

Security Design with Investor Private Information

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ABSTRACT

I study the security design problem of a firm when *investors* rather than managers have private information about the firm. I find that it is often optimal to issue information-sensitive securities such as equity. The “folklore proposition of debt” from traditional signaling models only goes through if the firm can vary the face value of debt with investor demand. When the firm has several assets, debt backed by a pool of assets is optimal when the degree of competition among investors is low, while equity backed by individual assets is optimal when competition is high.

MOST INFORMATION-BASED THEORIES of security design, starting with Myers and Majluf (1984), make the assumption that the issuing firm has private information about the value of its assets. This leads to a problem for the firm when it attempts to raise capital. Since investors will worry about being sold a lemon, a genuinely good firm that cannot signal its quality properly to the market will face underpricing when it issues securities. Accordingly, the firm will design the securities to minimize this underpricing. This theory has been applied fruitfully to explain the financing behavior of firms. One of the more robust predictions that emerges is the “folklore proposition of debt,” which posits that the firm should issue the least information-sensitive security possible, namely, standard debt.¹ Another prediction that arises is that if the firm has access to several assets, issuing securities backed by the entire pool of assets may also help to reduce underpricing (see DeMarzo (2005)).

A problem that has received much less attention is the one that corresponds to the case in which *investors*, rather than the firm itself, have private information about the prospects of the issuer. I argue that investor private information is often just as important a friction in financial markets as firm private

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¹ In a pooling equilibrium, debt has the lowest underpricing (see, e.g., Nachman and Noe (1994)). In a separating equilibrium, where a firm can signal its quality by costly retention as in DeMarzo (1997), issuing debt makes mimicking by low quality firms untempting.

information. For example, a start-up company seeking financing usually has to raise money from professional intermediaries such as venture capitalists or banks. These investors, based on their industry expertise and long experience in financing, may be better at evaluating the likely success of the firm than the entrepreneur. There is also evidence consistent with the existence of investor private information when the firm issues securities at later stages in its life cycle, such as in an IPO. In a typical book-building procedure, more “informative” bids are rewarded a higher allocation (Cornelli and Goldreich (2001)), and thus the persistent underpricing in IPOs could be interpreted as informational rents captured by investors.

Asset-backed securities markets are another setting in which investor private information may be a driving force for security design. Consider the sale of assets performed by the Resolution Trust Corporation (RTC) in the 1990s. The RTC was set up as a government agency in 1989 under the Financial Institutions Reform, Recovery, and Enforcement Act to dispose of the assets of failed savings and loans institutions. It was clear that the RTC had very little expertise in valuing these assets, as opposed to the eventual buyers who were sometimes the original owners of the assets. Initially, the RTC conducted mostly individual sales of assets, generating very little volume. Through the use of pooled asset auctions and securitized issues, however, the RTC had sold \$455 billion worth of assets by 1995 (see Vandell and Riddiough (1992) and Watkins (1992)).²

The goal of this paper is to characterize what implications investor private information has for the optimal security design of the firm. In particular, I try to answer the following questions: Does the folklore proposition of debt still hold when the asymmetric information problem is reversed, or are there some circumstances in which the firm should issue a more informationally sensitive security? When the firm has access to several assets, should it issue securities backed by a pool of assets or by individual assets? When investors no longer are price takers who make zero profits, as they are assumed to be in the signaling literature, how does the degree of competition among investors affect the optimal security design?

Just as in firm private information models, securities will be underpriced when investors have private information. This is because investors will only purchase securities at a price that is favorable to them. In both cases, the role of

² In other asset-backed securities markets, such as the market for mortgage-backed securities, it may be harder to argue that investors have superior knowledge about the quality of the actual assets (such as an individual real estate property). However, there may be other sources of private information that are important. Buyers in these markets are generally large investment banks, brokerage firms, and institutional investors, who have substantial expertise in valuing the securities issued through their knowledge of secondary market conditions or their access to proprietary valuation models. In analyzing data from an auction of collateralized mortgage obligations in which the participants were major investment firms, Bernardo and Cornelli (1997) find that the dispersion of bids in the auction was surprisingly high. This indicates that investors had significant differences in valuation *ex ante*, and Bernardo and Cornelli provide evidence that these differences were mainly attributable to private information regarding valuation rather than, for example, differences in the utility of holding the assets.

security design is to minimize underpricing while raising the capital needed for investments. The models are also similar in that debt is the least information-sensitive security and therefore has the least amount of underpricing per dollar of capital raised. The difference is that when the firm knows less than investors, it does not know how much capital a certain security will raise, or how much capital it actually needs. The crucial new role of security design is therefore to make sure that the amount of capital raised covaries correctly with investor information.

I show that this difference can lead to the optimality of information-sensitive securities. An information-sensitive security such as equity will raise more money in good states than in bad states relative to debt. If raising capital is also more valuable in good states than in bad, this can be sufficiently beneficial to outweigh the higher underpricing per dollar raised. I identify two separate reasons for why raising more capital in good states can be valuable. First, if the asymmetric information problems are smaller in good states, the cost of raising capital goes down. Second, if the investment opportunity of the firm is more valuable in good states, the benefit of raising capital goes up. Debt will then tend to raise too much capital in bad states, where it is not sufficiently needed to justify the cost of raising capital, and too little in good states, where the marginal productivity of capital more than outweighs the cost.

I also show that the net benefit of information-sensitive securities is larger when the market is more competitive. When markets are more competitive, the underpricing is smaller for all securities so that the cost disadvantage of information sensitivity goes down. Also, a more competitive market leads to more informative prices, which translates into a bigger difference in capital raising between debt and information-sensitive securities. Therefore, the capital raising advantage of information sensitivity goes up as the market becomes more competitive.

The results above are derived for the case in which the firm has to design the security at an *ex ante* stage. When the firm can vary the security design as a function of investor information, for example, by issuing several securities with different reserve prices or by issuing securities sequentially, the folklore proposition of debt is recovered. By letting the face value of debt be determined by investor demand, the firm can still ensure that it raises the right amount of capital in each state.

Next I study the case in which the firm has many assets. I show that when the number of assets is large enough, it is optimal to issue debt backed by the whole pool of assets rather than to issue securities backed by single assets. Pooling dilutes private information by averaging information across many assets, and hence helps to reduce underpricing. Debt is optimal because when assets are pooled, the amount of capital raised with a security becomes more predictable, so that the capital raising benefit of information-sensitive securities disappears.

However, if the degree of competition is high relative to the number of assets, issuing information-sensitive securities backed by individual assets can be optimal. When the degree of competition is high, the price will be set by investors from the upper tail of the information distribution. Underpricing depends on

how much information asymmetry there is between these investors. In the sale of a single asset, these top investors are more likely to have the same information, since they only need to agree about one asset. In the sale of a pool, information about all assets is relevant, which increases the potential for differences in information. Separate sales will dominate, and information-sensitive securities will be beneficial, since they tend to raise most of their capital exactly in the favorable state in which information asymmetries are small.

The theory I develop delivers several empirical predictions, and may shed light on some observed phenomena in financial markets. First, it shows that equity issues should be more commonly observed for firms with highly variable growth opportunities, and in markets where the degree of competition is high. Second, it explains the role of state contingent security design such as sequential debt issues or allowing investors to bid with securities. Third, it may explain why debt dominates in asset-backed securities markets where large pools of assets back securities, while equity issues are more prevalent when the asset base is more focused (as is the case for individual firms). In the conclusion I discuss how the results developed here may also be helpful in understanding life-cycle patterns of firm financing, the difference in financing between bank-oriented and market-oriented systems, and the role of financial intermediation.

Related Literature

It is interesting to contrast the results in my paper a bit further with the results developed in the signaling literature, especially in the paper by DeMarzo (2005) that studies the same pooling and security design problem when the *seller* is endowed with private information. As in my model, DeMarzo assumes that retaining cash flow is costly for the seller. Issuers signal their quality by their rate of retention. Low-quality issuers gain the least by mimicking a debt issue by a better firm, and therefore debt is the optimal security since it allows a good firm to retain as little as possible without being mimicked. Because firms signal through retention, the face value of debt will decrease in the quality of the issuer. In my setting, in contrast, the face value of debt often increases in the quality of the firm. Also, in DeMarzo's setup, there is no role for equity, since the firm knows exactly how much capital will be raised, and the degree of competition is irrelevant. However, the two models have the same prediction that when the asset base is large enough, and assets are not too correlated, pooling backed by debt is an optimal security design.

Perhaps most closely related to my paper is the independently developed study by DeMarzo, Kremer, and Skrzypacz (2005). They investigate a situation in which buyers bid with securities to obtain the right to invest in and run a project. Although they do not study *ex ante* security design by the issuer, which is the main focus of my paper, their setup is very similar to my analysis in Section III, where I let the security design depend on investor information. As in my paper, they find that debt with varying face value is the optimal security for this case. The primary difference is that giving cash to the seller up front in exchange for securities is never optimal in their model, since the seller is not liquidity constrained, whereas in my analysis the purpose of selling securities

is to raise cash. The models also differ in that I study variable-scale investment opportunities, whereas they study fixed-scale investments.³

Garmaise (1997) also examines an auction model with informed investors. In his model, two informed investors offer securities to a firm in a first-price auction. The results are driven by the fact that investors view the entrepreneur as being either overly optimistic or overly pessimistic, leading to debt in the first case and equity in the second case. There is no analysis of how the degree of competition affects security design.

Apart from the studies mentioned above, the investor private information problem has not received much attention in the security design literature. Comprehensive surveys of the security design literature can be found in Harris and Raviv (1991) and in Allen and Winton (1995). Neither of these surveys discusses the investor private information problem. Note that the idea that outside investors may have private information is not completely new to the corporate finance literature. However, as opposed to the view in this paper, the focus has mostly been on the positive role of informed investors. Allen (1993) and Habib and Johnsen (2000), for example, emphasize the benefits of security price information in guiding the investment decisions of the firm. Somewhat more closely related to this paper, Boot and Thakor (1993) and Fulghieri and Lukin (2001) extend the signaling literature to allow some investors to acquire information about the value of assets, although the security design motive in these papers ultimately stems from signaling considerations. Good firms want to separate themselves from bad firms, and might therefore issue equity securities to encourage information production about their assets. None of these papers discuss the role of security design in screening investors.

In a trading context, Gorton and Pennachi (1993) and Subrahmanyam (1991) use Kyle-type models to explain the existence of basket securities in stock markets. They show that liquidity traders can avoid getting picked off by informed traders by trading in pooled securities such as stock index futures. Even though their models do not include an issuer conducting security design, the intuition for these results is similar to the intuition for why pooling may help reduce underpricing in my model. Similarly, Gorton and Pennachi (1990) show that firms or financial intermediaries have an incentive to split cash flows into debt and equity so that uninformed traders can protect themselves against losses to informed traders. None of these papers study the impact of an increased degree of competition on security design.

The remainder of the paper is organized as follows. Section I presents the basic model setup. In Section II I study the security design problem when the firm has to specify the security *ex ante*. In Section III I allow the security design to depend on investor information. In Section IV I discuss the robustness of the security design results to the choice of sales mechanism. In Section V I

³ On the other hand, their analysis is more general along other dimensions. They allow for both private and common values, whereas I study a pure common value setting. They also allow for varying degrees of seller commitment power, and show that when the seller is unable to commit to a certain class of securities, call options will be used instead of debt, giving the lowest possible revenue to the seller.

extend the analysis to the multiple asset case, and in Section VI I present the conclusion. All proofs are in the Appendix.

I. Model Setup

There are two time periods, 0 and 1. Everyone is risk neutral and the discount rate is normalized to one. A single seller owns an asset with stochastic payoff $Z \in [0, \bar{z}]$ in period 1. The random variable Z has a cumulative distribution function $G(z)$ with associated density $g(z) > 0$.^{4,5}

The seller also has access to a constant returns-to-scale project that pays off $\pi * (1 + r(Z))$ in or after period 1 if π is invested in period 0. I allow for the possibility that the investment opportunity can co-vary with the payoff of the asset, as captured by the dependence of r on Z . The typical case is that $r(Z)$ is nondecreasing. I also allow for the possibility that $r(Z)$ can be negative over some range. This captures the notion that there can be a marginal transaction cost for raising money on top of the informational rents given up, such as a proportional brokerage fee.

The seller is liquidity constrained but can raise money by issuing securities backed by the cash flow of the asset. I write a security as a payoff function $w(Z)$ that depends on the realization of the asset cash flow. The security is sold at price π in an auction procedure defined below, where the price depends both on the security design and the information of investors.

The private information of investors is modeled as follows. There are N bidders in the auction of the security. Each bidder $n \in \{1, \dots, N\}$ draws a privately observed signal X_n , which is informative about the value of the asset. Conditional on the realized value of the asset, signals are distributed identically and independently according to the probability density $f(x|z) > 0$. I denote the associated cumulative distribution function by $F(x|z)$. I assume that the signal distribution satisfies the monotone likelihood ratio property:

Monotone Likelihood Ratio Property (MLRP): $\frac{f(x|z)}{f(x'|z)} \geq \frac{f(x|z')}{f(x'|z')}$ if $x > x'$ and $z > z'$, with strict inequality for some x, x' .

The MLRP assumption ensures that signals are informative, and that a higher signal leads to a more optimistic view of the value of the underlying asset.⁶

Inherently discrete information structures, where a signal A can take on integer values from 0 to H with probability $A = a$ given by $P(a|z)$, are captured

⁴ Throughout, I separate between random variables and their realizations by denoting random variables with capital letters and outcomes with lowercase letters.

⁵ All results go through when the density does not exist (e.g., when Z is discrete-valued).

⁶ Given the assumption that signals are drawn independently conditional on the realization of the cash flow z , the MLRP assumption is equivalent to the standard assumption in auction theory that the random variables of the model are *affiliated* (see Milgrom and Weber (1982)). The model is a special case of the "general symmetric model" of Milgrom and Weber. The most important restriction relative to the more general model is the assumption that bidders have the same valuation of realized cash flows. If bidders differ in their marginal utility of cash flow in different states, which would be the case, for example, if risk-sharing in the economy was incomplete, there would be an extra incentive for the firm to design securities that facilitate risk sharing. For models that analyze the risk-sharing motive of security design, see Allen and Gale (1988) and Duffie and Jackson (1989).

in this continuous framework by letting $X \in [0, H + 1)$ and setting $f(x|z) = f(x'|z) = P(a|z)$ if $x, x' \in [a, a + 1)$.⁷

The maximization problem of the seller can now be stated as

$$\max_{w(Z)} E((1+r)\pi - w) = \max_{w(Z)} E(r\pi) - E(w - \pi), \quad (1)$$

subject to

EQUILIBRIUM: w, π constitutes an equilibrium in the sales mechanism.

LIMITED LIABILITY: $0 \leq w(z) \leq z$ for all z .

MONOTONICITY: $w(z)$ and $z - w(z)$ are nondecreasing in z .

The term $E(r\pi)$ in the maximand is the expected social surplus generated by investing π at the excess return r . The second term is the expected underpricing of the security, and can be viewed as the informational rent given to investors.

The constraint $0 \leq w(z)$ is a limited liability constraint for bidders, which says that once the security is bought, they are not obliged to provide any additional cash flow in period 1. The constraint $w(z) \leq z$ is a limited liability constraint on the seller, reflecting his limited wealth outside of the cash flow generated by assets.⁸ The monotonicity assumption is standard in the security design literature and can be formally justified on grounds of moral hazard in period 1.⁹

II. Ex Ante Security Design

I first study the situation in which the security w is fixed in advance and so its design is not contingent on any investor information that may be elicited in the sales mechanism. I refer to this as ex ante security design. A Treasury auction, in which the government decides to sell a pre-specified amount of a bond with pre-specified coupons and maturity, can be viewed as an example of ex ante security design. An initial public offering where a firm pre-registers equity is another example.

A. Sales Mechanism

I assume that securities are sold in sealed bid, uniform price, K -unit auctions that work as follows. Investors submit bids, and the K investors with the highest bids get an equal allocation of the security at the market clearing price, which

⁷ The continuous representation of a discrete signal space can be viewed as the original signal plus a draw from a bidder's mixing strategy. (See, e.g., Pesendorfer and Swinkels (1997) for a similar way of representing signals.) This representation is useful because strategies are often mixed in a discrete signal space, but are pure in the continuous signal X .

⁸ For the main part of the paper, I assume that the seller cannot pledge any of the cash flow from the *new* project to investors, because of unmodeled moral hazard or information problems. This restriction is purely for expositional and analytical convenience. The qualitative nature of the results is unchanged when cash flows from the new project backs securities, as shown in Section VB.

⁹ Ruling out nonmonotonic securities is also convenient for technical reasons. With monotonic securities, MLRP on the signal space with respect to security cash flows is preserved, which allows me to use standard auction theory results for the characterization of equilibria.

equals the highest losing bid (the $K + 1$:st highest bid). Spreading the allocation out over several investors can reflect investor capital constraints. Alternatively, the number of units can be a choice variable of the issuer.

The uniform price auction is a standard auction format that closely resembles many allocation mechanisms in financial markets; for example, Treasury auctions. The IPO book-building process, whereby investors submit indications of interest followed by a posted price offering to the investors with the highest indications, is also a uniform price mechanism.¹⁰ Although the sales mechanism is standard, it should be pointed out that it is exogenously specified and may not be the optimal mechanism from the seller's point of view. I discuss the robustness of the security design results to other choices of mechanisms in Section IV.

The equilibrium of the uniform price auction was first characterized by Milgrom (1981). Denote by $\{Y_1, \dots, Y_N\}$ the set of investor signals X_n ordered in a decreasing order, so that $Y_1 \geq \dots \geq Y_N$. Equilibrium bids increase in the investor signal x , and since the price is equal to the $K + 1$:st highest bid, the equilibrium price is a function of the signal Y_{K+1} alone. Denoting by $\pi(w, y)$ security w 's price when $Y_{K+1} = y$, the equilibrium price is given as follows.

LEMMA 1: *Security w 's price in a sealed bid, uniform price, K -unit auction when $Y_{K+1} = y$ is given by*

$$\pi(w, y) = E(w \mid Y_K = Y_{K+1} = y).$$

Proof: This and all the following proofs are in the Appendix.

An investor will bid so that he is indifferent between winning and losing when he just marginally wins, which is when he ties with the marginal loser. Notice that this creates underpricing in equilibrium. Conditional on the signal $Y_{K+1} = y$, the expected value of a security is $E(w \mid Y_{K+1} = y)$, which is higher than the price $E(w \mid Y_K = Y_{K+1} = y)$ since the marginal winning investor typically ends up having a higher signal than the marginal loser. A major role for the security design is to minimize this underpricing.¹¹

¹⁰ See Spatt and Srivastava (1991), who show the equivalence between the book-building process and the sealed bid uniform price auction.

¹¹ It is worth pointing out that Lemma 1 also holds if the security is split into "sub-securities" that are sold in simultaneous auctions. This is immediate from the linearity of the price function, and the fact that the price-setting signal $Y_{K+1} = y$ is the same across auctions:

$$\begin{aligned} \pi(w_1 + w_2, y) &= E(w_1 + w_2 \mid Y_K = Y_{K+1} = y) \\ &= E(w_1 \mid Y_K = Y_{K+1} = y) + E(w_2 \mid Y_K = Y_{K+1} = y) \\ &= \pi(w_1, y) + \pi(w_2, y). \end{aligned}$$

Since the price is the same state-by-state, revenues are unchanged. Thus, there is no loss of generality from restricting attention to the issuance of one security, as long as securities can only be sold simultaneously.

This may not be true if securities are allowed to be auctioned sequentially, since information revealed in early auctions may affect the equilibrium in later auctions. I discuss sales mechanisms that utilize more information in Section IV.

B. Security Design

To further simplify the problem, it is convenient to decompose a security into its “smallest component” securities, where each component security pays one if its cash flow is above a certain threshold z and zero otherwise. I denote such a security by an indicator function $1_{Z \geq z}$. The following lemma states that any security design can be viewed as a portfolio of such component securities.

LEMMA 2: *Any security satisfying limited liability and monotonicity can be written as a portfolio of component securities,*

$$w(Z) = \int_0^{\bar{z}} 1_{Z \geq z} w'(z) dz,$$

where $w'(z) \in [0, 1]$, and any security written in this way satisfies limited liability and monotonicity.

The cutoff z is a measure of how “leveraged,” or information sensitive, a component security is. Debt includes all component securities with a cutoff below the face value D of debt: $w'(z) = 1_{z \leq D}$. A call option contains all component securities with a cutoff above the strike price S of the option: $w'(z) = 1_{z \geq S}$.

From the linearity of the pricing function, the maximization problem (1) can now be expressed as a decision over which component securities to auction off,

$$\begin{aligned} & \max_w E(r\pi(w, Y_{K+1})) - E(w - \pi(w, Y_{K+1})) \\ &= \int_0^{\bar{z}} \max_{w'(z) \in [0, 1]} [E(r\pi(1_{Z \geq z}, Y_{K+1})) - E(1_{Z \geq z} - \pi(1_{Z \geq z}, Y_{K+1}))] w'(z) dz. \end{aligned} \quad (2)$$

This problem has a simple solution: Include component security $1_{Z \geq z}$ in the security design (set $w'(z) = 1$) if the expected benefit given by $E(r\pi(1_{Z \geq z}, Y_{K+1}))$ exceeds the expected underpricing given by $E(1_{Z \geq z} - \pi(1_{Z \geq z}, Y_{K+1}))$; otherwise, do not include it. The following proposition summarizes the solution to the security design problem.

PROPOSITION 1: *The optimal ex ante security design is given by*

$$w'(z) = \begin{cases} 1 & \text{if } \phi(z) \geq 0 \\ 0 & \text{if } \phi(z) < 0, \end{cases} \quad (3)$$

where

$$\phi(z) \equiv \frac{E(r\pi(1_{Z \geq z}, Y_{K+1})) - E(1_{Z \geq z} - \pi(1_{Z \geq z}, Y_{K+1}))}{E(\pi(1_{Z \geq z}, Y_{K+1}))}.$$

The function $\phi(z)$ is the net benefit per expected dollar raised using component security $1_{Z \geq z}$. Debt is optimal if $\phi(z)$ is decreasing, while a call option is optimal if it is increasing.

To illustrate the forces that affect the choice of information sensitivity, it is useful to further decompose $\phi(z)$ into two parts as follows:

$$\phi(z) = \int \phi(z | y) q(z | y) dy,$$

where $\phi(z | y)$ is the *conditional net benefit* defined as

$$\phi(z | y) \equiv E(r | Y_{K+1} = y) - \frac{E(1_{Z \geq z} | Y_{K+1} = y) - \pi(1_{Z \geq z}, y)}{\pi(1_{Z \geq z}, y)}$$

and $q(z | y)$ is the *capital-raising density* for security $1_{Z \geq z}$ defined as

$$q(z | y) \equiv \frac{\pi(1_{Z \geq z}, y) f(Y_{K+1} = y)}{E(\pi(1_{Z \geq z}, Y_{K+1}))}.$$

These functions have the following interpretation. The capital-raising density $q(z | y)$ is a weighting function that shows what proportion of the expected capital raised with component security $1_{Z \geq z}$ comes from state $Y_{K+1} = y$. The conditional net benefit $\phi(z | y)$ is the seller's net gain per dollar raised in state $Y_{K+1} = y$ using security $1_{Z \geq z}$. The following lemma shows that for two securities that raise the same amount of capital in a given state, the less information-sensitive security always has a higher net benefit in that state.

LEMMA 3: $\phi(z | y)$ is decreasing in z .

The intuition is simple. If two securities raise the same amount of capital π in a given state, they produce the same revenues $r\pi$. However, the underpricing is higher for the more information-sensitive security, since the downward revision of the marginal winner's signal in the pricing function matters more the more information sensitive the security is. This is the driving force for reducing information sensitivity.

The potentially positive effect of information sensitivity is that more information-sensitive securities tend to raise more capital in high states relative to low states, as the following lemma shows.

LEMMA 4: For $z > z'$, $\frac{q(z | y)}{q(z' | y)}$ is increasing in y .

This can be beneficial for two reasons. First, if the conditional underpricing is decreasing in the indicated state, a security that raises more in high states may feature lower underpricing. Second, raising more in high states is also better if the benefit of raising capital $E(r | Y_{K+1} = y)$ is increasing in the state. Before giving general conditions for when information sensitive securities are optimal, I illustrate these effects in an example.

B.1. Example

Suppose cash flow Z is uniformly distributed on the unit interval, and investors get either high or low signals with $P(\text{High} | z) = z$, $P(\text{Low} | z) = 1 - z$. There are two bidders and one unit sold, so $Y_{K+1} = Y_2$. The price setting signal Y_2 is either high (when both bidders have high signals) or low (when at

least one bidder has a low signal).¹² Also, suppose the productivity of capital $E(r | Y_2 = y)$ is non-decreasing in y .

The following table gives the inputs necessary to calculate the optimal security design:

Y_2	$P(Y_2)$	$E(r Y_2)$	$\pi(1_{Z \geq z}, Y_2)$	$E(1_{Z \geq z} Y_2)$	$\phi(z Y_2)$	$q(z Y_2)$
High	$\frac{1}{3}$	r_H	$1 - z^3$	$1 - z^3$	r_H	$\frac{\frac{1}{3}(1-z^3)}{\frac{1}{3}(1-z^3) + \frac{2}{3}(1-z)^3}$
Low	$\frac{2}{3}$	r_L	$(1 - z)^3$	$(1 - z)^3 + \frac{3}{2}z(1 - z)^2$	$r_L - \frac{3}{2} \frac{z}{1-z}$	$\frac{\frac{2}{3}(1-z)^3}{\frac{1}{3}(1-z^3) + \frac{2}{3}(1-z)^3}$

Here, $r_H \equiv E(r | Y_2 = \text{High}) \geq r_L \equiv E(r | Y_2 = \text{Low})$.

Note that the underpricing $E(1_{Z \geq z} | y) - \pi(1_{Z \geq z}, y)$ is zero in the high state. If the loser's signal is high, the price is set as if both investors had high signals. Hence, since the winner must also have had a high signal, there is no underpricing. However, if the loser's signal is low, the price is set as if both investors had low signals while in actuality the winner might have had a high signal, which leads to underpricing. Note that this underpricing increases in z , which is the cost of information sensitivity.

The benefit of information sensitivity is that a higher proportion of the capital is raised in the high state, where underpricing is zero and productivity of capital is the highest. The unconditional benefit $\phi(z)$ can be calculated as

$$\begin{aligned}
 \phi(z) &= \phi(z | \text{High}) * q(z | \text{High}) + \phi(z | \text{Low}) * q(z | \text{Low}) \\
 &= r_L + (r_H - r_L) \frac{\frac{1}{3} + \frac{z}{(1-z)^2}}{1 + \frac{z}{(1-z)^2}} \\
 &\quad - \frac{1}{\frac{1}{(1-z)z} - 1}.
 \end{aligned}$$

The middle line increases in z , reflecting the productivity advantage of information sensitivity. The last line is the expected underpricing, which increases in z for $z \leq 0.5$ and decreases for $z \geq 0.5$.

The lower curve in Figure 1 shows $\phi(z)$ for two investors and $r_H = r_L = 0.1$, so that there is no productivity advantage in the high state. The optimal security design is found by aggregating the component securities for which the net benefit is positive, in this case for (approximately) $z \leq 0.1$ and $z \geq 0.9$. This corresponds to a debt component with face value 0.1 and a call option component

¹² For the case of two bidders and discrete signals, the equilibrium is in pure strategies, which is why I do not use the less transparent continuous representation of signals in the example. For more than two bidders, the equilibrium is mixed if signals are expressed in discrete form.

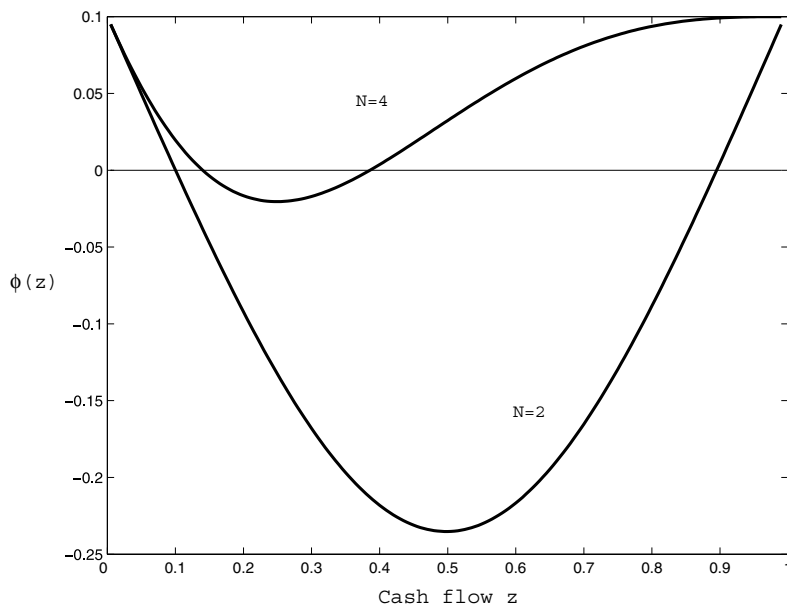


Figure 1. Net benefit function $\phi(z)$ of issuing component security $1_{Z \geq z}$. This figure illustrates the net benefit function $\phi(z)$ when Z is uniform on the unit interval, and signals can be high or low with probability $P(H | z) = z$. The number of units sold is $K = 1$ and the productivity of capital is $r = 0.1$. The optimal security design should have $w'(z) = 1$ if $\phi(z) \geq 0$ and $w'(z) = 0$ otherwise.

with strike price 0.9. For low z , there is so little underpricing in the low state that it does not matter that a lot of capital is raised. For high z , so little capital is raised in the low state that it does not matter that the percentage of underpricing is high. The total expected underpricing is highest for intermediate component securities, which are therefore retained by the seller.

Figure 1 also shows $\phi(z)$ for four investors, illustrating the effect of increasing the degree of competition. There is no underpricing as long as the second-highest bidder gets a high signal, which is increasingly likely as N goes up. Therefore, as N increases, expected underpricing goes down for any component of security, so less is retained.

Underpricing happens only in the case in which the winner gets a high signal and all other bidders get low signals. Note that as N goes up, the cash flow is likely to be very low when only one bidder has a high signal. Therefore, the relative capital-raising benefit of information sensitivity goes up since information-sensitive securities such as call options are expected to raise almost none of their capital in this state and almost all of their capital in the state in which underpricing is zero. This explains why the call option component in Figure 1 becomes relatively more important as the degree of competition rises.

B.2. Underpricing Advantage of Information Sensitivity: General Properties

I now show that the characteristics of the optimal security design in the above example hold more generally. First, it is always optimal to include some debt component when r is strictly positive, since debt can be made virtually risk-free if the face value is low enough. Second, if the signal distribution is such that good states have zero underpricing, and if the fraction of winning investors $\frac{K}{N}$ is sufficiently small, then an option component should always be included. Third, as N grows, the option component becomes a more important part of the security design.

PROPOSITION 2: *Suppose the information structure is inherently discrete with a finite set of distinct signals, and r is independent of Z . Then the following is true:*

- (i) *The optimal ex ante security design always contains a senior debt component: There is a $D(r, N) > 0$ such that for all $r > 0$ and number of bidders N , $w'(z) = 1$ for $z \leq D(r, N)$.*
- (ii) *For a sufficiently large number of bidders N , the optimal security design also contains a call option: There is an $S(r, N) < \bar{z}$ and M such that for all $r > 0$ and $N \geq M$, $w'(z) = 1$ if $z \geq S(r, N)$ or $z \leq D(r, N)$, and $w'(z) = 0$ if $S(r, N) > z > D(r, N)$. Also, for each $r > 0$ and each $z > 0$, there is an M such that $S(r, N) \leq z$ for all $N \geq M$.*
- (iii) *For a sufficiently large number of bidders N , the call option component becomes relatively more important as the number of bidders grows: For each z , there is a sequence $r(N) > 0$ and an M such that for all $N > M$, we have $S(r(N), N) = z$ and $D(r(N), N) < z$, where $D(r(N), N) \rightarrow 0$ as $N \rightarrow \infty$. Also, $r(N) \rightarrow 0$ as $N \rightarrow \infty$.*

Although the assumptions underlying this result do not appear to be very strong, it is worth noting a few caveats. First, for r held constant, the underpricing for *all* component securities goes to zero as N goes to infinity so that selling the entire asset becomes optimal for large N . However, the underpricing for high cash flows goes to zero at a faster rate, which leads to the result in Proposition 2 that the option component will dominate as N grows and r is decreased at the appropriate rate.

Second, Proposition 2 employs the fact that there is little “informational distance” between the price-setting investor and winning investors when the price-setting investor gets a sufficiently high signal. However, there are standard distributions where this is not the case. For example, suppose cash flow Z is normally distributed and signals are distributed as

$$X = Z + \varepsilon,$$

where $\varepsilon \sim N(0, \sigma^2)$ and ε is independent of Z . Then, it turns out that the underpricing is not necessarily decreasing in the pivotal signal, and thus information-sensitive securities have no capital-raising advantage. Without this advantage, debt is optimal:

PROPOSITION 3: Suppose r is constant, Z is normally distributed, and $X = Z + \varepsilon$ with ε normally distributed and independent of Z . Then debt is optimal for all N .

B.3. Productivity Advantage of Information Sensitivity

I now study the case in which the investment opportunity is correlated with the asset being sold, so that $r(Z)$ increases. Now there is an added advantage of raising more capital in high states, since the productivity of capital is higher in those states. This advantage, in contrast to the cost advantage discussed above, does not depend on the characteristics of the signal distribution. When $r(Z)$ is increasing, it is easy to see that raising more capital in high states is beneficial if the cost of underpricing is ignored. Of course, if the cost is taken into account, debt may still be optimal. However, as the degree of competition increases, underpricing becomes less and less important for any signal distribution, and hence the productivity advantage of information sensitivity will dominate.

These results are stated formally in the following proposition:

PROPOSITION 4: If $r(Z)$ increases, then the (gross) benefit $\int E(r(Z)|Y_{K+1} = y)q(z|y)dy$ of selling component security $1_{Z \geq z}$ increases in z . Also, there is an M such that for $N > M$, the net benefit $\phi(z)$ increases in z , and if $E(r) < 0$ (non-trivial transaction cost) the optimal ex ante security design is a call option with strike price $S > 0$.

Note that as N becomes large, underpricing goes to zero and hence selling the full asset is optimal if $r(Z)$ is everywhere strictly positive. The last part of the proposition therefore deals with the case in which $r(Z)$ can be negative. For example, suppose that for each dollar raised in the auction, there is a brokerage fee c . Suppose further that the productivity of capital is proportional to Z so that $r(Z) = bZ - c$. If $E(r) < 0$, it is not worth issuing a risk-free security, since the expected productivity of capital will not make up for the brokerage fee. However, an information-sensitive security avoids the fee when productivity of capital is low, and so can be worth issuing.

B.4. Discussion

The Degree of Competition: In Propositions 2 and 4, I let the degree of competition in the auction grow by increasing the number of investors while holding the number of units sold constant. This means that the fraction of winning investors goes to zero. Another natural experiment is to let supply and demand grow together, while keeping the fraction of winning investors constant. It is easy to show that as the auction becomes large in this way, the underpricing goes to zero (see, for example, Axelson (2003)). Proposition 4 therefore goes through unchanged, since it only relies on the fact that the cost of information sensitivity disappears as the market becomes competitive. However, Proposition 2 hinges crucially on the fact that all winning bidders tend to bunch together at

the top of the signal distribution as N grows. The fraction of winning investors must therefore be sufficiently small for the result to go through.¹³

Note that the fraction of winning investors is treated as exogenous. In principle, it can be a choice variable of the firm. Axelson (2003) and Parlour and Rajan (2001) show that it is not always optimal to concentrate the allocation to the fewest possible winners in common-value auctions, even when bidders have a linear demand. However, it can be shown that when signals are distributed according to the assumption in Proposition 2 and the productivity of capital is constant, it is optimal to minimize the fraction of winning investors to take advantage of the low underpricing in the tail of the signal distribution. Therefore, if the seller is free to pick K optimally, the result in Proposition 2 goes through.

Finally, it would be natural to consider the case in which the number of investors does not grow, but the number of signals observed by each investor grows. This would be the case, for example, if groups of investors can collude and consolidate their bids as the market grows large, or if gathering information becomes cheaper and investors find it beneficial to become better informed. As the number of signals grows large, the underpricing goes to zero since investors become perfectly informed and information asymmetry goes to zero. Therefore, Proposition 4 will still go through for this case. However, Proposition 2 does not go through. There are two reasons for this. First, the fraction of winning investors does not become small as the number of signals increases. Second, the distribution of the aggregate signal of a bidder goes toward a normal distribution as the number of signals becomes large—hence, the cost advantage of information sensitivity disappears.¹⁴

The Monotonicity and Limited Liability Constraints: The monotonicity and limited liability constraints, which ensure that a security $w(z)$ starts at zero and has a derivative between zero and one, are important drivers of the shape of the optimal securities. The issuer will typically do better if these constraints can be relaxed, and sometimes we see examples of this in financial markets. For example, in asset-backed securities markets, issuers sometimes provide insurance against default of issued debt, which violates the limited liability assumptions. This makes securities less information-sensitive, reducing the informational rents of investors. Similarly, we sometimes see securities that violate monotonicity. A digital option, which has a discrete jump from paying off nothing to paying off a fixed amount if the cash flow of the underlying asset is above a certain cutoff, is one example. If a firm has outstanding equity, it can also issue a nonmonotonic claim by raising debt and use part of the proceeds

¹³ In particular, the fraction of winning investors must be smaller than the probability of getting the highest possible discrete signal. This will always be true if the number of winning investors K is held constant and the number of investors N goes to infinity.

¹⁴ Although the debt result for normally distributed signals in Proposition 3 may suggest that debt should become optimal, the proposition is not directly applicable to the case in which the number of signals goes up for two reasons. First, the distribution of Z is not necessarily normal. Second, when investors have multidimensional signals, the bidding equilibrium derived in Proposition 5 does not necessarily hold (see Section V for a further discussion of the multidimensional signal case).

for a share repurchase. The aggregate issued claim will then increase for low firm cash flows and decrease for high cash flows. In a previous version of the paper, I show that issuing such a claim can result in zero underpricing. The reason for this is that investors with low and high signals may have the same expected value for a nonmonotonic claim, so that their private information is no longer useful.¹⁵

III. Ex Post Security Design

The results above are derived under the assumption that the firm has to design the security without taking any investor information into account. However, there are many issuing strategies used by firms in financial markets in which the security design is contingent on investor information. Maybe the simplest example is an auction with a reserve price, where the security is not issued unless investor signals are high enough to make them bid above the reserve. Another example is when a firm issues several securities sequentially, and lets the design of later issues be contingent on the price received in earlier issues.

I now show that if the firm is allowed to make the security design contingent on investor information, which I refer to as ex post security design, the benefit of information-sensitive securities disappears and debt becomes optimal. By letting the face value of the debt increase with investor information, the firm can replicate the beneficial capital-raising characteristics of an information-sensitive security while still minimizing the amount of underpricing.

A. Sales Mechanism

I model ex post security design as a natural extension of the sealed bid, uniform price auction in which not only the price but also the security design is allowed to vary with the signal of the $K + 1$: st highest bidder. In particular, the sales mechanism works as follows:

- (i) The firm sets up a menu of securities $w(Z, y)$, where the issued security will be a function of the pivotal signal $Y_{K+1} = y$.
- (ii) In a book-building phase, the firm collects indications of interest in terms of reported signals $\{\hat{x}_1, \dots, \hat{x}_N\}$.
- (iii) The K investors with the highest indications of interest receive an equal allocation of security $w(Z, y)$, where y is the $K + 1$: st highest report, at the price $\pi(w, y)$ described in Lemma 1.

The following proposition shows that in equilibrium investors report their true signal, *regardless of what the menu of securities is*.

PROPOSITION 5: *Given any menu $w(Z, y)$ that satisfies monotonicity and limited liability with respect to Z , in equilibrium all investors report their signals truthfully.*

¹⁵ For a similar result in the signaling literature, see Brennan and Kraus (1987) and Constantinides and Grundy (1989).

The key to understanding this result is to realize that investors can only influence the price and security design with a deviation that changes the pivotal report given by the marginal loser. Suppose an investor wins an allocation in a truth-telling equilibrium. Now also suppose he could go back and lower his bid to try to lower the price of the security. To do this, he would have to bid lower than the previous marginal loser, and hence he would no longer get any allocation of the security. But since the pricing function ensures that winners earn positive rents in equilibrium regardless of the security, such a deviation is not profitable. Similarly, an investor who loses in a truth-telling equilibrium would have to overreport enough to win, which would increase the pivotal report. But given his own low signal, the pricing function then ensures that this deviation would lead to negative expected profits regardless of the security design.

B Security Design

The result in Proposition 5 shows that as long as the sealed bid, uniform pricing rule is used, the firm is free to pick whatever security design it wants in each state $Y_{K+1} = y$. Denoting by $w_1(z, y)$ the derivative of $w(z, y)$ with respect to z , the maximization problem becomes

$$\int_0^{\bar{z}} \max_{w_1(z, y) \in [0, 1]} [E(r | Y_{K+1} = y) \pi(1_{Z \geq z}, y) - (E(1_{Z \geq z} | Y_{K+1} = y) - \pi(1_{Z \geq z}, y)) w_1(z, y)] dz = \int_0^{\bar{z}} \pi(1_{Z \geq z}, y) \max_{w_1(z, y) \in [0, 1]} (\phi(z | y) w_1(z, y)) dz.$$

The optimal security design now only depends on the sign of the conditional net benefit $\phi(z | y)$. Since $\phi(z | y)$ decreases in z , debt is optimal.

PROPOSITION 6: *The optimal ex post security design when $Y_{K+1} = y$ is debt, with face value $D(y)$ determined as follows:*

- If $\phi(0 | y) < 0$, $D(y) = 0$ (nothing is sold).
- If $\phi(\bar{z} | y) \geq 0$, $D(y) = \bar{z}$ (the whole asset is sold).
- Otherwise, $D(y)$ is given by the unique solution to $\phi(D(y) | y) = 0$.

Thus, when ex post security design is used, a version of the folklore proposition of debt holds, just as in a model of firm private information. The intuition for the debt result is also similar in the two models: Underpricing increases in information sensitivity, and debt minimizes the information sensitivity. In fact, at the time of issuance, there is a sense in which the model is converted into a firm private information model as the firm learns the investor information in the first stage of the mechanism. However, the underpricing here is not due to a good firm being pooled with bad firms, but rather to the informational rent that the firm has to commit to investors in exchange for stating their signals truthfully in the first stage. The two models also lead to different implications

about the amount of debt that should be issued as a function of firm quality. DeMarzo (2005) shows that a privately informed firm of higher quality should retain more cash flow (issue less debt) to signal its quality, while in my model a firm of better quality will often end up issuing more debt, either because the underpricing is lower (see the example below), or because the productivity of capital is higher.

The book-building mechanism above may look esoteric, but it encompasses variants of issuance strategies commonly observed in financial markets. The following proposition gives two examples of issuance strategies, using reserve prices and sequential issuance, which can replicate the outcome of the mechanism.

PROPOSITION 7: *Suppose the optimal debt schedule $D(y)$ in Proposition 6 is increasing in y . Then, the outcome of the mechanism can be implemented by the following two strategies:*

(i) *Auction of tranches with reserve prices:*

- *The firm issues debt tranches of decreasing seniority, which are sold simultaneously in sealed bid, uniform price, K -unit auctions. Each auction has a reserve price such that the tranche is only issued if the $K + 1$: st highest bidder meets the reserve.*

(ii) *Sequential issuance:*

- *First, the firm runs a sealed bid, uniform price, K -unit auction of senior debt with face value $D_S \equiv \min_{y: D(y) > 0} D(y)$.*
- *The firm infers $Y_{K+1} = y$ from the resulting price π in the first auction, assuming that investors bid according to Lemma 1. The firm then sells junior debt with face value $D_J(y) \equiv D(y) - D_S$ in a second sealed bid, uniform price, K -unit auction.*

It is important to note that I force the strategies above to be within the class of sealed bid, K -unit, uniform price auctions, even though this may not be optimal for the seller. For example, in the dynamic implementation strategy, I do not allow a winner in the first auction to be excluded in the second, nor do I allow for any release of information other than the pivotal signal between the two auctions. I discuss the extent to which the security design results may generalize to other choices of sales mechanisms in Section IV.

B.1. Example Continued

Going back to the example developed in Section IIB.1, I now show how to derive the optimal ex post security design. It is optimal to sell the whole asset when both investors have high signals ($Y_2 = \text{High}$). When at least one investor has a low signal ($Y_2 = \text{Low}$), the optimal face value of debt is given by

$$D_S \equiv \frac{\frac{2}{3}r_L}{1 + \frac{2}{3}r_L},$$

since this sets $\phi(D_S | \text{Low}) = 0$. The optimal debt level is increasing in the productivity of capital r . Also, note that the debt level is lower when the pivotal signal is low, reflecting the higher underpricing.

The optimal schedule can be implemented with the strategies described in Proposition 7. With the sequential issuance strategy, the first security to issue is senior debt with face value D_S . This is sold in a standard uniform price auction. After deducing the pivotal signal y from this auction, the firm issues nothing more if $Y_2 = \text{Low}$, and junior debt with face value

$$D_J = 1 - \frac{\frac{2}{3}r}{1 + \frac{2}{3}r}$$

if $Y_2 = \text{High}$.

Alternatively, the firm could sell the securities in two simultaneous auctions. In one auction, the senior debt component is sold. In the second auction, the junior debt component is sold with a reserve price of $E(\min[D_J, \max(Z - D_S, 0)] | Y_1 = Y_2 = \text{High})$.

IV. Discussion: Other Sales Mechanisms

For simplicity, the security design results above are derived taking as given the sealed bid uniform price auction format. This is not necessarily an innocuous restriction. I now discuss how the security design results may be affected when other sales mechanisms are used.

The key ex ante security design results in Propositions 2 and 4 about the benefit of information sensitivity rely on two conditions: Information-sensitive securities raise relatively more capital in “good” states, and the net benefit of capital raising may be higher in good states. Both these conditions are satisfied for other standard auction formats such as the sealed bid discriminatory price auction and the open ascending price auction.

The open ascending price auction is of particular interest as it generates the highest expected revenues of the three standard auction formats (while the discriminatory price auction generates the lowest expected revenues). In an ascending price auction, the price is raised gradually until all but K bidders remain. As bidders drop out, their signals can be inferred from the price they drop out at. The equilibrium price in an ascending price auction is exactly the same as the price in a sealed bid uniform price auction, except that it depends on the signals of all losing bidders:

$$\pi(w, y_{K+1}, \dots, y_N) = E(w | Y_K = Y_{K+1} = y_{K+1}, \dots, Y_N = y_N).$$

The nature of underpricing is also the same—the price underestimates the signal of the marginal winner. One can show that the ex ante security design results in Propositions 2 and 4 and the ex post security design result in Proposition 6 can be extended to this case.

It is beyond the scope of the paper to answer the question of what the *optimal* sales mechanism is. However, for the specific distributional assumptions made in the examples above, it can be shown that a version of the ascending price auction that gives the *entire* allocation to the highest bidder ($K = 1$) is optimal among mechanisms that satisfy certain robustness conditions, such as an ex post individual rationality condition that allows investors to refuse an allocation after the price is observed. For this special case, the security design results in Propositions 2, 4, and 6 go through.¹⁶ The intuition for why the ascending price auction generates higher revenues and why the entire allocation should be given to the highest bidder is related to the well-known *linkage principle* of Milgrom and Weber (1982), which states that the more the information is used for setting the price (and in this paper, designing the security), the higher the revenues will be. The ascending price auction links the price to the information of all losing bidders; allocating everything to the highest bidder means that all but one investor signal is utilized. The ascending price auction can be viewed as a dynamic version of the sealed bid auction, and is in fact the optimal dynamic mechanism.

Although the security design results appear to hold for the standard, robust sales mechanisms discussed above, they can change dramatically if more general mechanisms are allowed. For a common value setting such as the one in this paper, Crémer and McLean (1985) show that a seller can design an interim incentive-compatible mechanism where all surplus is extracted from investors *independent of the security design used*. Thus, the security design becomes irrelevant. Although this is a disturbing result, the mechanisms required involve unattractive features that are not commonly observed in financial markets, such as large payments from losing investors.

V. Extensions

A. The Multiple Asset Case

The previous analysis assumes that bidders only have private information about one asset or one factor affecting the cash flow distribution. I now turn to the case in which the seller is endowed with multiple assets. The security design here involves two steps. First, the seller partitions the assets into pools. Then, he issues a security for each pool backed by the cash flow of the pool.¹⁷ In particular, suppose the seller has I assets to sell, and that the cash flow of asset i is given by

¹⁶ Proof available upon request.

¹⁷ Hence, I assume the pooling decision is done ex ante so that it does not depend on bidder information. An interesting extension would be to analyze ex post design at the pool level as well.

$$Z_i = \Phi_0 + \Phi_i, \quad i = 1, \dots, I$$

where Φ_0 is a systematic factor common to all assets and Φ_i is an unsystematic factor. Further, to make the analysis tractable, consider the following distributional assumptions:

ASSUMPTION A1:

- Factors Φ_i are independently identically distributed random variables with distribution function $G(\phi)$.
- Investors receive conditionally independent signals $X_{ni} \in \{0, 1\}$ about each factor, where $P(X_{ni} = 1 | \Phi_i = \phi)$ is increasing in ϕ .

These assumptions make it possible to collapse the multidimensional signal investors receive about idiosyncratic factors in a pool into a one-dimensional signal (the sum of the signals about each factor).¹⁸

The following proposition shows that debt backed by a pool of assets tends to be the optimal design when the number of assets is large enough.

PROPOSITION 8: (*Optimal pooling when the number of assets is large*):

- (i) *If there is no systematic factor, there is a J such that for all number of assets $I > J$, the optimal ex post and ex ante security design is debt backed by the whole pool of assets.*
- (ii) *If there is a systematic factor, as the number of assets $I \rightarrow \infty$, pooling is strictly better than individual sales, and the unique optimal pool-backed security design is given by a senior debt component $\min(Z, D)$ plus a component $w(\max(Z - D, 0))$, where w is the optimal security design backed by only the systematic factor Φ_0 .*

The effect of pooling is similar to the effect of issuing debt: It makes the security less sensitive to information, since high and low signals tend to cancel out in a large pool. As is shown in Axelson (2003), pooling without any retention of cash flows is always beneficial if the number of assets is large enough, even under much more general distributional assumptions than I have made here.

Case 1 of the proposition shows that the seller can do even better by retaining the riskiest cash flows in the pool. The reason is simple: Component securities $1_{Z \geq z}$ with a cutoff value below the mean become less risky the larger the number of assets in the pool. Hence, these securities feature little underpricing and should be sold. This is not true for component securities that pay off only if the average pool cash flow is above the mean. In this case component securities become very risky, are underpriced, and should be retained. Note that debt becomes optimal even when a more information-sensitive security is the optimal security backing a single asset.

When there is systematic risk, pooling is still optimal because it lowers the underpricing created by unsystematic factors. However, the optimal security

¹⁸ In the general multidimensional case it is not possible to find a sufficient statistic that also satisfies MLRP, which is the condition used to ensure the monotone equilibrium in Lemma 1.

design does not necessarily consist of debt exclusively. A senior debt component will always be issued, since the cash flow from the average unsystematic factors becomes virtually risk-free and should be sold. However, the systematic factor is not diversified in the pool. The optimal security design should therefore also consist of a component that looks like the optimal security issued *if the systematic factor were the only asset*.¹⁹

Proposition 8 focuses on the case in which the number of assets is large, especially relative to the number of bidders. It can be shown that for some signal distributions, such as the normal distribution, pooling is beneficial for as little as two assets, and for any degree of competition. However, this is not true for the discrete signal distribution I assume here. The following proposition shows that when the signal distribution is discrete and the degree of competition is sufficiently high, information-sensitive securities backed by individual assets is the optimal security design.

PROPOSITION 9: *Suppose there is no systematic factor. Then there exists an M such that for all number of investors $N > M$, individual sales are optimal and the optimal ex ante security design is to issue information-sensitive securities as given in Proposition 2.*

When the information structure is discrete, there is no underpricing when the pivotal bidder receives the highest possible signal about a pool, since winning investors cannot have better information. The fewer assets there are in the pool, the more likely this scenario is since there are fewer assets to receive a high signal about. This is what drives the result that separate sales can be preferable to pooling. For this effect to be important, the fraction of winning investors has to be sufficiently small, since the pivotal bidder is then more likely to come from the top of the distribution. The more competitive the setting, the more the capital raising becomes tilted toward this low underpricing state, making the effect even stronger. For sufficiently high N , separate sales are always optimal. From Proposition 2, we also know that information-sensitive securities are optimal under the same circumstances, which explains the last part of Proposition 9.²⁰

There are two noteworthy features of the results in this section. First, the pooling decision and the security design decision are linked. When pooling is optimal, debt also tends to be optimal, and when separate sales are optimal, more information-sensitive securities tend to be optimal. Second, the choice of security design depends on the number of assets relative to the degree of

¹⁹ If there are more than one systematic factors affecting cash flows, it may no longer be optimal to put all assets in the same pool. It is typically optimal to pool assets that are affected by the same systematic factors to get rid of informational rents created by unsystematic information. However, if different asset classes are subject to different systematic factors, whether one big pool should be created or whether there should be one pool for each class of assets will depend on the specifics of the signal distribution.

²⁰ Proposition 9 assumes there is no systematic factor. This is because there is typically no one-dimensional sufficient statistic that can act as an aggregate signal for a pool with a systematic factor, which makes it technically difficult to characterize the equilibrium.

competition. If the number of assets is large relative to the degree of competition, pooling with debt is optimal. If the opposite holds, separate sales with information-sensitive securities can be optimal.

B. Project-Backed Securities

For analytical and expositional simplicity, thus far I have assumed that the asset Z being sold is preexisting and that its cash flow is not affected by the investment of the firm. This rules out the important case in corporate finance where investment capital is raised by issuing a security backed by cash flows from the investment itself. I now show that the flavor of most results in the paper go through for this case as well.

Suppose that investors now receive signals about a factor Φ , where the signals satisfy MLRP with respect to the factor, and that Z denotes the cash flow from a new investment opportunity. Specifically, if the firm invests I in the project, it pays off $Q(I, \Phi)$ in period 1 with Q strictly increasing in both elements. Also,

$$\begin{aligned}\frac{\partial^2 Q(I, \Phi)}{\partial I^2} &< 0 \\ \frac{\partial^2 Q(I, \Phi)}{\partial I \partial \Phi} &\geq 0.\end{aligned}$$

The first assumption requires that the investment observes decreasing returns to scale, as opposed to the previous analysis. This is necessary to ensure that the maximization problem is well specified. The second assumption says that the marginal returns to investment is nondecreasing in the given state.

Also, I assume that there is a (possibly very small) proportional capital raising cost c for each dollar raised. Furthermore, if the firm raises π dollars and invests less than the full amount in the project, the remaining $\pi(1 - c) - I$ is put in a risk-free asset. This gives the cash flow backing securities as

$$Z = Q(I, \Phi) + \pi(1 - c) - I.$$

As investors bid, they must now take into account the fact that the amount raised and the investment policy of the firm will affect the value of their securities. I assume that the firm can commit ex ante to an investment policy as a function of the price-setting signal Y_{K+1} .²¹

It is easy to show that the equilibrium pricing function from Lemma 1 will still hold. The maximization problem of the firm can then be expressed as

²¹ Alternatively, one can introduce an ex post moral hazard problem where the firm maximizes its own stake after capital has been raised. This does not change the solution to the ex post security design problem. In the ex ante security design problem, it will push the security design in the direction of straight equity since straight equity aligns the ex post incentives of the firm and investors. Indeed, one can show that if $c = 0$, straight equity becomes the optimal ex ante security design as the number of investors becomes large.

$$\max_{I(y), w(Z, y)} E(Z - w(Z, y))$$

such that

$$\begin{aligned}\pi(w(Z, y), y) &= E(w(Z, y) | Y_K = Y_{K+1} = y) \\ I(y) &\leq \pi(w(Z, y), y)(1 - c).\end{aligned}$$

The following proposition shows that debt is still optimal in the ex post security design problem.

PROPOSITION 10: *The optimal ex post security design is to issue debt with face value $D(y)$. Everything is invested in the project and nothing in the risk-free asset, and the investment level $I(y)$ is below the first-best level.*

That the investment level is below the first-best is intuitive, since the firm suffers both underpricing and a transaction cost c when raising capital. A positive transaction cost also ensures that the firm is better off not raising any extra funds on top of what is needed for investment. If the transaction cost were zero, the firm would be indifferent between raising extra funds or not.

In the ex ante security design problem, information-sensitive securities can be optimal since they tend to raise more capital when the investment opportunity is more valuable. I show this by way of an example.

Suppose the investment opportunity is given by the decreasing returns-to-scale technology

$$Q(I, \Phi) = \Phi I^\alpha$$

for $\alpha < 1$. Then, if the firm can raise capital at a transaction cost c per dollar but with no underpricing, the first-best amount raised given state $Y_{K+1} = y$ is calculated as

$$\begin{aligned}\pi^*(y) &= \frac{(\alpha E(\Phi | Y_{K+1} = y))^{\frac{1}{1-\alpha}}}{1 - c} \\ &= \frac{\alpha}{1 - c} E(Z | Y_{K+1} = y),\end{aligned}$$

where the expected cash flow $E(Z | Y_{K+1} = y)$ is calculated by setting $Z = Q(\pi^*(y), \Phi)$. Now, suppose the firm issues straight equity $w(Z) = \frac{\alpha}{1-c} Z$ in an auction. Given $Y_{K+1} = y$, the equity will sell at price

$$\pi(y) = E\left(\frac{\alpha}{1-c} Z | Y_K = Y_{K+1} = y\right).$$

As the number of bidders goes to infinity, underpricing goes to zero and $\pi(y)$ goes to $\pi^*(y)$. Thus, straight equity achieves the first-best in the limit as $N \rightarrow \infty$. This is not true for other securities. They either raise too much capital in some contingencies, leading to unnecessary transaction costs, or too little, leading to too low an investment level. Thus, equity is the unique optimal security as $N \rightarrow \infty$. An empirical implication of this is that a firm that finances growth

opportunities with high uncertainty should issue equity instead of debt, which is consistent with the behavior of young growth firms.

With lower levels of competition, underpricing would play a role in determining the optimal security design, and so the characteristics of the signal distribution would become important.

The optimality of information sensitivity in this example relies on the characteristics of the production function. In particular, when investors have good information, proportionately more should be invested, and straight equity has exactly that characteristic. One can show that if the investment amount necessary to undertake the project is fixed, there is no need to introduce information sensitivity, and debt is optimal.

VI. Conclusion

In this paper I develop a theory of security design in which investors have private information about the prospects of a firm. In raising capital for investment, the firm chooses the form of the securities to minimize the informational rents of investors while ensuring that the right amount of capital is raised. If the firm has to set the security design in advance, information-sensitive securities such as equity can become optimal. This is especially true as the degree of competition increases. The benefit of information-sensitive securities is that they often raise relatively more capital in states in which capital is more valuable, and in states in which underpricing is smaller. The cost of an information-sensitive security is that the underpricing per dollar raised in any state is higher than that for debt.

If the firm is able to make the face value sufficiently contingent on investors' indication of interest, for example, by issuing several securities with different reserve prices or sequentially issuing securities, I show that debt is optimal.

In the multiple asset case, I show that debt backed by pooling is often an optimal security design when the number of assets is large relative to the number of bidders, while equity backed by individual assets can become optimal as the degree of competition increases.

It is worth pointing out a few limitations of the analysis. Just as the signaling literature completely ignores private information on the side of investors, this paper completely ignores private information held by the firm. The truth probably lies somewhere in between, and it would be a natural next step to analyze the double-sided private information problem. Unfortunately, treating both signaling and screening in the same model is challenging, and it is far from obvious how the analysis in this paper can be extended to that case.

Another limitation of the analysis is that I have assumed that the firm has the power to design securities, not the investors. DeMarzo, Kremer, and Skrzypacz (2005) show that the security design flips from debt to a call option when investors suggest securities and the firm cannot commit not to accept the best offer ex post. Interestingly, in a recent paper, Inderst and Mueller (2006) show that when the informed investor has monopoly power and designs the securities,

the opposite holds and debt is optimal if the investor must leave the firm with some fixed ex ante reservation utility. It would be useful to have a more general theory that delineates exactly how the bargaining power of the firm and investors affects the security design.

These limitations notwithstanding, the results of the model are broadly consistent with observed patterns of security design. For example, the prediction that a large asset base should be sold through debt issues backed by a pool is consistent with securitization patterns in the asset-backed securities markets, while equity issues are more prevalent when the asset base is more focused (as is the case for individual firms).

The theory developed here may also be applied to analyze a variety of other phenomena. For example, the results have implications for the life-cycle patterns in firm financing. A feature of the theory is that it can provide an explanation for both debt and equity within the same framework, and it provides conditions under which we would expect to observe one rather than the other. Over the firm's life cycle we might expect the degree of competition among investors and the availability of information about the company to be smaller at early stages, leading to a preference for debt securities. As the company grows older, information is more widely accessible, and more investors have their eyes on the company. At this stage it is possible for the company to issue equity through an IPO.

Similarly, it is possible to explain why different financial systems seem to favor different types of securities. In a bank-oriented system with little capital market competition for the financing of companies, debt should be more prevalent. In a stock market-based system, information acquisition is cheaper and the pool of investors larger, which makes it easier to issue equity. Improved information technology and increased investor participation in stock markets would also lead to an increased propensity of firms to issue equity at an earlier stage of their life cycle. The increased fraction of young firms in stock markets, both in the United States and in Europe, may be a reflection of this development, as is the decreased fraction of large diversified conglomerates.

Finally, the theory suggests a role for financial intermediation. As Proposition 8 shows, having access to a large number of assets can reduce the underpricing per asset by allowing the seller to construct pooled securities. If a seller has access to only a single asset, there is room for a financial intermediary to step in and synthesize pooling by bringing together assets of many sellers and financing them through a pooled issue.²² When information acquisition costs are high, we would expect such intermediation to arise endogenously. Similarly, the theory would predict that as information acquisition costs in the economy decline, we would expect to see *disintermediation* as single asset sales become

²² If it is infeasible to find a sufficiently large number of assets at any single point in time for this type of pooling to be beneficial, it may be possible for an intermediary to engage in intertemporal pooling. For example, an underwriter who is involved in a large number of IPOs over time may be able to synthesize pooling by giving preferential treatment to investors who take allocations in each IPO, making it harder for investors to "cherry pick" (see Benveniste and Spindt (1989) for a theoretical IPO model with this flavor and Cornelli and Goldreich (2001) for empirical evidence).

more beneficial. A more detailed investigation of these issues is left for future research.

Appendix: Proofs

Proof of Lemma 1: See Pesendorfer and Swinkels (1997) for a full proof. A sketch of the proof is as follows. The postulated equilibrium bid for an investor with signal x is

$$b(x) = E(w \mid Y_K = Y_{K+1} = x),$$

which is increasing in x from the MLRP assumption. I show that there is no incentive for a bidder to deviate from this strategy even after learning the realization of the price-setting signal $Y_{K+1} = y_{K+1}$, so it must be an optimal strategy ex ante. The expected payoff for an investor with $x > y_{K+1}$ under the equilibrium strategy is

$$E(w \mid X = x > Y_{K+1} = y_{K+1}) - E(w \mid Y_K = Y_{K+1} = y_{K+1}).$$

Note that this is positive and is independent of the investor's bid as long as he wins. Thus, the investor is happy to follow the prescribed equilibrium when he is among the winning bidders. Now suppose he is among losing bidders if he bids according to the equilibrium, so that his payoff is zero. If he deviates by increasing his bid so that he wins, the price will be set by the bidder with the K^{th} highest signal instead. Suppose $Y_K = y_K \geq y_{K+1}$. The expected payoff of the investor is

$$E(w \mid Y_K = y_K \geq Y_{K+1} = y > X = x) - E(w \mid Y_K = Y_{K+1} = y_K).$$

This is nonpositive. Therefore, there is no incentive to deviate. Q.E.D.

Proof of Lemma 2: Monotonicity together with the assumption of a continuous distribution for Z implies that $w(z)$ is continuous, since any jump down (up) would violate monotonicity of $w(z)(z - w(z))$. Thus, $w'(z)$ exists. There may be kinks in $w(z)$, in which case $w'(z)$ is taken to be the left derivative: $w'(z) = \lim_{z' \uparrow z} \frac{w(z) - w(z')}{z - z'}$. Given the existence of $w'(z)$, monotonicity is then equivalent to requiring $w'(z) \in [0, 1]$. Limited liability implies $w(0) = 0$. Given the bound on $w'(z)$, limited liability is then guaranteed for all z . Q.E.D.

Proof of Lemma 3: First, I show that $\phi(z \mid y)$ depends on z only through the hazard rate $\frac{f(y \mid z)}{1 - F(y \mid z)}$. I then show that the hazard rate decreases in z :

$$\begin{aligned} \phi(z \mid y) &= E(r \mid Y_{K+1} = y) - \frac{E(1_{Z \geq z} \mid Y_{K+1} = y) - \pi(1_{Z \geq z}, y)}{\pi(1_{Z \geq z}, y)} \\ &= E(r \mid Y_{K+1} = y) - \frac{P(Z \geq z \mid Y_{K+1} = y) - P(Z \geq z \mid Y_{K+1} = Y_K = y)}{P(Z \geq z \mid Y_{K+1} = Y_K = y)}. \end{aligned}$$

We have that

$$\frac{P(Z \geq z | Y_{K+1} = y)}{P(Z \geq z | Y_{K+1} = Y_K = y)} = \frac{\frac{\int_z^{\bar{z}} f(Y_{K+1} = y | v)g(v) dv}{f(Y_{K+1} = y)}}{\frac{\int_z^{\bar{z}} f(Y_{K+1} = Y_K = y | v)g(v) dv}{f(Y_{K+1} = Y_K = y)}},$$

so $\phi(z | y)$ depends on z through the term

$$-\frac{\int_z^{\bar{z}} f(Y_{K+1} = y | v)g(v) dv}{\int_z^{\bar{z}} f(Y_{K+1} = Y_K = y | v)g(v) dv}.$$

Note that $\phi(z | y)$ is continuous in z . The derivative of the expression above with respect to z is equal to

$$\frac{\int_z^{\bar{z}} (f(Y_{K+1} = y | z)f(Y_{K+1} = Y_K = y | v) - f(Y_{K+1} = Y_K = y | z)f(Y_{K+1} = y | v))g(v)g(z) dv}{\left(\int_z^{\bar{z}} f(Y_{K+1} = Y_K = y | v)g(v) dv\right)^2}.$$

Note that

$$f(Y_K = Y_{K+1} = y | v) = \frac{1}{2} \frac{N!}{K-1!(N-K-1)!} f(y | v)^2 (1 - F(y | v))^{K-1} F(y | v)^{N-K-1}$$

$$f(Y_{K+1} = y | v) = \frac{N!}{K!(N-K-1)!} f(y | v) (1 - F(y | v))^K F(y | v)^{N-K-1}.$$

We therefore have

$$f(Y_K = Y_{K+1} = y | v) = C * \frac{f(y | v)}{1 - F(y | v)} f(Y_{K+1} = y | v)$$

for some positive constant C that does not depend on v . Hence,

$$\begin{aligned} & \int_z^{\bar{z}} (f(Y_{K+1} = y | z)f(Y_{K+1} = Y_K = y | v) \\ & \quad - f(Y_{K+1} = Y_K = y | z)f(Y_{K+1} = y | v))g(v)g(z) dv \\ &= C * \int_z^{\bar{z}} \left(\frac{f(y | v)}{1 - F(y | v)} - \frac{f(y | z)}{1 - F(y | z)} \right) f(Y_{K+1} = y | v)f(Y_{K+1} = y | z)g(v)g(z) dv. \end{aligned} \tag{A1}$$

If I show that $\frac{f(y | z)}{1 - F(y | z)}$ is decreasing in z , it follows that $\frac{f(y | v)}{1 - F(y | v)} - \frac{f(y | z)}{1 - F(y | z)}$ is negative for $v > z$. The last line in expression (A1) is then nonpositive, which proves that $\phi(z | y)$ decreases in z . To see this, note that the inverse of $\frac{f(y | z)}{1 - F(y | z)}$ is given by

$$\frac{1 - F(y | z)}{f(y | z)} = \int_y^{\bar{x}} \frac{f(x | z)}{f(y | z)} dx,$$

which increases in z from MLRP. Therefore, $\frac{f(y | z)}{1 - F(y | z)}$ decreases in z , which proves the lemma. Also note the following properties:

COROLLARY 1:

- For all z , $\phi(z | y)$ strictly decreases in z at some y since the MLRP inequality holds strictly at least for some x, x' such that $x > x'$.
- For a given y , $\phi(z | y)$ is constant in z for all z if and only if there exists no $y' > y$ such that the MLRP inequality holds strictly for some pair z, z' . Q.E.D.

Proof of Lemma 4: I show that the derivative of $\frac{q(z | y)}{q(z' | y)}$ with respect to y is positive when $z > z'$:

$$\begin{aligned} \frac{\partial}{\partial y} \frac{q(z | y)}{q(z' | y)} &= \frac{E(\pi(1_{Z \geq z'}, Y_{K+1}))}{E(\pi(1_{Z \geq z}, Y_{K+1}))} \frac{\partial}{\partial y} \left(\frac{\pi(1_{Z \geq z}, y) g(z | Y_{K+1} = Y_K = y)}{\pi(1_{Z \geq z'}, y) g(z' | Y_{K+1} = Y_K = y)} \right) \\ &= \frac{\frac{E(\pi(1_{Z \geq z'}, Y_{K+1}))}{E(\pi(1_{Z \geq z}, Y_{K+1}))}}{\left(\frac{\pi(1_{Z \geq z'}, y)}{g(z' | Y_{K+1} = Y_K = y)} \right)^2} \\ &\quad * \left(\frac{\pi(1_{Z \geq z}, y)}{\frac{\partial}{\partial y} \frac{g(z | Y_{K+1} = Y_K = y)}{g(z | Y_{K+1} = Y_K = y)}} \frac{\pi(1_{Z \geq z'}, y)}{g(z' | Y_{K+1} = Y_K = y)} \right. \\ &\quad \left. - \frac{\pi(1_{Z \geq z'}, y)}{\frac{\partial}{\partial y} \frac{g(z | Y_{K+1} = Y_K = y)}{g(z | Y_{K+1} = Y_K = y)}} \frac{\pi(1_{Z \geq z}, y)}{g(z | Y_{K+1} = Y_K = y)} \right), \end{aligned}$$

which is proportional to

$$\begin{aligned} &\frac{\pi(1_{Z \geq z}, y)}{\frac{\partial}{\partial y} \frac{g(z | Y_{K+1} = Y_K = y)}{g(z | Y_{K+1} = Y_K = y)}} \pi(1_{Z \geq z'}, y) - \frac{\pi(1_{Z \geq z'}, y)}{\frac{\partial}{\partial y} \frac{g(z' | Y_{K+1} = Y_K = y)}{g(z' | Y_{K+1} = Y_K = y)}} \pi(1_{Z \geq z}, y) \\ &= \pi(1_{Z \geq z'}, y) \int_z^{\bar{z}} \frac{\frac{\partial}{\partial y} \frac{g(v | Y_{K+1} = Y_K = y)}{g(z | Y_{K+1} = Y_K = y)}}{\frac{\partial}{\partial y} \frac{g(z | Y_{K+1} = Y_K = y)}{g(z | Y_{K+1} = Y_K = y)}} dv - \pi(1_{Z \geq z}, y) \int_{z'}^{\bar{z}} \frac{\frac{\partial}{\partial y} \frac{g(v | Y_{K+1} = Y_K = y)}{g(z' | Y_{K+1} = Y_K = y)}}{\frac{\partial}{\partial y} \frac{g(z' | Y_{K+1} = Y_K = y)}{g(z' | Y_{K+1} = Y_K = y)}} dv. \end{aligned}$$

For $z > z'$, we have $\pi(1_{Z \geq z'}, y) > \pi(1_{Z \geq z}, y)$. Also, from MLRP, $\frac{g(v | Y_{K+1} = Y_K = y)}{g(z | Y_{K+1} = Y_K = y)}$ increases in y for $v > z$ and decreases in y for $v < z$. Hence, the term above is positive. Q.E.D.

Proof of Proposition 2: The result follows immediately if I can show that the following claims are true:

1. ϕ is continuous with $\phi(0) = r$ and for any $z > 0$, $\phi(z) < r$.
2. For any $z > 0$, there is an $M(z)$ such that $\phi(v)$ strictly increases for all $v \geq z, N \geq M$.
3. $r - \phi(z) \rightarrow 0$ as $N \rightarrow \infty$.

Claim 1 shows that some debt with face value larger than zero is always optimal, since ϕ is strictly positive at zero and is continuous, so that there exists $D > 0$ such that $\phi(z) \geq 0$ for $z \leq D$. Claim 2 shows that if $r(N) = \phi(z)$ for some $z > 0$ and $N > M(z)$, then a call option component with strike price z is also optimal to include. Also, for any z' such that $0 < z' < z$, for $N > M(z')$ and $r(N) = \phi(z)$, all component securities $1_{Z \geq v}$ such that $z' \geq v > z$ should be retained. Hence, as $N \rightarrow \infty$, the debt component becomes negligible. Claim 3 shows that $r(N) \rightarrow 0$.

I write a discrete finite signal structure as a signal $A \in \{0, 1, \dots, H\}$ for some integer $H \geq 1$, with probability distribution $P(A = a | Z = z)$ for any integer $0 \leq a \leq H$. The continuous representation of the signal structure is a signal $X \in [0, H + 1)$ with density $f(x | z)$ such that $f(x | z) = P(A = a | Z = z)$ for all z if $x \in [a, a + 1)$. I call an interval $[a, a + 1)$ an *equivalence interval*, since any signal X within the interval has the same informational content about Z . I assume that MLRP holds strictly except between signals in an equivalence interval. We have that

$$\begin{aligned}\phi(z) &= \int \phi(z | y) q(z | y) dy \\ &= r - \int_0^{H+1} \left(\frac{E(1_{Z \geq z} | Y_{K+1} = y)}{\pi(1_{Z \geq z}, y)} - 1 \right) q(z | y) dy \\ &= r - \int_0^{H+1} \left(\frac{P(Z \geq z | Y_{K+1} = y)}{P(Z \geq z | Y_K = Y_{K+1} = y)} - 1 \right) q(z | y) dy.\end{aligned}$$

Proof of Claim 1: $\phi(0) = r$ follows from the expression above since

$$P(Z \geq 0 | Y_{K+1} = y) = P(Z \geq 0 | Y_K = Y_{K+1} = y) = 1.$$

Also, since MLRP holds strictly for signals in different equivalence intervals, Corollary 1 in the proof of Lemma 3 shows that $\phi(z | y)$ strictly decreases in z for all $y < H$. Hence, $\phi(z) < r$ for all $z > 0$. That $\phi(z)$ is continuous follows since both $\phi(z | y)$ and $q(z | y)$ are continuous in z .

Proof of Claims 2 and 3: First, note that for $y \in [H, H + 1)$ (the marginal loser has the highest possible discrete signal), we have that $\phi(z | y)$ is constant in z from Corollary 1 in the proof of Lemma 3. Hence, $\phi(z | y) = r$ for $y \geq H$ (there is no underpricing) and we get

$$r - \phi(z) = \int_{y=0}^H \left(\frac{P(Z \geq z | Y_{K+1} = y)}{P(Z \geq z | Y_K = Y_{K+1} = y)} - 1 \right) q(z | y) dy.$$

I first show that for all $z > 0$ and $y < H$, we have $\frac{P(Z \geq z | Y_{K+1}=y)}{P(Z \geq z | Y_K=Y_{K+1}=y)} - 1 \in [a_1, a_2]$ for constants $0 < a_1 \leq a_2 < \infty$ independent of the number of bidders. Hence, we have that $r - \phi(z)$ goes to zero with $\int_{y=0}^H q(z | y) dy$. I show this as follows:

$$\begin{aligned} \frac{P(Z \geq z | Y_{K+1} = y)}{P(Z \geq z | Y_K = Y_{K+1} = y)} &= \frac{\int_z^{\bar{z}} g(v | Y_{K+1} = y) dv}{\int_z^{\bar{z}} g(v | Y_K = Y_{K+1} = y) dv} \\ &= \left(\frac{\int_0^{\bar{z}} f(Y_K = Y_{K+1} = y | Z = v) g(v) dv}{\int_0^{\bar{z}} f(Y_{K+1} = y | Z = v) g(v) dv} \right) \\ &\quad \times \left(\frac{\int_z^{\bar{z}} f(Y_{K+1} = y | Z = v) g(v) dv}{\int_z^{\bar{z}} f(Y_K = Y_{K+1} = y | Z = v) g(v) dv} \right), \end{aligned}$$

where the second step follows from Bayes's Law. We have that

$$\begin{aligned} &\frac{\int_z^{\bar{z}} f(Y_{K+1} = y | Z = v) g(v) dv}{\int_z^{\bar{z}} f(Y_K = Y_{K+1} = y | Z = v) g(v) dv} \\ &= \frac{\frac{N!}{K!(N-K-1)!}}{\frac{1}{2} \frac{N!}{K-1!(N-K-1)!}} \frac{\int_z^{\bar{z}} f(y | v) (1 - F(y | v))^K F(y | v)^{N-K-1} g(v) dv}{\int_z^{\bar{z}} f(y | v)^2 (1 - F(y | v))^{K-1} F(y | v)^{N-K-1} g(v) dv}. \end{aligned}$$

Dividing the numerator and denominator by $F(y | z)^{N-K-1}$ and taking the limit as $N \rightarrow \infty$, we then have that

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{2}{K} \frac{\int_z^{\bar{z}} f(y | v) (1 - F(y | v))^K \left(\frac{F(y | v)}{F(y | z)} \right)^{N-K-1} g(v) dv}{\int_z^{\bar{z}} f(y | v)^2 (1 - F(y | v))^{K-1} \left(\frac{F(y | v)}{F(y | z)} \right)^{N-K-1} g(v) dv} \\ &= \frac{2}{K} \frac{1 - F(y | z)}{f(y | z)}. \end{aligned}$$

This follows since $F(y | z)$ strictly decreases in z from MLRP. Therefore, as $N \rightarrow \infty$,

$$\frac{P(Z \geq z | Y_{K+1} = y)}{P(Z \geq z | Y_K = Y_{K+1} = y)} - 1 \rightarrow \frac{1 - F(y | z)}{f(y | z)} \frac{f(y | 0)}{1 - F(y | 0)} - 1.$$

Since the inverse hazard rate $\frac{1-F(y|z)}{f(y|z)}$ strictly increases in z for $y < H$, this is bounded and strictly greater than zero for all $z > 0$. Hence, $r - \phi(z)$ goes to zero with $\int_{y=0}^H q(z|y) dy$.

Next, we have that

$$\begin{aligned} q(z|y) &= \frac{\pi(1_{Z \geq z}, y) f(Y_{K+1} = y)}{E(\pi(1_{Z \geq z}, Y_{K+1}))} \\ &= \frac{\int_z^{\bar{z}} f(Y_{K+1} = Y_K = y | Z = v) g(v) dv}{E(\pi(1_{Z \geq z}, Y_{K+1}))}. \end{aligned}$$

Note that the unconditional expected price in the numerator goes to the true unconditional value of the security (see, e.g., Pesendorfer and Swinkels (1997)), which is a positive constant. Hence, $\int_{y=0}^H q(z|y) dy$ approaches zero with

$$\begin{aligned} &\int_{y=0}^H \int_z^{\bar{z}} f(Y_{K+1} = Y_K = y | Z = v) g(v) dv dy \\ &= \frac{1}{2} \int_{y=0}^H \int_z^{\bar{z}} \frac{N!}{K-1!(N-K-1)!} \\ &\quad \times f(y|v)^2 (1-F(y|v))^{K-1} F(y|v)^{N-K-1} g(v) dv dy. \end{aligned}$$

$F(y|z)$ strictly decreases in z from MLRP. Dividing the above by $F(H|z + \varepsilon)^{N-K-1}$ for some $\varepsilon > 0$, the integral goes to infinity as N goes to infinity. Dividing by $F(H|z - \varepsilon)^{N-K-1}$, the integral goes to zero. Since ε is arbitrary, we must have that $r - \phi(z)$ goes to zero at a rate proportional to $F(H|z)^{N-K-1}$. (Note that I have not pinpointed the exact convergence rate; all I need is that $r - \phi(z)$ goes to zero faster for a higher z). Since $F(H|z)$ decreases in z , we have that $\frac{r - \phi(z)}{r - \phi(z')}$ goes to infinity with N for $z' > z > 0$. Hence, for any $z > 0$, $\phi(v)$ decreases for all $v > z$ as $N \rightarrow \infty$. This proves Claims 2 and 3, and hence the proposition. Q.E.D.

Proof of Proposition 3: We have that $r - \phi(z)$ is given by

$$\frac{1 - G(z)}{E(1 - G(z | Y_K = Y_{K+1} = y))} - 1.$$

I show that the derivative of this with respect to z is nonnegative, which proves the result. The derivative is given by

$$\frac{E(g(z | Y_K = Y_{K+1} = y))(1 - G(z)) - g(z)E(1 - G(z | Y_K = Y_{K+1} = y))}{[E(1 - G(z | Y_K = Y_{K+1} = y))]^2}.$$

This has the same sign as

$$\begin{aligned} &\int_z^\infty \int_{-\infty}^\infty (g(z | Y_K = Y_{K+1} = y) g(v) \\ &\quad - g(z) g(v | Y_K = Y_{K+1} = y)) f(Y_{K+1} = y) dy dv. \end{aligned} \quad (\text{A2})$$

Using

$$\begin{aligned} g(z | Y_K = Y_{K+1} = y)g(v) \\ = \frac{(1 - F(y | z))^{K-1} f(y | z)^2 F(y | z)^{N-K-1} g(v)g(z)}{\int_{-\infty}^{\infty} (1 - F(y | v))^{K-1} f(y | v)^2 F(y | v)^{N-K-1} g(v) dv}, \end{aligned}$$

expression (A2) is proportional to

$$\begin{aligned} \int_z^{\infty} \int_{-\infty}^{\infty} (1 - F(y | z))^{K-1} f(y | z)^2 F(y | z)^{N-K-1} a(y) dy g(v) dv \\ - \int_z^{\infty} \int_{-\infty}^{\infty} (1 - F(y | v))^{K-1} f(y | v)^2 F(y | v)^{N-K-1} a(y) dy g(v) dv, \quad (\text{A3}) \end{aligned}$$

where $a(y)$ is given as

$$a(y) \equiv \frac{\int_{-\infty}^{\infty} \frac{1 - F(y | v)}{f(y | v)} (1 - F(y | v))^{K-1} f(y | v)^2 F(y | v)^{N-K-1} g(v) dv}{\int_{-\infty}^{\infty} (1 - F(y | v))^{K-1} f(y | v)^2 F(y | v)^{N-K-1} g(v) dv}.$$

Suppose $a(y)$ is decreasing. Then expression (A3) is positive:

$$\begin{aligned} \int_z^{\infty} \int_{-\infty}^{\infty} (1 - F(y | z))^{K-1} f(y | z)^2 F(y | z)^{N-K-1} a(y) dy g(v) dv \\ - \int_z^{\infty} \int_{-\infty}^{\infty} (1 - F(y | v))^{K-1} f(y | v)^2 F(y | v)^{N-K-1} a(y) dy g(v) dv \\ = \int_z^{\infty} \int_{-\infty}^{\infty} (1 - F(y | z))^{K-1} f(y | z)^2 F(y | z)^{N-K-1} a(y) dy g(v) dv \\ - \int_z^{\infty} \int_{-\infty}^{\infty} (1 - F(y + z - v | z))^{K-1} f(y + z - v | v)^2 \\ \times F(y + z - v | v)^{N-K-1} a(y) dy g(v) dv \\ = \int_z^{\infty} \int_{-\infty}^{\infty} (1 - F(y | z))^{K-1} f(y | z)^2 F(y | z)^{N-K-1} a(y) dy g(v) dv \\ - \int_z^{\infty} \int_{-\infty}^{\infty} (1 - F(y | z))^{K-1} f(y | v)^2 F(y | v)^{N-K-1} a(y + v - z) dy g(v) dv \end{aligned}$$

$$\begin{aligned}
&= \int_z^\infty \int_{-\infty}^\infty (1 - F(y|z))^{K-1} f(y|z)^2 F(y|z)^{N-K-1} \\
&\quad \times (a(y) - a(y+v-z)) dy \frac{g(v)}{1 - G(z)} dv \\
&> 0.
\end{aligned}$$

I now verify that $a(y)$ is decreasing. Note that

$$\begin{aligned}
a(y') &= \frac{\int_{-\infty}^\infty \frac{1 - F(y'|v)}{f(y'|v)} (1 - F(y'|v))^{K-1} f(y'|v)^2 F(y'|v)^{N-K-1} g(v) dv}{\int_{-\infty}^\infty (1 - F(y'|v))^{K-1} f(y'|v)^2 F(y'|v)^{N-K-1} g(v) dv} \\
&= \frac{\int_{-\infty}^\infty \frac{1 - F(y|v)}{f(y|v)} (1 - F(y|v))^{K-1} f(y|v)^2 F(y|v)^{N-K-1} g(v + y' - y) dv}{\int_{-\infty}^\infty (1 - F(y|v))^{K-1} f(y|v)^2 F(y|v)^{N-K-1} g(v + y' - y) dv}.
\end{aligned}$$

Therefore, $a(y) - a(y')$ is given by

$$\begin{aligned}
&\frac{\int_{-\infty}^\infty \frac{1 - F(y|v)}{f(y|v)} (1 - F(y|v))^{K-1} f(y|v)^2 F(y|v)^{N-K-1} g(v) dv}{\int_{-\infty}^\infty (1 - F(y|v))^{K-1} f(y|v)^2 F(y|v)^{N-K-1} g(v) dv} \\
&- \frac{\int_{-\infty}^\infty \frac{1 - F(y|v)}{f(y|v)} (1 - F(y|v))^{K-1} f(y|v)^2 F(y|v)^{N-K-1} g(v + y' - y) dv}{\int_{-\infty}^\infty (1 - F(y|v))^{K-1} f(y|v)^2 F(y|v)^{N-K-1} g(v + y' - y) dv},
\end{aligned}$$

which has the same sign as

$$\begin{aligned}
&\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1 - F(y|v)}{f(y|v)} (1 - F(y|v))^{K-1} f(y|v)^2 F(y|v)^{N-K-1} \\
&\quad * (1 - F(y|z))^{K-1} f(y|z)^2 F(y|z)^{N-K-1} g(v) g(z + y' - y) dv dz \\
&\quad - \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1 - F(y|z)}{f(y|z)} (1 - F(y|v))^{K-1} f(y|v)^2 F(y|v)^{N-K-1} \\
&\quad * (1 - F(y|z))^{K-1} f(y|z)^2 F(y|z)^{N-K-1} g(v) g(z + y' - y) dv dz \\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty b(v, z) g(v) g(z + y' - y) dv dz,
\end{aligned}$$

where

$$\begin{aligned} b(v, z) \equiv & \left(\frac{1 - F(y | v)}{f(y | v)} - \frac{1 - F(y | z)}{f(y | z)} \right) \\ & * (1 - F(y | v))^{K-1} f(y | v)^2 F(y | v)^{N-K-1} \\ & \times (1 - F(y | z))^{K-1} f(y | z)^2 F(y | z)^{N-K-1}. \end{aligned}$$

Note that $b(v, z)$ is positive if $v > z$ since $\frac{1-F(y|v)}{f(y|v)}$ increases in v . Also, $b(v, z) = -b(z, v)$. We can then write

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b(v, z) g(v) g(z + y' - y) dv dz \\ &= \int_{-\infty}^{\infty} \int_z^{\infty} (b(v, z) g(v) g(z + y' - y) + b(z, v) g(z) g(v + y' - y)) dv dz \\ &= \int_{-\infty}^{\infty} \int_z^{\infty} b(v, z) (g(v) g(z + y' - y) - g(z) g(v + y' - y)) dv dz, \end{aligned}$$

where $b(v, z)$ is always positive in the last double integral since $v > z$. Therefore, $a(y) - a(y') > 0$ for $y' > y$, so $a(y)$ decreases if

$$g(v) g(z + y' - y) - g(z) g(v + y' - y) > 0.$$

Equivalently, this holds if $\frac{g(z)}{g(z + y' - y)}$ increases in z for $y' > y$. We have that

$$\begin{aligned} \frac{g(z)}{g(z + y' - y)} &= \frac{e^{-\frac{(z-\mu)^2}{2}}}{e^{-\frac{(z+y'-y-\mu)^2}{2}}} \\ &= e^{\frac{(z+y'-y-\mu)^2 - (z-\mu)^2}{2}} \\ &= e^{\frac{((y-y')+\mu)^2 - \mu^2}{2}} e^{\frac{2z(y'-y)}{2}}, \end{aligned}$$

which indeed increases in z for $y' > y$. Q.E.D.

Proof of Proposition 4: To show the first part of the proposition, I show that $\int E(r | Y_{K+1} = y) * (q(z | y) - q(z' | y)) dy$ is positive for $z > z'$. Note that

$$\int (q(z | y) - q(z' | y)) dy = 0.$$

This follows since $\int q(z | y) dy = 1$ for all z . Using this result and the fact that $\frac{q(z | y)}{q(z' | y)}$ increases in y from Lemma 4, there must exist some y^* such that $q(z | y) - q(z' | y) < 0$ if $y < y^*$ and $q(z | y) > q(z' | y)$ if $y > y^*$. We have

$$\begin{aligned}
& \int E(r | Y_{K+1} = y)(q(z | y) - q(z' | y)) dy \\
&= \int_{y \leq y^*} E(r | Y_{K+1} = y)(q(z | y) - q(z' | y)) dy \\
&\quad + \int_{y > y^*} E(r | Y_{K+1} = y)(q(z | y) - q(z' | y)) dy \\
&> \int_{y \leq y^*} E(r | Y_{K+1} = y^*)(q(z | y) - q(z' | y)) dy \\
&\quad + \int_{y > y^*} E(r | Y_{K+1} = y^*)(q(z | y) - q(z' | y)) dy \\
&= 0.
\end{aligned}$$

The last inequality follows since $E(r(Z) | Y_{K+1} = y)$ increases in y when $r(z)$ increases, so that $E(r | Y_{K+1} = y^*) \geq E(r | Y_{K+1} = y)$ over the range $y^* \geq y$, where $q(z | y) - q(z' | y) \leq 0$, and $E(r | Y_{K+1} = y^*) \leq E(r | Y_{K+1} = y)$ over the range $y^* < y$ where $q(z | y) - q(z' | y) \geq 0$. This shows the first part of the proposition. To show the last part, note that as $N \rightarrow \infty$, expected underpricing goes to zero as is shown in, for example, Kremer (2002). Thus, $\phi(z) \rightarrow \int E(r | Y_{K+1})q(z | y) dy$, which increases in z . Furthermore, $\int E(r | Y_{K+1})q(0 | y) dy = E(r) < 0$, so the optimal security is a call option with strike price larger than zero. Q.E.D.

Proof of Proposition 5: An investor with signal $X = x$ solves the following maximization problem in the mechanism

$$\begin{aligned}
& \max_{x'} \int_0^{x'} (E(w(Z, y) | X = x, Y_{K:N-1} = y) \\
& \quad - E(w(Z, y) | Y_K = Y_{K+1} = y)) f(Y_{K:N-1} = y | X = x) dx,
\end{aligned}$$

where x' is his reported signal and $Y_{K:N-1}$ is the K :th order statistic among his $N - 1$ opponents. Since the integrand is positive for $y < x$ and negative for $y > x$, truth telling is the optimal strategy. Q.E.D.

Proof of Proposition 6: The proof follows immediately from Lemma 3 and the discussion in the text. Q.E.D.

Proof of Proposition 7:

(i) Auction of tranches with reserve prices:

First, consider the auctioning of a single security w with a reserve price r . I first show that the equilibrium-bidding strategies postulated in the proof of Lemma 1 are unchanged. The postulated bidding strategy is

$$b(w, x) = E(w | Y_{K+1} = Y_K = x).$$

The reserve price defines a cutoff signal $Y_{K+1} = y^*$ below which the security is not issued:

$$b(w, y^*) = r.$$

Suppose an investor with signal $X = x$ bids according to the equilibrium. If his signal is among the K highest and $Y_{K+1} = y$, his expected payoff is

$$(E(w | X = x > Y_{K+1} = y) - E(w | Y_K = Y_{K+1} = y)) * 1_{y \geq y^*}.$$

Note that this is positive and is independent of the investor's bid. Thus, the investor is happy to follow the prescribed equilibrium when he is among the winning bidders. Now suppose he were among the losing bidders if he bid according to the equilibrium but deviates by increasing his bid and wins when $x < Y_K$. Then the price will be set by the bidder with the K :th highest signal instead, and the expected payoff of the investor is

$$(E(w | X = x < Y_K = y) - E(w | Y_K = Y_{K+1} = y)) * 1_{y \geq y^*}.$$

The above expression is negative. Therefore, there is no incentive to deviate. Note also that the price is the same as before whenever the reserve is met. Finally, note that if several such securities are auctioned simultaneously, the equilibrium is unaffected.

Now suppose the firm issues M tranches with decreasing seniority, so that tranche m with face value D_m is defined by

$$w_m(Z) = \min \left(D_m, \max \left(Z - \sum_{n=1}^{m-1} D_n, 0 \right) \right).$$

Also, suppose the reserve price r_m for each tranche is set such that the cutoff signal $Y_{K+1} = y_m^*$ defined above increases in m . Suppose these tranches are auctioned simultaneously. Then, for a certain realization $Y_{K+1} = y$, all tranches $m < m^*$ and no tranches $m \geq m^*$ will be issued, where m^* is defined by

$$y_{m^*}^* > y$$

$$y_{m^*-1}^* \leq y.$$

The resulting revenues raised are equal to

$$\begin{aligned} \sum_{m=1}^{m^*-1} \pi(w_m(Z), y) &= \sum_{m=1}^{m^*-1} E \left(\min \left(D_m, \max \left(Z - \sum_{n=1}^{m-1} D_n, 0 \right) \right) \middle| Y_K = Y_{K+1} = y \right) \\ &= E \left(\min \left(\sum_{m=1}^{m^*-1} D_m, Z \right) \middle| Y_K = Y_{K+1} = y \right), \end{aligned}$$

which is the equivalent outcome of selling debt with face value $\sum_{m=1}^{m^*-1} D_m$.

Since $D(y)$ was assumed to be increasing, by issuing enough tranches and setting the reserve prices the right way, the outcome of the optimal debt schedule in the book-building mechanism can be implemented.

(ii) Sequential Issuance:

I first show that the equilibrium-bidding strategies postulated in the proof of Lemma 1 are unchanged in both auctions in the sequence. Look at deviations in the last stage. Under the equilibrium strategy, the previous auctions reveal $Y_{K+1} = y$. However, the bidding strategy is optimal even contingent on knowing $Y_{K+1} = y$, as the proof of Proposition 5 shows, so it is still an equilibrium regardless of what price was observed in the first stage.

Now look at deviations in the first stage. Since an action in this stage will not impact the pricing in the second stage, there is no incentive to deviate from the static equilibrium to affect prices. However, a deviation could affect the choice of security in the second stage. In the second stage, if junior debt with face value D_J is issued, a winning investor with signal x has expected payoff,

$$E(\min(D_J, \max(Z - D_S, 0)) | X = x > Y_{K+1} = y) \\ - E(\min(D_J, \max(Z - D_S, 0)) | Y_K = Y_{K+1} = y).$$

Note that this is positive and increases in D_J . Since $D_J = D(y') - D(\text{low})$, where y' is the inferred value of Y_{K+1} from the first auction, investors with signal $x > y$ would prefer a higher inferred y' . However, an investor with $x > y$ cannot increase the perceived y' in the first stage by deviating from the equilibrium strategy, since increasing his bid does not change the price. An investor who would lose under the equilibrium strategy in the first stage will lose in the second stage and receive a payoff of zero, so he has no incentive to deviate in the first stage by increasing the price. Thus, the static equilibrium is also an equilibrium in the first stage.

Since this is the case, the firm will be able to invert the price in the first stage for $Y_{K+1} = y$, and will be able to set D_J as postulated in the proposition. This replicates the outcome of the bookbuilding mechanism. Q.E.D.

Proof of Proposition 8: For this and all other proofs in this section, I start by establishing the following aggregation result:

LEMMA 5: Suppose a pool of assets is given by

$$Z = \sum_{j=1}^I \Phi_j$$

(no systematic factor). Denote the aggregate signal X_n about the pool as

$$X_n \equiv \sum_{j=1}^I X_{nj} + U_n,$$

where U_n is a uniform random variable on $[0, 1]$. Then, equilibrium and underpricing in the auction of an admissible security $w(Z)$ are given as in Proposition 5 using X_n as the one-dimensional signal observed by bidder n .

Proof: Following Milgrom and Weber (1982), I say that the random variables in the vector a are *affiliated* if the joint density function $f(a)$ satisfies

$$f(a \vee a')f(a \wedge a') \geq f(a)f(a'), \quad (\text{A4})$$

where $a \vee a'$ denotes the component-wise maximum of a and a' , and $a \wedge a'$ denotes the component-wise minimum. A function f is affiliated if it satisfies (A4). I need to show that X_n is a sufficient statistic for $w(Z)$, and that the variables $\{X_1, \dots, X_N, \Phi_1, \dots, \Phi_I\}$ are affiliated. Using Theorem 5 from Milgrom and Weber, one can show that $E(w | X_n = x, Y_K^{-n} = y)$ increases in x and y , which is sufficient for the equilibrium in Proposition 5 to be valid.

It is easy to show that X_n is a sufficient statistic for Z using the formal definition of a sufficient statistic,²³ but it is also obvious from the symmetry of signals and assets: To predict Z , it does not matter for which assets you obtain a high signal, only how many high signals you obtain (that is, X_n). Since $w(Z)$ is a deterministic function of Z , X_n is therefore also a sufficient statistic for signals about $w(Z)$.

It remains to be shown that $f(x_1, \dots, x_N, \phi_1, \dots, \phi_I) \equiv f(X_1 = x_1, \dots, X_N = x_N, \Phi_1 = \phi_1, \dots, \Phi_I = \phi_I)$ is affiliated. I can write the density as $f(x_1, \dots, x_N, \phi_1, \dots, \phi_I) = f(x_1 | \phi_1, \dots, \phi_I) * \dots * f(x_N | \phi_1, \dots, \phi_I) * g(\phi_1) * \dots * g(\phi_I)$, since signals are conditionally independent and assets are independent. From Theorem 1 of Milgrom and Weber (1982), it is enough to show that all functions in this product are affiliated. This holds trivially for $g(\phi)$, so all I need to show is that $f(x_n | \phi_1, \dots, \phi_I)$ is affiliated. It is enough to show that for x'_n in a higher equivalence interval than x_n and for $\phi_i > \phi'_i$, we have

$$\begin{aligned} & f(x_n + 1 | \phi_1, \dots, \phi_i, \dots, \phi_I) f(x_n | \phi_1, \dots, \phi'_i, \dots, \phi_I) \\ & \leq f(x_n + 1 | \phi_1, \dots, \phi'_i, \dots, \phi_I) f(x_n | \phi_1, \dots, \phi_i, \dots, \phi_I) \end{aligned}$$

for all i . From the symmetry of assets, it is enough to show this for ϕ_1 , or that

$$\frac{f(x_n + 1 | \phi_1, \dots, \phi_I)}{f(x_n | \phi_1, \dots, \phi_I)}$$

increases in ϕ_1 . I show this by induction. That is, if it holds for $\phi_1, \dots, \phi_i$, it also holds for $\phi_1, \dots, \phi_i, \phi'$ for some extra arbitrary component ϕ' . Note that

$$f(x_n | \phi_1, \dots, \phi_I) = f(x_n | \phi_2, \dots, \phi_I) * P(0 | \phi_1) + f(x_n - 1 | \phi_2, \dots, \phi_I) * P(1 | \phi_1),$$

²³ See, for example, DeGroot (1986).

so

$$\begin{aligned}
 & \frac{f(x_n + 1 | \phi_1, \dots, \phi_I)}{f(x_n | \phi_1, \dots, \phi_I)} \\
 &= \frac{f(x_n + 1 | \phi_2, \dots, \phi_I) * P(0 | \phi_1) + f(x_n | \phi_2, \dots, \phi_I) * P(1 | \phi_1)}{f(x_n | \phi_2, \dots, \phi_I) * P(0 | \phi_1) + f(x_n - 1 | \phi_2, \dots, \phi_I) * P(1 | \phi_1)} \\
 &= \frac{\frac{f(x_n + 1 | \phi_2, \dots, \phi_I)}{f(x_n | \phi_2, \dots, \phi_I)} + \frac{P(1 | \phi_1)}{P(0 | \phi_1)}}{1 + \frac{f(x_n - 1 | \phi_2, \dots, \phi_I)}{f(x_n | \phi_2, \dots, \phi_I)} * \frac{P(1 | \phi_1)}{P(0 | \phi_1)}}.
 \end{aligned}$$

That this increases in $\phi_2, \dots, \phi_I$ is implied by the induction hypothesis. Since ϕ_1 is arbitrary, it increases in all ϕ s. Thus, it holds for all x_n between 1 and $I - 2$. It remains to be checked that it also holds for $x_n = 0$ and $x_n = I - 1$. Toward this end, note that

$$\begin{aligned}
 \frac{f(I | \phi_1, \dots, \phi_I)}{f(I - 1 | \phi_1, \dots, \phi_I)} &= \frac{\prod_{i=1}^I P(1 | \phi_i)}{\sum_{i=1}^I P(0 | \phi_i) \prod_{j \neq i} P(1 | \phi_j)} \\
 &= \frac{1}{\sum_{i=1}^I \frac{P(0 | \phi_i)}{P(1 | \phi_i)}} \\
 \frac{f(1 | \phi_1, \dots, \phi_I)}{f(0 | \phi_1, \dots, \phi_I)} &= \frac{\sum_{i=1}^I P(1 | \phi_i) \prod_{j \neq i} P(0 | \phi_j)}{\prod_{i=1}^I P(0 | \phi_i)} \\
 &= \sum_{i=1}^I \frac{P(1 | \phi_i)}{P(0 | \phi_i)}.
 \end{aligned}$$

Since $\frac{P(1 | \phi_i)}{P(0 | \phi_i)}$ increases in ϕ_i , both of these increase in ϕ_i for all i and we are done. Q.E.D.

To prove the first part of Proposition 8, divide the pool of assets $\sum_{i=1}^I Z_i$ into $\sqrt{I}$ identical securities $Z = \frac{\sum_{i=1}^I Z_i}{\sqrt{I}}$. Asymptotically, the distribution of $Z - \sqrt{I}E(Z_i)$ is normal with zero mean and standard deviation $\sigma^2(Z_i)$. It is straightforward to show that the renormalized sum of signals $\frac{\sum_{i=1}^I X_{in}}{\sqrt{I}} - E(\frac{\sum_{i=1}^I X_{in}}{\sqrt{I}})$ also reaches a nondegenerate asymptotic normal distribution, both unconditionally and conditional on Z . As I goes to infinity, the auction of a security $Z = \frac{\sum_{i=1}^I Z_i}{\sqrt{I}}$ will therefore be associated with a certain fixed amount of underpricing $C > 0$. When selling the whole asset pool, the underpricing therefore grows at rate $\sqrt{I}$ with I (since the whole pool of assets consists of $\sqrt{I}$ such securities with

fixed underpricing). The underpricing per asset in the pool goes to zero, so this dominates separate sales. The optimality of debt follows from Proposition 3.

To prove the second part of Proposition 8, I start by establishing the optimal security design if the entire pool is issued. Consider the problem of selling I identical securities backed by the normalized pool of assets $Z = \frac{\sum_{i=1}^I Z_i}{I}$. As I goes to infinity, we have that $\frac{\sum_{i=1}^I \Phi_i}{I} \rightarrow E(\Phi_i)$. Thus, $Z \rightarrow \Phi_0 + E(\Phi_i)$. Then, $\phi(z) \rightarrow r$ for all $z \leq E(\Phi_i)$, so it is always optimal to include a senior debt component. The remaining cash flow $Z - E(\Phi_i) = \Phi_0$, so the optimal security design of the remaining cash flow will be identical to the single asset problem where Φ_0 is the asset.

It remains to be shown that this is better than selling securities backed by individual assets. Suppose I securities backed by individual assets are sold. From the linkage principle in Milgrom and Weber (1982), we know that revealing any information affiliated with the value of the security being sold prior to the auction increases revenues. Thus, for any security design, suppose the true value of each unsystematic component is revealed prior to the auction. This increases revenues. Now suppose the security design is allowed to be done after the information has been revealed. This increases revenues even further. In this case, the optimal security design with respect to the sale of asset i when $\Phi_i = \phi_i$ will be to issue a security $w(Z) = \min(Z, \phi_i) + w^*(Z - \phi_i)$, where $w^*(\cdot)$ is the optimal security design that would have obtained for an asset with cash flow Φ_0 . Aggregating across assets, this policy yields exactly the same revenue as for the pool as $I \rightarrow \infty$, and hence the pool dominates separate sales. Q.E.D.

Proof of Proposition 9: The beginning of this proof is identical to the proof of Claims 2 and 3 of Proposition 2 except for the derivation of the limit of $\frac{P(Z \geq z | Y_{K+1}=y)}{P(Z \geq z | Y_K=Y_{K+1}=y)}$. In this context, I now have to condition on a vector of realizations $\{v_i\}$ instead of a scalar z , where $\{v_i\} \equiv \{v_1, \dots, v_J\}$ is the vector of components that make up a certain pool. Here, we have that

$$\begin{aligned} & \frac{\int_z^{\bar{z}} f(Y_{K+1} = y | v) g(v) dv}{\int_z^{\bar{z}} f(Y_K = Y_{K+1} = y | v) g(v) dv} \\ &= \frac{\frac{N!}{K!(N-K-1)!}}{\frac{1}{2} \frac{N!}{K-1!(N-K-1)!}} \\ & \times \frac{\int_{\{v_i\}: \sum v_i \geq z} f(y | \{v_i\}) (1 - F(y | \{v_i\}))^K F(y | \{v_i\})^{N-K-1} g(\{v_i\}) d\{v_i\}}{\int_{\{v_i\}: \sum v_i \geq z} f(y | \{v_i\})^2 (1 - F(y | \{v_i\}))^{K-1} F(y | \{v_i\})^{N-K-1} g(\{v_i\}) d\{v_i\}}, \end{aligned}$$

where $\int_{\{v_i\}:\sum v_i \geq z}$ denotes a multiple integration over all asset vectors that sum up to at least z .²⁴ Define the function $F_{\max}(y, z)$ as $F_{\max}(y, z) \equiv \arg \max_{\{v_i\}:\sum v_i = z} F(y | \{v_i\})$, with associated “density” $f_{\max}(y, z)$ defined as $f_{\max}(y, z) \equiv f(y | \arg \max(F_{\max}(y, z)))$. Since $F(y | \{v_i\})$ decreases in each v_i , $F_{\max}(y, z)$ strictly decreases in z . Dividing numerator and denominator of the expression above by $F_{\max}(y, z)^{N-K-1}$ and taking the limit as $N \rightarrow \infty$, we get

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{2}{K} \frac{\int_{\{v_i\}:\sum v_i \geq z} f(y | \{v_i\})(1 - F(y | \{v_i\}))^K \left(\frac{F(y | \{v_i\})}{F_{\max}(y, z)} \right)^{N-K-1} g(\{v_i\}) d\{v_i\}}{\int_{\{v_i\}:\sum v_i \geq z} f(x | \{z_i\})^2 (1 - F(y | \{v_i\}))^{K-1} \left(\frac{F(y | \{v_i\})}{F_{\max}(y, z)} \right)^{N-K-1} g(\{v_i\}) d\{v_i\}} \\ = \frac{2}{K} \frac{1 - F_{\max}(y, z)}{f_{\max}(y, z)}. \end{aligned}$$

Therefore, as $N \rightarrow \infty$, $\frac{P(Z \geq z | Y_{K+1}=y)}{P(Z \geq z | Y_K=Y_{K+1}=y)} - 1 \rightarrow \frac{f(y | 0)}{1 - F(y | 0)} \frac{1 - F_{\max}(y, z)}{f_{\max}(y, z)} - 1$. Since the inverse hazard rate $\frac{1 - F(y | \{v_i\})}{f(y | \{v_i\})}$ strictly increases in each v_i for $y < H$, and since $\arg \max_{\{v_i\}:\sum v_i = z} F(y | \{v_i\}) > \{0\}$ for $z > 0$, this is strictly greater than zero for all $z > 0$. Hence, $r - \phi(z)$ again goes to zero with $\int_{y=0}^H q(z | y) dy$.

Now, we have that

$$q(z | y) = \frac{\int_{\{v_i\}:\sum v_i \geq z} f(Y_{K+1} = Y_K = y | \{v_i\}) d\{v_i\}}{E(\pi(1_{Z \geq z}, Y_{K+1}))},$$

where again the unconditional expected price in the denominator goes to a positive constant. Hence, $\int_{y=0}^H q(z | y) dy$ goes to zero with

$$\begin{aligned} \int_{y=0}^H \int_{\{v_i\}:\sum v_i \geq z} f(Y_{K+1} = Y_K = y | \{v_i\}) d\{v_i\} \\ = \frac{1}{2} \int_{y=0}^H \int_z^{\bar{z}} \frac{N!}{K-1!(N-K-1)!} f(y | \{v_i\})^2 \\ \times (1 - F(y | \{v_i\}))^{K-1} F(y | \{v_i\})^{N-K-1} g(\{v_i\}) d\{v_i\} dy. \end{aligned}$$

Again, note that $F(y | \{v_i\})$ decreases in each v_i . Dividing the above by $F_{\max}(H, z + \varepsilon)^N$ for some $\varepsilon > 0$, the integral goes to infinity as N goes to infinity. Dividing by $F_{\max}(H, z - \varepsilon)^N$, the integral goes to zero. Since ε is arbitrary, we must have that $r - \phi(z)$ goes to zero at a rate proportional to $F_{\max}(H, z)^N$. Since $F_{\max}(H, z)^N$ strictly decreases in z , we again have that $\frac{r - \phi(z)}{r - \phi(z)}$ goes to infinity with N for

²⁴ I cannot write $f(Y_{K+1} = y | v) = \frac{N!}{K!(N-K-1)!} f(y | v)(1 - F(y | v))^K F(y | v)^{N-K-1}$ anymore, as it is no longer necessarily true that the aggregate signal X for bidder n is independent of signal X' of bidder m conditional on the sum of values $z = \sum v_i$: $f(X = x, X' = x' | z) \neq f(x | z)f(x' | z)$. Only if I condition on the whole vector $\{v_i\}$ does conditional independence hold.

$v' > v > 0$, and hence a call option dominates just like in the single asset case (except for a vanishingly small range around $v = 0$). The rate of convergence of a call option that starts paying off at z is proportional to $F_{\max}(H, z)^N$. Note that the smaller $F_{\max}(H, z)$, the faster is the convergence. Therefore, the larger $f_{\max}(H + 1, z)$, the faster is the convergence.

I now show that if the seller decides to raise a certain amount of capital C per asset, underpricing is larger for pooled sales than individual sales for N large. This proves the result, since it shows that more capital can be raised for the same amount of underpricing with individual sales. I demonstrate this result for the case of a pool of two components $Z = \frac{V_1 + V_2}{2}$ versus a sale of the individual components; the proof generalizes easily to the case of more components. To raise C per component when selling the pool Z , the optimal security for N sufficiently large is a call option with strike price z' approximately close to solving

$$\int_{z'}^{\bar{z}} (v - z') g\left(\frac{V_1 + V_2}{2} = v\right) dv = C. \quad (\text{A5})$$

Similarly, for the sale of the individual component V_1 (and symmetrically for V_2), the optimal security is a call option with strike price z such that

$$\int_z^{\bar{z}} (v_1 - z) g(v_1) dv_1 = C. \quad (\text{A6})$$

We know that the underpricing of the pooled asset goes to zero at a rate proportional to

$$f(H + 1 | \{\hat{v}_1, \hat{v}_2\}) = P(Y_{n1} = 1 | \hat{v}_1) P(Y_{n2} = 1 | \hat{v}_2),$$

where $\{\hat{v}_1, \hat{v}_2\}$ minimize $f(H + 1 | \{v_1, v_2\})$ subject to $\frac{v_1 + v_2}{2} = z'$, while the underpricing for the individual asset goes to zero at rate $P(Y_{n1} = 1 | z)$. If v_1 in $f(H + 1 | \{v_1, v_2\})$ can be set lower or equal to z while still having $\frac{v_1 + v_2}{2} = z'$, the result follows immediately since $P(Y_{n1} = 1 | v_1)$ increases in v_1 . This obtains since in that case we have $P(Y_{n1} = 1 | \hat{v}_1) P(Y_{n2} = 1 | \hat{v}_2) \leq P(Y_{n1} = 1 | v_1) P(Y_{n2} = 1 | v_2) < P(Y_{n1} = 1 | z)$, so that the underpricing for the individual asset goes to zero faster. For this not to hold, it is necessary that $z' > \frac{z + \bar{z}}{2}$. I show that this is not possible while preserving the capital-raising conditions (A5) and (A6). Equivalently, I show that for $z' = \frac{z + \bar{z}}{2}$, the expected value of the capital raised in the pool is smaller than the expected value of the capital raised in the individual sale, or that for all z , we have

$$\int_{\frac{z + \bar{z}}{2}}^{\bar{z}} \left(v - \frac{z + \bar{z}}{2}\right) g\left(\frac{V_1 + V_2}{2} = v\right) dv \leq \int_z^{\bar{z}} (v_1 - z) g(v_1) dv_1. \quad (\text{A7})$$

To show this, I rewrite the convolution density $g(\frac{V_1+V_2}{2} = v)$ for the relevant case $v > \frac{\bar{z}}{2}$:

$$g\left(\frac{V_1+V_2}{2} = v\right) = 2 \int_{2v-\bar{z}}^{\bar{z}} g(v_1)g(2v-v_1)dv_1.$$

Using Fubini's theorem and a change of variables, I can then rewrite the left-hand side of (A7) as

$$\begin{aligned} & \int_{\frac{z+\bar{z}}{2}}^{\bar{z}} \left(v - \frac{z+\bar{z}}{2}\right) \left(2 \int_{2v-\bar{z}}^{\bar{z}} g(v_1)g(2v-v_1)dv_1\right) dv \\ &= \int_z^{\bar{z}} \int_{\frac{z+\bar{z}}{2}}^{\frac{z_1+\bar{z}}{2}} 2 \left(v - \frac{z+\bar{z}}{2}\right) g(2v-v_1)dv g(v_1)dv_1 \\ &= \int_z^{\bar{z}} \int_{z+\bar{z}-v_1}^{\bar{z}} \frac{1}{2}(u - \bar{z} + v_1 - z)g(u)dug(v_1)dv_1, \end{aligned}$$

so that (A7) becomes

$$\int_z^{\bar{z}} \left(\int_{z+\bar{z}-v_1}^{\bar{z}} \frac{1}{2}(u - \bar{z} + v_1 - z)g(u)du - (v_1 - z) \right) g(v_1)dv_1 \leq 0. \quad (\text{A8})$$

But this always holds, since the expression in parentheses is strictly negative for $z < \bar{z}$:

$$\begin{aligned} \int_{z+\bar{z}-v_1}^{\bar{z}} \frac{1}{2}(u - \bar{z} + v_1 - z)g(u)du &< \int_{z+\bar{z}-v_1}^{\bar{z}} \frac{1}{2}(v_1 - z)g(u)du \\ &\leq (v_1 - z). \end{aligned}$$

The first relation follows from $u \leq \bar{z}$. This shows that individual sales dominate. The rest of the proposition follows directly from Proposition 2. Q.E.D.

Proof of Proposition 10: Again, the maximization can be done pointwise for each y . Suppose you fix π and $I \leq \pi(1-c)$. Then, an optimal security has to solve

$$\max_{w(Z,y):\pi(w(Z,y),y)=\pi} E(Z - w(Z, y)),$$

which is equivalent to solving the Lagrangian

$$\begin{aligned} & \min_{w_1(z,y) \in [0,1]: \pi(w(Z,y),y)=\pi} E(w(Z, y) | Y_{K+1} = y) \\ & - \lambda(\pi - E(w(Z, y) | Y_{K+1} = Y_K = y)). \end{aligned}$$

Rewriting this in terms of component securities gives

$$\min_{w_1(z,y) \in [0,1]} \int w_1(z, y)(E(1_{Z \geq z} | Y_{K+1} = y) - \lambda E(1_{Z \geq z} | Y_{K+1} = Y_K = y)) + \lambda \pi,$$

such that

$$\pi = \int w_1(z, y) E(1_{Z \geq z} | Y_{K+1} = Y_K = y) dz.$$

The solution is to include component security $1_{Z \geq z}$ if

$$\frac{E(1_{Z \geq z} | Y_{K+1} = y)}{E(1_{Z \geq z} | Y_{K+1} = Y_K = y)} \leq \lambda,$$

or, since Z is deterministic and strictly increasing in Φ , if

$$\frac{E(1_{\Phi \geq Q^{-1}(I, z + I - \pi(1-c))} | Y_{K+1} = y)}{E(1_{\Phi \geq Q^{-1}(I, z + I - \pi(1-c))} | Y_{K+1} = Y_K = y)} \leq \lambda.$$

Since we know from Proposition 6 that this ratio increases in z , it follows that debt is optimal.

Now, suppose an investment policy $I(y)$ is fixed but the firm needs to decide how much capital $\pi \geq \frac{I(y)}{1-c}$ to raise. Suppose the firm decides to raise an extra dollar. This increases the firm's cash flow by a safe $1 - c$ dollars. But $D(y)$ has to go up by at least one dollar to compensate investors. Hence, the firm's retained stake must go down. Thus, $\pi = \frac{I(y)}{1-c}$. A similar argument taking the derivative with respect to the optimal investment level shows that it is below the first-best level. Q.E.D.

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