{k=1}^{N} b^k A^k e + u\right)\right)\right].$$

Proposition 2 details the first-order conditions for b^i, γ^i, and λ. To ease presentation, we define

$$bA \equiv (b^1 \quad \ldots \quad b^N)\begin{pmatrix} A^1 \\ \vdots \\ A^N \end{pmatrix}.$$

It also helps to take the portion of net order flow $\sum_{k=1}^{N} b^k s^k + u$ that excludes order flow due to the speculators' filtering of prices, and to define Ψ as the inner product of this signal-based order flow,

$$\Psi \equiv (bA\, I)\begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix}\begin{pmatrix} A'b' \\ I \end{pmatrix} = bA\Sigma_e A'b' + \Sigma_u. \tag{9}$$

PROPOSITION 2:

$$b^i: \quad A^i\Sigma_e A'b'\Gamma'\lambda'(I+\gamma^i) + A^i\Sigma_e(A^{i'}b^{i'} + A'b'\gamma^{i'})\lambda\Gamma = A^i\Sigma_e V'(I+\gamma^i) \tag{10}$$

$$\gamma^i: \quad \gamma^{i'} = -\Psi^{-1} bA\Sigma_e A^{i'} b^{i'} \tag{11}$$

$$\lambda: \quad \Gamma'\lambda' = \Psi^{-1} bA\Sigma_e V'. \tag{12}$$

The first-order condition for b^i, equation (10), characterizes the weights that speculator i places on each private signal for each of his trades. The equation lacks multiplicative symmetry because of the cross-correlation of information. Because of this structure, equation (10) alone cannot be solved directly; the array of first-order conditions across all speculators must be constructed.

The matrix equation (11) for $\gamma^{i'}$ is the matrix of regression coefficients of the projections of speculator i's orders based on private signals onto the vector of total net order flows. This equation captures the strategic interactions between speculators via price. The negative sign indicates that speculators *subtract* these projections from their trades on private signals to avoid paying a quadratic cost on the portions of their trades on signals that are revealed through prices.

The pricing matrix equation (12) is the matrix of regression coefficients from the projection of the vector of values onto the vector of total net order flows. In equilibrium, market makers have correct conjectures about $\{\gamma^k\}$, that is, how speculators trade on net order flows. Inspecting the pricing equation reveals that the "effective" pricing filter for market makers, $\mu \equiv \lambda\Gamma = \lambda(I + \sum_k \gamma^k)$, depends only on the "primitive" b trading intensities for speculators: Market makers see the same net order flow as speculators, and hence can unwind the $\{\gamma^k\}$ filtering by speculators to filter directly the signal-based portion, $\sum_{k=1}^{N} b^k s^k + u$. It follows that the observability of prices ultimately affects profits *only* through its effect on the strategic behavior of speculators.

Finally, note that b^i and γ^i enter nonquadratically into speculator i's objective. However, the speculator's objective can still implicitly be decomposed into two quadratic parts, one in b^i and the other in γ^i. This is because γ^i is the projection of $b^i s^i$ onto the vector of net order flows and (from Proposition 1) speculator i's vector of orders is orthogonal to the vector of net order flows. Even so, the associated first-order conditions characterizing optimization are nonlinear because the $\{b^k\}$ enter $\{\gamma^k\}$, which, in turn, multiply the $\{b^k\}$. This reflects the strategic use by traders of the information in prices. The practical consequence is that we cannot just invert a matrix to solve for equilibrium trading strategies. Rather, we must iterate, first solving for the best responses to conjectured trading strategies and pricing, and then iteratively substituting these best responses in as the basis for a new round of conjectures.

Stacking the first-order conditions for b', we can write the recursion as

$$(a\Sigma_e A')b' = \begin{pmatrix} \sum_{k\neq 1} A^1 \Sigma_e A^{k'} b^{k'} \mu(\mu'(I+\gamma^1) + (I+\gamma^{1'})\mu)^{-1} \\ \vdots \\ \sum_{k\neq N} A^N \Sigma_e A^{k'} b^{k'} \mu(\mu'(I+\gamma^N) + (I+\gamma^{N'})\mu)^{-1} \end{pmatrix} + \begin{pmatrix} A^1 \Sigma_e V'(I+\gamma^1)(\mu'(I+\gamma^1) + (I+\gamma^{1'})\mu)^{-1} \\ \vdots \\ A^N \Sigma_e V'(I+\gamma^N)(\mu'(I+\gamma^N) + (I+\gamma^{N'})\mu)^{-1} \end{pmatrix}. \tag{13}$$

The left-hand side represents the vector of best-responses by speculators to conjectured strategies of other speculators and the associated pricing. The conjectured b' shows up on the right-hand side both directly and in the $\{\gamma^k\}$ projections. That is, a conjectured b' implies values of $\{\gamma^k\}$ and hence $\lambda\Gamma$ on the right-hand side; and the left-hand side is the best-response to those conjectures. We use an iterative algorithm to solve for the equilibrium values of b. We first conjecture a value of b', use this value to compute $\{\gamma^k\}$ and μ, and then substitute these values into the right-hand side of (13). We then compute the best response to these conjectures. Finally, we use this vector of best responses as the new vector of conjectured strategies, and iterate.

II. Theoretical Characterizations under Symmetry

The theoretical analysis is more complicated than the discussion above suggests. The iterative mapping might not be stable: Each speculator responds to an entire matrix of trading intensities, and also to a pricing matrix. Because his response influences the pricing matrix, a speculator's best response might be to overreact to the trading intensities of rivals. Iteration on this overreaction would vitiate the equilibrium. We now demonstrate that the iterated best responses are in fact stable. Specifically, we prove in symmetric settings that for sufficiently small correlations in asset values, this best-response mapping is a contraction on the appropriate space of matrices. Further, our numerical analysis strongly suggests that the contraction mapping property extends to asymmetric settings and arbitrary cross-asset correlations. It follows that there is a unique linear equilibrium. We present the contraction mapping theorem in the two-asset, two-speculator setting that we focus on numerically, in which each speculator sees one of the two innovations to each asset, and the innovation that he sees for one asset has correlation θ with the innovation to the other asset that he does not see. The proof considers a more general symmetric case,[4] and the analytical argument extends to environments that are "close" to symmetric, due to the slackness in the approximation bounds that we derive. We state the simpler case because it lets us simplify presentation by redefining notation:

PROPOSITION 3: *Let there be two assets, $j = 1, 2$, and two speculators, $i = 1, 2$. The value of asset j is $v_j = e_j^1 + e_j^2$, for $j = 1, 2$. Speculator i sees innovations e_1^i and e_2^i, $i = 1, 2$, where e_j^1 and e_j^2 are identically and independently distributed according to a normal $N(0, 1)$ distribution, and $corr(e_1^j, e_2^i) = \theta \leq \frac{1}{3}$, for $i \neq j$. Let liquidity trade be distributed according to an arbitrary symmetric variance–covariance matrix, Σ_u. Then the mapping T implicit in equation (13) is a contraction mapping, implying that there is a unique symmetric linear equilibrium.*

The proof approach is novel. Because γ^i and μ are highly nonlinear in b, the direct recursion is highly nonlinear. To prove the contraction mapping property, one must transform the direct recursion in b into a recursion of a quadratic form in b, that is, $bA\Sigma_e A'b'$. With symmetry, $b^1 = b^2$, and we write the recursion in $b^i\left(\begin{smallmatrix}1 & \theta\\ \theta & 1\end{smallmatrix}\right)b^{i\prime} \equiv b^i(I+p)b^{i\prime} \equiv Q$, which is well behaved:

$$Q = Q(I+p)^{-1}p(I+\gamma^i)^{-1} + \frac{1}{N}\Sigma_u. \tag{14}$$

Using (14), we establish the contraction property by showing that the norm of the coefficient term $(I+p)^{-1}p(I+\gamma^i)^{-1}$ is a fraction. The norm of the factor $(I+p)^{-1}p$ can be made arbitrarily small by shrinking θ. Even though $(I+\gamma^i)^{-1}$ is endogenous, a lemma establishes that its norm is bounded. With the contraction property established, we recover the fixed point of b^i by finding the Cholesky factor of the fixed point of the mapping, and then multiplying that factor by the

[4] The analysis also applies to the symmetric N-asset, N-speculator environment set out in example 1, in which speculator i alone sees the sole innovation to asset i.

inverse of the Cholesky factor of $I+p$, that is, by multiplying by

$$\begin{pmatrix} 1 & \theta \\ 0 & (1-\theta^2)^{\frac{1}{2}} \end{pmatrix}^{-1}.$$

Unobservable prices. This theoretical analysis also provides an easy way to derive equilibrium outcomes if speculators cannot see prices. With unobservable prices, we obtain equilibrium trading intensities by setting $\gamma^i = 0$ for all i. In symmetric settings such as Proposition 3, this allows us to extend Caballe and Krishnan to obtain closed-form solutions: Substituting $\gamma^i = 0$ into (14) yields

$$Q = Q(I+p)^{-1}p + \frac{1}{N}\Sigma_u.$$

Rearranging and substituting back for Q yields

$$b^i(I+p)b^{i'}(I-(I+p)^{-1}p) = \frac{1}{N}\Sigma_u.$$

Expanding the left-hand side yields

$$\begin{aligned} b^i(I+p)b^{i'}(I-(I+p)^{-1}p) &= b^i(I+p)b^{i'}(I+p)^{-1}(I+p-p) \\ &= b^i(I+p)b^{i'}(I+p)^{-1} = b^i b^{i'}. \end{aligned}$$

Hence, $b^i b^{i'} = \frac{1}{N}\Sigma_u$. When liquidity trade is uncorrelated across assets, the following is immediate.

COROLLARY 1: *If speculators do not see prices, then in symmetric settings in which liquidity trade is uncorrelated across assets, speculators only trade directly on their signals—b is a diagonal matrix with identical entries* $\frac{\sigma_u}{\sqrt{N}}$.

Concretely, Corollary 1 says that in symmetric settings where speculators do not see prices, each speculator uses his Merck signal only to trade Merck independently of the correlation of the Merck signal with the value of Glaxo. This remarkable outcome arises because speculators understand that a market maker for Merck will use order flow for Glaxo to price Merck, estimating the value of Merck via a projection onto the order flows for both stocks.

In contrast, when speculators see prices, they can account for the market maker's projections. The γ's in trading strategies are (minus) the coefficients of the projections of the speculators' direct trades onto order flows (equivalently, prices). Speculators subtract these projections, so that their total net trades are effectively the market makers' forecast errors of their direct trades on signals. Hence, the same direct trading intensities convey less information to market makers, mitigating the total price impacts of their trades. In turn, this encourages speculators to trade more aggressively on the remaining forecast error components of private information. In particular, in contrast to the unobservable price setting, when asset values are positively correlated, a positive signal for Merck will lead a speculator to buy *both* Merck and Glaxo.

Extending Corollary 1, we will show numerically that when information or liquidity trade is divided asymmetrically between assets, and speculators do not see prices, then speculators trade *against* their signal from the more liquid or more volatile stock. For example, when speculators do not see prices, and one asset has more than half of the liquidity trade, speculators trade *against* signals about the more liquid stock in the less liquid market: The direct trading losses are more than offset by the beneficial price impact on the more liquid stock. In symmetric settings these two effects have offsetting impacts on profits, so that speculators only use "Merck" signals to trade Merck. So, too, with unobservable prices, if one speculator has access to more private information than other speculators, the speculator with more information trades against his "cross-asset signal," selling Merck when he has a positive signal about Glaxo (and asset values are positively correlated). In the observable price setting, asymmetries have similar qualitative marginal effects on cross-asset trading intensities. However, because speculators can exploit the observability of prices to subtract the projections of their direct trades, the asymmetries must be extreme in order to induce a speculator to trade against cross-asset signals. For example, (ceteris paribus) there must be *far* less liquidity trade in the Merck market than the Glaxo market for a speculator to sell Merck when he receives a positive signal about Glaxo (about 1/8th as much as in our numerical examples). These observations emphasize how radically the observability of prices alters trading strategies.

A. Trading Strategies

We say that an environment is *fully symmetric* if information is symmetrically distributed across both assets and speculators, and liquidity trade is symmetrically distributed across assets. With this added structure, we now show that trading strategies take a form that mirrors their structure in standard single-asset settings: Trading strategies are proportional to the ratio of the standard deviation of liquidity trade to the standard deviation of speculator signals (see, for example, Kyle (1985)).

PROPOSITION 4: *In the fully symmetric, two-speculator observable price environment of Proposition 3, speculative trading intensities on intrinsic information are characterized by*

$$b \sim (A\Sigma_e A')^{-1/2}\Sigma_u^{1/2}.$$

In sharp contrast, with unobservable prices, Corollary 1 immediately reveals that trading strategies are *not* proportional to $(A\Sigma_e A')^{-1/2}$.

B. Cross-asset Correlations

We now derive a result of the unraveling by market makers of the filtering by traders of order flow.

PROPOSITION 5: *In the fully symmetric, two-speculator observable price environment of Proposition 3, the covariance matrix of price changes, $\lambda\Gamma\Psi\Gamma'\lambda'$, is independent of Σ_u, that is, of the distributional properties of liquidity trade. Only the variance–covariance structure of the innovations to asset values, Σ_e, affects the correlation structure of price changes.*

The economic force driving this result is that if the correlation structure of noise trade affected the correlation structure of prices, then prices would be *systematically* driven by processes other than value—market makers would be *systematically* mispricing stocks—and speculators would exploit this. Formally, using $b \sim \Sigma_u^{1/2}$, we have that the inner product of net order flow, $\Psi = bA\Sigma_e A'b' + \Sigma_u \sim \Sigma_u$; and $\lambda \sim \Sigma_u^{-1/2}$ unwinds the correlation structure of liquidity trade, leaving only the impact of the variance–covariance structure of intrinsic information. Our numerical investigations indicate that this result extends to arbitrary numbers of speculators and to asymmetric settings.

We next prove that in fully symmetric environments, the strategic behavior by speculators causes price changes to be *more correlated* than the underlying fundamentals. Our numerical investigations strongly suggest that this result extends more generally to asymmetric environments.

PROPOSITION 6: *In the fully symmetric, two-speculator observable price environment of Proposition 3, price changes (returns) are more correlated than the underlying fundamentals. For small correlations in fundamentals, θ, the price change correlation is approximately $\frac{5}{3}\theta$.*

Corollary 1 revealed that in symmetric settings, unobservable prices cause speculators to trade only on their direct signals; whereas with observable prices, speculators weight positively their signal for one asset when trading the other asset. Proposition 6 just showed that observable prices cause price changes to be more correlated than the underlying fundamentals; and we find numerically that price changes are more correlated when prices are observable than when they are not. These findings would seem to suggest that the cross-asset correlations in net order flow should be higher when speculators see prices than when they do not. We now show that this intuition is completely misplaced. The variance–covariance matrix of net order flows is $\Gamma(bA\Sigma_e A'b' + \Sigma_u)\Gamma'$. The next result shows that this matrix takes a simple form.

PROPOSITION 7: *The variance–covariance matrix of net order flows is $\Sigma_u\Gamma'$. In the fully symmetric, two-speculator observable price environment of Proposition 3, only the variance–covariance structure of liquidity trade, and not the correlation structure of the innovations to asset values, Σ_e, affects the correlation structure of order flows.*

Numerical analysis suggests that this result extends more generally to asymmetric environments. The economic force driving this result is analogous to that for the correlation structure of pricing—were net order flows to reflect the correlation structure of speculators' private information, market makers would exploit this in their pricing, reducing speculator profits. The upshot of Propositions 5 and 7 is that the correlation structures of order flow and prices are dual to each other in the following sense: The covariance structure of prices is driven only by the covariance structure of value, while the covariance structure of order flows is driven only by the covariance structure of liquidity trade.

The next proposition derives the surprising consequences of Proposition 7 for the correlation structure of net order flows when liquidity trade is uncorrelated so that only the intrinsic correlation in fundamentals and strategic speculative behavior affect the order flow correlation.

PROPOSITION 8: *Consider the fully symmetric, two-speculator environment of Proposition 3, and suppose that liquidity trade is uncorrelated across assets and asset values are positively correlated, that is, $\theta > 0$. Then, with observable prices, cross-asset net order flows are negatively correlated. If, instead, prices are unobservable, cross-asset net order flows are positively correlated.*

With observable prices, the negative correlation in cross-asset net order flows is immediate: With uncorrelated liquidity trade, Σ_u is diagonal, so the cross-asset correlation takes the sign of the off-diagonal term in $\Gamma = \Gamma'$, γ_{ij}, which is negative. Even though speculators trade directly on their cross-asset signals, they also subtract out the projections of these direct trades onto net order flows, and with uncorrelated liquidity trade this latter effect dominates, leading to negative cross-correlations in net order flows.

In contrast, with unobservable prices, speculator 1's signal for asset 1 is correlated with speculator 2's signal for asset 2, and both trade with the same intensity on those signals. As a result, even though speculators only trade on direct signals, the cross-asset correlation in order flow is positive. Hence, with uncorrelated liquidity trade, observable and unobservable prices generate opposing predictions—but they are the opposite of what a casual contemplation of direct cross-asset trading intensities suggests. This result indicates that with observable prices, some correlation in liquidity trade is needed to reconcile the positive cross-asset correlation in order flows found in the data.

Numerical investigation reveals that with observable prices and modest correlation in liquidity trade across assets (relative to the correlation in private signals), net order flows become positively correlated across assets, but remain less correlated than liquidity trade. In contrast, with unobservable prices, net order flows are always more correlated than liquidity trade.

C. Factor Structure

Hasbrouck and Seppi (2001) apply a principal component decomposition of cross-asset order flows and returns, and find that commonality in order flows explains roughly two-thirds of the commonality in returns. We can apply the same decomposition in our model to cross-asset order flows and price changes (our analogue of returns). This factor analysis reveals the dimensions along which our stylized model can reconcile the data. Moreover, the primitive parameters that we generate—the correlation in information of speculators and the correlation in liquidity trade—provide a theoretical underpinning for Hasbrouck and Seppi's purely empirical findings.

Hasbrouck and Seppi note that the principal common factors of cross-asset order flow and cross-asset returns are the eigenvectors of the covariance matrix of the relevant variable. The eigenvectors are ordered by the magnitudes of the corresponding eigenvalues. Hasbrouck and Seppi initially regress returns on the first principal component of order flow P^{X+u}. This results in a set of residuals w_t. They next calculate the principal components of those residuals, P^w. Finally, they show that together P^{X+u} and P^w explain 21.7% of the total variance of returns, of which P^{X+u} comprises 14.6%, or about two-thirds.

We can carry out a parallel principal components analysis in our model. To ease presentation, we focus on a two-asset, two-speculator, fully symmetric setting. With symmetry, the covariance matrices of order flow and price changes take the form $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$. The associated eigenvalues are $a-b$ and $a+b$, and eigenvectors are $2^{-1/2}\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$ and $2^{-1/2}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. The scaling factor $2^{-1/2}$ can be ignored as it drops out in the calculations. We focus on the empirically relevant case where (a) cross-asset order flows and (b) cross-asset returns (price changes) are each positively correlated. With positive correlation, $a+b$ is the larger eigenvalue and $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ is the corresponding eigenvector.

The analogue of Hasbrouck and Seppi's first-step regression analysis in our model is

$$p_i = cP^{X+u} + w_i,$$

where c is a regression coefficient and w_i is a residual. Exploiting symmetry, we have

$$c = \frac{\text{cov}(p_i, q_1+q_2)}{\text{var}(q_1+q_2)} = \frac{\text{cov}(\lambda_{i1}q_1 + \lambda_{i2}q_2, q_1+q_2)}{\text{var}(q_1+q_2)} = \frac{\lambda_{i2}+\lambda_{i1}}{2}.$$

The covariance matrix of the residuals is

$$\text{cov}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \text{cov}\begin{pmatrix} p_1 - cP^{X+u} \\ p_2 - cP^{X+u} \end{pmatrix} = \text{cov}\begin{pmatrix} p_1 - c(q_1+q_2) \\ p_2 - c(q_1+q_2) \end{pmatrix}.$$

Under symmetry, the eigenvalues again take the form $a-b$ and $a+b$. The relevant principal component is

$$w_1 - w_2 = p_1 - c(q_1+q_2) - (p_2 - c(q_1+q_2)) = q_1 - q_2,$$

as it must be orthogonal to P^{X+u}.[5]

Following Hasbrouck and Seppi, we next calculate the regression coefficients for w_i on P^w:

$$w_i = d_i P^w + \epsilon_i.$$

[5] The other principal component is $(1-2c)(q_1+q_2)$, which is perfectly correlated with P^{X+u}.

With the principal component $(q_1 - q_2)$, we obviously have $d_2 = -d_1$, where

$$d_1 = \frac{\mathrm{cov}(p_1 - c(q_1+q_2), q_1 - q_2)}{\mathrm{var}(q_1 - q_2)} = \frac{\mathrm{cov}((\lambda_{11} - c)q_1 + (\lambda_{12} - c)q_2, q_1 - q_2)}{\mathrm{var}(q_1 - q_2)}.$$

Substituting for $\lambda_{11} - c = \dfrac{\lambda_{11} - \lambda_{12}}{2}$ and $\lambda_{12} - c = \dfrac{\lambda_{12} - \lambda_{11}}{2}$, and rearranging yields

$$d_1 = \frac{\mathrm{cov}\left(\dfrac{\lambda_{11} - \lambda_{12}}{2}(q_1 - q_2), q_1 - q_2\right)}{\mathrm{var}(q_1 - q_2)} = \frac{\lambda_{11} - \lambda_{12}}{2}.$$

The respective shares of the total variation attributed to P^{X+u} and P^w are

$$\begin{aligned} S(P^{X+u}) &= 2c^2 \frac{\mathrm{var}(P^{X+u})}{\mathrm{var}(p_1) + \mathrm{var}(p_2)} = 2c^2 \frac{\mathrm{var}(q_1 + q_2)}{\mathrm{var}(p_1) + \mathrm{var}(p_2)} \\ &= 2c^2 \frac{\mathrm{var}(q_i) + \mathrm{cov}(q_1, q_2)}{(\lambda_{ii}^2 + \lambda_{ij}^2)\mathrm{var}(q_i) + 2\lambda_{ii}\lambda_{ij}\mathrm{cov}(q_i, q_j)} \\ S(P^w) &= 2d_i^2 \frac{\mathrm{var}(P^w)}{\mathrm{var}(p_1) + \mathrm{var}(p_2)} = 2d_i^2 \frac{\mathrm{var}(q_1 - q_2)}{\mathrm{var}(p_1) + \mathrm{var}(p_2)} \\ &= 2d_i^2 \frac{\mathrm{var}(q_i) - \mathrm{cov}(q_1, q_2)}{(\lambda_{ii}^2 + \lambda_{ij}^2)\mathrm{var}(q_i) + 2\lambda_{ii}\lambda_{ij}\mathrm{cov}(q_i, q_j)}, \end{aligned}$$

which we can calculate numerically.

Calibration. Our model is sufficiently stylized that it is not directly comparable to the Hasbrouck and Seppi framework. To appreciate the stylization, note that in our model only two factors drive pricing (reflecting the two net order flows),[6] whereas Hasbrouck and Seppi find three significant factors. Further, in our two-asset model, the ratio of the first two eigenvalues is *entirely* driven by the correlation structure of the matrices, whereas in the data, the correlation structure is not the sole determinant. We also consider a symmetric setting, whereas Hasbrouck and Seppi average over heterogeneous stocks.

Our findings in Propositions 4 and 5 that (a) speculative trade is proportional to $(A\Sigma_e A')^{-1/2}\Sigma_u^{1/2}$ and (b) the correlation of price changes is unaffected by the cross-asset correlation of liquidity trade η together imply that as long as the correlation in liquidity trade is high enough that cross-asset order flows are positively correlated, then $\frac{S(P^{X+u})}{S(P^{X+u})+S(P^w)}$ depends only on the cross-asset signal correlation θ, and not on either η or the scales of asset value innovations and liquidity trade. Raising θ increases the fraction explained by commonality in order flow. We find that a relatively small signal correlation of $\theta = 0.2$ generates

[6] We would have more factors if we added noise from public information arrival or order flow from additional heterogeneous assets.

the two-thirds share ratio that Hasbrouck and Seppi find.[7] For $\theta = 0.2$, the cross-asset correlation in liquidity trade must be at least $\eta = 0.135$ for net order flows to be positively correlated (and, for example, $\eta = 0.25$ generates a net order flow correlation of 0.12).

Taking our model further, we can compare the ratios of the first two eigenvalues in the covariance matrices of order flow and price changes with their empirical counterparts. Fixing $\theta = 0.2$, a liquidity trade correlation of $\eta = 0.5$ allows us to match the 2.183 eigenvalue ratio for order flows that Hasbrouck and Seppi find. Because the correlation of price changes is unaffected by the correlation of liquidity trade, η only influences the eigenvalue ratio for order flow; and $\theta = 0.2$ implies an eigenvalue ratio for price changes of 1.968, which is about one-third of the ratio for returns that Hasbrouck and Seppi find.

Summarizing, we highlight three points. First, our model can generate the result that the primary principal factor for order flow is also the main influence on price changes or returns. This result requires a relatively small and believable cross-asset correlation in innovations to asset values. Second, we require some minimal level of cross-asset correlation in liquidity trade,[8] and a relatively high correlation in liquidity trade to match the eigenvalue ratio that Hasbrouck and Seppi find. Third, with unobservable prices, one needs a slightly higher correlation in value innovations, $\theta = 0.236$, to match $\frac{S(P^{X+u})}{S(P^{X+u})+S(P^{w})} = 2/3$; $\eta = 0.283$ allows us to match the eigenvalue ratio for order flow; and $\theta = 0.236$ implies an eigenvalue ratio for price changes of 2.071.

III. Numerical Characterizations

We now explore the quantitative properties of our model further. To isolate the strategic effects due to information linkages across assets, we normalize parameter values so that if (a) a market maker only sees the net order flow for his own stock and (b) speculators do not see prices, then (c) total expected speculator profits do not change as we vary the parameters that describe the economy. It follows that any variations in speculator profits in an integrated stock market are due to the interaction between the economic environment and market information structure.

If speculators only see their private signals and do not see prices, and a market maker only sees the net order flow for his stock, it is well known (Kyle (1985)) that trading strategies for each asset are proportional to the ratio of the standard deviation of liquidity trade to the standard deviation of informed

[7] In our model, $S(P^{X+u}) + S(P^{w}) = 1$ because our model has no noise, so that the order flows of the two assets perfectly determine pricing—simply adding a third heterogeneous asset would change this.

[8] This should not be confused with standard "measures" of liquidity—the inverses of the estimated λ_{ii}—which are constants in our model, reflecting the amounts of intrinsic private information relative to liquidity trade. In our model, correlation of liquidity trade is reflected in the use by market makers of cross-asset order flow in pricing—that is, the off-diagonal elements of the λ matrix.

signals, $\frac{\sigma_u}{\sigma_e}$. Hence, pricing is proportional to $\frac{\sigma_e}{\sigma_u}$, and expected speculator profits are proportional to $\sigma_u \sigma_e$. Therefore, multiplying the variance–covariance matrix of signals by a constant simply scales trading intensities and profits, so we can normalize these totals without loss of generality; the variance–covariance matrix of liquidity trade can be normalized for the same reason.

We perturb the economy in three ways.

Division of Information between Speculators. We impose symmetry across asset values and liquidity trade, and vary the share of the variance to asset values that one speculator observes. This isolates how the division of private information between speculators affects equilibrium outcomes. The normalization that preserves total expected speculator profits when a market maker only observes own order flow holds constant the total variance of each asset and preserves the cross-asset value correlation structure. Within this setting, we characterize the impact of increased competition by varying the number of speculators who receive the same set of signals.

Division of Liquidity Trade between Assets. We impose symmetry across asset values and speculator signals, and vary the share of liquidity trade between assets. This isolates the effects of relative liquidity levels between assets. With independently distributed liquidity trade, the appropriate normalization holds constant the sum of the *standard deviations* of liquidity trade across assets.

Division of Uncertainty between Assets. We impose symmetry across speculators and liquidity trade and vary the variance share of asset values between assets. This isolates the effects of relative levels of private information between assets on equilibrium outcomes. The normalization holds constant the sum of the *standard deviations* of asset values and preserves the correlation structure.

A. Division of Information between Speculators

Our numerical analysis focuses on the economic environment of Example 2, in which speculators see one signal for each asset. In our base two-speculator, two-asset setting, liquidity trade is independently and normally distributed with $u_i \sim N(0, 1)$. Using the notation of Proposition 3, the value of asset j is $v_j = e_j^1 + e_j^2$. Speculator i sees signal $s_1^i = e_1^i$ for asset 1 and signal $s_2^i = e_2^i$ for asset 2. The signals e_j^1 and e_j^2 that the two speculators receive for asset j are independently and normally distributed with mean zero. However, even though speculator 1 does not see e_1^2, we allow cross-asset signal correlations to be positive, $\mathrm{corr}(e_1^j, e_2^i) = \theta$, for $i \neq j$, so that speculator 1's signal e_2^1 about asset 2 also provides him information about the innovation e_1^2 to asset 1 that he does not see. The variance–covariance matrix for asset value innovations is

$$
\begin{array}{cc}
 & \begin{array}{cccc} e_1^1 & e_1^2 & e_2^1 & e_2^2 \end{array} \\
\begin{array}{c} e_1^1 \\ e_1^2 \\ e_2^1 \\ e_2^2 \end{array} &
\begin{pmatrix}
\nu & 0 & 0 & \theta\xi \\
0 & (1-\nu) & \theta\xi & 0 \\
0 & \theta\xi & \nu & 0 \\
\theta\xi & 0 & 0 & (1-\nu)
\end{pmatrix}
\end{array},
$$

where $\xi = \sqrt{\nu(1-\nu)}$ is the normalization factor that preserves the correlation structure as we vary the information that speculator 1 has relative to speculator 2. We vary ν from 0.1 (so that speculator 1 sees 10% of the variance of each asset value) to $\nu = 0.9$. Because of the cross-asset correlation structure, speculator 1's share of the private information is

$$\frac{\mathrm{var}(e_1^1) + \mathrm{var}(E\left[e_1^2|e_2^1\right])}{\mathrm{var}(e_1^1) + \mathrm{var}(E\left[e_1^2|e_2^1\right]) + \mathrm{var}(E\left[e_1^1|e_2^2\right]) + \mathrm{var}(e_1^2)} = \frac{\nu + \theta^2(1-\nu)}{1+\theta^2}.$$

We focus on a high correlation of $\theta = 0.75$, as higher correlations largely serve to magnify the qualitative impacts.[9]

The top line in Figure 1 represents expected total speculator profits when market makers only see "own" net order flow and speculators do not see prices. Their constancy reflects that we have normalized informational parameters to leave profits in this setting unaffected by the division of information between speculators. The middle line plots profits in the Caballe–Krishnan environment in which market makers see stock prices, but speculators do not. Finally, the bottom line plots profits in the integrated market that we study, where both market makers and speculators see all prices.

Figure 1 shows how the observability of prices amplifies competition between speculators. The added channel of prices through which speculators gain and release information intensifies competition because each speculator internalizes the fact that as he increases an order, other speculators see the price impacts and reduce their orders. This "order reduction" *raises* the speculator's profit. In turn, speculators raise trading intensities, competing away more of the potential profits.

This figure also highlights that the observability of prices amplifies the sensitivity of total expected speculator profits to the division of information between speculators, especially when their information is highly correlated. Profits are far higher when one speculator has most of the information than when information is evenly divided. In turn, comparing the Caballe and Krishnan setting with the isolated markets benchmark surprisingly reveals that the observability of all prices by market makers only modestly reduces speculator profits when information is unevenly divided, but significantly substitutes for the observability of prices by speculators when information is evenly divided.

Figure 2 shows how a speculator's expected trading profit rises with his share of private information, ν. The figure reveals that the smaller is ν, the more a trader is hurt by the observability of prices (the profit difference narrows with ν). This is because the smaller is ν, the less the speculator contributes to total order flow, and hence, the less he can back out from prices.

Figure 3 shows how trading intensities vary with a speculator's share of private information. The figure highlights that trading intensities are far higher if prices are observable. In part, this reflects that when speculators

[9] Lowering correlations raises aggregate profits for any ν, reduces cross-asset trading, and shrinks the consequences of market design.

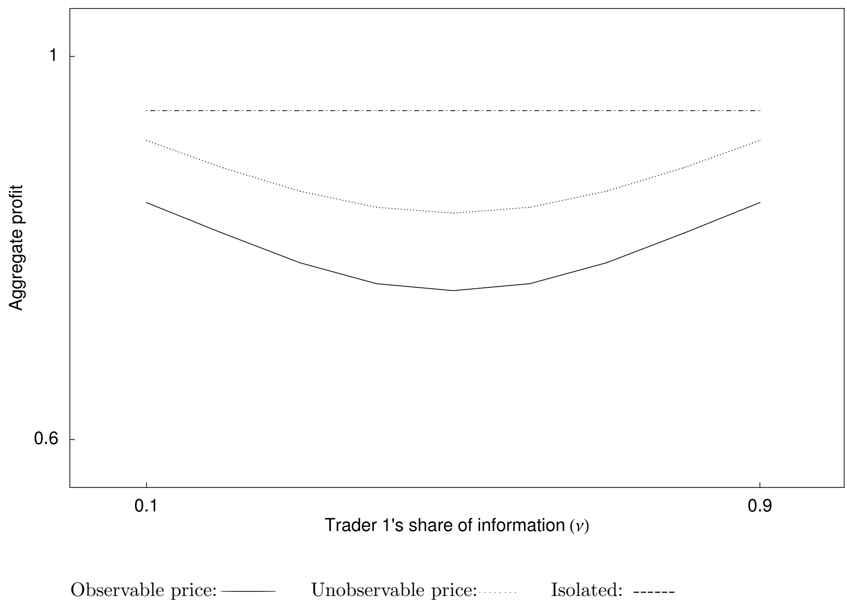

Figure 1. Total speculator profit as a function of speculator 1's share of the total private information. There are two speculators. Speculator 1 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance ν; speculator 2 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance $1-\nu$. The asset i fundamentals are distributed independently across speculators. The covariance of the fundamental to each asset that one speculator sees with the fundamental to the other asset that the other speculator sees is scaled to generate a cross-asset correlation of $\theta = 0.75$. Liquidity trade for asset $i = 1, 2$ is independently normally distributed with zero mean and unit variance.

see prices, they offset the impact of their trades on private signals by subtracting their projections on net order flows via γ^i. In addition, trading intensities with observable prices are higher due to the strategic "order-reduction" effect: Speculators have incentives to increase trading intensities to influence price, which causes other agents to scale down orders, thereby mitigating price impacts.

The first panel of Figure 3 reveals that trade on direct information first rises rapidly with a speculator's information share, before declining as ν is further increased. The second panel shows that indirect trading intensities exhibit an entirely different pattern: When a speculator has little private information, he uses his asset 1 signal to trade asset 2 *extremely* aggressively. In fact, when prices are observable and $\nu = 0.1$, speculator 1 uses his noisy signal e_1^1 of e_2^2 to trade asset 2 *more aggressively* ($b_1^2 = 1.56$) than speculator 2, who sees e_2^2($b_2^2 = 1.15$). This reflects that when ν is small, the expected (squared) size of speculator 1's trade on indirect information, $(b_1^2)^2\nu$, is far less than $(b_2^2)^2(1-\nu)$. As ν rises, indirect trading intensities fall *sharply*. Indeed, when prices are unobservable, the speculator with more information trades *against* his asset 2

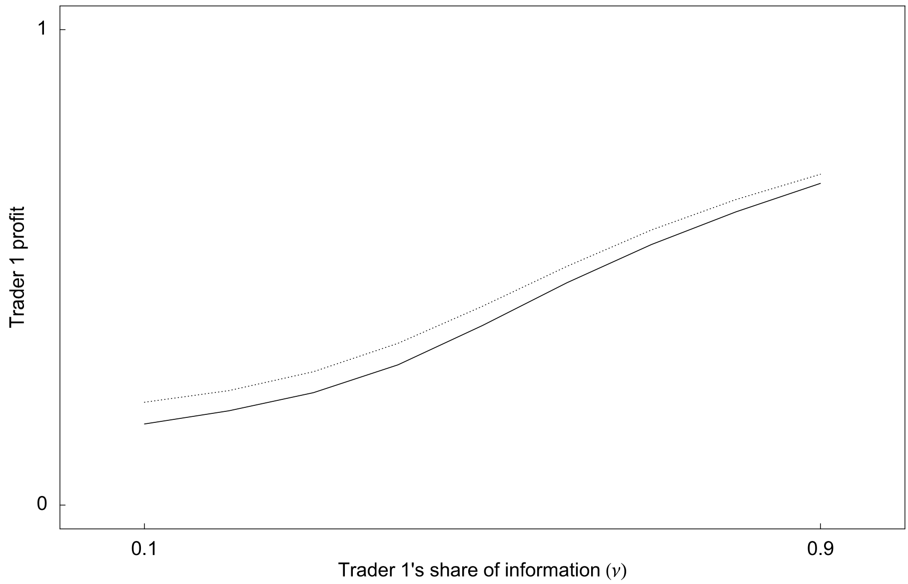

Figure 2. Speculator 1's expected profit as a function of his share of the total private information. There are two speculators. Speculator 1 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance ν; speculator 2 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance $1 - \nu$. The asset i fundamentals are distributed independently across speculators. The covariance of the fundamental to each asset that one speculator sees with the fundamental to the other asset that the other speculator sees is scaled to generate a cross-asset correlation of $\theta = 0.75$. Liquidity trade for asset $i = 1, 2$ is independently normally distributed with zero mean and unit variance.

signal when trading asset 1 (see Corollary 1). By doing so, the speculator gains more through the influence on the price of asset 2 than he loses through the (small) quadratic loss in market 1 due to his trade against his asset 2 signal.

Figure 4 reveals the "direct" trading weights on the net order flow for the asset being traded first fall slightly (in absolute value) as the speculator's share of private information rises and then more than triple in magnitude as ν rises from 0.4 to 0.9. That is, the speculator with most of the private information more extensively exploits the information in prices in his trading. The opposite pattern arises for "indirect" trading intensities on net order flow, that is, when traders use information in the order flow for asset 2 to trade asset 1. Indeed, when ν is large, the sign *switches*. This result reflects that when speculator 1 has most of the private information, speculator 2 trades very intensively on his noisy indirect signals of speculator 1's information. From 1's perspective, speculator 2 so overtrades that it changes the sign of the projection.

The trades by speculators on information in prices are predictable by market makers—market makers unwind their impact to filter directly the signal-based

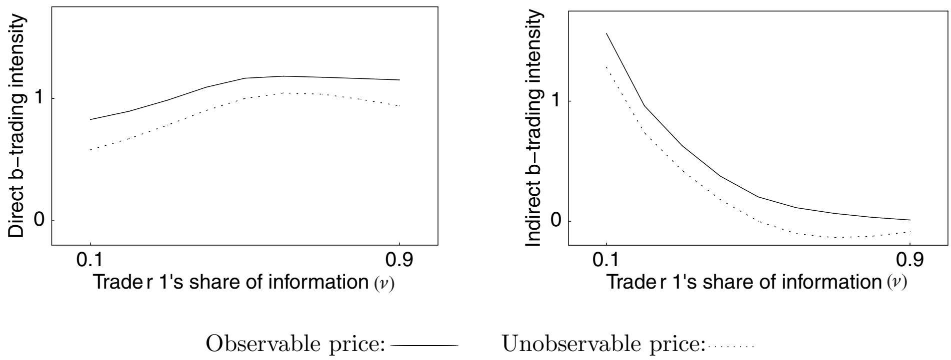

Figure 3. Speculator 1's trading intensities on his intrinsic asset signals as a function of his share of the total private information. The first panel is speculator 1's direct trading intensity on the fundamental to asset i of asset i. The second panel is speculator 1's indirect trading intensity on the fundamental to asset i of asset $-i$. There are two speculators. Speculator 1 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance ν; speculator 2 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance $1-\nu$. The asset i fundamentals are distributed independently across speculators. The covariance of the fundamental to each asset that one speculator sees with the fundamental to the other asset that the other speculator sees is scaled to generate a cross-asset correlation of $\theta = 0.75$. Liquidity trade for asset $i = 1, 2$ is independently normally distributed with zero mean and unit variance.

portion of order flows. Figure 5 reveals how significantly market makers use information in the order flow for other assets. In particular, if information is evenly distributed between speculators, the weight a market maker places on order flow for the "other" asset is almost half that on "own" asset order flow; and if information is unevenly distributed, the weight on "other" order flow is about 25% of that on "own" order flow. In sum, market makers heavily use the information contained in the prices of other assets.

Increased Competition. To uncover the impact of greater competition, we double the number of speculators who see the same signals, that is, we consider four speculators, with two speculators seeing the same signal pair. For simplicity, we focus on the symmetric setting with $\nu = 0.5$. We find that doubling competition causes direct trading intensities to fall marginally to 1.08 from 1.17, while indirect trading intensities almost double from 0.20 to 0.38. That is, *aggregate* direct trading intensities almost *double*, because there are now two of each speculator type, and aggregate indirect trading intensities almost *quadruple*. This percentage increase in trading intensities is more than twice as great as that when prices are unobservable.

The increased competition between speculators when prices are observable drives their total profit down by 36%, from 0.76 to 0.56. As a benchmark comparison, going from a monopoly to a symmetric duopoly in the canonical linear demand oligopoly output game reduces total profits by only 12%. This impact is *overwhelmingly* due to the observability of prices; when speculators do not see prices, doubling competition reduces their profits by only 6% from 0.84 to 0.79.

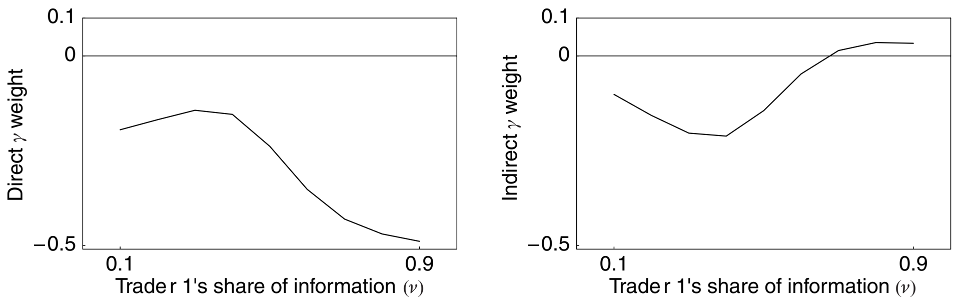

Figure 4. Speculator 1's trading intensities on prices as a function of his share of total private information. The first panel is speculator 1's trading intensity on price p_i of asset i. The second panel is speculator 1's indirect trading intensity on price p_i of asset $-i$. There are two speculators. Speculator 1 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance ν; speculator 2 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance $1-\nu$. The asset i fundamentals are distributed independently across speculators. The covariance of the fundamental to each asset that one speculator sees with the fundamental to the other asset that the other speculator sees is scaled to generate a cross-asset correlation of $\theta = 0.75$. Liquidity trade for asset $i = 1, 2$ is independently normally distributed with zero mean and unit variance.

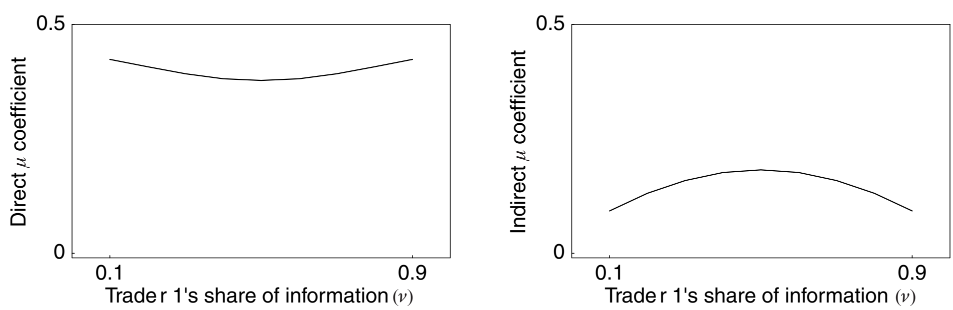

Figure 5. The coefficients of the market maker's effective pricing filter, $\mu = \lambda\Gamma$, as a function of the share of total private information that speculator 1 has. The first panel is the coefficient on net order flow of asset i for the pricing of asset i. The second panel is the coefficient on net order flow of asset i for the pricing of asset $-i$. There are two speculators. Speculator 1 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance ν; speculator 2 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance $1-\nu$. The asset i fundamentals are distributed independently across speculators. The covariance of the fundamental to each asset that one speculator sees with the fundamental to the other asset that the other speculator sees is scaled to generate a cross-asset correlation of $\theta = 0.75$. Liquidity trade for asset $i = 1, 2$ is independently normally distributed with zero mean and unit variance.

The differential impacts of increased competition on speculator profit when prices are and are not observable reflect the differential impacts on trading intensities.

Price Correlations. Figure 6 displays how the cross-asset correlation in price changes varies with ν. In contrast to the correlation of information across

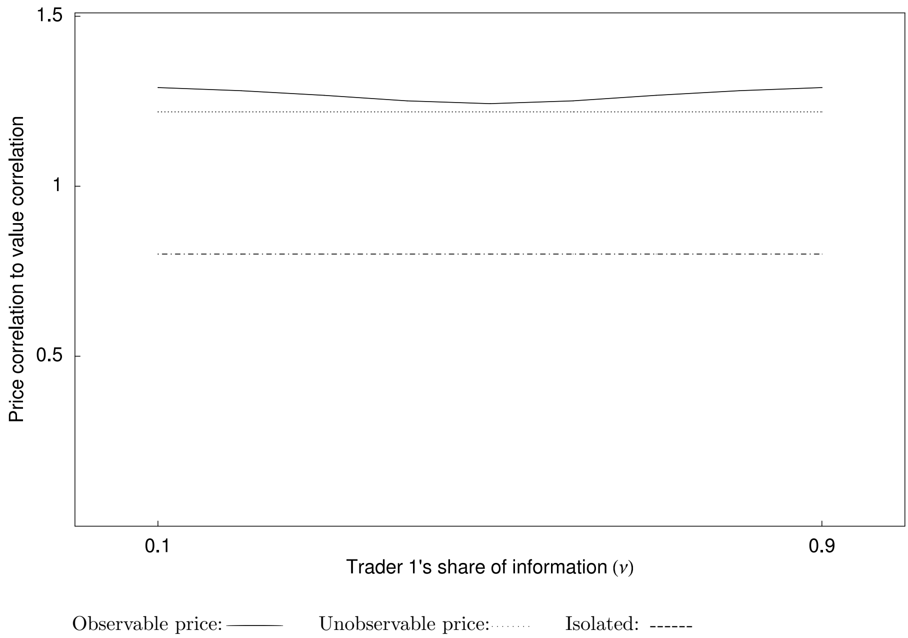

Figure 6. The ratio of the correlation in asset prices to the underlying correlation in asset values. There are two speculators. Speculator 1 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance ν; speculator 2 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and variance $1 - \nu$. The asset i fundamentals are distributed independently across speculators. The covariance of the fundamental to each asset that one speculator sees with the fundamental to the other asset that the other speculator sees is scaled to generate a cross-asset correlation of $\theta = 0.75$. Liquidity trade for asset $i = 1, 2$ is independently normally distributed with zero mean and unit variance.

speculators, which is fixed at 0.75, the cross-asset correlation $\theta\sqrt{\nu(1-\nu)}/2$ rises as $\nu \to 1/2$. Consequently, in the figure we normalize price correlations by dividing by the asset value correlation.

Strikingly, in integrated stock markets, prices are far more correlated than the underlying asset values, but in isolated markets, prices are far less correlated. This reflects that market makers in integrated markets digest the information in order flows for all assets when pricing. This cross-asset correlation raises the volatility of prices in integrated markets.[10] Cross-asset correlations are higher yet when speculators see prices, because speculators trade more aggressively on their cross-asset information. The price volatility consequences of the informational environment are substantial. For example, when information is evenly divided between speculators, the variance of price changes is 19% higher when prices are observed by speculators than when they are not.

[10] This indicates that failing to account for market integration on an individual stock basis biases predicted price variances downward.

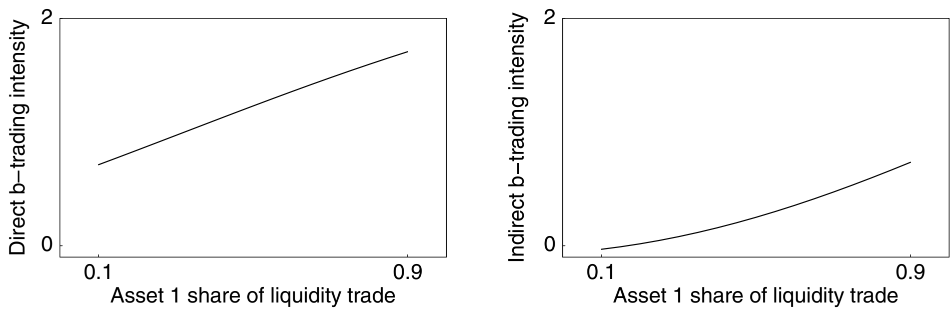

Figure 7. Direct and indirect trading intensities of asset 1 as a function of market 1's share of the total standard deviation of liquidity trade across assets. There are two speculators. Speculator 1 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and standard deviation $\frac{1}{2}$; speculator 2 sees asset fundamentals for each of the two stocks that are normally distributed with mean zero and standard deviation $\frac{1}{2}$. The asset i fundamentals are distributed independently across speculators. The covariance of the fundamental to each asset that one speculator sees with the fundamental to the other asset that the other speculator sees is scaled to generate a cross-asset correlation of $\theta = 0.75$. The total standard deviation of liquidity trade across the two markets is 2.

B. Division of Liquidity Trade across Assets

We now explore how the division of liquidity trade affects outcomes. That is, we address what happens when one asset has more liquidity trade behind which speculators can hide their trades. To do so, we consider symmetrically distributed asset values and symmetrically situated speculators. We then vary the share of the standard deviation of liquidity trade between the assets.

We find that total expected speculator profits are remarkably insensitive to the division of liquidity trade between assets. Maintaining a total standard deviation of liquidity trade across assets of 2.0, going from a setting in which each asset has liquidity trade of 1.0 to a very unbalanced distribution of liquidity trade where asset 1 has a standard deviation of liquidity trade of 1.45, raises speculator profits by only 2%. That is, in stark contrast to the impact of the division of information between speculators, the division of liquidity trade between assets has trivial consequences for speculator profits, even when speculators see prices.

However, Figure 7 shows that the division of liquidity trade affects *how* speculators trade on their signals. Increasing asset 1's share of liquidity trade from 0.1 to 0.9 causes speculators to double direct trading intensities on asset 1 signals; when asset 1 has almost all of the liquidity trade, speculators use their asset 2 signal to trade both assets equally aggressively, and they trade *against* asset 1 signals in the illiquid asset 2 market. Still, liquidity trade must be very asymmetrically divided before speculators trade against their indirect signals in the less liquid market. In contrast, if speculators do not see prices, they *always* trade against their indirect signals in the less liquid market. When

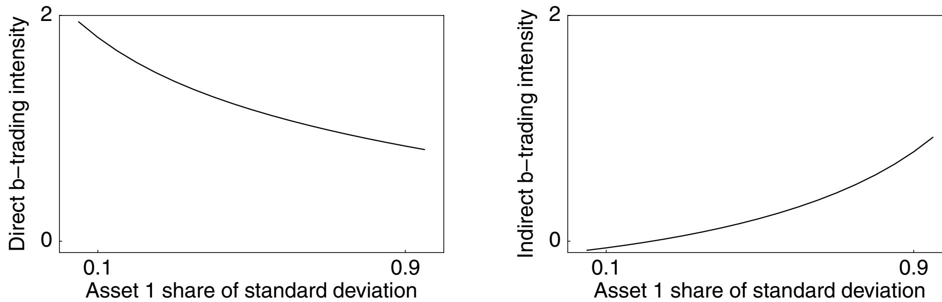

Figure 8. Direct and indirect trading intensities of asset 1 as a function of asset 1's share of the total standard deviation of private information across assets. There are two speculators. The total standard deviation of private information across assets is 2. The speculators each see a fundamental to asset 1 that is distributed normally and independently of the other speculator's signal of asset 1, with a mean of zero and a standard deviation that varies between 1 and 1.45. The speculators each see a fundamental to asset 2 that is distributed normally and independently of the other speculator's signal, with a mean of zero and a standard deviation such that the sum across assets is 2. The covariance of the fundamental to each asset that one speculator sees with the fundamental to the other asset that the other speculator sees is scaled to generate a cross-asset correlation of $\theta = 0.75$. Liquidity trade for asset $i = 1, 2$ is independently normally distributed with zero mean and unit variance.

speculators see prices, they are more willing to trade in the same direction as their signals because they can unwind the trade's projection onto net order flows, mitigating price impacts.

C. Division of Uncertainty between Assets

We next explore how the division of uncertainty across assets affects outcomes. That is, we address what happens when there is more private information about asset 1's value than asset 2's. To do so, we assume that speculators are symmetrically situated, and the distribution of liquidity trade is the same across assets. We fix the total standard deviation of liquidity trade to 2.0 and then progressively increase the standard deviation of asset 1 from 1.0 to 1.45.

We find that total expected speculator profits are as insensitive to the division of private information between assets as they are to the division of liquidity trade. More asymmetric divisions of private information between assets lead to higher profits, but there is only about a 2% difference between profits when uncertainty is evenly and very unevenly divided.

Figure 8 reveals that speculators trade more aggressively on their signal for the less volatile asset both directly and indirectly. Also, as with very unevenly distributed liquidity trade, if information is very unevenly divided between assets, speculators trade *against* their more volatile asset signal of the less volatile asset's value. In sum, the division of information between assets matters in complicated ways for trading strategies, but not aggregate outcomes.

IV. Conclusion

This paper develops a method to solve for strategic speculative outcomes in a multi-asset stock market when speculators see prices and internalize how their trades affect the informational content of prices. We thereby build on Admati's (1985) noisy rational expectations model to allow for strategic behaviour by speculators. In particular, speculators in our model internalize the fact that they are informationally large and that their trades have impacts on prices that convey information to other speculators. Because asset values are correlated, a signal about one asset contains information about other assets, and because there are multiple speculators, each speculator uses the information in prices to try to unravel the signals of others. We prove that there is a unique linear equilibrium by developing an iterative best-response mapping and showing that it is a contraction.

We then offer several testable predictions. We generate predictions about the covariance structures of prices changes (returns) and order flows across stocks that are driven solely by strategic behaviour. In particular, we find that the covariance structure of prices is driven by the covariance structure of asset fundamentals, while the covariance structure of order flows is driven by the covariance structure of liquidity trade. We also show that in integrated markets, the use of cross-asset information by market makers causes prices to be more correlated than the underlying fundamentals, and that this, in turn, raises the volatility of prices. Finally, we provide theoretical underpinnings for the Hasbrouck and Seppi (2001) finding that commonality in order flows explains two-thirds of the commonality in returns, generating this in our model with a moderate cross-asset correlation in asset value fundamentals plus a slight positive correlation in liquidity trade across assets.

While one must be careful translating predictions from a stylized model to the real world, the qualitative implications are clear. The transparency of pricing and order flows across assets has greatly increased in recent years. This suggests that the market has become "closer" to our observable price setting. Accordingly, (a) the volatility of returns and net order flows should have increased, and (b) the contemporaneous correlation in returns (e.g., within industries) should have risen, but (c) the contemporaneous correlation in order flows should not.

Our companion numerical analysis provides insights into the quantitatively important aspects of (a) the informational structure of the market and (b) the distributions over fundamentals, private information, and liquidity trade. The observability of prices sharply lowers speculator profits, especially when more speculators compete over the same information. This is because prices provide additional channels through which speculators receive and transmit information, giving each speculator incentives to raise trading intensities to influence the trades of other speculators. As a result, the observability of prices enhances the sensitivity of total speculator profits to the division of information between speculators. In sharp contrast, the divisions of liquidity trade and uncertainty across assets have little effect on aggregate speculative profits. In sum, the division of private information between *speculators* matters for aggregate outcomes, but the division of information and liquidity between *assets* does not.

Appendix

Proof of Proposition 1: Use the law of recursive projections to expand the first term in the first-order condition (5),

$$\begin{aligned} E[v|s^i, X+u] &= E[v|X+u] + E[v - E[v|X+u]|s^i - E[s^i|X+u]] \\ &= p + E[v - E[v|X+u]|s^i - E[e|X+u]], \end{aligned}$$

where the last equality reflects that $E[v|(X+u)] = p$. By construction, the vector of net order flows $X+u$ is orthogonal to the vector of forecast errors $s^i - E[s^i|X+u]$, so that

$$E[E[v|X+u]|s^i - E[s^i|X+u]] = 0.$$

Hence, we can remove the conditioning on $X+u$ from the first-order condition (5), which reduces to

$$\begin{aligned} 0 &= E[(p + E[v|s^i - E[s^i|X+u]] - p) - q^{i\prime}\lambda' x^i|s^i] \\ &= E[E[v|s^i - E[s^i|X+u]] - q^{i\prime}\lambda' x^i|s^i]. \end{aligned}$$

Solving for x^i yields

$$x^i = (\lambda')^{-1}(q^{i\prime})^{-1}E[E[v|s^i - E[s^i|X+u]]].$$

Q.E.D.

Proof of Lemma 1: Substitute for γ^i and Γ into the first-order condition (5). Recognizing that we do not need to condition on net order flow, u drops out and the first-order condition becomes

$$0 = E\left[v - \lambda\Gamma\sum_{k=1}^{N} b^k s^k - q^{i\prime}\lambda'\left(b^i s^i + \gamma^i \sum_{k=1}^{N} b^k s^k\right)\middle| s^i\right].$$

The goal is to remove $q^i = (I - \sum_{k\neq i} B^k)^{-1}$ from the first-order condition. To do this, we use

$$\begin{aligned} \Gamma = I + \sum_i \gamma^i &= I + \sum_i B^i\left(I - \sum_k B^k\right)^{-1} \\ &= \left(I - \sum_k B^k + \sum_k B^k\right)\left(I - \sum_k B^k\right)^{-1} \qquad \text{(A1)} \\ &= I\left(I - \sum_k B^k\right)^{-1} \end{aligned}$$

and

$$\begin{aligned} I+\gamma^i = I + B^i\left(I-\sum_k B^k\right)^{-1} &= \left(I-\sum_k B^k + B^i\right)\left(I-\sum_k B^k\right)^{-1} \\ &= \left(I-\sum_{k\neq i} B^k\right)\left(I-\sum_k B^k\right)^{-1} \\ &= (q^i)^{-1}\Gamma. \end{aligned} \tag{A2}$$

Hence, if we multiply the first-order condition by $I+\gamma^{i'} = \Gamma'(q^{i'})^{-1}$, the q^i terms cancel. Q.E.D.

Proof of Proposition 2: First-order condition for $\boldsymbol{\lambda}$: Exploiting the fact that market maker expected profits are a scalar, and hence equal to its trace, we first rewrite the market maker's least squares prediction problem as

$$\min_{\lambda} \operatorname{tr}\left\{\begin{pmatrix} V' - \sum_{k=1}^N \left(b^k A^k + \gamma^k \sum_{\ell=1}^N b^\ell A^\ell\right)'\lambda' \\ -\Gamma'\lambda' \end{pmatrix} \times \begin{pmatrix} V - \sum_{k=1}^N \left(b^k A^k + \gamma^k \sum_{\ell=1}^N b^\ell A^\ell\right)\lambda & -\Gamma\lambda \end{pmatrix} \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix}\right\},$$

where we have commuted under the trace. The associated first-order condition is

$$\begin{aligned} 0 &= \operatorname{tr}\left\{\begin{pmatrix} V' - \sum_{k=1}^N \left(b^k A^k + \gamma^k \sum_{\ell=1}^N b^\ell A^\ell\right)'\lambda' \\ -\left(\sum_{k=1}^N \gamma^k + I\right)'\lambda' \end{pmatrix} \right. \\ &\qquad \left. \times \begin{pmatrix} -\sum_{k=1}^N \left(b^k A^k + \gamma^k \sum_{\ell=1}^N b^\ell A^\ell\right) & -\left(\sum_{k=1}^N \gamma^k + I\right) \end{pmatrix} \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix}\right\} \\ &= \operatorname{tr}\left\{\begin{pmatrix} -\sum_{k=1}^N \left(b^k A^k + \gamma^k \sum_{\ell=1}^N b^\ell A^\ell\right) & -\left(\sum_{k=1}^N \gamma^k + I\right) \end{pmatrix} \right. \\ &\qquad \left. \times \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} V' - \sum_{k=1}^N \left(b^k A^k + \gamma^k \sum_{\ell=1}^N b^\ell A^\ell\right)'\lambda' \\ -\left(\sum_{k=1}^N \gamma^k + I\right)'\lambda' \end{pmatrix}\right\}, \end{aligned} \tag{A3}$$

where we commuted under the trace. We now compute the trace to rewrite (A3) compactly as

$$\begin{aligned}
&(-\Gamma bA \quad -\Gamma)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}V' - A'b'\Gamma'\lambda'\\ -\Gamma'\lambda'\end{pmatrix} = 0\\
&\Rightarrow \Gamma(bA \quad I)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}A'b'\\ I\end{pmatrix}\Gamma'\lambda' = \Gamma(bA \quad I)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}V'\\ 0\end{pmatrix}.
\end{aligned} \tag{A4}$$

The speculator's optimization problem. We first rewrite the speculator's objective, commuting under the trace to obtain

$$\begin{aligned}
&\max_{b^i,B^i} E\left[\operatorname{tr}\left\{\left(V - \lambda\Gamma\sum_{k=1}^{N} b^k A^k \quad -\lambda\Gamma\right)'\left(b^i A^i + \gamma^i\left(\sum_{k=1}^{N} b^k A^k\right) \quad \gamma^i\right)\begin{pmatrix}e\\ u\end{pmatrix}(e' \; u')\right\}\right]\\
&= \max_{b^i,B^i} \operatorname{tr}\left\{\begin{pmatrix}V' - \sum_{k=1}^{N} A^{k'}b^{k'}\Gamma'\lambda'\\ \Gamma'\lambda'\end{pmatrix}\left(b^i A^i + \gamma^i\left(\sum_{k=1}^{N} b^k A^k\right) \quad \gamma^i\right)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\right\}.
\end{aligned} \tag{A5}$$

First-order conditions for $\mathbf{b}^i$.

$$\begin{aligned}
0 &= \operatorname{tr}\left\{\begin{pmatrix}V' - \sum_{k=1}^{N} A^{k'}b^{k'}\Gamma'\lambda'\\ -\Gamma'\lambda'\end{pmatrix}(A^i + \gamma^i A^i \quad 0)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\right.\\
&\qquad \left. - \begin{pmatrix}A^{i'}\Gamma'\lambda'\\ 0\end{pmatrix}\left(b^i A^i + \gamma^i \sum_{k=1}^{N} A^k b^k \quad \gamma^i\right)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\right\}\\
&= \operatorname{tr}\left\{\begin{pmatrix}V' - \sum_{k=1}^{N} A^{k'}b^{k'}\Gamma'\lambda'\\ -\Gamma'\lambda'\end{pmatrix}(A^i + \gamma^i A^i \quad 0)\Sigma_e - \begin{pmatrix}A^{i'}\Gamma'\lambda'\\ 0\end{pmatrix}\right.\\
&\qquad \left. \times \left(\left(b^i A^i + \gamma^i \sum_{k=1}^{N} A^k b^k\right)\Sigma_e \quad \gamma^i \Sigma_u\right)\right\}.
\end{aligned}$$

The off-diagonals drop out due to the trace operation, leaving

$$0 = \operatorname{tr}\left\{\left(V' - \sum_{k=1}^{N} A^{k'}b^{k'}\Gamma'\lambda'\right)(A^i + \gamma^i A^i)\Sigma_e - (A^{i'}\Gamma'\lambda')\left(b^i A^i + \gamma^k \sum_{k=1}^{N} A^k b^k\right)\Sigma_e\right\}.$$

We now manipulate the first-order condition to isolate the $\{b^k\}$ terms. The trace is preserved under transposition: Taking the transpose of the second term and then commuting under the trace yields

$$\begin{aligned}
0 = &\ \operatorname{tr}\left\{A^i\Sigma_e\left(V' - \sum_{k=1}^{N} A^{k'}b^{k'}\Gamma'\lambda'\right)(I+\gamma^i) - \Sigma_e\left(b^iA^i + \gamma^i\sum_{k=1}^{N}A^kb^k\right)'\lambda\Gamma A^i\right\}\\
= &\ \operatorname{tr}\left\{A^i\Sigma_e\left(V' - \sum_{k=1}^{N} A^{k'}b^{k'}\Gamma'\lambda'\right)(I+\gamma^i) - A^i\Sigma_e\left(b^iA^i + \gamma^i\sum_{k=1}^{N}A^kb^k\right)'\lambda\Gamma\right\}\\
\Rightarrow &\ A^i\Sigma_e\sum_{k=1}^{N}A^{k'}b^{k'}\Gamma'\lambda'(I+\gamma^i) + A^i\Sigma_e\left(A^{i'}b^{i'} + \sum_{k=1}^{N}A^{k'}b^{k'}\gamma^{i'}\right)\lambda\Gamma\\
= &\ A^i\Sigma_eV'(I+\gamma^i).
\end{aligned}$$

First-order condition for $\mathbf{B}^{\mathrm{i}}$. We first note that

$$\frac{\partial\gamma^i}{\partial B^i} = \left(I - \sum_k B^k\right)^{-1} + B^i\left(I - \sum_k B^k\right)^{-2} = (I+\gamma^i)\Gamma \quad\text{and}\quad \frac{\partial\Gamma}{\partial B^i} = \Gamma^2.$$

Differentiating (18) with respect to B^i and setting it equal to zero yields

$$\begin{aligned}
0 = \operatorname{tr}&\left\{\begin{pmatrix} V' - \sum_{k=1}^{N}\left(b^kA^k + \gamma^k\sum_{\ell=1}^{N}b^\ell A^\ell\right)'\lambda' \\ -\left(\sum_{k=1}^{N}\gamma^k + I\right)'\lambda'\end{pmatrix}\frac{\partial\gamma^i}{\partial B^i}\begin{pmatrix}\sum_{k=1}^{N}b^kA^k & I\end{pmatrix}\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\right\}\\
-\operatorname{tr}&\left\{\begin{pmatrix}\sum_{\ell=1}^{N}(b^\ell A^\ell)'\frac{\partial\Gamma'}{\partial B^i}\lambda' \\ \frac{\partial\Gamma'}{\partial B^i}\lambda'\end{pmatrix}\begin{pmatrix}b^iA^i + \gamma^i\sum_{k=1}^{N}b^kA^k & \gamma^i\end{pmatrix}\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\right\}.
\end{aligned}$$

Using the fact that the transpose is the identity under the trace, we rewrite this as:

$$\begin{aligned}
0 = \operatorname{tr}&\left\{\begin{pmatrix} V' - \sum_{k=1}^{N}\left(b^kA^k + \gamma^k\sum_{\ell=1}^{N}b^\ell A^\ell\right)'\lambda' \\ -\left(\sum_{k=1}^{N}\gamma^k + I\right)'\lambda'\end{pmatrix}\frac{\partial\gamma^i}{\partial B^i}\begin{pmatrix}\sum_{k=1}^{N}b^kA^k & I\end{pmatrix}\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\right\}\\
-\operatorname{tr}&\left\{\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}\left(b^iA^i + \gamma^i\sum_{k=1}^{N}b^kA^k\right)' \\ \gamma^{i'}\end{pmatrix}\lambda\frac{\partial\Gamma}{\partial B^i}\begin{pmatrix}\sum_{\ell=1}^{N}(b^\ell A^\ell) & I\end{pmatrix}\right\}.
\end{aligned}$$

Commuting under the trace, we have

$$
\begin{aligned}
0 = \operatorname{tr}&\left\{ \frac{\partial \gamma^i}{\partial B^i} \begin{pmatrix} \sum_{k=1}^N b^k A^k & I \end{pmatrix} \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} V' - \sum_{k=1}^N \left(b^k A^k + \gamma^k \sum_{\ell=1}^N b^\ell A^\ell \right)' \lambda' \\ - \left(\sum_{k=1}^N \gamma^k + I \right)' \lambda' \end{pmatrix} \right\} \\
&- \operatorname{tr}\left\{ \begin{pmatrix} \sum_{\ell=1}^N b^\ell A^\ell & I \end{pmatrix} \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} \left(b^i A^i + \gamma^i \sum_{k=1}^N b^k A^k \right)' \\ \gamma^{i'} \end{pmatrix} \lambda \frac{\partial \Gamma}{\partial B^i} \right\} \\
&= \frac{\partial \gamma^i}{\partial B^i} \begin{pmatrix} \sum_{k=1}^N b^k A^k & I \end{pmatrix} \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} V' - \sum_{k=1}^N \left(b^k A^k + \gamma^k \sum_{\ell=1}^N b^\ell A^\ell \right)' \lambda' \\ - \left(\sum_{k=1}^N \gamma^k + I \right)' \lambda' \end{pmatrix} \\
&- \begin{pmatrix} \sum_{\ell=1}^N b^\ell A^\ell & I \end{pmatrix} \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} \left(b^i A^i + \gamma^i \sum_{k=1}^N b^k A^k \right)' \\ \gamma^{i'} \end{pmatrix} \lambda \frac{\partial \Gamma}{\partial B^i}.
\end{aligned}
$$

We rewrite this expression more compactly as

$$
\begin{aligned}
0 &= \frac{\partial \gamma^i}{\partial B^i} (bA \quad I) \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} V' - A'b'\Gamma'\lambda' \\ -\Gamma'\lambda' \end{pmatrix} \\
&\quad - (bA \quad I) \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} A^{i'} b^{i'} + A'b'\gamma^{i'} \\ \gamma^{i'} \end{pmatrix} \lambda \frac{\partial \Gamma}{\partial B^i} \\
&= (I + \gamma^i)\Gamma (bA \quad I) \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} V' - A'b'\Gamma'\lambda' \\ -\Gamma'\lambda' \end{pmatrix} \\
&\quad - (bA \quad I) \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} A^{i'} b^{i'} + A'b'\gamma^{i'} \\ \gamma^{i'} \end{pmatrix} \lambda \Gamma^2.
\end{aligned}
$$

Inspection of the first term reveals that it is $(I + \gamma^i)$ times the market maker's first-order condition (A4), and hence equal to zero. Thus, the first order condition for B^i simplifies to

$$
\begin{aligned}
&(bA \quad I) \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} A^{i'} b^{i'} + A'b'\gamma^{i'} \\ \gamma^{i'} \end{pmatrix} \lambda \Gamma^2 = 0 \\
\Rightarrow &(bA \quad I) \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} A'b' \\ I \end{pmatrix} \gamma^{i'} \lambda \Gamma^2 = -(bA \quad I) \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} A^{i'} b^{i'} \\ 0 \end{pmatrix} \lambda \Gamma^2.
\end{aligned}
$$

Carrying out the matrix multiplications yields the solution in (11):

$$
\gamma^{i'} = -(bA\Sigma_e A'b' + \Sigma_u)^{-1} bA\Sigma_e A^{i'} V'.
$$

Proof of Proposition 3: Defining $p = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$, routine algebra reveals that

$$\sum_{j \neq i} A^i \Sigma_e A^{j'} = p, \quad \text{and} \quad A^i \Sigma_e V' = I + p.$$

Because each b^i is identical, symmetric, and countersymmetric in the sense that the diagonal elements are identical, we can focus on a generic row of this recursion, which is a matrix equation:

$$\begin{aligned}(I+p)b^{i'} = {} & pb^{i'}\mu(\mu'(I+\gamma^i) + (I+\gamma^{i'})\mu)^{-1} + (I+p)(I+\gamma^i)(\mu'(I+\gamma^i) \\ & + (I+\gamma^{i'})\mu)^{-1}.\end{aligned} \tag{A6}$$

Before proving the contraction mapping property we need two lemmas.

LEMMA 1: *Given symmetry across assets and agents, $\|\gamma^i\| < \frac{1}{N}$, where N is the number of agents.*

Proof:

$$\begin{aligned}\gamma^{i'} &= -\left((bA \quad I)\begin{pmatrix}\Sigma_e & 0 \\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}A'b' \\ I\end{pmatrix}\right)^{-1}(bA \quad I)\begin{pmatrix}\Sigma_e & 0 \\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}A^{i'}b^{i'} \\ 0\end{pmatrix} \\ &= -\left((bA \quad I)\begin{pmatrix}\Sigma_e & 0 \\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}A'b' \\ I\end{pmatrix}\right)^{-1}(bA \quad I)\begin{pmatrix}\Sigma_e & 0 \\ 0 & \Sigma_u\end{pmatrix}\frac{1}{N}\begin{pmatrix}A'b' \\ 0\end{pmatrix} \\ &= -\frac{1}{N}(I - (J'J)^{-1}\Sigma_u).\end{aligned}$$

$J'J$ has positive eigenvalues because it is positive definite, and since it is the sum of two positive definite terms the eigenvalues of Σ_u must be positive but smaller than the eigenvalues of $J'J$. Hence, $(J'J)^{-1}\Sigma_u$ has fractional eigenvalues, and $(I - (J'J)^{-1}\Sigma_u)$ must also have fractional eigenvalues (Dhrymes (1978, p. 74.)). Q.E.D.

LEMMA 2: *Let Q and R be symmetric 2×2 matrices that are also countersymmetric, that is, they have identical elements on the diagonals as well. Then $QR = RQ$.*

Proof: Direct computation.

We now bound the nonlinear terms.

LEMMA 3: $\|\mu(\mu'(I+\gamma^i) + (I+\gamma^{i'})\mu)^{-1}\| < 1$.

Proof: Using Lemma 2,

$$\mu(\mu'(I+\gamma^i) + (I+\gamma^{i'})\mu)^{-1} = \mu(2(I+\gamma^i)\mu)^{-1} = \frac{1}{2}(I+\gamma^i)^{-1}.$$

Because an upper bound on γ^i is $-\frac{1}{N}I$, it follows that

$$\left\| \frac{1}{2}(I+\gamma^i)^{-1} \right\| < \frac{1}{2}\left\| \frac{N}{N-1} \right\| < 1. \qquad \text{Q.E.D.}$$

We now establish the contraction property of the mapping. Using symmetry across agents

$$bA\Sigma_e A'b' = b^i\,(I \quad \cdots \quad I)\,A\Sigma_e A' \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} b^{i'} = b^i N(I+p)b^{i'}.$$

Next note that using Lemma 2,

$$(I+\gamma^i)(\mu'(I+\gamma^i)+(I+\gamma^{i'})\mu)^{-1} = \frac{1}{2}\mu^{-1} = \frac{1}{2}(bA\Sigma_e V')^{-1}(bA\Sigma_e A'b' + \Sigma_u),$$

where the last equality follows from the definition of μ. Substituting for $bA\Sigma_e A'V'$ and $bA\Sigma_e A'b'$ into μ^{-1} yields

$$\frac{1}{2}\mu^{-1} = \frac{1}{2}(Nb^i(I+p))^{-1}(Nb^i(I+p)b^{i'} + \Sigma_u).$$

Now substitute for $\frac{1}{2}\mu^{-1}$ into the first-order condition for agent i, equation (A6), to obtain

$$\begin{aligned} &(I+p)b^{i'} = pb^{i'}\frac{1}{2}(I+\gamma^i)^{-1} + \frac{1}{2}(I+p)(Nb^i(I+p))^{-1}(Nb^i(I+p)b^{i'} + \Sigma_u) \\ \Rightarrow\ & b^{i'} = (I+p)^{-1}pb^{i'}\frac{1}{2}(I+\gamma^i)^{-1} + \frac{1}{2}(Nb^i(I+p))^{-1}(Nb^i(I+p)b^{i'} + \Sigma_u). \end{aligned}$$

Multiplying both sides by $Nb^i(I+p)$ yields

$$Nb^i(I+p)b^{i'} = Nb^i(I+p)(I+p)^{-1}pb^{i'}\frac{1}{2}(I+\gamma^i)^{-1} + \frac{1}{2}(Nb^i(I+p)b^{i'} + \Sigma_u).$$

Next subtract $\frac{1}{2}Nb^i(I+p)b^{i'}$ from both sides to obtain

$$\frac{1}{2}Nb^i(I+p)b^{i'} = Nb^i(I+p)(I+p)^{-1}pb^{i'}\frac{1}{2}(I+\gamma^i)^{-1} + \frac{1}{2}\Sigma_u.$$

Since p is diagonal and $b^{i'}$ is symmetric and countersymmetric, then again by Lemma 2 we can commute $(I+p)^{-1}pb^{i'}$ to $b^{i'}(I+p)^{-1}p$ and divide by $\frac{N}{2}$ to obtain

$$b^i(I+p)b^{i'} = b^i(I+p)b^{i'}(I+p)^{-1}p(I+\gamma^i)^{-1} + \frac{1}{N}\Sigma_u. \qquad \text{(A7)}$$

Defining the 2×2 matrix $Q \equiv b^i(I+p)b^{i'}$, we can write equation (A7) as

$$Q = Q(I+p)^{-1}p(I+\gamma^i)^{-1} + \frac{1}{N}\Sigma_u.$$

From Lemma 1

$$\|(I+\gamma^i)^{-1}\| < 2.$$

It is straightforward to establish that the maximum singular value of $(I+p)^{-1}p$ is $\frac{\theta}{1-\theta}$. For $\theta < \frac{1}{3}$, we then have $\frac{\theta}{1-\theta} < \frac{1}{2}$, so that

$$\|(I+p)^{-1}p(I+\gamma^i)^{-1}\| < 1,$$

implying that the mapping of Q is a contraction. We recover the fixed point of b^i by finding the Cholesky factor of the fixed point of Q and multiplying that factor by the inverse of the Cholesky factor of $I+p$. The Cholesky factor of $I+p$ is

$$\begin{pmatrix} 1 & \theta \\ 0 & (1-\theta^2)^{\frac{1}{2}} \end{pmatrix}.$$

Proof of Proposition 4: Matrices that are both symmetric and countersymmetric commute with each other. Exploiting that γ^i is symmetric and countersymmetric and identical across agents we multiply the first-order condition for b^i, equation (10), by b^i and add across i to obtain

$$\begin{aligned} &bA\Sigma_e A'b'\Gamma'\lambda'(I+\gamma^i) + 2b^iA^i\Sigma_e A^{i'}b^{i'}\lambda\Gamma + bA\Sigma_e bA'b'\gamma^{i'}\lambda\Gamma \\ &\quad = bA\Sigma_e V'(I+\gamma^i). \end{aligned} \tag{A8}$$

Using the definition of Ψ in (9), we next substitute $\Psi - \Sigma_u$ in the first term to rewrite (A8) as

$$\begin{aligned} &(\Psi-\Sigma_u)\Gamma'\lambda'(I+\gamma^i) + 2b^iA^i\Sigma_e A^{i'}b^{i'}\lambda\Gamma + bA\Sigma_e A'b'\gamma^{i'}\lambda\Gamma = bA\Sigma_e V'(I+\gamma^i) \\ &\quad = \Psi\Gamma'\lambda'(I+\gamma^i), \end{aligned}$$

where the last equality follows from the first-order condition for λ, equation (12). Subtracting $(\Psi-\Sigma_u)\Gamma'\lambda'(I+\gamma^i)$ from both sides yields

$$2b^iA^i\Sigma_e A^{i'}b^{i'}\lambda\Gamma + bA\Sigma_e A'b'\gamma^{i'}\lambda\Gamma = \Sigma_u\Gamma'\lambda'(I+\gamma^i) = \Sigma_u(I+\gamma^i)\Gamma\lambda,$$

where the final equality follows from the symmetric-countersymmetric structure of $\Gamma'\lambda'$ and $I+\gamma^i$, which allows us first to transpose and then commute. Dividing by $\lambda\Gamma$ then yields

$$2b^iA^i\Sigma_e A^{i'}b^{i'} + bA\Sigma_e A'b'\gamma^{i'} = \Sigma_u(I+\gamma^i). \tag{A9}$$

We now use the first-order condition for γ, equation (11), to simplify (A9). Multiplying both sides of (11) by $\Psi = bA\Sigma_e A'b' + \Sigma_u$ and rearranging yields

$$bA\Sigma_e A'b'\gamma^{i'} = -(\Sigma_u\gamma^{i'} + bA\Sigma_e A^{i\prime}b^{i\prime}). \tag{A10}$$

We substitute (A10) for $bA\Sigma_e A'b'\gamma^{i'}$ in (A9) and rearrange to obtain

$$2b^i A^i \Sigma_e A^{i'} b^{i'} - bA\Sigma_e A^{i\prime}b^{i\prime} = \Sigma_u(I + 2\gamma^i) = \Sigma_u\Gamma.$$

Subtracting the common $b^i A^i \Sigma_e A^{i'} b^{i'}$ on the left-hand side gives us

$$b^i A^i \Sigma_e A^{i'} b^{i'} - b^{j\neq i}A^{j\neq i}\Sigma_e A^{i\prime}b^{i\prime} = \Sigma_u\Gamma = \Psi^{-1}\Sigma_u^2,$$

where the final equality follows from substituting for Γ using the first- order condition for γ^i, equation (11). Multiplying both sides by Ψ^{-1} then yields

$$\Psi(b^i A^i \Sigma_e A^{i'} b^{i'} - b^{j\neq i}A^{j\neq i}\Sigma_e A^{i\prime}b^{i\prime}) = \Sigma_u^2. \tag{A11}$$

We now substitute the structure from the setting of Proposition 2,

$$A^i\Sigma_e A^{i'} = I \qquad \text{and} \qquad A^{j\neq i}\Sigma_e A^{i'} = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix} \equiv p,$$

to simplify the inner term on the left-hand side of equation (A11),

$$\begin{aligned} b^i A^i \Sigma_e A^{i'} b^{i'} - b^{j\neq i}A^{j\neq i}\Sigma_e A^{i\prime}b^{i\prime} &= b^i(I-p)b^{i'} \\ &= 2b^i(I+p)b^{i'}\frac{1}{2}(I+p)^{-1}(I-p) \\ &= (\Psi - \Sigma_u)\frac{1}{2}(I+p)^{-1}(I-p). \end{aligned} \tag{A12}$$

The first equality follows because with two agents, $b^i = b^{j\neq i}$. The second follows from commuting $I - p$ with $b^{i\prime}$ using symmetry-countersymmetry. The third uses $2b^i(I+p)b^{i'} = bA\Sigma_e A'b' = \Psi - \Sigma_u$. Now substitute for the inner term on the left-hand side of (A11) using (A12) to obtain

$$\Psi(\Psi - \Sigma_u)\frac{1}{2}(I+p)^{-1}(I-p) = \Sigma_u^2.$$

Multiplying both sides by $(\Psi^{-1})^2$ yields

$$(I - \Psi^{-1}\Sigma_u)\frac{1}{2}(I+p)^{-1}(I-p) = (\Psi^{-1}\Sigma_u)^2$$

or

$$(I - \Psi^{-1}\Sigma_u)C = (\Psi^{-1}\Sigma_u)^2, \tag{A13}$$

where

$$\begin{aligned} C \equiv \frac{1}{2}(I+p)^{-1}(I-p) &= \frac{1}{2(1-\theta^2)}(I-p)^2 \\ &= \frac{1}{2(1-\theta^2)}\begin{pmatrix} 1+\theta^2 & -2\theta \\ -2\theta & 1+\theta^2 \end{pmatrix}. \end{aligned} \tag{A14}$$

Equation (A13) is a quadratic in $\Psi^{-1}\Sigma_u$ (or equivalently Γ), and its solution takes the general form

$$\Psi\Sigma_u^{-1} = f(C).$$

Substituting for $\Psi = bA\Sigma_e A'b' + \Sigma_u$ and rearranging yields

$$bA\Sigma_e A'b' = f(C)\Sigma_u - \Sigma_u = (f(C) - I)\Sigma_u,$$

which establishes the result.

Proof of Proposition 5: Using Proposition 4, we have

$$\Psi = bA\Sigma_e A'b' + \Sigma_u \sim \Sigma_u^{1/2}((A\Sigma_e A')^{-1/2} A\Sigma_e A'(A\Sigma_e A')^{-1/2} + I)\Sigma_u^{1/2}.$$

Prices are $\lambda(X + u) = \lambda\Gamma(bAe + u)$, so that the variance–covariance matrix of prices is

$$\begin{aligned}\lambda\Gamma(bA\Sigma_e A'b' + \Sigma_u)\Gamma'\lambda' &= \lambda\Gamma\Psi\Gamma'\lambda' \\ &= V\Sigma_e A'b'\Psi^{-1}\Psi\Psi^{-1}bA\Sigma_e V' \\ &= V\Sigma_e A'b'\Psi^{-1}bA\Sigma_e V' \\ &= \left(V\Sigma_e A'b'\Sigma_u^{-1/2}\right)\left(\Sigma_u^{1/2}\Psi^{-1}\Sigma_u^{1/2}\right)\left(\Sigma_u^{-1/2}bA\Sigma_e V'\right),\end{aligned}$$

where the first equality follows from substituting for $\lambda\Gamma$ from the first-order condition (12), and the final equality follows from multiplying and dividing by $\Sigma_u^{1/2}$. The middle term can be written as

$$\left(\Sigma_u^{-1/2}\Psi\Sigma_u^{-1/2}\right)^{-1} = \left(\Sigma_u^{-1/2}(bA\Sigma_e A'b' + \Sigma_u)\Sigma_u^{-1/2}\right)^{-1}.$$

From Proposition 4, $b \sim (A\Sigma_e A')^{-1/2}\Sigma_u^{1/2}$, so the $\Sigma_u^{1/2}$ terms cancel, leaving only the effect of Σ_e. Q.E.D.

Proof of Proposition 6: Substitute for $\lambda\Gamma$ using (12) to write the price change covariance matrix as

$$V\Sigma_e A'b'\Psi^{-1}\Psi\Psi^{-1}bA\Sigma_e V' = V\Sigma_e A'b'\Psi^{-1}bA\Sigma_e V'. \tag{A15}$$

We have

$$\begin{aligned}&V\Sigma_e A'b' = 2\begin{pmatrix}1 & \theta\\ \theta & 1\end{pmatrix} b^i = 2(I+p)b^i, \qquad V\Sigma_e V = 2(I+p),\\ &\text{and} \qquad bA\Sigma_e A'b' = 2b^i(I+p)b^{i'}.\end{aligned}$$

Exploiting the symmetric-countersymmetric structure, we commute and write (A15) as

$$(V\Sigma_e V')(bA\Sigma_e A'b')\Psi^{-1}, \tag{A16}$$

where $V\Sigma_e V'$ is the value covariance matrix. Now substitute

$$bA\Sigma_e A'b' = \Psi - \Sigma_u = (f(C) - I)\Sigma_u \qquad \text{and} \qquad \Psi = f(C)\Sigma_u \tag{A17}$$

from the proof of Proposition 4 into (A16) to obtain

$$(V\Sigma_e V')(f(C) - I)\Sigma_u(f(C)\Sigma_u)^{-1} = (V\Sigma_e V')(f(C) - I)f(C)^{-1}. \tag{A18}$$

To calculate the correlation, we divide by the square root of the product of the diagonal terms: In this symmetric environment, to prove that the correlation in price changes exceeds that in values, we must prove that the ratio of an off-diagonal element to a diagonal element exceeds θ. To do so, we use the structure of the solution *f(C)* for $\Psi^{-1}\Sigma_u = \Gamma$ from equation (A13). Writing (A13) as

$$C(I - \Gamma) = \Gamma^2, \tag{A19}$$

we complete the square,

$$\Gamma^2 + C\Gamma + \frac{1}{4}C^2 = C + \frac{1}{4}C^2,$$

and take the square root to solve for

$$\Gamma = -\frac{1}{2}C + C^{1/2}\left(I + \frac{1}{4}C\right)^{1/2}. \tag{A20}$$

We next use

$$C(I - \Gamma) = \Gamma^2 \Rightarrow -(\Gamma - I) = C^{-1}\Gamma^2$$

and substitute into (A18) to obtain

$$\begin{aligned}(V\Sigma_e V')(f(C) - I)f(C)^{-1} &= 2(I + p)(\Gamma - I)\Gamma^{-1} = -2(I + p)C^{-1}\Gamma \\ &= 2(I + p)C^{-1}\left(-\frac{1}{2}C + C^{1/2}\left(I + \frac{1}{4}C\right)^{1/2}\right) \\ &= (I + p)(-I + C^{-1/2}(4I + C)^{1/2}).\end{aligned} \tag{A21}$$

We now use the definition of C in (A14) to calculate the components of (A21):

$$C^{-1} = \frac{2}{1-\theta^2}\begin{pmatrix} 1+\theta^2 & 2\theta \\ 2\theta & 1+\theta^2 \end{pmatrix} \quad \text{and} \quad 4I + C = \frac{1}{2(1-\theta^2)}\begin{pmatrix} 9-7\theta^2 & -2\theta \\ -2\theta & 9-7\theta^2 \end{pmatrix}$$

$$\Rightarrow C^{-1/2} = \left(\frac{2}{1-\theta^2}\right)^{1/2}\begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix} \quad \text{and} \quad (4I + C)^{1/2} = \left(\frac{1}{2(1-\theta^2)}\right)^{1/2}\begin{pmatrix} -a & \dfrac{\theta}{a} \\ \dfrac{\theta}{a} & -a \end{pmatrix},$$

where $a^2 = \dfrac{9 - 7\theta^2 + [(9 - 7\theta^2)^2 - 4\theta^2]^{1/2}}{2}$, with $a > 0$ the relevant solution to

$a^4+\theta^2-(9-7\theta^2)a^2=0$. Substituting into (A21) yields

$$
\begin{aligned}
&(V\Sigma_e V')(f(C)-I)f(C)^{-1}\\
&=\begin{pmatrix}1&\theta\\ \theta&1\end{pmatrix}\left(-I+\left(\frac{2}{1-\theta^2}\right)^{1/2}\begin{pmatrix}1&\theta\\ \theta&1\end{pmatrix}\left(\frac{1}{2(1-\theta^2)}\right)^{1/2}\begin{pmatrix}-a&\dfrac{\theta}{a}\\ \dfrac{\theta}{a}&-a\end{pmatrix}\right)\\
&=\begin{pmatrix}1&\theta\\ \theta&1\end{pmatrix}\left(-I+\left(\frac{1}{1-\theta^2}\right)\begin{pmatrix}1&\theta\\ \theta&1\end{pmatrix}\begin{pmatrix}a&-\dfrac{\theta}{a}\\ -\dfrac{\theta}{a}&a\end{pmatrix}\right)\\
&\sim\begin{pmatrix}1&\theta\\ \theta&1\end{pmatrix}\left(\begin{pmatrix}-(1-\theta^2)&0\\ 0&-(1-\theta^2)\end{pmatrix}+\begin{pmatrix}1&\theta\\ \theta&1\end{pmatrix}\begin{pmatrix}a&-\dfrac{\theta}{a}\\ -\dfrac{\theta}{a}&a\end{pmatrix}\right)\\
&\sim\begin{pmatrix}1&\theta\\ \theta&1\end{pmatrix}\left(\begin{pmatrix}-(1-\theta^2)+a-\dfrac{\theta^2}{a}&a\theta-\dfrac{\theta}{a}\\ a\theta-\dfrac{\theta}{a}&-(1-\theta^2)+a-\dfrac{\theta^2}{a}\end{pmatrix}\right)\\
&\sim\begin{pmatrix}-(1-\theta^2)+a-\dfrac{\theta^2}{a}+a\theta^2-\dfrac{\theta^2}{a}&\theta\left(-(1-\theta^2)+2a-\dfrac{\theta^2}{a}-\dfrac{1}{a}\right)\\ \theta\left(-(1-\theta^2)+2a-\dfrac{\theta^2}{a}-\dfrac{1}{a}\right)&-(1-\theta^2)+a-\dfrac{\theta^2}{a}+a\theta^2-\dfrac{\theta^2}{a}\end{pmatrix}.
\end{aligned}
$$

Hence, the off-diagonal/diagonal ratio exceeds θ if

$$
\begin{aligned}
&-(1-\theta^2)+2a-\frac{\theta^2}{a}-\frac{1}{a}>-(1-\theta^2)+a-\frac{\theta^2}{a}+a\theta^2-\frac{\theta^2}{a}\Leftrightarrow a-\frac{1}{a}\\
&\quad>\theta^2\left(a-\frac{1}{a}\right),
\end{aligned}
$$

which holds as long as assets are not perfectly correlated. Substituting $\theta\to 0$ into the off-diagonal/diagonal ratio yields $\frac{5}{3}\theta$, which our numerical analysis confirms. Q.E.D.

Proof of Proposition 7: The variance–covariance matrix of net order flows is $\Gamma(bA\Sigma_e A'b'+\Sigma_u)\Gamma'$. We first simplify the solution for Γ. From equation (11), we have

$$
\begin{aligned}
\Gamma=I+\sum_i\gamma^i=I-&\left(\begin{pmatrix}bA&I\end{pmatrix}\begin{pmatrix}\Sigma_e&0\\ 0&\Sigma_u\end{pmatrix}\begin{pmatrix}A'b'\\ I\end{pmatrix}\right)^{-1}\begin{pmatrix}bA&I\end{pmatrix}\\
&\times\begin{pmatrix}\Sigma_e&0\\ 0&\Sigma_u\end{pmatrix}\begin{pmatrix}A'b'\\ 0\end{pmatrix}.
\end{aligned}
$$

Adding and subtracting $((bA \quad I)\binom{\Sigma_e \quad 0}{0 \quad \Sigma_u}\binom{A'b'}{I})^{-1}\Sigma_u$, we write Γ as

$$\begin{aligned}
\Gamma &= I - \left((bAI)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}A'b'\\ I\end{pmatrix}\right)^{-1}(bAI)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}A'b'\\ I\end{pmatrix}\\
&\quad + \left((bA \quad I)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}A'b'\\ I\end{pmatrix}\right)^{-1}\Sigma_u\\
&= I - I + \left((bAI)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}A'b'\\ I\end{pmatrix}\right)^{-1}\Sigma_u\\
&= \left((bAI)\begin{pmatrix}\Sigma_e & 0\\ 0 & \Sigma_u\end{pmatrix}\begin{pmatrix}A'b'\\ I\end{pmatrix}\right)^{-1}\Sigma_u.
\end{aligned}$$

We use this solution for Γ to simplify the variance–covariance matrix of net order flows,

$$\Gamma(bA\Sigma_e A'b' + \Sigma_u)\Gamma' = \Gamma(bA\Sigma_e A'b' + \Sigma_u)\Sigma_u^{-1}\Sigma_u\Gamma' = \Gamma\Gamma^{-1}\Sigma_u\Gamma' = \Sigma_u\Gamma'.$$

In the fully symmetric two-speculator setting, Γ does not depend on Σ_e (see Proposition 6).

Proof of Proposition 8 (Unobservable prices): The variance–covariance matrix of net order flows is $(bA\Sigma_e A'b' + \Sigma_u)$: With uncorrelated liquidity trade Σ_u is diagonal, so that the off-diagonal terms equal their values in $bA\Sigma_e A'b'$. From Corollary 1, with symmetry, speculators only trade on their direct signals and they trade with the same intensity on each direct signal. Hence,

$$\begin{aligned}
bA\Sigma_e A'b' &= \begin{pmatrix} b_{11} & 0 & b_{11} & 0\\ 0 & b_{11} & 0 & b_{11}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & 0 & \frac{\theta}{2}\\ 0 & \frac{1}{2} & \frac{\theta}{2} & 0\\ 0 & \frac{\theta}{2} & \frac{1}{2} & 0\\ \frac{\theta}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}\\
&\quad \times \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix} b_{11} & 0\\ 0 & b_{11}\\ b_{11} & 0\\ 0 & b_{11}\end{pmatrix}\\
&= \begin{pmatrix} b_{11} & 0 & b_{11} & 0\\ 0 & b_{11} & 0 & b_{11}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & 0 & \frac{\theta}{2}\\ 0 & \frac{1}{2} & \frac{\theta}{2} & 0\\ 0 & \frac{\theta}{2} & \frac{1}{2} & 0\\ \frac{\theta}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix} b_{11} & 0\\ 0 & b_{11}\\ b_{11} & 0\\ 0 & b_{11}\end{pmatrix}\\
&= (b_{11})^2\begin{pmatrix}1 & \theta\\ \theta & 1\end{pmatrix}.
\end{aligned}$$

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