

Model Uncertainty and Option Markets with Heterogeneous Beliefs

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ABSTRACT

This paper provides option pricing and volume implications for an economy with heterogeneous agents who face model uncertainty and have different beliefs on expected returns. Market incompleteness makes options nonredundant, while heterogeneity creates a link between differences in beliefs and option volumes. We solve for both option prices and volumes and test the joint empirical implications using S&P500 index option data. Specifically, we use survey data to build an Index of Dispersion in Beliefs and find that a model that takes information heterogeneity into account can explain the dynamics of option volume and the smile better than can reduced-form models with stochastic volatility.

THE NOTION THAT OPTIONS CAN BE REPLICATED via a dynamic trading strategy in the underlying asset (Black and Scholes (1973)) has been one of the most influential innovations in the theory of financial markets. That concept allowed the development of pricing and risk management models of option and other complex financial instruments. However, the traditional no-arbitrage approach with deterministic volatility is silent with respect to option trading volume: Since options are redundant securities, agents are indifferent about holding them and thus their trading volume is indeterminate. On the other hand, the last two decades have witnessed an impressive proliferation of options and other derivative securities, with option open interest in all main option exchanges increasing tenfold in 15 years. This suggests that options are far from redundant, that is, they provide an economic value that at least exceeds the cost of maintaining option exchanges. The main contribution of this paper is to link heterogeneity in beliefs to option open interest with a model that delivers realistic *joint* restrictions on both option open interest and prices.

Our model investigates a simple generalization of the standard general equilibrium Lucas economy with rational agents characterized by identical preferences and endowments but incomplete and heterogeneous information. We

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allow the dividend growth rate to be stochastic. Thus, agents need to form expectations about future dividends to form their optimal portfolios. We assume that agents address this form of model uncertainty by rationally using all available information to update their initial beliefs about the dividend growth rate. When agents have different beliefs, they select different optimal portfolios and trading occurs. In general equilibrium, this has important implications for both the market risk premium and the option-implied volatility smile. Agents with a more pessimistic posterior of the dividend growth rate demand state-contingent insurance protection from the optimist, which can be achieved by means of out-of-the-money (OTM) put options. Similarly, agents with more optimistic posterior estimates of the dividend growth rate demand OTM call options from the pessimists. Because the marginal utility of both agents is higher (lower) in bad (good) states of the world, differences in beliefs generate an option-implied volatility smile: The cost of an OTM put is higher than an OTM call option. Another important implication of the model is that although the instantaneous dividend volatility is deterministic, the endogenous equilibrium volatility of the underlying stock price is stochastic and driven by the level of difference in beliefs via the posterior mean of the dividend growth rate. The model therefore provides a structural underpinning for reduced-form stochastic volatility models.

Our work is motivated by both the growing interest in the asset pricing implications of model uncertainty and the behavior of derivative markets during periods of high uncertainty. The events that unfolded subsequent to August 17, 1998, when Russia defaulted on the GKOs/OFZs debt, provide a useful example. The 2 months that followed were characterized by great uncertainty about the possibility of contagion. This uncertainty grew even more pronounced when Long-Term Capital Management (LTCM) disclosed large financial losses to the extent that on September 23rd, 1998, Alan Greenspan (the Federal Reserve Bank Chairman) was prompted to initiate and coordinate the rescue of LTCM.¹ While the actual number of institutions involved in these two defaults and their real exposure were unclear, a flare of rumors, readily reported by the financial press, contributed to the uncertainty surrounding the state of the economy. One of these rumors included the speculation of an imminent bankruptcy protection filing of a major investment bank.² The derivative markets reacted. On October 8, 1998, the implied volatility peaked at 48.56%, up from an historic average of 19%. Even more interestingly, however, both the spread between the implied and realized volatility and the open interest on S&P500 options grew to record levels. Most option models find it particularly difficult to explain these empirical observations for this period. This paper attempts to provide a rationale for

¹ Just one week before the rescue, Alan Greenspan testified to the United States Congress that "risk in hedge fund lending was well under control" (Gretchen Morgenson, "Hedge Fund Bailout Rattles Investors and Markets," *New York Times*, September 25, 1998).

² On September 11, Lehman's already-battered stock declined in a single day by 7% amid a rumor that the firm would file for Chapter 11 bankruptcy protection. Senior officials scrambled to assure clients that Lehman was secure. The rumors even induced the New York Stock Exchange to examine Lehman's books. Lehman passed the review. Two days later, Moody's Investors Services confirmed Lehman's debt ratings and said the firm's ratings outlook was "stable."

the behavior of option markets that goes beyond the simple assumption of large (exogenous) changes in volatility.

The assumption of informational heterogeneity is also motivated by a growing body of empirical evidence that shows options are used for informationally driven trades. For instance, Pan and Poteshman (2006) find that the put-call volume ratio of buyer-originated trades predicts future returns in the underlying asset, Amin and Lee (1997) show that good (bad) earnings news is positively correlated with a pre-announcement increase in call (put) open interest, Cao, Chen, and Griffin (2005) show that the option volume in the pre-announcement period predicts takeover premia, and moreover Easley, O'Hara, and Srinivas (1998) find a link between option volume and stock returns, even independent of take-over announcements.³ These studies focus on individual stock options and conjecture that some investors have private information on the underlying stock, which they then try to exploit by trading in the option market. In the case of *index* options, however, private information is a much less compelling motive for trading. Thus, we explore a different source of informationally driven trades, namely, the heterogeneous beliefs of fully rational agents with learning.

The model derives the optimal portfolio holdings in the underlying asset and in options as a function of the difference in beliefs. We then estimate the structural model using moment conditions on both option prices and open interest. Using the standard deviation of the beliefs distribution that we obtain from the Survey of Professional Forecasters and the Consumer Confidence Survey, we build a Difference in Beliefs Index (ψ) on market fundamentals and demonstrate the existence of an important link between heterogeneous beliefs on market fundamentals and (a) open interest in index options, (b) future realized and implied volatility, and (c) the shape of the implied volatility smile. More specifically, we address the following questions:

First, what is the extent to which differences in beliefs on market fundamentals explain the dynamics of option trading and open interest? Unlike models with time-varying but deterministic volatility, the Difference in Beliefs (DB) model generates testable restrictions on the dynamics of option volume and open interest. We use a panel data of options to run a Chi-square test to assess the extent to which changes in the differences in beliefs explain option trading. We fail to reject the overidentifying restrictions on the option open interest. Moreover, we find that the link between changes in the differences in beliefs and option open interest is economically significant: A one standard deviation increase in the Difference in Beliefs Index increases option open interest by 20%. We find that the relationship is nonlinear.

Second, how does the model fare against traditional reduced-form option pricing models in terms of hedging errors? We find that the Difference in Beliefs model generates lower hedging errors than both Heston (1993) and Black and

³ Other important empirical contributions on the informational role of options include Manaster and Rendleman (1982), Stephan and Whaley (1990), Figlewski and Webb (1993), Mayhew, Sarin, and Shastri (1995), and Chakravarty, Gulen, and Mayhew (2002).

Scholes (1973). We test the model by using a subset of options to estimate the structural parameters and then using the pricing errors on different subsets to build a Chi-square test statistic of the overidentifying restrictions. We fail to reject the model when using both out-of-the-money and in-the-money subsets of options.

Third, how do the differences in beliefs affect the shape of the volatility smile? We find that the implied volatility smile is sensitive to the Differences in Beliefs Index. For low Index levels, the smile level is about 15%. For high Index levels, it exceeds 20%. Moreover, we find that changes in the Index affect the steepness of the smile: The greater the Difference in Beliefs Index, the steeper the implied volatility smile.

Fourth, to what extent can differences in beliefs incorporate future realized and implied volatility? We find that current levels in the Differences in Beliefs Index have positive and statistically significant predictive power for the future realized volatility, even after controlling for the implied volatility.

Fifth, do differences in beliefs explain violations of textbook arbitrage bounds? Bakshi, Cao, and Chen (2000) find that the delta of a call option is often negative or larger than 1. We run a Logit regression to assess the extent to which the Differences in Beliefs Index can explain these no-arbitrage violations. We find that the differences in beliefs slope coefficients are positive and statistically significant. An increase in the Index increases the probability that the Black and Scholes delta is negative or above 1. This suggests that the heterogeneity in beliefs is an important pricing factor, generating sufficient market incompleteness to yield deviations from arbitrage bounds based on one-factor models.

Given these five sets of results, we conclude that both the information heterogeneity and the belief structure of the economy have important option pricing and risk management implications.

Our model draws from several contributions in the incomplete information and rational learning literature. David and Veronesi (2002) develop an incomplete information option pricing model in which investors' uncertainty about the drift of firms' dividends affects option prices through its impact on stock volatility. Arguing that stock return volatility is stochastic because of fluctuations in "uncertainty," they derive a Fourier Transform option pricing formula and show that the time-varying correlation between returns and volatility induced by uncertainty in dividends is related to higher-order moments of the return distribution. A contribution of their approach is to circumvent the deterministic variance property of the estimation error of Gaussian models. Our approach is related to theirs in that, like them, we assume that the drift of the dividends is not observable, but rather stochastic, and that the belief dynamics are implied by rational learning. However, an important difference between our respective approaches is that we allow for heterogeneity and derive equilibrium portfolio holdings and implications in terms of both option open interest and prices. Moreover, we model two forms of incomplete information; the first affects the dividend drift, and the second affects a signal that is correlated with

dividends. Both forms are described by a continuous-time process as opposed to a switching regime model.

Other option pricing models with uncertainty include Campbell and Li (2001), who study the implications of uncertainty in volatility regimes, and Garcia, Luger, and Renault (1999), who study a regime switching model. Similar to our approach, Guidolin and Timmermann (2000) model learning in an economy in which prices are endogenous. Their assumption of an independent and identically distributed (i.i.d.) drift process, however, makes option prices converge to Black-Scholes values in the continuous time limit. The original models of asset pricing in economies with incomplete information include Detemple (1986), Dothan and Feldman (1986), Genotte (1986), and Detemple and Murthy (1994). More recent contributions include Brennan (1998), Brennan and Xia (2000), David (1997), and Veronesi (1999, 2000).

Our work is also related to the literature that studies the role of options in incomplete markets. In the Black and Scholes (1973) and Merton (1973) models of derivative pricing, options are redundant securities as their payoffs can be replicated using financial claims. Thus, these models yield no empirical testable restrictions in terms of volume. To overcome this empirical shortcoming, which is shared by any model with time-varying but deterministic volatility, several articles investigate the role of options in economies in which they are not redundant. Grossman and Zhou (1996) consider a model in which the demand of options is generated by the utility function of portfolio insurers. In their economy, some agents (portfolio insurers) receive infinite disutility if their wealth level approaches a given lower bound, and thus these agents use options as optimal risk sharing contracts. Bates (2001) considers an economy in which crashes can occur and agents differ in their risk aversion. In his model, the less crash-averse agents insure the more crash-averse agents through options. In both models, options help to complete the markets. Our model differs from this literature because it does not assume that agents have different risk aversion coefficients and market incompleteness is exclusively generated by model uncertainty when agents have different priors. In this respect, our modeling approach is related to Detemple and Murthy (1994) and Zapatero (1998), who explore the implications of differences in beliefs on financial innovations and interest rate volatility, and Basak (2000), who investigates the existence of a sunspot equilibrium in which an extraneous process that is uncorrelated with market fundamentals affects asset prices when agents have different beliefs. However, these models do not investigate option pricing and trading implications, which constitute instead the focus of this article. Related work on this latter topic includes Brennan and Solanki (1981) and Leland (1980).

Other sources of uncertainty that make options nonredundant have been studied by Franke, Stapleton, and Subrahmanyam (1998), who examine a one-period economy in which agents are exposed to different nonhedgeable background risks that lead to market incompleteness. Options trading also arises in the presence of additional risk factors such as stochastic volatility and jumps (Hull and White (1987), Wiggins (1987), Heston (1993), Liu and Pan (2003)) or because of various informational reasons (Back (1993), Biais and Hillion (1994), Brennan and Cao (1996), John et al. (2000)). Buraschi and Jackwerth

(2000) develop an empirical test of the dynamic spanning properties of option contracts and find that option returns embed a risk premium that cannot be explained simply by the risk–return trade-off generated by the diffusion process of the underlying asset. Their empirical evidence yields support for asset pricing models in which options are not redundant such as in the case of models with undiversifiable stochastic volatility and/or jump risk. Similarly, Bakshi and Kapadia (2003) study delta-hedged gains of European options and find evidence of a negative market volatility risk premium.

The rest of the paper is organized as follows. Section I describes the model. We describe the data sets in Section II. Section III presents the model's estimation and the empirical evidence. The estimation differs from the literature as it uses joint information on both option prices and open interest. In particular, we employ a GMM test for the overidentifying restrictions of the model. Thus, Section III discusses these results, with the rest of the empirical results articulated in the following four subsections: Subsection A discusses the model performance in terms of the fitting errors for option volume and open interest, Subsection B discusses the model performance in terms of hedging errors, Subsection C discusses the time series implications for the model-implied difference in beliefs and shows evidence that the Differences in Beliefs Index predicts future realized volatility even after controlling for the current implied volatility, and Subsection D shows that the Differences in Beliefs Index can explain, at least in part, the Bakshi et al. (2000) no-arbitrage violation puzzle. Section IV concludes. All proofs are contained in Appendix A.

I. The Model

We study a finite-horizon general equilibrium economy in which two types of agents are endowed with shares in a production technology that generates a dividend flow. Agents have identical preferences and endowments but differ in their beliefs about the dividend growth rate (see also Detemple and Murthy (1994)). This simple generalization of the standard Lucas model is sufficient to make options nonredundant and help hedge sources of risk.

ASSUMPTION 1 (Preferences): *The economy is populated by two sets of constant relative risk aversion (CRRA) agents who maximize finite lifetime utility, that is,*

$$\max E^n \left[\int_0^T \frac{c_n(t)^\gamma}{\gamma} dt \middle| \mathcal{F}_t^n \right], \quad n = 1, 2. \quad (1)$$

The two sets of agents differ in terms of their beliefs, which affect their expectations.

Utility is maximized subject to a budget constraint. Let $\xi^n(t)$ be the equilibrium state price density of agent n . When a sufficient number of option contracts are traded that agents can span the state-space, it is possible to use the results in Cox and Huang (1989) and Karatzas, Lehoczky, Shreve (1987) and express the intertemporal budget constraint in its martingale form:

$$E^n \left[\int_0^T \xi^n(t) c_n(t) dt \middle| \mathcal{F}_t^n \right] = X_0^n. \quad (2)$$

The initial wealth is $X_0^n = e^n P_1(0)$, where e^n is the number of stock units with which agent n is initially endowed and $P_1(0)$ is the (endogenous) stock price at time zero. The martingale representation of the budget constraint in (2) says that in equilibrium, the value of the future consumption stream is equal to the current value of the asset(s) of each individual. Because the individuals have different initial beliefs, the stochastic discount factor $\xi^n(t)$ must be agent specific for (2) to be satisfied in equilibrium.

ASSUMPTION 2 (Dividend Process): *The dividend process follows a geometric Brownian motion with stochastic drift, that is*

$$\begin{aligned} d \ln \delta(t) &= \mu_\delta(t) dt + \sigma_\delta dW_\delta(t) \\ d\mu_\delta(t) &= (a_{0\delta} + a_{1\delta}\mu_\delta(t))dt + n_\delta dW_{\mu_\delta}(t). \end{aligned} \quad (3)$$

Agents also observe a process $z(t)$, which contains a signal for the growth rate of dividends:

$$\begin{aligned} dz(t) &= (\alpha\mu_\delta(t) + \beta\mu_z(t))dt + \sigma_z dW_z(t) \\ d\mu_z(t) &= (a_{0z} + a_{1z}\mu_z(t))dt + n_z dW_{\mu_z}(t). \end{aligned}$$

While both agents are aware of the dynamics of $\delta(t)$ and $z(t)$, they do not know the current value of the stochastic drifts $\mu_\delta(t)$ and $\mu_z(t)$. Agents have different initial prior beliefs about the value of $\mu_\delta(0)$ and they rationally update their estimates of $\mu_\delta(t)$ and $\mu_z(t)$ given the observed history of the dividend process $\delta(t)$ and the signal $z(t)$. With no loss of generality, we assume that $E(dW_\delta(t)dW_z(t)) = 0$.

The two stochastic drifts $\mu_\delta(t)$ and $\mu_z(t)$ are not observable, so agents are exposed to model uncertainty. Agents can observe the realizations of the dividend process $\delta(t)$ and signal $z(t)$ and rationally compute posterior estimates of the drifts, given their initial priors $\mu_\delta^n(0)$ and $\mu_z^n(0)$ and all available information.⁴ The larger the parameter β is relative to α , the larger the importance of the signal $z(t)$ to estimate the drift of the fundamental process $\delta(t)$. For convenience, let us rewrite our economy using vector notation: $Y_t = [\ln \delta_t, z_t]'$ and $\mu_t = [\mu_\delta(t), \mu_z(t)]'$, so that⁵

$$\begin{aligned} dY_t &= (A_0 + A_1\mu_t)dt + BdW_Y(t) \\ d\mu_t &= (a_0 + a_1\mu_t)dt + b dW_\mu(t). \end{aligned}$$

Given the difference in prior expected drifts and $m_Y^n = E[\mu(t) | F_{Y_t}^n]$, each agent's beliefs are affected by different innovation processes dW_Y^n . More specifically,

⁴ For a discussion on how to endogenize the differences in beliefs, see Morris (1995).

⁵ The matrix-form representation is obtained by setting $A_0 = [0, 0]'$, and $A_1 = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$, $B = [\sigma_\delta, \sigma_z] \times I_2$, $dW_Y(t) = [dW_\delta(t), dW_z(t)]'$, $a_0 = [a_{0\delta}, a_{0z}]$, $a_1 = [a_{1\delta}, a_{1z}] \times I_2$, $b = [n_\delta, n_z] \times I_2$, $dW_\mu(t) = [dW_{\mu_\delta}(t), dW_{\mu_z}(t)]$.

since Y_t is commonly observable, the perceived innovations dW_t^n and drifts $m^n(t)$ must be such that $Y_t^n = Y_t$, so that

$$\begin{aligned} dW_Y^n(t) &= B^{-1} [dY_t - (A_0 + A_1 m^n(t)) dt] \\ &= dW_Y(t) + B^{-1}(\mu_t - m^n) dt, \quad n = 1, 2. \end{aligned} \quad (4)$$

which implies in turn that

$$dW_Y^2(t) = dW_Y^1(t) + \psi_t dt, \quad \psi_t = B^{-1}(m_t^1 - m_t^2). \quad (5)$$

The process ψ_t is the disagreement between the two agents about the growth rates of the observable processes scaled by their respective volatilities. It plays an important role in the characterization of the equilibrium because it parameterizes the difference in the way the two agents process new information. That is, agents may interpret the same innovation in the signal $z(t)$ differently; those with low estimates $m_z(t)$ may interpret a positive shock to $z(t)$ as good news for the dividend growth rate, while those with high estimates for $m_z(t)$ may revise their estimates of the dividend growth downwards if the change in the signal is not large enough. The stochastic process ψ_t also has another useful interpretation. It is the Radon-Nykodim derivative that describes the change in the subjective probability measures of the two agents.

Market Completeness. Uncertainty with respect to the dividend growth rate and the correlation between the signal $z(t)$'s drift and the $\delta(t)$'s drift makes observed innovations in $z(t)$ affect the posterior estimate of the dividend process. Thus, the state-space that affects security pricing is generated by a two-dimensional Brownian process dW_δ^n, dW_z^n . This implies that a market with continuous trading in the stock and a risk-free bond is incomplete. In this framework, uncertainty generates an equilibrium demand for another asset. We introduce options and study how changes in the differences in beliefs ψ_t (on either the dividend or the signal) generate trading and affect equilibrium asset prices. Thus, in what follows, we assume that agents can trade in three assets: a riskless bond, a stock and an option. The bond price follows the process $dB(t) = B(t)r(t)dt$. The stock and option prices evolve according to the general stochastic dynamics

$$\begin{aligned} dP(t) &= P(t) [\mu_p(t) dt + \sigma_p(t)' dW_Y] \\ dO(t) &= O(t) [\mu_o(t) dt + \sigma_o(t)' dW_Y], \end{aligned} \quad (6)$$

where the expected return and volatility of the securities are endogenously determined in equilibrium. Due to uncertainty, agents have different perceived innovation processes dW_Y^n , thus they confront different *subjective* price processes with drifts μ_p^n and μ_o^n , which have the Brownian innovation dW_Y^n . Since in equilibrium agents need to agree on security prices and the Brownian innovations dW_Y^n affect both dY_t and security prices, agents must disagree on security expected returns. However, the difference in the two agents' security expected returns is closely linked to the differences in beliefs on fundamentals

$\psi(t)$. By substituting equation (4) into (6), it is easy to show that the difference in the perceived rates of return is

$$\mu_p^1(t) - \mu_p^2(t) = \sigma_p' \psi(t)$$

$$\mu_o^1(t) - \mu_o^2(t) = \sigma_o' \psi(t).$$

The differences in expected asset returns are equal to the differences in beliefs multiplied by the asset volatility. This clearly suggests that the differences in beliefs must also affect equilibrium risk premia. When option markets are open, the economy is dynamically complete under each perceived security price process. Thus, there exists a *unique* stochastic discount factor $\xi^n(t)$ for each of the two agents. Under no arbitrage, it follows that

$$\frac{d\xi^n(t)}{\xi^n(t)} = -r(t)dt - \theta^n(\delta(t), z(t))dW_Y^n.$$

The market prices of the risk factor θ^n are individually specific and depend on both $\delta(t)$, as in a Lucas tree economy, and $z(t)$. Because agents are uncertain about dividend growth rates, they use the signal to make inferences. However, because the nature of the relationship between the signal and the dividend growth rate is stochastic, agents are uncertain about their interpretation of the signal. This uncertainty is priced in equilibrium: θ^n is a function of $z(t)$. This implies that asset prices can be volatile even if the dividend process is smooth, which is a desirable property given the well-known difficulty in reconciling the volatility of the stochastic discount factor with market fundamentals, as discussed in the risk premium puzzle literature (see, e.g., Hansen and Jagannathan (1997)).

Let us define $\pi_n(t) = (\pi_{n1}(t), \pi_{n2}(t))^T$ as the units of the consumption good that agent n invests in each security (stock and option respectively). We define the equilibrium concept as follows.

DEFINITION 1 (Equilibrium): *An equilibrium is a collection of price processes $(P(t), O(t), \text{ and } r(t))$ along with individual consumption and portfolio choices $(c_n(t), \pi_n(t))$ such that:*

- (i) *Given equilibrium prices $(P(t), O(t), \text{ and } r(t))$, consumption and portfolio holdings $(c_n(t), \pi_n(t))$ maximize each individual utility function subject to the budget constraints.*
- (ii) *The consumption good, the stock and the option markets clear, that is,*

$$c_1(t) + c_2(t) = \delta(t), \quad \pi_{11}(t) + \pi_{21}(t) = 1, \quad \pi_{21}(t) + \pi_{22}(t) = 0.$$

where $\pi_{n1}(t)$ are the stock holdings and $\pi_{n2}(t)$ are the option holdings for agent $n = 1, 2$.

To compute the equilibrium, we use the approach discussed in Detemple (1986). First, we solve for the optimal learning problem of each individual. Given this solution, each agent acts as a standard utility maximizer in a world with complete information. Second, we use the individual-specific filtered dynamics

of $\delta(t)$ and $z(t)$ to solve for the optimal portfolio problem. Finally, we aggregate the two optimal portfolio solutions and use the market clearing conditions to obtain equilibrium asset prices.

Since the drifts of $\delta(t)$ and $z(t)$ are unobservable, individuals rationally compute their best estimates $m_\delta^n(t)$ and $m_z^n(t)$ using their priors and all available information. Using standard results in filtering theory (see Theorems 12.6 and 12.7 in Liptser and Shiryaev (2001)), it is possible to prove the following results.

LEMMA 1 (Learning):

- (a) Let $m(t) = E[\mu(t) | \mathcal{F}_t^Y]$ and $\gamma(t) = E[(\mu_t - m_t)(\mu_t - m_t)' | \mathcal{F}_t^Y]$. Under some technical regularities, m_t and γ_t are unique, continuous $\mathcal{F}_t^Y$ -measurables for any t solutions of the system of equations

$$\begin{aligned} dm_t &= [a_0 + a_1 m_t] dt + \gamma_t A_1' (BB')^{-1} [dY_t - (A_0 + A_1 m_t) dt] \\ \dot{\gamma}_t &= a_1 \gamma_t + \gamma_t a_1' + bb' - \gamma_t A_1' (BB')^{-1} A_1 \gamma_t \end{aligned} \quad (7)$$

with initial conditions $m_0 = E(\mu_0 | \mathcal{F}_0^Y)$ and $\gamma_0 = E[(\mu_0 - m_0)(\mu_0 - m_0)' | \mathcal{F}_0^Y]$. If the matrix γ_0 is positive definite, then the matrices γ_t , $0 \leq t \leq T$, will be positive definite as well.

- (b) Moreover, if $\sum_{i=1}^2 E(\mu_i)^4 < \infty$ and if the initial belief $P(\mu_0 \leq a | Y_0)$ on the drift μ_t is conditionally Gaussian $N(m_0, \gamma_0)$, then for any t_j , $0 \leq t_0 < t_1 \cdots < t_n \leq t$, the conditional distribution $P(\mu_{t_0} \leq a_0, \dots, \mu_{t_n} \leq a_n | \mathcal{F}_t^Y)$ is Gaussian.

The posterior belief on the dividend growth rate is stochastic and depends on future realizations of the observable variables Y_t . The sensitivity of the revision in the posterior belief depends on the “confidence” level γ_t . The lower the γ_t , the higher the confidence in the initial prior and thus the lower the revision of the posterior, given new observations on Y_t . The reverse is true for the less confident agent. Note that we can express the estimated drift dynamics as

$$\begin{aligned} dm_t &= (a_0 - GA_0 + (a_1 - GA_1)m_t) dt + G dY_t \\ G &= \gamma_t A_1' (BB')^{-1}. \end{aligned} \quad (8)$$

A convenient solution for γ can be obtained using the stationary solution S of γ , which satisfies $\dot{\gamma}_t = 0$ for $\gamma_t = S$. This solution satisfies a well known system of Riccati equations that can be solved using standard methods.

Remark 1: In the special case in which agents have heterogeneous initial beliefs $m^i(0) \neq m^j(0)$ but identical prior variances, that is $\gamma^i(0) = \gamma^j(0)$, then the differences in beliefs $\psi(t) = B^{-1}(m_t^i - m_t^j)$ as defined in (5) follow an ordinary differential equation (ODE). Using the expression in (8), we have

$$d\psi_t = B^{-1}(dm_t^i - dm_t^j) = B^{-1}(a_1 - GA_1)B\psi_t dt.$$

The stochastic components of the stochastic differential equation in (8) then cancel out, so that

$$d\psi(t) = H\psi(t)dt$$

with $H = B^{-1}(a_1 - GA_1)B$, given the initial conditions $\psi_\delta(0)$ and $\psi_z(0)$. Hence, the difference in beliefs process satisfies a linear system of ordinary differential equations with constant coefficients. The solution is therefore

$$\psi(t) = C_1 h_1 e^{\phi_1 t} + C_2 h_2 e^{\phi_2 t},$$

where ϕ_1 and ϕ_2 are eigenvalues of the matrix H , $h_1 = [h_{1,\delta}, h_{1,z}]'$ and $h_2 = [h_{2,\delta}, h_{2,z}]'$ are its eigenvectors, and C_1 and C_2 are constants, determined by the initial conditions $\psi_\delta(0)$ and $\psi_z(0)$ according to

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{pmatrix} h_{1,\delta} & h_{2,\delta} \\ h_{1,z} & h_{2,z} \end{pmatrix}^{-1} \begin{bmatrix} \psi_\delta(0) \\ \psi_z(0) \end{bmatrix}. \quad (9)$$

Note, however, that when agents have different prior variances, that is $\gamma^1(0) \neq \gamma^2(0)$, the stochastic components in (8) do not cancel out and the difference in beliefs process is stochastic, following

$$d\psi(t) = B^{-1}(a_1\psi_t - [G_t^1 A_1 m_t^1 - G_t^2 A_1 m_t^2]) + (G_t^1 - G_t^2) dY_t, \quad (10)$$

where $G^i = \gamma^i A_1' (BB')^{-1}$.

Remark 2: Our model specification is closely related to behavioral models that investigate the role of overconfidence in asset prices. For instance, Scheinkman and Xiong (2003) study a partial equilibrium, two-date model in which two sets of agents have the same initial prior on dividend growth but have different degrees of confidence in the accuracy of the signal. Thus, even if at time $t = 0$ the two agents have common priors, the different interpretation of the signal induce them to obtain different posteriors at any time $t > 0$. In our setting, this can be modeled by assuming $m^A(0) = m^B(0)$ but a different set of parameters β and/or γ (thus A_1 and b). If agent A assumes a low value of β , then he will take less into account realizations of the signal in computing $m^A(t)$, for $t > 0$. A similar effect is also generated when agent A assumes a higher signal volatility b relative to B . In the behavioral finance literature this agent is called “overconfident” and his posterior is less sensitive to realizations of the signal. Our approach is, however, different from Scheinkman and Xiong (2003) because we consider a dynamic model in which the agent can dynamically hedge and implement state contingent risk sharing contracts. We characterize the effect of differences in opinions in the absence of short-selling constraints.

Given the differences in beliefs $\psi_\delta(t)$ and $\psi_z(t)$ and the process dY_t , we can solve for the equilibrium. Note, however, that because agents have heterogeneous beliefs about the dynamics of the two underlying factors, they have

different state price densities. Each state price density is a function of the individual specific beliefs. To solve for the equilibrium it is convenient to use the following aggregation technique of Cuoco and He (1994) and Karatzas and Shreve (1998) in which the representative agent utility function is constructed by taking a (state-dependent) weighted average of individual utilities:

$$U(c, \lambda) = \max_{c_1+c_2=c} c_1(t)^\gamma / \gamma + \lambda_t (c_2(t)^\gamma / \gamma). \quad (11)$$

The first-order conditions for the optimal consumption plan of agent n give $c_n(t) = (y_n \xi^n(t))^{1/\gamma-1}$, where y_n is the (constant) Lagrange multiplier associated with the budget constraint (2). Thus, each agent's Lagrange multiplier y_n must satisfy

$$E^n \left[\int_0^T \xi^n(t) (y_n \xi^n(t))^{1/\gamma-1} dt \middle| \mathcal{F}_t^n \right] = X_0^n, \quad (12)$$

with $X_0^n = e^n P_1(0)$ being the (endogenous) value of the initial stock endowment. Imposing the market clearing condition $c_1(t) + c_2(t) = \delta(t)$, we obtain the equilibrium consumption allocation of the two agents and the two stochastic discount factors. We summarize the results in the following proposition.

PROPOSITION 1 (Equilibrium): *In equilibrium, the individual state price densities are equal to:*

$$\xi^1(t) = \frac{1}{y_1} [\delta(t)]^{\gamma-1} \lambda_t \left(1 + \lambda_t^{\frac{1}{\gamma-1}}\right)^{1-\gamma} \quad \xi^2(t) = \frac{1}{y_2} [\delta(t)]^{\gamma-1} \left(1 + \lambda_t^{\frac{1}{\gamma-1}}\right)^{1-\gamma}, \quad (13)$$

and the relative weight of the second agent λ_t is state dependent and equal to $\lambda_t = \frac{y_1}{y_2} \eta_t$, with $\eta_t = \xi^1(t)/\xi^2(t)$. The two constants y_n satisfy the individual static budget constraints (12). The ratio of the two agents' state price densities evolves according to

$$\frac{d\eta(t)}{\eta(t)} = -\psi_\delta(t) dW_\delta^1(t) - \left(a\psi_\delta(t) \frac{\sigma_\delta}{\sigma_z} + b\psi_z(t) \right) dW_z^1(t). \quad (14)$$

Note that $\frac{d\lambda_t}{\lambda_t} = \frac{d\eta_t}{\eta_t}$. Finally, the individual optimal consumption allocations are given by

$$c_1(t) = \delta(t) \frac{(y_1 \eta(t)/y_2)^{\frac{1}{\gamma-1}}}{\left(1 + (y_1 \eta(t)/y_2)^{\frac{1}{\gamma-1}}\right)}; \quad c_2(t) = \delta(t) \frac{1}{\left(1 + (y_1 \eta(t)/y_2)^{\frac{1}{\gamma-1}}\right)}. \quad (15)$$

Proof: See Appendix A.

The main result above is that the relative consumption share of the two agents is stochastic even if agents have identical CRRA preferences and the dividend volatility is deterministic. The consumption share $c_1(t)/\delta(t)$ of the first agent

depends on λ_t , which is proportional to the ratio of the two state price densities $\eta(t)$. Thus, it is a function of the difference in beliefs ψ_δ and ψ_z and is stochastic, with a law of motion that is described by (14).

To gain intuition about the role of differences in beliefs on the dynamics of asset prices, consider the case of a negative dividend shock. Let $\psi_\delta > 0$, that is $m_\delta^1(t) > m_\delta^2(t)$, so that the first agent is more optimistic about the dividend growth rate than is the second agent. A negative dividend shock $d\hat{W}_\delta^n(t) < 0$ has two effects. First, the overall consumption level decreases. Second, $\eta(t)$ increases, making the second (pessimist) agent's weight larger and increasing his relative consumption share. This second effect plays a crucial role in our analysis as differences in beliefs have direct, real effects on intertemporal risk-sharing among agents and, therefore, on asset prices.

To investigate the asset pricing implications, it is instructive to describe the market prices of risk. The following proposition describes the main result.

PROPOSITION 2: *In equilibrium, agents are averse to both dividend and signal shocks. The two agent-specific prices of risk are equal to*

$$\begin{aligned}\theta_\delta^1(t) &= (1 - \gamma)\sigma_\delta + \frac{\psi_\delta(t)(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}}{(1 + (y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}})}; \\ \theta_\delta^2(t) &= (1 - \gamma)\sigma_\delta - \frac{\psi_\delta(t)}{(1 + (y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}})} \\ \theta_z^1(t) &= \frac{(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}} \left(a\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\psi_z(t) \right)}{(1 + (y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}})}; \quad \theta_z^2(t) = -\frac{\left(a\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\psi_z(t) \right)}{(1 + (y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}})}.\end{aligned}\tag{16}$$

Proof: See Appendix A.

The price of dividend risk. The market price of dividend risk $\theta_\delta^1(t)$ is equal to the sum of two terms. The first term is standard and is equal to the product between the dividend volatility and the coefficient of relative risk aversion. The second term is new. When agents have different beliefs, the Pareto weighting process is stochastic and its volatility is proportional to the difference in beliefs ψ_δ . From equation (14), we see that when ψ_δ is positive, a negative shock to dividends, $dW_\delta(t)$, increases the Pareto weight of the second (pessimistic) agent. The effect is proportional to the differences in beliefs ψ_δ . This means that the consumption of the first agent is reduced not only because aggregate dividends decline but also because his relative consumption allocation is reduced: The first (optimistic) agent ex ante insures the second (pessimistic) agent. The extent of this effect is proportional to the difference in beliefs multiplied by the consumption share of the second agent, that is $\frac{c_2(t)}{\delta(t)}\psi_\delta(t)$.

The price of the signal risk. In an economy with homogeneous agents, that is $\psi_\delta = \psi_z = 0$, the signal is used by each agent to update the estimate of the dividend growth rate. However, it is not priced. The reason is that in such an economy, $\text{Cov}(dW_z, d\eta) = 0$: Unexpected changes in the signal are not correlated with marginal utilities. When people have heterogeneous beliefs, however, the economy supports an equilibrium in which the signal itself has a positive market price of risk. This may seem surprising at first since the signal does not separately affect consumption. However, agents are averse to innovations in the volatility since it changes the state-price density ratio and therefore their consumption share. The market price of signal risk of the first agent is proportional to the consumption share of the second agent and the level of the differences in beliefs, that is, $\theta_z^1(t) = \frac{c_2(t)}{\delta(t)}[\alpha\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(t)]$.

It is also instructive to note that the difference in the two prices of risk for each factor is proportional to the difference in beliefs, that is,

$$\theta_\delta^1(t) - \theta_\delta^2(t) = \psi_\delta(t); \quad \theta_z^1(t) - \theta_z^2(t) = \left(a\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\psi_z(t) \right).$$

When $\psi_\delta = 0$, the dividend price of risk becomes agent independent. When both $\psi_\delta = 0$ and $\psi_z = 0$, the signal is no longer a priced source of risk and $\theta_z^1(t) = \theta_z^2(t) = 0$.

A. Asset Pricing

Given the stochastic discount factor $\xi^1(s)$, the stock price is equal to

$$P(t) = \frac{1}{\xi^1(t)} E_t^1 \left[\int_t^T \xi^1(s) \delta(s) ds \right].$$

In equilibrium, asset prices can be computed using the expectation operator and the stochastic discount factor of either agent. For simplicity we consider the pricing from the first agent's perspective. European stock options expiring at time H with final payoffs equal to $C(H, \delta(H)) = \max(P(H) - K, 0)$ can be valued using the same stochastic discount factor, so that

$$C(t, H) = E_t^1[\xi^1(H) \max(P(H) - K, 0)]/\xi^1(t).$$

The term structure of bond prices paying a unit of the numéraire at time H is given by $Q(t, H) = E_t^1[\xi^1(H)]/\xi^1(t)$.

The pricing kernel depends on both the cash flow process $\delta(t)$ and the difference in beliefs $\psi(t)$, which in turn affects the stochastic weight process $\eta(t)$. Asset prices can be obtained either numerically using the results in Proposition 1 or in closed-form under the assumption that agents have common variance prior distributions. In this case, the difference in beliefs follows an ODE (Remark 1) and it is possible to show (see Lemma A.1. in the Appendix) that the joint distribution of $\delta(t)$ and $\eta(t)$ is log normal. Using this result and Proposition 1,

we obtain the representation for stock, option, and bond prices as given in Proposition 3.

PROPOSITION 3 (Asset Pricing): *Let $N_2(z)$ be a standardized bivariate normal distribution. When the agents' prior distributions have common variance, the equilibrium stock price is given by*

$$P(t) = \frac{1}{\xi^1(t)} \int_{-\infty}^{+\infty} \left[\int_t^T F_\eta(t, s, Z_\eta, Z_\delta) F_\delta(t, s, Z_\eta, Z_\delta)^\gamma ds \right] d\mathcal{N}_2(Z).$$

European call option prices with time to maturity H are equal to

$$C(t, H)$$

$$= \frac{1}{\xi^1(t)} \int_{-\infty}^{+\infty} [F_\eta(t, H, Z_\eta, Z_\delta) F_\delta(t, H, Z_\eta, Z_\delta)^{\gamma-1} \max(P(H) - K, 0)] d\mathcal{N}_2(Z)$$

The bond price is equal to

$$Q(t, H) = \frac{1}{\xi^1(t)} \int_{-\infty}^{+\infty} [F_\eta(t, H, Z_\eta, Z_\delta) F_\delta(t, H, Z_\eta, Z_\delta)^{\gamma-1}] d\mathcal{N}_2(Z),$$

where $F_\eta(t, s, z)$ and $F_\delta(t, s, z)$ are deterministic functions of the 2×1 vector Z . See the Appendix for their description.

Proof: See Appendix A.

Given the functional form of F_η and F_δ , asset prices are deterministic integrals that can be computed at any desired level of accuracy using standard numerical integration methods.

B. Optimal Portfolio Holdings and Volume

We can obtain optimal portfolio holdings by equating the dynamic budget constraint to its martingale representation (Cox and Huang (1989)). First we calculate the first agent's optimal position. Then, we derive that of the second agent from the asset market clearing conditions (the stock's net supply is one unit, while the option's is zero). The dynamic budget constraint satisfies the stochastic differential equation

$$\begin{aligned} dX^1(t) = & -c_1(t)dt + X^1(t)r(t)dt + \pi_1(t)^\top (\mu_i^1(t) - r(t))dt \\ & + \pi_1(t)^\top \sigma(t) dW_Y^1(t), \end{aligned} \tag{17}$$

where $\pi_1(t) = (\pi_{11}(t), \pi_{12}(t))^\top$ is the number of units of the consumption good invested in each security (stock and option, respectively) by the first agent and $dW_Y^1(t) = (dW_\delta^1(t), dW_z^1(t))^\top$. Given the equilibrium stochastic discount factor, at any time $t < T$ the wealth level must also satisfy the static budget constraint

$X^1(t) = \frac{1}{\xi^1(t)} E^1[\int_t^T c_1(s) \xi^1(s) ds | \mathcal{F}_t^1]$. Since $c^n(t)$ and $\xi^n(t)$ are functions of the state variables $\delta(t)$ and $\eta(t)$, Ito's Lemma applied to the static budget constraint yields

$$dX^1(t) - E[dX^1(t) | \mathcal{F}_t^1] = \frac{\partial X^1}{\partial \delta} \sigma_\delta \delta(t) dW_\delta^1(t) - \frac{\partial X^1}{\partial \eta} \psi_\delta(t) \eta(t) dW_\delta^1(t) - \frac{\partial X^1}{\partial \eta} \left(a \psi_\delta(t) \frac{\sigma_\delta}{\sigma_z} + b \psi_z(t) \right) \eta(t) dW_z^1(t). \quad (18)$$

The stochastic differential equations (17) and (18) are Ito representations of the same wealth process with respect to the same vector of Brownian motions. Thus, the factor loading on the Brownian motions must be identical in any state of the world. This yields the following system of two equations with two unknowns:

$$\pi_1(t)^\top \sigma(t) = \begin{bmatrix} \frac{\partial X^1}{\partial \delta} \sigma_\delta \delta(t) - \frac{\partial X^1}{\partial \eta} \psi_\delta(t) \eta(t), & -\frac{\partial X^1}{\partial \eta} \left(a \psi_\delta(t) \frac{\sigma_\delta}{\sigma_z} + b \psi_z(t) \right) \eta(t) \end{bmatrix},$$

where $\sigma(t) = \begin{pmatrix} \frac{\partial P(t)}{\partial \delta} \sigma_\delta \delta(t) - \frac{\partial P(t)}{\partial \eta} \psi_\delta(t) \eta(t) & -\frac{\partial P(t)}{\partial \eta} \left(a \psi_\delta(t) \frac{\sigma_\delta}{\sigma_z} + b \psi_z(t) \right) \eta(t) \\ \frac{\partial O(t)}{\partial \delta} \sigma_\delta \delta(t) - \frac{\partial O(t)}{\partial \eta} \psi_\delta(t) \eta(t) & -\frac{\partial O(t)}{\partial \eta} \left(a \psi_\delta(t) \frac{\sigma_\delta}{\sigma_z} + b \psi_z(t) \right) \eta(t) \end{pmatrix}.$

Solving for the portfolio holding, we obtain the following result for the first agent's optimal position.

PROPOSITION 4 (Stock and Option Holding): *The optimal portfolio holding is*

$$\pi_1(t) = \begin{bmatrix} \pi_{1,1}(t) \\ \pi_{1,2}(t) \end{bmatrix} = [\sigma(t)_{(4 \times 4)}^\top]^{-1} \begin{bmatrix} \frac{\partial X^1}{\partial \delta} \sigma_\delta \delta(t) - \frac{\partial X^1}{\partial \eta} \psi_\delta(t) \eta(t) \\ -\frac{\partial X^1}{\partial \eta} \left(a \psi_\delta(t) \frac{\sigma_\delta}{\sigma_z} + b \psi_z(t) \right) \eta(t) \end{bmatrix}. \quad (19)$$

The partial derivatives of the first agent's wealth, stock price, and option price with respect to the dividend and stochastic weights are fully characterized in the Appendix.

Proof: See Appendix A.

Equation (19) shows that the optimal portfolio holdings are functions of both $\delta(t)$ and $\eta(t)$ even when the difference in beliefs ψ is deterministic (see Remark 1). Moreover, differences in beliefs ψ_δ and ψ_z directly affect the optimal portfolio composition. A positive trading volume can obtain, even in the absence of dividend shocks, following innovations in either ψ_δ or ψ_z . Given $\pi(t)$, we can calculate the trading volume in each security as the changes over time in the optimal portfolio holdings. For convenience, let us define the open interest

as the total value invested in options scaled by the underlying asset's value, $\widehat{OI}_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i) = \pi_{1,2}(t)/P(t)$, where $\pi_{1,2}(t)$ is the portfolio holdings (in consumption units) invested in options as derived in equation (19).

To gain intuition on the pricing implications of the model, we calibrate μ_δ and σ_δ to the growth rate and volatility of the S&P 500 dividend from 1950 to 2000, obtaining 3% and 5%, respectively, and plot equilibrium asset pricing properties as a function of the difference in beliefs. We assume that the volatility of the signal equals that on dividends, that is, $\sigma_z = 5\%$. Moreover, we calibrate the volatility of the stochastic drift of the dividend and signal processes, which are a proxy of model uncertainty, to half the volatility of their levels, that is $n_\delta = n_z = 2.5\%$.⁶ We assume a relatively small value for the risk aversion coefficient, that is $\gamma = -\frac{1}{2}$. In a log-utility economy, $\gamma = 0$ and differences in beliefs do not affect stock prices. When $\gamma < 0$, agents are more risk averse than a log-utility investor.

B.1. Optimal Portfolio Holdings

Figure 1, Panels A and B show how changes in the differences in beliefs impact the optimal portfolio holdings of at-the-money call options. When both differences in beliefs ψ_δ and ψ_z are zero, agents are identical and each holds his initial stock endowment. When the differences in beliefs increase, the first agent's growth rate estimate increases compared to that of the second agent. An increase in the difference in beliefs about the dividend from zero to one increases stock holdings by 8% and options holdings from 0% to 2% of wealth. This range of values is comparable to an actual market capitalization of S&P 500 options of about 3% of the S&P 500 market capitalization. A change in the signal difference in beliefs from zero to 1 increases the option holdings by 0.5%. Options holding reacts less to the signal difference in beliefs ψ_z than to the dividend difference in beliefs ψ_δ . It is possible to see that the difference in beliefs creates a natural demand for insurance. When $\mu_\delta^1 > \mu_\delta^2$, the first agent is more optimistic about the dividend growth rate and acts as an insurer of the second (pessimist) agent's consumption stream by writing OTM puts in exchange for OTM calls. The open interest in the option market signals the extent of risk-sharing between agents with different beliefs.

Our results generalize Leland (1980) and Brennan and Solanki (1981) to an incomplete markets general equilibrium economy with intertemporal consumption. In a complete markets Black-Scholes world, Leland (1980) shows that the optimal strategy of an investor maximizing terminal wealth utility is convex in the terminal wealth if the agent is more optimistic than the aggregate investor. This can be equivalently achieved either by holding a leveraged position in the reference portfolio and buying put options, or by investing directly in call options. In an incomplete market general equilibrium setting, however, when investors care about intertemporal consumption this is no longer true

⁶ This parameter is calibrated from the values obtained by Xia (2001) using a VAR approach.

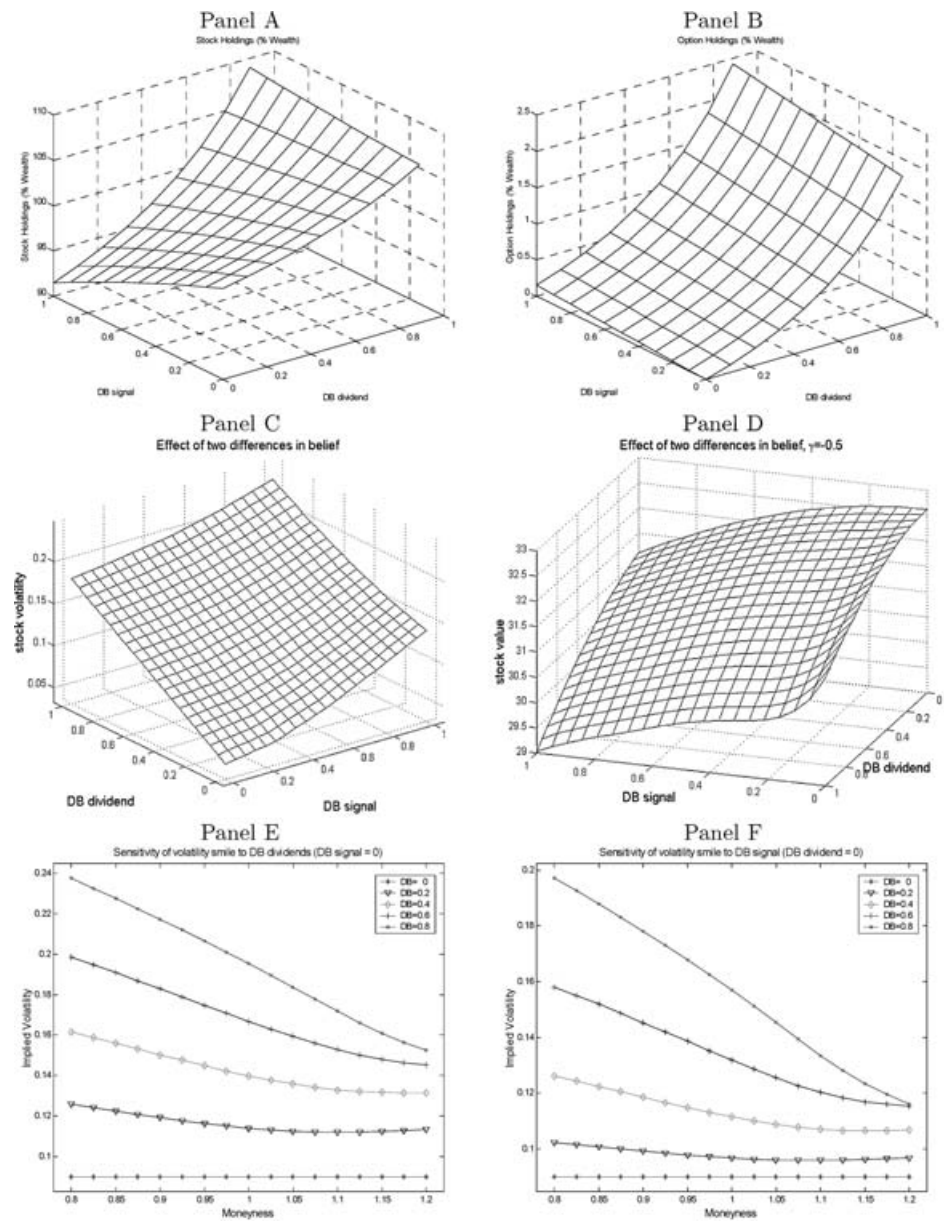


Figure 1. Effect of difference in beliefs (DB). The figure presents how differences in beliefs of both ψ_z , the signal DB, and ψ_δ , the dividend DB, affect the optimal portfolio holdings (Panels A and B), the stock volatility (Panel C), the stock price (Panel D), and the implied volatility smiles (Panels E and F). In Panels E and F, we fix one of the differences in beliefs at zero to illustrate the convergence to Black-Scholes (1973) and change the other difference in beliefs to a value between zero and 1, solving for the price of options with different moneyness. Time to maturity is fixed at 1 month. In Panels A and B, we change both differences in beliefs, resolve for the equilibrium solution, and present the portfolio holdings. Both holdings are calculated for the first agent (optimist) and are expressed in terms of wealth. Option holdings are expressed in terms of the option notional value multiplied by option delta.

since they can engage in state-dependent risk-sharing. This is assumed away in both Leland (1980) and Brennan and Solanki (1981) settings. Since in equilibrium the expected marginal utilities of the two agents must equate, that is $E^A[U'(c_A(t)) | \mathcal{F}_t^A] = E^B[U'(c_B(t)) | \mathcal{F}_t^B]$, the optimist needs to consume proportionally more in good states of the world and less in bad states of the world. The resulting strategy is neither convex nor concave since it involves, if the market is completed by options, purchasing an OTM call option and writing an OTM put option.

B.2. Stock Volatility and Equity Premium

State-dependent risk-sharing has additional pricing implications in equilibrium as the time variation in the difference in beliefs generates endogenous stochastic volatility. We obtain the formal relationship between the stock volatility and $\psi(t)$ by applying Ito's Lemma to Proposition 3.⁷ Figure 1, Panel C shows that when both differences in beliefs are zero, the stock volatility equals the dividend volatility, which is 5%. An increase in both differences in beliefs from zero to 0.5, however, increases the stock volatility from 5% to 13%, which is close to the average volatility of the S&P 500 index. A further increase in both differences in beliefs to a value of one increases the stock volatility to 25%. The difference in beliefs about the dividend more strongly affects the stock volatility than that about the signal. The economic intuition linking the differences in beliefs and endogenous stochastic volatility is as follows: Because differences in beliefs make each agent have their own specific stochastic discount factor $\xi^n(t)$ and the relative weight $\eta(t)$ of the two agents in the representative agent utility function is equal to the ratio of the two stochastic discount factors, the volatility of $\eta(t)$ is proportional to the difference in beliefs (see equation (14)). The consumption process and the investment decision in the stock is directly affected by the difference in beliefs even in the absence of dividend shocks (see equation (15)). This implies that the stock volatility is a function of the volatility of $\eta(t)$ and thus of the levels of ψ_δ and ψ_z , that is, $\sigma_p = \frac{\partial P}{\partial \eta} \sigma_\eta$.

The time variation in the difference in beliefs can reconcile relatively high stock volatility with low dividend volatility. However, can the model match the observed excess returns? The required excess return on a risky security is $\sigma_{i\delta}(t)\theta_\delta^n(t) + \sigma_{iz}(t)\theta_z^n(t)$. When both differences in beliefs are zero, the only priced source of risk is the dividend innovation. Furthermore, the price of risk is linear in the coefficient of risk aversion $-\gamma$, as in the standard single-agent CRRA utility model. In this case, the model can generate an excess return of 8% with the low observed sample dividend and price volatilities only at the cost of assuming

⁷ $dP(t)/P(t) = \mu_p^1(t)dt + \sigma_{p\delta}(t)dW_\delta^1(t) + \sigma_{pz}(t)dW_z^1(t)$, where $\sigma_{p\delta}(t)$ equals $(\partial P/\partial \delta)\delta(t)\sigma_\delta - (\partial P/\partial \eta)\psi_\delta(t)\eta(t)$ and $\sigma_{pz}(t)$ equals $(\partial P/\partial \eta)(a\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\psi_z(t))\eta(t)$. Thus, the total stock volatility is equal to

$$\sigma^2\left(\frac{dP}{P}\right) = \left(\frac{\partial P}{\partial \delta}\delta(t)\sigma_\delta - \frac{\partial P}{\partial \eta}\psi_\delta(t)\eta(t)\right)^2 + \left(\frac{\partial P}{\partial \eta}\left(a\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\psi_z(t)\right)\eta(t)\right)^2.$$

a coefficient of risk aversion of $\gamma = -24$. When the difference in beliefs increases from zero to 1, however, the price of dividend risk θ_δ increases six-fold and the sample equity risk premium can be matched even with $\gamma = -2$. This occurs because a negative dividend realization not only reduces the cash flow stream, but it also affects the marginal valuation through the stochastic weight process $\eta(t)$. The higher the difference in beliefs, the higher the effect of a dividend shock on $\eta(t)$ and the higher the price of risk.⁸

B.3. Stock Prices

Figure 1, Panel D shows that the difference in beliefs negatively affects the stock price level: The greater the difference in beliefs is (either about the dividend or about the signal), the smaller the stock price is. An increase in the difference in beliefs about the dividend (or signal) from zero to 1 drives the value of the stock down by about 10%, from 33 to 30 consumption units. The intuition for this is that an increase in the difference in beliefs generates both a higher expected stock growth rate and a higher stock price volatility. When $\gamma < 0$, at the calibrated parameters the higher dividend growth rate does not compensate for the increase in volatility. Furthermore, the equilibrium stock price declines as the stock price becomes more heavily influenced by the (pessimistic) agent whose estimate of the dividend growth rate is lower than the actual value. The situation reverses for positive values of γ (lower risk aversion than log-utility).

B.4. Option Prices

The difference in beliefs generates an option-implied smile. Figure 1, Panels E and F illustrate how the difference in beliefs impacts the option-implied volatility surface. Setting both differences in beliefs to zero yields a standard one-factor Black-Scholes (BS) economy. The volatility smile is flat since in this case the stock price transition density is log normal (see the Appendix). When only one of the two differences in beliefs is nonzero, options become nonredundant. The effect of the differences in beliefs is asymmetric, however. In-the-money calls (OTM puts) are more expensive than OTM calls (in-the-money puts). When the difference in beliefs about the dividend process increases from 0 to 0.6, the smile level (at-the-money volatility) increases from 7% to 17%. The

⁸ In equilibrium, the required excess return is equal to

$$\mu_P^1(t) - r(t) = (1 - \gamma) \text{cov} \left(\frac{dP(t)}{P(t)}, \frac{d\delta(t)}{\delta(t)} \right) - \frac{(y_1 \eta(t)/y_2)^{\frac{1}{1-\gamma}}}{(1 + (y_1 \eta(t)/y_2)^{\frac{1}{1-\gamma}})} \text{cov} \left(\frac{dP(t)}{P(t)}, \frac{d\eta(t)}{\eta(t)} \right).$$

The first term is the standard consumption CAPM, where the security's risk premium is proportional to the covariance between the stock return and the aggregate consumption. In the second term, the risk premium is decreasing in the covariance between this asset and the weight process. When $\eta(t)$ is high, the representative agent puts less weight on the first agent. An asset negatively correlated with changes in $\eta(t)$ is more valuable to the first agent, therefore requiring a smaller risk premium. For the second agent the situation is reversed.

slope of the volatility smile is significant and similar to the one commonly observed in the market: a 20% OTM put is priced at 20% implied volatility while the OTM call is priced at 15% implied volatility. The difference in beliefs affects the signal on the volatility smile similarly, though with smaller magnitude. The smile level increases from 7% to 14% when the difference in beliefs about the signal increases from 0 to 0.6. The corresponding smile slope is about 3.5%. The difference in beliefs creates a natural demand for insurance. When $\mu_\delta^1 > \mu_\delta^2$, so that $\psi_\delta > 0$, trade occurs and the first agent (optimist) insures the consumption stream of the second agent (pessimist) writing OTM puts in exchange for OTM calls. The market value of the two options is not symmetric, however. When $\psi_\delta > 0$, a negative dividend shock $dW_\delta < 0$ increases $\eta(t)$. Thus, it increases the relative consumption share of the pessimist. This implies that negative dividend shocks increase the marginal utility of the optimist more than that of the pessimist. Thus, in equilibrium, the optimist requires a higher price to write OTM puts and hedge the consumption stream of the pessimist than the pessimist requires to write OTM calls. This generates a (stochastic) smile whose slope depends on the extent of the differences in beliefs.

B.5. Skewness

An important stream of the literature analyzes the role of negative skewness in equity returns and the option smile. Ait-Sahalia and Lo (1998), Bakshi, Cao, and Chen (1997), Bates (2000), Duffie, Pan, and Singleton (2000), Madan, Carr, and Chang (1998), Pan (2002), and Rubinstein (1994) investigate the asymmetry in the underlying stock return risk-neutral distribution. Bakshi, Kapadia, and Madan (2003) describe the links among risk-neutral skew, risk aversion and higher-order moments of the physical distribution. Our model generates an endogenous volatility skew under both the risk-neutral and physical measures. Figure 2 summarizes the results. The value of the risk-neutral skewness is close to -1.09 , as reported by Bakshi et al. (2003) for the OEX index. Moreover, we find that the model-implied risk-neutral skewness is higher than the physical skewness, consistent with Bakshi et al. (2003). The magnitude of the physical skewness ranges from zero, when both differences in beliefs are zero, to -0.8 , when both differences in beliefs are equal to 1. This is comparable to the sample historical skewness, which is equal to -0.7 for S&P500 monthly returns.

The intuition for the model-implied skewness is similar to that for the negative slope of the implied volatility smile. When negative stock returns follow a negative dividend shock, the first agent (optimist) receives an additional negative reallocation shock ($d\eta(t) > 0$). His consumption falls, his marginal utility increases, and he becomes more risk averse, demanding a larger risk premium to hold the stock. This reduces the stock price by more than what would be caused simply by a dividend reduction, and this effect generates an asymmetry in the risk-neutral distribution that makes the left tail extend more towards negative values. This is known as negative skewness.

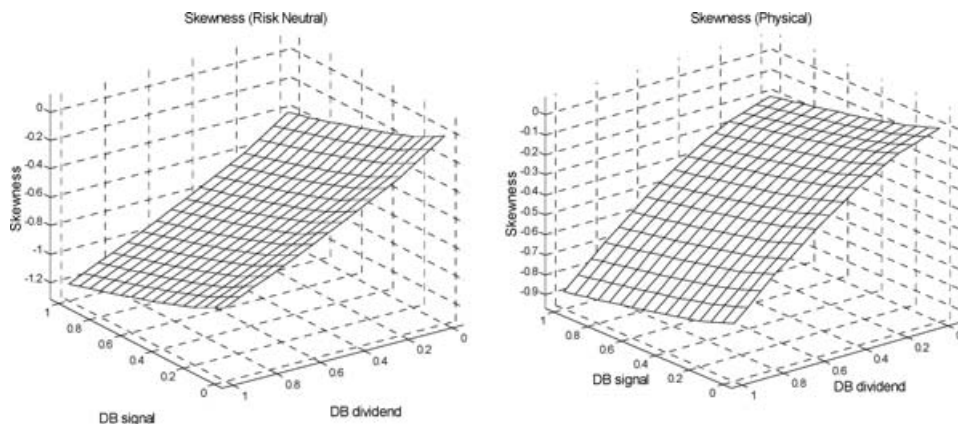


Figure 2. Skewness. These two figures present the skewness of the physical and risk-neutral distributions of stock returns as a function of both ψ_z , the signal difference in beliefs (DB), and ψ_δ , the dividend DB.

II. The Data Set

A. Options Data

The data set includes daily information on the S&P500 index options from October 1986 to August 1996. It contains option trading prices, bid-ask quotes, volumes, open interest, and the underlying index price. A key feature of the data set is the detailed open interest information.

S&P 500 options are among the most actively traded derivative securities in the world. They are European and have no wild card features (see Fleming and Whaley (1994)), they normally expire on the third Friday of the contract month, and the expiration dates follow the March quarterly cycle (March, June, September, December). Originally, these options expired only at the market close and were denoted by the ticker SPX. In June 1987, the CBOE introduced a second set of options that expire at the market open, denoted by ticker NSX. On August 24, 1992 the CBOE reversed the ticker symbols of the two option contracts. Our sample contains SPX options throughout, that is close-expiry options until August 24, 1992, and open-expiry thereafter. Following Dumas, Fleming, and Whaley (1998), for the first subperiod (close-expiry) we measure the option's time to expiration using the number of calendar days between the trade date and the expiration date. For the second subperiod (open-expiry) we use the number of calendar days minus 1. To avoid the problem of bid-ask bounce we follow Ait-Sahalia and Lo (1998) and use the midpoint of the bid and ask quotes.

Three main issues affect any option data set when used for asset pricing purposes. First, in-the-money options are much more illiquid than at- or out-of-the-money options. This reflects the demand of portfolio insurers for out-of-the-money puts. When the volume of in-the-money options is small, the recorded quoted prices are known to be noisy. Second, because the S&P 500 index futures

are traded on the Chicago Mercantile Exchange (CME) while options are traded at the CBOE, reported quotes may not be perfectly synchronized across the two markets. Third, the ex ante future dividend rate is not known since the S&P 500 only provides an ex post series of realized dividends. We address these issues using the same approach as in Ait-Sahalia and Lo (1998), which consists of using the put-call parity and the spot-futures parity to infer the synchronous futures price and the implied future dividend rate, respectively. We obtain interest rates from the LIBOR market. We refer the reader to Ait-Sahalia and Lo (1998) for specific details on these three filters. Finally, we delete any option that violates the basic arbitrage restrictions

$$Se^{-dT} \geq C_i \geq \max(0, Se^{-dT} - Ke^{-rt}) \quad \text{and} \\ e^{-rT}K \geq P \geq \max(e^{-rT}K - Se^{-dT}, 0).$$

These deleted options constitute less than 1% of the data set. We then calculate implied volatilities and eliminate all options with an implied volatility over 100%.

Table I shows the summary statistics for option volume and open interest. Most of the option open interest is concentrated in short-maturity near-the-money contracts beyond which the volume declines quickly. For instance, about 75% of option trading volume is in options with moneyness between 0.98 and 1.02. Table I describes the annual dynamics of the option trading volume for different moneyness levels. Because the trading volume contains a deterministic time trend, we follow Gallant, Rossi, and Tauchen (1992) and detrend the volume time series assuming a quadratic deterministic component.

B. The Difference in Beliefs Index

We use both the Survey of Professional Forecasters⁹ and the Consumer Confidence Survey to construct a proxy for the difference in beliefs about fundamentals sampled at the monthly frequency.

B.1. Survey of Professional Forecasters

The Federal Reserve Bank of Philadelphia conducts a survey of economic variable forecasts (including output, inflation, and interest rates) prepared by private sector economists. On average, 30 forecasters participate in each survey, with the composition relatively stable. The participants in the survey are asked to forecast approximately 27 economic variables over the subsequent five quarters. We focus on GDP, the GDP implicit price deflator, corporate profits after tax, civilian unemployment, industrial production, and the start of new housing units. These are the variables most related to our definition of economic fundamentals.

⁹ David and Veronesi (2001) use the Survey of Professional Forecasters data to approximate the estimation error of the representative agent and the level of uncertainty in their model.

Table I
Open Interest Summary Statistics

This table presents the summary statistics for the put and call option volume and open interest on an annual basis. The average daily volume is measured in terms of the number of contracts traded per day. The average open interest is the average number of contracts outstanding per day. Put option volume and open interest is given in parentheses.

	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995
SPX Trading Volume: Calls (Puts)										
0.78 < K/S < 0.92	328 (432)	899 (2,054)	329 (543)	877 (2,129)	495 (3,953)	1,269 (4,314)	784 (3,929)	236 (1,900)	397 (2,205)	1,262 (5,528)
0.92 < K/S < 0.98	828 (1,466)	2,342 (3,907)	936 (1,148)	2,287 (3,111)	3,391 (6,165)	5,041 (7,898)	3,154 (8,934)	2,982 (14,364)	2,930 (21,218)	6,027 (24,492)
0.98 < K/S < 1.00	609 (677)	2,233 (2,338)	1,494 (1,128)	2,019 (2,074)	3,397 (3,919)	4,434 (5,053)	4,045 (5,475)	5,507 (9,309)	9,044 (18,407)	9,769 (16,401)
1.00 < K/S < 1.02	696 (364)	2,235 (1,371)	1,636 (973)	2,564 (1,732)	3,942 (2,550)	3,924 (2,424)	4,241 (2,612)	8,422 (4,787)	1,3342 (10,437)	1,3366 (5,986)
1.02 < K/S < 1.08	1,180 (271)	2,822 (843)	2,112 (447)	2,810 (666)	6,226 (3,282)	5,295 (1,390)	5,511 (1,134)	8,639 (1,695)	17,749 (4,180)	11,512 (1,241)
1.08 < K/S < 1.20	553 (63)	1,474 (759)	814 (1,270)	835 (335)	2,476 (1,047)	1,165 (241)	715 (169)	258 (61)	1,531 (239)	1,219 (268)
SPX Open Interest: Calls (Puts)										
0.78 < K/S < 0.92	1,111 (1,243)	1,891 (2,429)	1,026 (1,819)	2,257 (4,823)	1,036 (5,782)	2,864 (9,461)	1,559 (8,206)	865 (7,375)	996 (7,597)	4,565 (12,336)
0.92 < K/S < 0.98	1,581 (1,436)	2,224 (2,349)	2,467 (2,435)	3,293 (4,409)	3,061 (5,386)	5,304 (8,018)	3,505 (6,684)	3,610 (9,677)	3,539 (13,323)	7,115 (12,595)
0.98 < K/S < 1.00	1,290 (1,112)	2,249 (1,806)	4,081 (3,023)	4,520 (4,171)	5,315 (6,168)	7,649 (7,928)	6,174 (6,234)	8,152 (9,481)	9,522 (15,166)	11,035 (10,004)
1.00 < K/S < 1.02	1,209 (909)	2,094 (1,415)	3,680 (2,285)	4,875 (3,453)	5,887 (5,083)	6,472 (4,601)	6,267 (3,589)	9,387 (5,613)	12,623 (10,817)	10,807 (3,982)
1.02 < K/S < 1.08	890 (440)	1,667 (936)	2,605 (1,252)	3,368 (1,433)	5,346 (4,113)	5,837 (2,096)	5,396 (1,318)	6,674 (1,698)	11,569 (4,210)	6,731 (1,207)
1.08 < K/S < 1.20	821 (201)	1,877 (1,263)	1,782 (3,002)	3,124 (1,863)	4,531 (3,003)	5,346 (1,640)	4,124 (1,003)	1,069 (306)	4,183 (655)	3,038 (542)

B.2. Consumer Confidence Index

Every month, the Conference Board of the NFO Research Company conducts a consumer survey. Questionnaires are mailed to a nationwide-representative sample of 5,000 households. The data have been available since June 1977. The Index is based on responses to five survey questions with respect to: (1) the appraisal of current business conditions; (2) expectations regarding business conditions 6 months hence; (3) the appraisal of current employment conditions; (4) expectations regarding employment conditions 6 months hence; and (5) expectations regarding total family income 6 months from the time of the survey. For each of these five questions, participants' answers can be categorized into three groups: positive, negative, and neutral. The response proportions to each question are seasonally adjusted. The relative value is defined as the percentage of the positive answer with respect to the sum of the positive and negative, and it is initialized at the beginning of 1985. The Consumer Confidence Index

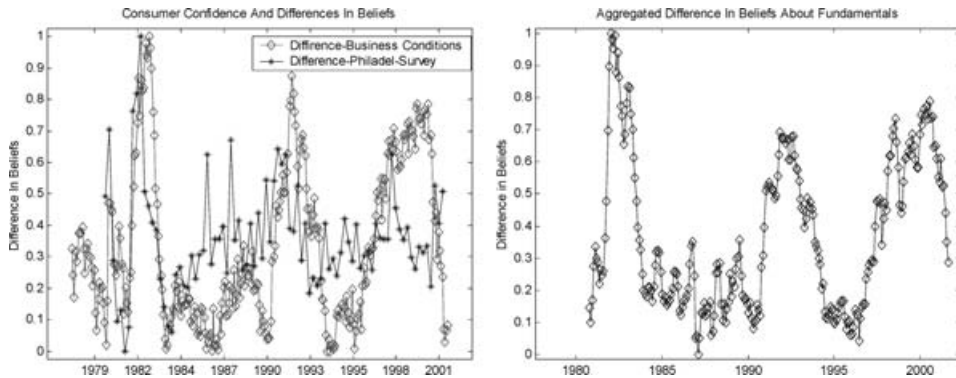


Figure 3. Differences in beliefs. The left figure shows the dynamics of the difference in beliefs implied by the consumer confidence survey and the survey of professional forecasters data. The right figure shows the Difference in Beliefs Index about the dividend process. This is obtained by aggregating information from the two surveys.

is the average of all five indices. The Present Situation Index averages the indices for questions 1 and 3, while the Expectations Index averages the indices for questions 2, 4, and 5.

We construct a Dispersion in Beliefs Index (ψ_δ) about market fundamentals based on both surveys. Each serves different goals. The Survey of Professional Forecasters has the advantage of being based on a restricted pool of market professionals, while the Consumer Confidence Index is available at a monthly frequency. Thus, we compute the time series of the cross-sectional standard deviations of the beliefs from both surveys and then aggregate information from the two time series by computing the first principal component. This gives us a proxy sampled at a monthly frequency. We find that this explains about 80% of the original time-series variations. Figure 3 (left panel) shows the dynamics of the difference in beliefs implied from the Consumer Confidence Survey and from the Survey of Professional Forecasters. Figure 3 (right panel) shows the Difference in Beliefs Index.

III. Empirical Results

The empirical analysis is organized as follows. We estimate the DB model using its key insight that option prices and open interest are simultaneously influenced by differences in beliefs. Thus, we first characterize the joint moment conditions on option prices and open interest, and use panel data from the CBOE to estimate the model. Second, we assess the model's performance using out-of-sample (cross-sectional) option pricing errors and open interest. Third, we compare these results to an alternative reduced-form model with exogenous stochastic volatility derived from Heston (1993) and Liu and Pan (2003). Fourth, we describe the performance of the model in terms of hedging errors at different (out-of-sample) horizons. Fifth, we study how current

changes in the difference in beliefs affect future stock volatility and the shape of the option-implied volatility smile.

Let $C_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)$ and $OI_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)$ be the model-implied call option prices and open interest, respectively, with Θ being the structural parameters of the model and K_i the strike prices. Let the empirical values of the option prices and open interest be $C_t(K_i/S, T_i)$ and $OI_t(K_i/S, T_i)$, respectively. We estimate the structural model by minimizing a GMM quadratic criterion defined in terms of the estimation errors for the option prices, open interest, and the first two moments of the dividend process. For easier interpretation and better numerical properties, we standardize the estimation errors by the observed level of the endogenous variables. Thus, the estimation errors are percentage deviations, rather than absolute dollar deviations. The objective function is therefore as follows:

$$\min \sum_{t=2}^T h_t(\hat{\Theta})' \sum_{t=2}^{-1} h_t(\hat{\Theta}),$$

where

$$h_t(\hat{\Theta}) = \begin{bmatrix} \frac{C_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)}{C_t(K_i/S, T_i)} - 1 \\ \frac{OI_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)}{OI_t(K_i/S, T_i)} - 1 \\ \left[\ln \left(\frac{\delta_{t+1}}{\delta_t} \right) \right] - \mu_{\delta} \\ \left[\ln \left(\frac{\delta_{t+1}}{\delta_t} \right) \right]^2 - \left[\sigma^2 \left(\frac{d\delta}{\delta} \right) + \mu_{\delta}^2 \right] \end{bmatrix}.$$

The last two entries of the vector $h_t(\hat{\Theta})$ are specification errors for the first two moments of the dividend process. This enforces the structural parameters to be empirically directly related to the dynamics of the actual dividend process. Note that because the dividend drift is stochastic, the second moment of the dividends is also affected by the drift volatility.¹⁰ The process $\hat{\psi}_{\delta,t}$ is the Difference in Beliefs Index that we obtain directly from survey data as we describe earlier. At each time step t , $\hat{\psi}_{z,t}$ is treated as a latent variable and obtained by minimizing $h_t(\hat{\Theta})' \Sigma^{-1} h_t(\hat{\Theta})$.¹¹ Clearly, this reduces the degrees of freedom of the GMM function by one.¹² The weighting matrix Σ is the Newey-West (1987) covariance matrix of the estimation errors.

¹⁰ See the Appendix for a formal derivation of the second moments: $\sigma^2(\frac{d\delta}{\delta}) = n_{\delta}^2/2 + \sigma_{\delta}$.

¹¹ Alternatively, one could select $\psi_z(t)$ to match the time series of one particular option's open interest or price. While this alternative approach gives an implied time series that is independent of the weighting matrix Σ , it has the disadvantage of being more sensitive to the observation errors of the selected option.

¹² The Lagrange multipliers y_1 and y_2 are obtained from the budget constraint (2) by substituting the equilibrium consumption process obtained in Proposition 2. Option prices are then obtained numerically from Proposition 3, noting that $\xi^1(t) = \delta(t)^{\gamma-1}(1 + (y_1\eta(t)/y_2)^{1/(1-\gamma)})^{1-\gamma}$.

Table II
Estimated Parameters of the Models

This table reports the parameter estimates for the three option pricing models: the Black-Scholes model, the Heston stochastic volatility model, and the Difference in Beliefs (DB) model. The Black-Scholes and the Heston models are estimated cross-sectionally so that the parameter values reported are the sample averages of these cross sections. The standard deviations are given in parentheses.

Parameter Estimates (DB Model)		Parameter Estimates (Heston Model)	
γ	-0.95 (0.45)	θ_v	0.042 (0.023)
σ_δ	0.038 (0.01)	k_v	1.294 (1.402)
$a_{0\delta}$	0.0118 (0.005)	σ_v	0.431 (0.252)
$a_{1\delta}$	-0.020 (0.01)	ρ	-0.634 (0.383)
n_δ	0.037 (0.02)		
α	0.65 (0.11)		
β	0.35 (0.13)		
σ_z	0.061 (0.02)		
a_{0z}	0.017 (0.005)		
a_{1z}	-0.03 (0.02)		
n_z	0.025 (0.05)		
Implied average ψ_z	0.61 (0.19)		
Implied average ψ_δ	0.64 (0.1)		

Parameter Estimates. Table II summarizes the parameter estimates. The values of a and b are 0.65 and 0.35, with standard deviations of 0.11 and 0.13, respectively, suggesting that both differences in beliefs are important in explaining the joint dynamics of option prices and open interest. The estimated values of the signal and dividend volatility are 6.1% and 3.8%, respectively. The estimated dividend volatility is below its sample counterpart. However, the difference is not statistically significant, with a p -value of 0.24. Because in the model the stochastic dividend growth rate affects the overall dividend volatility, the estimate of the instantaneous volatility σ_δ should be expected to be lower than the overall sample dividend volatility. The estimated dividend growth rate is 5.9%, higher than the empirical dividend growth rate. Again, the difference is not statistically significant, however with a p -value of 0.35.

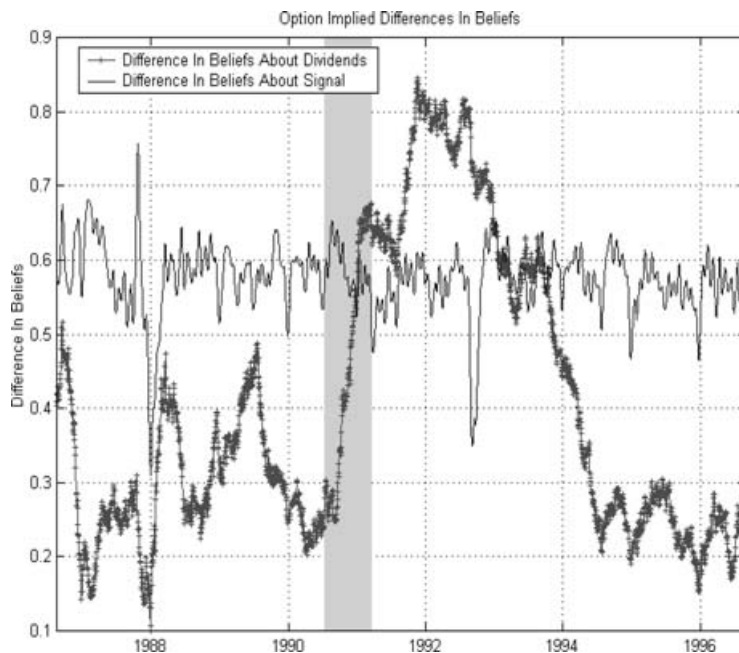


Figure 4. Option-implied difference in beliefs. This figure shows the dynamics of the difference in beliefs about the signal and dividend processes implied by the option market. The difference in beliefs about the dividend risk is estimated directly from survey data and is model independent. The signal difference in beliefs is estimated as a latent variable using the model pricing restrictions.

Finally, the volatility of the stochastic drift of the dividend and signal are 1.3% and 2.5%, respectively.

The risk aversion coefficient γ is -0.95 . This level of risk aversion is higher than the logarithmic utility case but still reasonable. The average level of the signal difference in beliefs $\psi_z(t)$ is 0.61. This is comparable to an observed average level of dividend difference in beliefs $\psi_\delta(t)$ equal to 0.64. The standard deviations of the dividend difference in beliefs is 0.1. Figure 4 presents the dynamics of the two differences in beliefs. The difference in beliefs about the dividend process is quite persistent, increasing substantially during the recession in the early 1990s from 0.20 to about 0.80, then decreasing to historically low levels after 1992, a period characterized by a strong rally in the stock market. The time variation of the difference in beliefs on the signal is less persistent.

To gain insight on the dynamics of the dispersion in beliefs on $z(t)$, Figure 5 gives a nonparametric scatter plot of the difference in beliefs on $z(t)$ with respect to the daily S&P 500 index return (Panel C) and the intraday volatility of the S&P 500 index (Panel D). We find that differences in beliefs on the signal are negatively correlated with the stock market. Large dispersions in beliefs on the signal take place during large market declines. When the difference in beliefs is one standard deviation above its long-term mean, the average annualized stock market return is -13% . The relationship is asymmetric: When the difference

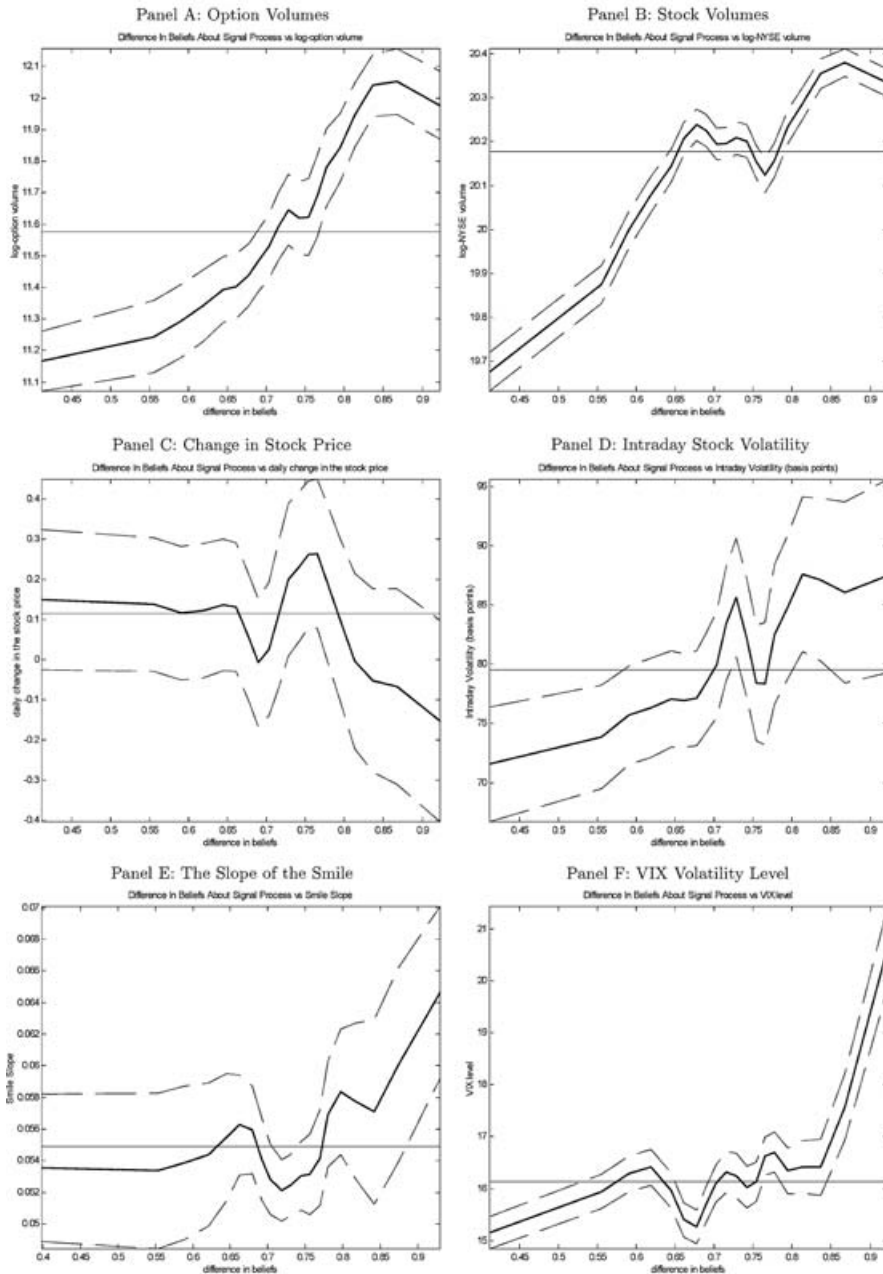


Figure 5. The impact of differences in beliefs: A nonparametric view. Panel A shows the relation between the logarithm of option volume and the signal difference in beliefs $\psi_{z,t}$. Panel B shows the relation between the logarithm of the NYSE stock volume and $\psi_{z,t}$. Panel C shows the relation between changes in the stock price and $\psi_{z,t}$. Panel D shows the relation between intraday volatility and $\psi_{z,t}$. Panel E shows the relation between the slope of the smile and $\psi_{z,t}$. Panel F shows the relation between the VIX Volatility index and $\psi_{z,t}$. The solid line is the nonparametric estimate, while the dotted lines show the 95% confidence interval.

in beliefs is smaller than its long-term mean, the average daily change in the stock market is not particularly sensitive to changes in beliefs, whereas when the difference in beliefs is large, the average stock market sensitivity is high. Moreover, we find that large dispersions in beliefs about the signal are correlated with high intraday volatility of the S&P 500 index (Panel D). The relationship is both statistically and economically significant. When the difference in beliefs is one standard deviation below the long-term mean, the intraday volatility is about 70 basis points. For values one standard deviation above the long-term mean the intraday volatility is 85 basis points on average.

We test the overidentifying restrictions of the structural model using a GMM test that takes advantage of the large cross-sectional information contained in the data set. To do this, we first estimate the structural parameters using a subset of options with moneyness $K_i/S = \{0.97, 1.03\}$ and maturities $T_i = \{45, 135\}$ days. We then compute the pricing errors on the remaining set of options with moneyness levels $K_i/S = \{0.92, 1.00, 1.08\}$ and maturities $T_i = \{30, 180\}$ days. We use these pricing errors to construct an out-of-sample GMM test statistic.¹³ Using different sets of options for estimation and testing allows us to increase the test's power. More specifically, let $C_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)$ and $OI_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)$ be the model-implied call option prices and open interest conditional on the vector of structural parameters Θ , the differences in beliefs $\psi_{z,t}$, and $\psi_{\delta,t}$, the strike price K_i , and the time to maturity T_i . Let the sample counterparts of the previous variables be $C(t, K_i/S, T_i)$ and $OI(t, K_i/S, T_i)$, respectively.

Pricing Errors. The out-of-sample pricing errors for the remaining subset of options expressed in percentage terms are defined as

$$\varepsilon_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i) = \frac{C_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)}{C(t, K_i/S, T_i)} - 1.$$

If the expected value of the pricing errors is zero, then, under some regularity conditions, its sample counterpart converges to zero. Thus, we consider the test statistic

$$J = \left(\frac{1}{T} \sum_{t=1}^T \varepsilon_t \right)' \Sigma^{-1} \left(\frac{1}{T} \sum_{t=1}^T \varepsilon_t \right).$$

The variance-covariance matrix is estimated using a Newey-West (1987) correction with 10 lags to adjust for potential heteroskedasticity and autocorrelation in the pricing errors. Because these options are not used in the estimation, under the null hypothesis of zero mean errors, this test statistic is asymptotically distributed as a Chi-square with six degrees of freedom.

Table III summarizes the GMM test results for different option moneyness and maturity parameters. The joint test on all options, independent of moneyness and maturity, gives a J -statistic of 7.02 with a p -value of 32%. The overidentifying restrictions of the model are not rejected. We therefore explore

¹³ The same out-of-sample test is used in David and Veronesi (2002).

Table III
Specification Test for Option Prices

This table presents the results for the model specification test. Let the estimation error be

$$\varepsilon_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i) = \frac{C_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)}{C(t, K_i/S, T_i)} - 1.$$

The structural parameters of the model are estimated using option prices and open interest for options with $K_i/S = \{0.97, 1.03\}$ and $T_i = \{45, 90\}$ days. Then, we use the pricing errors of six options with moneyness levels $K_i/S = \{0.92, 1.00 \text{ and } 1.08\}$ and maturity $T_i = \{30, 135\}$ to compute a Chi-square test. Let $C_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)$ and $OI_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)$ be the model-implied call option price and open interest, conditional on the vector of structural parameters Θ , the strike price K_i , and the time to maturity T_i . The sample counterparts of the previous variables are $C(t, K_i/S, T_i)$ and $OI(t)$, respectively. Consider the following J_T statistics:

$$J_T = \left(\frac{1}{T} \sum_{t=1}^T \varepsilon_t \right)' \Sigma^{-1} \left(\frac{1}{T} \sum_{t=1}^T \varepsilon_t \right).$$

The variance-covariance matrix Σ is estimated using Newey and West (1987) with 10 lags to adjust for potential heteroskedasticity and autocorrelation in the pricing errors. The asymptotic distribution of the test statistics is a Chi-square with six degrees of freedom. Panel A reports the level of the J_T statistics with the associated p -values in parentheses. In Panel B, we compute the J_T -statistic on open interest estimation errors based on a out-of-sample set of options. The fitting errors for the remaining subset of options is expressed in percentage terms.

$$\varepsilon_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i) = \frac{OI_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)}{OI(t, K_i/S, T_i)} - 1.$$

Panel A: Option Prices			
Moneyness (K/S)	Maturity		
	30 Days	135 Days	All Maturities
0.92	0.874 (0.35)	0.950 (0.33)	1.468 (0.48)
1.00	0.609 (0.43)	0.656 (0.42)	1.227 (0.54)
1.08	2.176 (0.14)	2.340 (0.13)	4.587 (0.10)
All strikes	3.209 (0.36)	2.856 (0.41)	7.015 (0.32)
Panel B: Open Interest			
Moneyness (K/S)	Maturity		
	30 Days	180 Days	All Maturities
0.97	1.065 (0.302)	1.470 (0.225)	3.384 (0.184)
1.03	1.904 (0.168)	2.413 (0.12)	4.129 (0.127)
All strikes	2.960 (0.228)	3.266 (0.195)	6.088 (0.193)

how the model performance depends on the option characteristics by stratifying the sample with respect to the moneyness and maturity parameters. We find that the price dynamics of out-of-the-money options, that is $K/S = 1.08$, is the most difficult to fit. However, for these options, the p -values still range between 10% and 14%. The aggregate p -value is 10%. At-the-money options, that is, $K/S = 1$, have the highest p -values, ranging between 44% and 54%, while in-the-money calls, that is $K/S = 0.92$, have p -values that range between 33% and 48%. Looking at the option maturity, we find that the p -value is 36% for short-term options and 41% for long-term options. The model is never rejected.

Open Interest. We now turn to the *open interest* assessment of the structural model. The spirit of the test is similar to that for option prices. We first estimate the structural parameters of the model using prices of a subset of options. Then we compute the errors for the open interest on the remaining set of options with moneyness levels $K_i/S = \{0.97, 1.03\}$ and maturities $T_i = \{45, 135\}$ days. The out-of-sample open interest fitting errors for the remaining subset of options are expressed in percentage terms as

$$\varepsilon_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i) = \frac{OI_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)}{OI(t, K_i/S, T_i)} - 1.$$

We construct the J_T -statistic $J_T = (\frac{1}{T} \sum_{t=1}^T \varepsilon_t)' \Sigma^{-1} (\frac{1}{T} \sum_{t=1}^T \varepsilon_t)$ in the same fashion as for the pricing errors. Table III summarizes the results of the GMM test performed on open interest data. The joint test on all options, independent of moneyness and maturity, have a J -statistic of 6.09, with a p -value of 19%. The overidentifying restrictions of the model are not rejected. We stratify the sample for different maturity and moneyness levels to study how the model performance depends on these characteristics. Similar to the pricing errors GMM tests, we find that the dynamics of the out-of-the-money option open interest are the most difficult to fit. However, for these options the p -value is still between 12% and 17%, with an aggregate p -value equal to 13%. The overidentifying restrictions of the model are not rejected. In-the-money options have the highest p -values ranging between 22% and 30%. We also compare the model performance with respect to the option maturity and find that the p -value is 23% for short-term options and 20% for long-term options.

A. Option Volume and Open Interest

The DB model generates implications for the dynamics of the option open interest as a function of the Difference in Beliefs Index.¹⁴ This contrasts with generalized deterministic models, which, by construction, can fit any cross-section of option prices at any desired level of accuracy (in the sample), but leave the dynamics of the option open interest indeterminate. In this section, we quantify the extent to which the DB model explains the option open interest.

¹⁴ Other important related studies that assume a positive link between trading volume and difference in beliefs include Varian (1989), Bessembinder, Chan, and Seguin (1996), Harris and Raviv (1993), and Shalen (1993).

Table IV
Open Interest

This table presents the mean absolute percentage error for the open interest. We consider the open interest implication for two models: the general equilibrium DB model and the partial equilibrium model with stochastic volatility and jumps of Liu and Pan (2003) and Heston (1993). We fit both models to all option prices and to one open interest with $K_i/S = \{0.97, 1.00, 1.03\}$ and maturity $T_i = \{45, 135\}$ days. The open interest fitting errors are defined as

$$\varepsilon_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i) = \text{abs} \left[\frac{OI_t(\Theta, \psi_{z,t}, \psi_{\delta,t}, K_i/S, T_i)}{OI(t, K_i/S, T_i)} - 1 \right].$$

Moneyness (K/S)	Maturity		
	Model	45 Days	135 Days
$K/S = 0.97$	SVJ-open interest	33.27%	14.93%
	DB-open interest	18.94%	13.11%
$K/S = 1$	SVJ-open interest	24.75%	15.86%
	DB-open interest	25.12%	10.13%
$K/S = 1.03$	SVJ-open interest	31.90%	21.22%
	DB-open interest	27.30%	19.11%

The Difference in Beliefs Index is positively correlated with option open interest. During the 1987 market crash, the Difference in Beliefs Index drops from 0.40 to 0.15. At the same time, the open interest to the S&P 500 capitalization ratio declines to reach the lowest value in 1987, about 2.5 times lower than in March 1985. During the high growth expansionary period of the 1990s, we again find a positive correlation between the two time series. The open interest to S&P 500 capitalization ratio peaks during the 1991 to 1992 recession, then it steadily declines. In April 1992, the open interest is about twice what it is in January 1995. At the same time, in April 1992, the Dispersion in Beliefs Index reaches 0.80 and then declines to 0.25 in January 1995.

Table IV summarizes the average open interest fitting error for different maturities and moneyness levels. For the sake of comparison, we also estimate and compute fitting errors based on the Liu and Pan (2003) model, wherein they generalize Heston (1993) in the context of a partial equilibrium model in which options are held to hedge unexpected changes in volatility and jumps and they solve the dynamic portfolio problem. We refer to this model as SV-J and use it as a benchmark, since in models with generalized deterministic volatility, such as Derman and Kani (1994), options are redundant and open interest is undetermined. We find that for all option classes, the DB model produces smaller open interest fitting errors than the SV-J model. The largest differences are for short-term in-the-money (ITM) and long-term at-the-money (ATM) options. In the first case, the fitting errors are 33.27% for the SV-J model and 18.94% for the DB model. In the second case, the fitting errors are 15.86% for the SV-J model and 10.13% for the DB model. For OTM call options, the pricing errors of the two models are not significantly different (30% for short-term options and 20% for long-term options).

To study the economic significance of the link between open interest and the difference in beliefs, we plot the logarithm of option trading volume as a function of the difference in beliefs (see Figure 5, Panel A).¹⁵ We find that the option trading volume is positively correlated with the difference in beliefs. For instance, a one standard-deviation change in the difference in beliefs from 0.45 to 0.65 results in a 20% increase in the option trading volume. The relationship is nonlinear and steeper at both higher and lower levels of the difference in beliefs. For instance, an increase in the difference in beliefs from 0.65 to 0.85 results in a 60% increase in the level of the option trading volume. We find a similar relationship for the total NYSE stock trading volume. An increase in the difference in beliefs from 0.45 to 0.65 results in a 30% increase of the NYSE trading volume.

The time variation in the DB generates endogenous stochastic volatility. As such, it nests some features of the Heston (1993) and Liu and Pan (2003) models. The DB model, therefore, should not be interpreted as an alternative to these reduced-form specifications. Rather, it offers a structural explanation for the stochastic and time-varying endogenous volatility. To study in more detail the marginal contribution of changes in the difference in beliefs with respect to the stochastic volatility, we regress the option trading volume on the difference in beliefs for both the non-dividend and dividend processes, controlling for the stochastic volatility implied by the DB model. Moreover, because in the DB model volatility is stochastic and endogenously driven by changes in $\psi_{z,t}$ and $\psi_{\delta,t}$, we decompose the total observable volatility at time t , V_t , into two components: (a) the part that is predicted by the DB model, $\hat{V}_t(\psi)$, and (b) the volatility that is unexplained by changes in the difference in beliefs, $V_t - \hat{V}_t(\psi)$. This second component captures the portfolio reallocation decision driven by volatility changes due to non-informational reasons. We consider the following regression:

$$\begin{aligned} \log(\text{OptVl}m_t) = & \beta_0 + \beta_1 \log(\hat{V}_t(\psi)) + \beta_2[V_t - \hat{V}_t(\psi)] \\ & + \beta_3 \log(\psi_{z,t}) + \beta_4 \log(\psi_{\delta,t}) + e_t, \end{aligned}$$

where $\text{OptVl}m_t$ is the option trading volume at time t . This regression allows us to assess the relative importance of the informational component for the trading

¹⁵ We use a standard kernel regression to find the exact nonparametric relationship between the two variables. Let $\psi_z(t)$ be the option-implied difference in beliefs and y be an observed financial variable. For each level of $\psi_z(t)$, the nonparametric function $y_t = y(\psi_z(t))$ is estimated as

$$y(x_0) = \frac{\sum_{t=1}^T \left[\frac{1}{h} K\left(\frac{x_0 - x_t}{h}\right) y_t \right]}{\sum_{t=1}^T \left[\frac{1}{h} K\left(\frac{x_0 - x_t}{h}\right) \right]}; \quad x_t = \psi_z(t).$$

The function $K(\cdot)$ is a kernel function, the scalar h is the bandwidth, and x_j are the grid-points. As a kernel function we select the Epanechnikov kernel function $K(x) = \frac{3}{4\sqrt{5}}(1 - \frac{1}{5}x^2)I_{|x| \leq \sqrt{5}}$. See Silverman (1986) for a discussion of its accuracy with respect to the more traditional Gaussian kernel. The bandwidth is selected via standard cross-validation procedures.

Table V
Option-Implied Difference in Beliefs: Regressions

This table reports the results of the regression of S&P 500 Index options and NYSE spot trading volume on information flow proxies and differences in beliefs. We run the following regressions:

$$\log(\text{OptVlm}_t) = \beta_0 + \beta_1 \log(V_t) + \beta_2(V_t - \hat{V}_t) + \beta_3 \log(\psi_{z,t}) + \beta_4 \log(\psi_{\delta,t}) + e_t,$$

where OptVlm_t stands for option volume at time t , V_t stands for implied volatility, $\hat{V}_t$ is the volatility implied by the model, $\psi_{z,t}$ is the difference in beliefs about the signal, and $\psi_{\delta,t}$ is the difference in beliefs about dividends. We also run the equivalent regression for the stock trading volume:

$$\log(\text{StockVlm}_t) = \beta_0 + \beta_1 \log(V_t) + \beta_2(V_t - \hat{V}_t) + \beta_3 \log(\psi_{z,t}) + \beta_4 \log(\psi_{\delta,t}) + e_t.$$

The Newey-West (1987) heteroskedasticity and autocorrelation adjusted t -statistics are reported in parentheses.

	log(Option Volume)	log(Spot Volume)
Intercept	10.16	18.14
t -stats	(3.14)	(4.57)
$V_t - \hat{V}_t(\psi)$	0.25	0.53
t -stats	(2.49)	(2.44)
$\hat{V}_t(\psi)$	0.32	0.51
t -stats	(3.92)	(2.67)
$\psi_{z,t}$	0.96	0.38
t -stats	(1.95)	(2.57)
$\psi_{\delta,t}$	0.14	-0.16
t -stats	(1.45)	(0.34)
R^2	22%	13%

activity, with respect to other exogenous components that affect changes in volatility. We run the equivalent regression for the stock trading volume. We detrend trading volumes using the method described in the data section to ensure stationarity; moreover, the standard errors are corrected for heteroskedasticity and autocorrelation using Newey and West (1987).

Table V gives the results. Consistent with the literature, we find that changes in volatility are positively correlated with the option open interest. A 10% increase in the model-implied volatility $\hat{V}_t(\psi)$ increases the option volume by 3.2%. The residual change in volatility that is unexplained by the model, that is $V_t - \hat{V}_t(\psi)$, is statistically significant. The size of its slope coefficient, however, is about one third of the slope coefficient for the model-implied volatility. This suggests that the model is effective in capturing a large component of the realized volatility. The main result is that even after controlling for the two measures of volatility, the two Difference in Beliefs Indices are still positive and statistically significant. Of these two indices, the largest coefficient is for the difference in beliefs on the signal, $\psi_{z,t}$. A 10% increase in the difference in belief increases the option volume by 9.6%. The effect is almost as strong as that to the model-implied volatility. The R^2 of the regression is 22%. We conclude that even after controlling for portfolio reallocation decisions driven by

changes in volatility, a large component of trading activity is directly driven by informationally related factors.

B. Hedging Errors

We follow Bakshi et al. (1997) and David and Veronesi (2002) and use the hedging errors to assess the model's pricing performance. We use this metric, as opposed to in-sample pricing errors, because Dumas et al. (1998) show that several of the option pricing models that perform well in terms of in-sample pricing errors display large out-of-sample pricing errors. They interpret this result as evidence of model overfitting in the context of generalized deterministic volatility function models. Thus, we consider the time- t problem of hedging a short position in a call option with τ periods to maturity and the strike price K . In both Heston and the DB models, the volatility of the underlying asset is stochastic. In the DB model this is due to changes in the difference in beliefs. Thus, the replication portfolio returns require a third security to hedge the risk of changes in the difference in beliefs, which has asset pricing effects. Let $x_s(t)$ be the model-implied quantity of the underlying assets, $x_0(t)$ be the bond position, and $x_c(t)$ be the quantity invested in another call option with the same maturity but different strike price K_1 . For the DB model, the option with strike price K_1 is used to hedge the risk of unexpected changes in the difference in beliefs. The optimal hedge ratio can be obtained from the derivative of the option price with respect to the state variables S_t and η_t :

$$x_c(t) = \frac{\partial C(t, K)/\partial \eta_t}{\partial C(t, K_1)/\partial \eta_t}$$

$$x_s(t) = \frac{\partial C(t, K)}{\partial S_t} - \frac{\partial C(t, K_1)}{\partial S_t} x_c(t).$$

The initial bond amount $x_0(t)$ is selected so that the value of the portfolio at time t is equal to the value of the option to be hedged. For comparison, we also present results based on the Heston (1993) model. In this case, the optimal hedging portfolio weights are given by

$$x_c(t) = \frac{\partial C(t, K)/\partial \sigma}{\partial C(t, K_1)/\partial \sigma}$$

$$x_s(t) = \frac{\partial C(t, K)}{\partial S_t} - \frac{\partial C(t, K_1)}{\partial S_t} x_c(t).$$

For completeness, we consider a delta-vega hedging strategy based on the Black-Scholes model. This strategy is clearly inconsistent since, in the Black-Scholes model, the volatility is not time-varying. Nonetheless, it is one of the most widely adopted hedging strategies used by risk managers.

The hedge requires continuous rebalancing to replicate the option return. Ideally, one would like the rebalancing period to be as frequent as possible to minimize discretization errors. However, because we observe the difference in

beliefs at a monthly frequency, while the available option data are at a daily frequency, we reestimate the model at a daily frequency assuming a constant rate of change ψ_δ between consecutive observations. Although this approach is a pragmatic empirical compromise, the assumption is clearly quite restrictive. It is important to note that such a restriction affects the empirical results by reducing the hedging performance of our model with respect to the alternative benchmark specification. We select the rebalancing period to be one day and calculate the hedging errors at time $t + \Delta t$ as

$$H(t + \Delta t) = x_s(t)S(t + \Delta t) + x_0(t)e^{r(t)\Delta t} + x_c(t)C(t + \Delta t, \tau - \Delta t, K_1) - C(t + \Delta t, \tau - \Delta t, K).$$

We calculate the hedging errors at a 1-week (5 trading days) horizon. Thus, we reestimate the model at a daily frequency and model ψ_z as a latent variable. The main empirical question that we investigate is whether high frequency joint observations on both option prices and open interest add significant information for hedging purposes. Table VI reports the average hedging errors.

Across all maturities and moneyness levels, the hedging error of the DB model is \$0.305, versus \$0.32 for the Heston (1993) model and \$0.353 for a

Table VI
Hedging Errors

This table reports the hedging performance of the model. For comparison, we report the equivalent hedging errors generated by the BS and Heston models. We consider the time- t problem of hedging a short position in a call option with τ periods to maturity and a strike price K . The hedging errors are defined as

$$H(t + \Delta t) = x_s(t)S(t + \Delta t) + x_0(t)e^{r(t)\Delta t} + x_c(t)C(t + \Delta t, \tau - \Delta t, K_1) - C(t + \Delta t, \tau - \Delta t, K),$$

where $x_S(t)$, $x_0(t)$, and $x_C(t)$ are the model-implied portfolio weights for the underlying asset, the cash account and a call option, respectively, with the same maturity but different strike prices. In the case of Black-Scholes (1973), we consider delta-vega hedging. We construct the desired hedge and calculate the hedging errors at time $t + \Delta t$. The estimation is updated at each time step. We calculate the average pricing errors for each moneyness and maturity.

Moneyness (K/S)	Maturity			
	Model	<45 Days	45–180 Days	All Maturities
$K/S < 0.95$	Black Scholes Delta-Vega	0.361	0.382	0.367
	Heston	0.365	0.348	0.359
	DB	0.316	0.406	0.333
$0.95 < K/S < 1.05$	Black Scholes Delta-Vega	0.313	0.306	0.311
	Heston	0.296	0.288	0.293
	DB	0.271	0.298	0.280
$K/S > 1.05$	Black Scholes Delta-Vega	0.361	0.397	0.373
	Heston	0.313	0.289	0.304
	DB	0.331	0.351	0.321
Average	Black Scholes Delta-Vega	0.348	0.364	0.353
	Heston	0.325	0.309	0.320
	DB	0.301	0.339	0.305

Black-Scholes delta-vega hedging strategy with continuous recalibration. For short-dated options, that is less than 45 days, the difference in hedging performance is even larger: \$0.30 for the DB model, \$0.325 for the Heston (1993) model, and \$0.35 for the Black-Scholes delta-vega strategy. The percentage difference is remarkable. For short-dated options, the DB model generates hedging errors that are 10% lower than the Heston (1993) model and almost 20% lower than the Black-Scholes delta-vega strategy.

To study the performance difference with respect to other option characteristics, we stratify the sample according to moneyness level. We find that, independent of the moneyness level, the DB model always produces lower hedging errors than both the Heston (1993) model and the Black-Scholes delta-vega strategy. The biggest difference is for short-term ITM options. In this case, the hedging errors of the DB model are \$0.316, 15% lower than the \$0.365 of the Heston (1993) model and 14% lower than the \$0.361 of the Black-Scholes delta-vega strategy. For OTM options, however, the hedging errors of the DB model are larger than Heston (1993), but still 16% better than the Black-Scholes delta-vega hedging strategy. The previous results suggest that the explicit use of open interest information is an effective forward-looking measure of the dynamics of the priced risk factors that affect option prices. As such, it helps to reduce the hedging errors from replication strategies.

C. Difference in Beliefs, the Smile, and the Future Volatility

Differences in beliefs are statistically significant to help explain both the cross section and the time series of option prices. We now examine in more detail why this model is successful. In particular, we examine the relation between difference in beliefs and the shape of the contemporaneous implied volatility smile. Then, we examine whether differences in beliefs predict future realized price volatility.

We find that the signal difference in beliefs affects option prices in a nonlinear way. For low difference in beliefs levels, the volatility smile is 15%. Larger levels of heterogeneity in beliefs induce a substantial shift in the option-implied volatility smile. For high levels the average level of the smile is above 20%. The difference is economically large and statistically significant. Figure 5, Panel E shows the effect of the signal difference in beliefs on the slope of the volatility smile. The correlation is positive for values of the signal difference in beliefs above 0.70.

Figure 6 shows the shape of the smile for various levels of the dividend difference in beliefs. In this figure, we keep the signal difference in beliefs equal to zero and change only the dividend difference in beliefs. The level of the differences in beliefs is positively related to the slope of the implied volatility smile. For small amounts of dispersion the smile is almost negligible. For large amounts the volatility smile is quite steep. For example, when the level of the difference in beliefs is 0.80, the corresponding slope of the smile is approximately 7%. This corresponds to an implied volatility of 23% for in-the-money

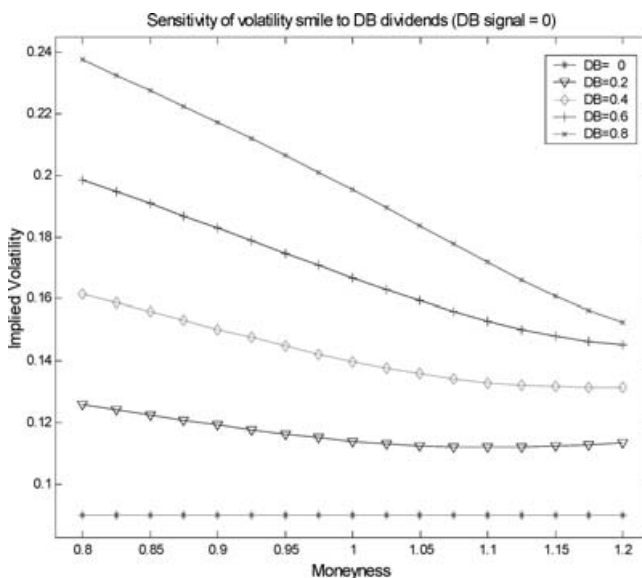


Figure 6. Implied volatility smile and differences in beliefs. This figure presents the shape of the model-implied volatility smile for different levels of the signal difference in beliefs, $\psi_{z,t}$.

calls and 16% for out-of-the-money calls. When the difference in beliefs is 0.20, the slope of the smile is less than 1%.

Does the current level of the difference in beliefs predict future volatility? To address this question we regress the future realized τ -period-ahead volatility $V_{t+\tau}$ onto the current level of the option-implied difference in beliefs. The empirical literature documents substantial evidence that the current level of the implied volatility is a significant, albeit imperfect, predictor of the future realized volatility (Christensen and Prabhala (1998)). Thus, in order to correctly assess the marginal contribution of the difference in beliefs, we control for the implied volatility. Moreover, because volatility is persistent, we also control for the current realized volatility. The forecasting regression is as follows.

$$V_{t+\tau} = \alpha_0 + \alpha_1 IV_t + \alpha_2 V_t + \alpha_3 \psi_{z,t} + \alpha_4 \psi_{\delta,t},$$

where $V_{t+\tau}$ is the future realized volatility τ periods ahead and IV_t is the option-implied volatility at time t . We select the forecasting horizon to range between 1 week and 6 months. Table VII gives the results. We find that the null hypothesis that the option-implied difference in beliefs about the signal predicts the future realized volatility is not rejected, with t -statistics ranging between 1.6 and 3.4, depending on the horizon. The slope coefficient is positive: The higher the current level of the signal difference in beliefs, the higher the future realized volatility. Consistent with the literature, for short horizons the lagged

Table VII
Predicting Future Volatility

This table shows the results of the volatility regressions. We run the regressions

$$V_{t+\tau} = \alpha_0 + \alpha_1 IV_t + \alpha_2 V_t + \alpha_3 \psi_{z,t} + \alpha_4 \psi_{\delta,t},$$

where $V_{t+\tau}$ is the realized τ -period-ahead volatility, IV_t is the option-implied volatility, $\psi_{z,t}$ is the difference in beliefs about the signal, and $\psi_{\delta,t}$ is the difference in beliefs about dividends. We analyze the forecasting power for horizons from 1 week to 6 months. The standard errors are adjusted for overlapping errors and the t -statistics are shown in parentheses.

	Forecast Horizon					
	5 Days	2 Weeks	1 Month	2 Months	3 Months	6 Months
Intercept	0.018	0.034	0.035	0.036	0.029	0.057
t -stats	(1.08)	(3.14)	(3.07)	(3.96)	(5.65)	(4.39)
IV_t	0.650	0.780	0.752	0.573	0.446	0.385
t -stats	(5.54)	(5.21)	(4.32)	(2.39)	(3.05)	(2.11)
V_t	0.345	0.270	0.245	0.294	0.190	0.167
t -stats	(2.43)	(1.34)	(1.54)	(1.21)	(0.83)	(0.74)
$\psi_{z,t}$	0.018	0.037	0.031	0.044	0.035	0.021
t -stats	(1.56)	(1.71)	(2.16)	(3.36)	(3.38)	(1.86)
$\psi_{\delta,t}$	0.039	0.032	0.060	0.131	0.190	0.134
t -stats	(0.98)	(0.17)	(0.24)	(0.89)	(1.33)	(1.25)
R^2	24%	36%	36%	41%	43%	39%

volatility is statistically significant. At horizons longer than one week, however, the lagged volatility is not significant. Moreover, we find that differences in beliefs about the drift of the dividend process have large and positive slope coefficients. The t -statistics are small, however. The R^2 of the regressions ranges between 23% at a 1-week horizon and 43% at a 3-month horizon. Consistent with the literature, we find that the current implied volatility has a positive and significant slope coefficient. We also find that the relationship between differences in beliefs and future volatility is *economically* significant. A change of one standard deviation in the difference in beliefs, that is, from 0.61 to 0.80, increases the expected future volatility in 2 weeks' time by 0.7%, for example, from 15% to 15.7%.

The difference in beliefs improves the forecast of the future realized volatility. This may explain why the DB model performs better than the Heston model in terms of hedging errors. The DB model is fit to information on both the open interest and the implied volatility smile. Andersen (1996) provides evidence that the joint dynamics of volatility and volume are likely driven by a common factor, which he labels “information flow.” Hence, a model, like the one we discuss in this article, that links the structural relation between information flow to both prices and volumes more accurately accounts for the future dynamics of volatility, and generates better hedging performance.

D. Violations of One-Factor Option Pricing Models

We now examine the model's performance with respect to a different dimension. Several studies document that the dynamics of option prices are not consistent with one-factor models. Bakshi et al. (2000) explore simple static arbitrage violations of option pricing models that assume a one-factor structure such as Black and Scholes. For instance, in these models an increase in the underlying asset value implies an increase (decrease) in the value of a call (put) option. The value of a call option is monotonically increasing in the value of the underlying asset.

DEFINITION 2: *In a frictionless one-factor economy the delta of a call (put) must be positive (negative) and lower (greater) than one (minus 1) in order for prices to be consistent with the Principle of No-Arbitrage.¹⁶ That is,*

- *Violation 1: $\Delta S \Delta C < 0$ for call options and $\Delta S \Delta P > 0$ for put options*
- *Violation 2: $\Delta S \neq 0$ and $\Delta C / \Delta S > 1$ for call options and $\Delta P / \Delta S < -1$ for put options,*

where S is the stock (underlying) price, C is the call option price, and P is the put option price.

Violations of these restrictions can be interpreted either as evidence of frictions or of a second (priced) stochastic factor. In what follows, we first explore the empirical violation frequency of the one-factor no-arbitrage restrictions in our sample. We find (see Table VIII) that violations of type-1 occur with a frequency between 17% and 24% for call options and between 15% and 22% for put options. Violations of type-2 occur with a frequency between 4% and 11% for call options and between 2% and 4% for put options. The dynamics of the violation frequencies are relatively stable across years, and are not specific to a particular subperiod. The magnitudes of violations are close to those reported in Bakshi et al. (2000), who interpret these violations as evidence against one-factor models of option pricing. We explore whether the difference in beliefs can be the additional second stochastic factor that explains these violations.

In Table IX, we compare the empirical (Panel A) violation frequencies with those obtained by simulating (Panel B) the model at the estimated parameter values. We find that type-1 violation frequencies are relatively stable across moneyness and maturity, varying from 12% for short maturity out-of-the-money calls to 17% for long-maturity in-the-money calls. The type-2 violations frequencies show strong dependence on moneyness, varying from 26% for ITM short-maturity calls to 3% for OTM short-maturity calls.

The model generates frequencies of type-1 violations that are very close to the empirical ones, across all moneyness and maturities. For instance, for short-term OTM calls the model-implied type-1 violation frequency is 12.01% compared to an observed empirical frequency of 12.65%. For long-term OTM

¹⁶ Bakshi, Cao, and Chen (2000) consider two other violations related to the fact that the option delta should be different from zero. We do not explore these issues.

Table VIII
Violation Frequencies for One Factor Models

The table shows the empirical frequency of type-1 and type-2 violations (Definition 2). The violation frequency is reported on an annual basis.

Year	Number of Obs.	Violation 1	Violation 2
CBOE-CALLS			
1986	9,984	18.90%	7.97%
1987	17,283	20.97%	8.25%
1988	13,760	24.06%	4.16%
1989	15,529	17.78%	7.31%
1990	19,720	20.25%	4.92%
1991	17,260	21.29%	5.04%
1992	18,939	23.10%	6.05%
1993	19,711	16.93%	5.80%
1994	21,118	18.68%	4.08%
1995	21,796	20.37%	10.87%
1996	14,346	21.24%	5.50%
CBOE-PUTS			
1986	9,984	19.16%	1.51%
1987	17,283	15.45%	2.00%
1988	13,760	20.80%	4.30%
1989	15,529	15.41%	2.71%
1990	19,720	19.30%	2.07%
1991	17,260	20.84%	1.63%
1992	18,939	22.07%	2.73%
1993	19,711	19.30%	2.02%
1994	21,118	17.54%	3.46%
1995	21,796	18.61%	2.66%
1996	14,346	21.26%	3.58%

options, the difference is still small, at less than 2%. The biggest discrepancy is for the long-maturity ITM calls for which the model-implied violation frequency is 9.45% whereas the observed empirical frequency is 16.50%. The type-2 violation frequencies are harder to replicate. The model is able to generate a frequency of about 2%, while the empirical frequency varies from 3% to 25%. However, as Bakshi et al. (2000) observe, violations of type-2 are more likely due to the tick size, as opposed to the effect of a second stochastic factor.

An alternative explanation for the observed large violations of the one-factor no-arbitrage restrictions is that volatility is stochastic. We therefore directly examine the marginal contribution of the difference in beliefs dynamics with respect to the dynamics of the volatility to explain the observed pattern of violations. Because the violations have an inherent binary structure, that is equal to 1 if the violation has occurred and zero if the violation has not occurred,

Table IX
Empirical and Model-Implied Violation Frequency

The table summarizes the violation frequency of one-factor no-arbitrage restrictions. The violations are classified as in Definition 2. Panel A shows the observed empirical frequencies. Panel B shows the violation frequencies implied by the model using Monte Carlo simulations. For each set of parameters (moneyness, time to maturity) we calculate the option price. Then, we step ahead one day and simulate 1,000 realizations for the price of the underlying asset. For each realization, we calculate the corresponding option price. Finally, we compute the violation frequency.

Moneyness (K/S)		<45 Days	45–180 Days
Panel A: Empirical Frequency of Violations			
$K/S < 0.95$	Violation 1	16.06%	16.50%
	Violation 2	25.96%	14.74%
$0.95 < K/S < 1.05$	Violation 1	14.30%	16.61%
	Violation 2	9.35%	11.22%
$K/S > 1.05$	Violation 1	12.65%	15.07%
	Violation 2	3.41%	5.17%
Panel B: Model-Implied Frequency of Violations			
$K/S < 0.95$	Violation 1	13.10%	8.53%
	Violation 2	2.19%	2.12%
$0.95 < K/S < 1.05$	Violation 1	8.98%	9.74%
	Violation 2	1.95%	1.32%
$K/S > 1.05$	Violation 1	12.01%	14.30%
	Violation 2	1.67%	1.84%

we use logit techniques¹⁷ to regress violation events onto the explanatory variables. Let $y_{it}^{(j)}$ be the occurrence of a violation of type- j for option i at time t . The probability that a violation event occurs can be specified as

$$\Pr(y_{it}^{(j)} = 1) = F(\beta_0 + \beta_1 \log(\hat{V}_{ti}(\psi)) + \beta_2(V_{ti} - \hat{V}_{ti}(\psi)) + \beta_3 \log(\psi_{z,t}) + \beta_4 \log(\psi_{\delta,t})),$$

where $\hat{V}_{ti}$ is the implied volatility from the model at time t for option i , which is stochastic and a function of ψ , while V_{ti} is the actual observed implied volatility. On the right-hand side, we also include option-specific characteristics such as moneyness and time to maturity to ensure that $\psi_{z,t}$ is not capturing any spurious effects. We estimate the model using maximum likelihood and summarize the results in Table X. The t -statistics of the coefficients are given in parentheses.

The probability of violations of type-1 and -2 is positively related to the level of $\psi_{z,t}$ for both puts and calls. In Panel B, we calculate the average marginal impact on the violation probability given a one-standard deviation change in

¹⁷ The probit results are similar and are not reported to save space.

Table X
Logit Analysis of Violations

This table summarizes the Logit regression results for the violation frequency according to Definition 2.

$$\Pr\left(y_{it}^{(j)} = 1\right) = F(\beta_0 + \beta_1 \log(\hat{V}_{ti}(\psi)) + \beta_2(V_{ti} - \hat{V}_{ti}(\psi)) + \beta_3 \log(\psi_{z,t}) + \beta_4 \log(\psi_{\delta,t})).$$

Panel A reports the parameter estimates. Panel B reports the marginal impact of a one standard deviation change in $\psi_{z,t}$, $\psi_{\delta,t}$, and the implied volatility onto the probability of violation. For money-ness and maturity, the marginal impact is computed given a 1% change in the exogenous variables. V_{ti} is the implied volatility, $\hat{V}_t(\psi)$ is the stochastic model implied volatility.

Panel A: Logit Parameter Estimates				
	Violation 1 Calls	Violation 2 Calls	Violation 1 Puts	Violation 2 Puts
Constant	−0.42 (3.45)	−0.66 (6.77)	−1.51 (11.22)	−1.48 (4.07)
Moneyiness (K/S)	1.02 (2.85)	−0.99 (6.09)	−0.56 (0.83)	−0.42 (5.13)
Maturity	0.30 (7.29)	−0.25 (5.95)	0.20 (5.22)	−0.11 (2.41)
$\psi_{z,t}$	0.32 (10.39)	0.14 (2.58)	0.42 (5.36)	0.11 (1.43)
$\psi_{\delta,t}$	0.36 (1.62)	−0.16 (1.12)	0.32 (0.92)	0.52 (0.98)
$\hat{V}_{ti}(\psi)$	1.30 (9.15)	0.21 (3.02)	0.55 (5.35)	0.16 (1.43)
$V_{ti} - \hat{V}_{ti}(\psi)$	0.85 (1.64)	0.10 (1.25)	0.20 (1.18)	0.10 (1.09)
R^2	16%	13%	11%	9%

Panel B: Marginal Impact of Each Variable					
	Average Change	Effect on the Probability of Violations			
		Violation 1 Calls	Violation 2 Calls	Violation 1 Puts	Violation 2 Puts
Moneyiness (K/S)	0.1	2.00%	−1.61%	−0.83%	−0.50%
Maturity	0.02	0.12%	−0.08%	0.06%	−0.03%
$\psi_{z,t}$	0.25	1.56%	0.57%	1.55%	0.35%
$\psi_{\delta,t}$	0.1	0.69%	−0.26%	0.47%	0.63%
$\hat{V}_{ti}(\psi)$	0.05	1.27%	0.17%	0.41%	0.10%
$V_{ti} - \hat{V}_{ti}(\psi)$	0.02	0.33%	0.03%	0.06%	0.02%

the explanatory variables. The marginal effect is computed at the average value of the explanatory variables.

The results show that a 0.1 change in moneyiness (from 1 to 1.10) increases the probability of type-1 violations by 2.0% for call options, keeping the other variables constant. A one-standard deviation change in $\psi_{z,t}$ (from the average

value of 0.72 to 0.97) increases the probability of type-1 violations by 1.56% for call options and 1.55% for put options. An equivalent change in the volatility increases the probability of type-1 violations by 1.27% and 0.41% for call and put options, respectively. Thus, we find that the change in the difference of beliefs is at least as important as the change in the volatility. The size of the effect is significant if one considers that the average violation frequency ranges over the years between 17% and 24%. The positive coefficient means that a high level in the difference in beliefs makes options behave more as nonredundant securities.

For type-2 violations, the coefficient of $\psi_{z,t}$ is positive and statistically significant, although it is smaller than that of type-1. A one-standard deviation change in $\psi_{z,t}$ yields a 0.57% increase in the probability of type-2 violations (Calls), while that of the stochastic volatility yields a 0.17% increase. The difference in beliefs about the dividend process $\psi_{\delta,t}$ does not have a significant effect on the probability of type-2 violations for call options.

The marginal effect of changes in the implied volatility that are not due to changes in the differences in beliefs, $V_{ti} - \hat{V}_{ti}$, is substantially smaller than both the direct and indirect effect of changes in ψ . For instance, in the case of type-1 violations for call options, the marginal effect of changes in $\hat{V}_{ti}(\psi)$ is 1.27%, the direct effect of $\psi_{z,t}$ is 1.56%, but the effect of $(V_{ti} - \hat{V}_{ti})$ is only 0.33%.

To summarize, we find that the two Difference in Beliefs Indices are significant explanatory variables when describing the probability of violations of the no-arbitrage bounds implied by one-factor models. The significance of the difference in beliefs persists even after controlling for changes in the stochastic volatility.

IV. Conclusion

In this paper, we study both theoretically and empirically an economy in which options are not redundant. Agents with heterogeneous beliefs are uncertain about the stochastic drifts of the risk factors. We characterize the way in which differences in beliefs have intertemporal risk-sharing implications that affect both option prices and volume. In particular, we model two forms of beliefs: (a) those about the drift of the dividend process, and (b) those about a factor (signal) that is correlated with the drift of the dividend process. We then estimate and test the structural model. Using survey data to construct a Difference in Beliefs Index that serves as a direct proxy for the differences in beliefs about market fundamentals, we find the following.

First, we test the structural overidentifying pricing restrictions of the model by using an out-of-sample GMM test. We fail to reject the model, obtaining a p -value of 31.9% when using restrictions on option prices and 19.3% when using restrictions on open interest. The results are robust to different subsamples. We also fail to reject the model when we consider only ITM, OTM, short-term, or long-term options.

Second, for all option classes the Difference in Beliefs model produces open interest fitting errors that are smaller than a stochastic volatility model that

abstracts from the heterogeneity in beliefs, as in Liu and Pan (2003). The largest differences between the two models stem from short-term ITM and long-term ATM options. In the first case, the fitting errors difference is 14.3%; in the second case it is 5.7%.

Third, the Difference in Beliefs model generates lower hedging errors at a 1-week horizon than both Heston (1993) and Black and Scholes (1973). Across all maturities and moneyness levels, the hedging error of the DB model is 5% lower than the Heston (1993) model and about 17% lower than a Black-Scholes delta-vega hedging strategy. For short dated options, which is less than 45 days, the difference in hedging performance is even larger, with hedging errors that are 8% lower than the Heston (1993) model and 16% lower than the Black-Scholes delta-vega strategy.

Fourth, current levels of the index of dispersion in beliefs have positive and statistically significant predictive power for the future realized volatility, even after controlling for current implied volatility. The R^2 of the regression ranges between 23% for a 1-week horizon, to 43% for a 3-month horizon. Both the current and future implied volatility smiles are very sensitive to our Difference in Beliefs Index: The greater the dispersion of beliefs, the steeper the implied volatility smile.

Fifth, we use the DB model to address the puzzle highlighted by Bakshi et al. (2000). They document evidence of violations of basic arbitrage bounds implied by one-factor option pricing models. For instance, in most of these models, an increase in the value of the underlying asset implies an increase (decrease) in the value of a call (put) option: The option delta is restricted to values between zero and $+1(-1)$. Thus, we run a Logit regression to assess the extent to which the index of dispersion in beliefs can explain these no-arbitrage violations. We find that an increase in the index substantially increases the probability that the Black and Scholes call delta is negative or above $+1$. The results are both economically and statistically significant. We find that most arbitrage violations, and the extent of these violations, are correlated with abnormal changes in the Dispersion in Beliefs.

The empirical evidence gives strong support to the role played by differences in belief in the dynamics of option prices and volume.

Appendix A: Proofs

Proof of Proposition 1 (Equilibrium): The result follows easily from the first-order and market clearing conditions. Then, using Ito's lemma we can obtain the diffusion process of $d\eta(t)$ with $\eta(t)$ being the ratio of the two stochastic discount factors, that is, $\eta(t) = \frac{\xi^1(t)}{\xi^2(t)}$. We outline the proof for the equilibrium proposition in four steps: (a) we characterize how individuals update their beliefs about the dynamics of the dividend and signal processes, $\delta(t)$ and $z(t)$, respectively; (b) we compute the price of risk as a function of these beliefs; (c) we derive the aggregation properties of the economy and characterize the market prices of risk; and (d) we derive the equilibrium consumption allocation. To solve for

the equilibrium in the economy with heterogeneous agents, we use the representative agent technique with stochastic weights discussed by Cuoco and He (1994).

By simple substitution, the agent-specific Brownian motions are related to the true innovations as follows:

$$\begin{aligned} dW_\delta(t) &= \frac{m_\delta^n(t) - \mu_\delta(t)}{\sigma_\delta} dt + dW_\delta^n(t) \\ dW_z(t) &= \left[\alpha \frac{m_\delta^n(t) - \mu_\delta(t)}{\sigma_z} + \beta \frac{m_z^n(t) - \mu_z(t)}{\sigma_z} \right] dt + W_z^n(t). \end{aligned}$$

Let asset prices follow the diffusion processes

$$dP_i(t) = P_i(t)[\mu_i(t)dt + \sigma_{i\delta}(t)dW_\delta(t) + \sigma_{iz}(t)dW_z(t)],$$

with $P_1(t) = P(t)$ and $P_2(t) = O(t)$. Substituting the agent-perceived Brownian motions, we obtain

$$\begin{aligned} dP_i(t) &= P_i(t)[\mu_i^n(t)dt + \sigma_{i\delta}(t)dW_\delta^n(t) + \sigma_{iz}(t)dW_z^n(t)] \\ \mu_i^n(t) &= \mu_i(t) + \sigma_{i\delta}(t)\frac{\mu_\delta^n(t) - \mu_\delta(t)}{\sigma_\delta} + \sigma_{iz}(t)\left[\alpha\frac{\mu_\delta^n(t) - \mu_\delta(t)}{\sigma_\delta} + \beta\frac{\mu_z^n(t) - \mu_z(t)}{\sigma_z}\right], \end{aligned}$$

where $\mu_i^n(t)$ is the expected instantaneous return for asset i from the perspective of agent n and $W_z^n(t)$ is agent n 's individual Brownian motion. Thus, the difference in expected returns from the perspective of two different agents is given by

$$\begin{aligned} \mu_i^1(t) - \mu_i^2(t) &= \sigma_{i\delta}(t)\frac{\mu_\delta^1(t) - \mu_\delta^2(t)}{\sigma_\delta(t)} \\ &\quad + \sigma_{iz}(t)\left[\alpha\frac{\mu_\delta^1(t) - \mu_\delta^2(t)}{\sigma_z} + \beta\frac{\mu_z^1(t) - \mu_z^2(t)}{\sigma_z}\right] \end{aligned} \quad (A1)$$

$$= \sigma_{i\delta}(t)\psi_\delta(t) + \sigma_{iz}(t)\left(\alpha\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(t)\right). \quad (A2)$$

Let us define $\theta_\delta^n(t)$ to be agent n 's price of dividend risk and $\theta_z^n(t)$ to be his price of signal risk. By no-arbitrage, excess returns need to satisfy

$$\mu_i^n(t) - r(t) = \sigma_{i\delta}(t)\theta_\delta^n(t) + \sigma_{iz}(t)\theta_z^n(t). \quad (A3)$$

From the previous two equations we have

$$\begin{aligned} &\sigma_{i\delta}(t)[\theta_\delta^1(t) - \theta_\delta^2(t)] + \sigma_{iz}(t)[\theta_z^1(t) - \theta_z^2(t)] \\ &= \sigma_{i\delta}(t)\psi_\delta(t) + \sigma_{iz}(t)\left(\alpha\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(t)\right). \end{aligned}$$

Because the last equation has to hold for any $\sigma_{i\delta}(t)$ and $\sigma_{iz}(t)$, it follows that

$$\begin{aligned}\theta_\delta^1(t) - \theta_\delta^2(t) &= \psi_\delta(t) \\ \theta_z^1(t) - \theta_z^2(t) &= \left(\alpha \psi_\delta(t) \frac{\sigma_\delta}{\sigma_z} + \beta \psi_z(t) \right).\end{aligned}$$

Representative agent with stochastic weight. Because the economy is dynamically complete with trading in a stock, option, and riskless security, each agent's stochastic discount factor is unique. Thus, it is convenient to study the equilibrium properties for a representative agent economy. Clearly, due to the difference in the filtrations perceived by the two agents, the relative Pareto weights of the two agents are stochastic (see Karatzas and Shreve (1998) and Cuoco and He (1994)). The optimal program is

$$\max_{\substack{c_1, c_2 \\ s.t. c_1 + c_2 = \delta}} \frac{c_1^\gamma}{\gamma} + \lambda \frac{c_2^\gamma}{\gamma}. \quad (\text{A4})$$

In equilibrium, $u'_1(c_1) = \lambda u'_2(c_2)$. Solving for the optimal allocation of the aggregate consumption between the two agents, we obtain $c_1 = \delta \lambda^{\frac{1}{\gamma-1}} / (1 + \lambda^{\frac{1}{\gamma-1}})$ and $c_2 = \delta / (1 + \lambda^{\frac{1}{\gamma-1}})$. Substituting, we obtain the indirect utility function

$$V(\delta; \lambda) = \frac{\delta^\gamma}{\gamma} \lambda (1 + \lambda^{\frac{1}{\gamma-1}})^{1-\gamma}.$$

The first-order conditions of each agent require $u'_i(c_i) = y_i \xi^i$ and $V'_\delta(\delta; \lambda) = u'(c_1) = \lambda u'(c_2)$, thus

$$\xi_t^1 = \frac{1}{y_1} \delta_t^{\gamma-1} \lambda_t (1 + \lambda_t^{\frac{1}{\gamma-1}})^{1-\gamma} \quad \text{and} \quad \xi_t^2 = \frac{1}{y_2} \delta_t^{\gamma-1} (1 + \lambda_t^{\frac{1}{\gamma-1}})^{1-\gamma},$$

with $\lambda_t = \frac{y_1}{y_2} \xi_t^1 / \xi_t^2 = \frac{y_1}{y_2} \eta_t$, so that

$$c_1(t) = \delta(t) \frac{(y_1 \eta(t) / y_2)^{\frac{1}{\gamma-1}}}{1 + (y_1 \eta(t) / y_2)^{\frac{1}{\gamma-1}}}; \quad c_2(t) = \delta(t) \frac{1}{1 + (y_1 \eta(t) / y_2)^{\frac{1}{\gamma-1}}}. \quad (\text{A5})$$

The constants $\lambda(0)$, y_1 , and y_2 solve the static individual first-order conditions and budget constraints $E^n[\int V'_\delta \cdot (y_n \xi_t^n)^{\frac{1}{\gamma-1}} dt] = e_n P(0)$, which imply

$$\begin{aligned}E^1 \left[\int \delta_t \left(\frac{(y_1 \eta(t) / y_2)^{1/\gamma-1}}{(1 + (y_1 \eta(t) / y_2)^{1/\gamma-1})} \right)^\gamma dt \right] &= e_1 P(0) \\ E^2 \left[\int \delta_t \frac{y_1}{y_2} \eta(t) \left(\frac{1}{(1 + (y_1 \eta(t) / y_2)^{1/\gamma-1})} \right)^\gamma dt \right] &= e_2 P(0).\end{aligned}$$

Using Ito's rule, λ_t satisfies the differential equation $\frac{d\lambda_t}{\lambda_t} = \frac{d\eta_t}{\eta_t} = -(\theta_\delta^1 - \theta_\delta^2) dW_\delta^1 - (\theta_z^1 - \theta_z^2) dW_z^1$. Thus

$$\frac{d\lambda_t}{\lambda_t} = -\psi_\delta dW_\delta^1 - \left(a\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\psi_z(t) \right) dW_z^1. \quad \text{Q.E.D.}$$

Proof of Proposition 2 (Market Prices of Risk): From the first-order conditions for the optimal consumption choice for agent n , we have $c_n(t) = (y_n \xi^n(t))^{1/(\gamma-1)}$. Moreover, $\frac{d\xi^n(t)}{\xi^n(t)} = -r(t)dt - \theta^n(\delta(t), z(t))dW_Y^n$. Thus, applying Ito's Lemma we obtain

$$dc_n(t) = \mu_{c_n}^n(t)dt - \frac{c_n(t)}{\gamma-1}\theta_\delta^n(t)dW_\delta^n(t) - \frac{c_n(t)}{\gamma-1}\theta_z^n(t)dW_z^n(t).$$

The market clearing condition for consumption requires that $c_1(t) + c_2(t) = \delta(t)$, thus the diffusion process for the sum of individual consumptions should be identical to the diffusion process for the dividend. Hence, the following restrictions follow. First,

$$\frac{c_1(t)}{1-\gamma}\theta_\delta^1(t) + \frac{c_2(t)}{1-\gamma}\theta_\delta^2(t) = \sigma_\delta\delta(t).$$

From $\theta_\delta^1(t) - \theta_\delta^2(t) = \psi_\delta(t)$, the price of dividend risk is equal to $\theta_\delta^1(t) = [\frac{c_1(t)}{1-\gamma} + \frac{c_2(t)}{1-\gamma}]^{-1}[\sigma_\delta\delta(t) + \frac{\psi_\delta(t)}{1-\gamma}]$. Simplifying the terms and using the market clearing condition, we obtain the price of the dividend risk for each agent as

$$\theta_\delta^1(t) = (1-\gamma)\sigma_\delta + \psi_\delta(t)\frac{c_2(t)}{\delta(t)}, \quad \theta_\delta^2(t) = (1-\gamma)\sigma_\delta - \psi_\delta(t)\frac{c_1(t)}{\delta(t)}.$$

Substituting the solution for the individual consumptions, we obtain the solution for the dividend price of risk

$$\begin{aligned} \theta_\delta^1(t) &= (1-\gamma)\sigma_\delta + \frac{\psi_\delta(t)}{(1 + (y_1\eta(t)/y_2)^{\frac{1}{\gamma-1}})}, \\ \theta_\delta^2(t) &= (1-\gamma)\sigma_\delta - \frac{\psi_\delta(t)(y_1\eta(t)/y_2)^{\frac{1}{\gamma-1}}}{(1 + (y_1\eta(t)/y_2)^{\frac{1}{\gamma-1}})}. \end{aligned}$$

Second, for the individual price of signal risk the restriction is

$$\frac{c_1(t)}{1-\gamma}\theta_z^1(t) + \frac{c_2(t)}{1-\gamma}\theta_z^2(t) = 0.$$

Because the difference of the individual prices of signal risk is equal to $\theta_z^1(t) - \theta_z^2(t) = (\alpha\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(t))$, the prices of signal risk are equal to

$$\theta_z^1(t) = \frac{c_2(t)}{\delta(t)} \left(\alpha\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(t) \right), \quad \theta_z^2(t) = -\frac{c_1(t)}{\delta(t)} \left(\alpha\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(t) \right).$$

Substituting the solution for individual consumptions into the equation above, we obtain the solution for the signal price of risk

$$\theta_z^1(t) = \frac{1}{1 + (y_1\eta(t)/y_2)^{\frac{1}{\gamma-1}}} \left(\alpha\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(t) \right),$$

$$\theta_z^2(t) = -\frac{(y_1\eta(t)/y)^{\frac{1}{\gamma-1}}}{1 + (y_1\eta(t)/y_2)^{\frac{1}{\gamma-1}}} \left(\alpha\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(t) \right). \quad \text{Q.E.D.}$$

LEMMA 2 (Joint Distribution of Dividend and SPD Ratio): *When agents' prior distributions have common variance, the conditional joint distribution of the stochastic weight process and dividend process $(\eta(t), \delta(t))$, is bivariate log normal.*

- (a) *Marginal distribution of $\eta(s)$:* The dynamics of the stochastic weight process $\eta(t)$ are given by

$$\frac{d\eta(t)}{\eta(t)} = -\psi_\delta(t)dW_\delta^1(t) - \left(\alpha\psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(t) \right) dW_z^1(t),$$

so that

$$\eta(s) = \eta(t) \exp \left\{ -\int_t^s \psi_\delta(u)dW_\delta^1(u) - \int_t^s \left(\alpha\psi_\delta(u)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(u) \right) dW_z^1(u) \right\}$$

$$\times \exp \left\{ \int_t^s \left[-\frac{1}{2}[\psi_\delta(u)]^2 - \frac{1}{2} \left[\alpha\psi_\delta(u)\frac{\sigma_\delta}{\sigma_z} + \beta\psi_z(u) \right]^2 \right] du \right\}.$$

From Remark 1, it follows that under the maintained assumption of a common variance in the priors' distribution, the functions $\psi_\delta(u)$ and $\psi_z(u)$ are deterministic. Thus, the previous Ito integrals are normally distributed. It follows that the stochastic weight process $\eta(s)$, conditional on time t , is log normally distributed with mean $M_\eta(s, t)$ and variance $V_\eta(s, t)$, which can be easily computed as solutions of two deterministic integrals, that is,

$$\eta(s) = \eta(t) \exp \left(M_\eta(s, t) - \sqrt{V_\eta(s, t)} Z_\eta \right), \quad Z_\eta \sim N(0, 1).$$

- (b) *Marginal distribution of $\delta(s)$:* The dynamics of the log-dividend process $\delta(s)$ follow $d \ln \delta(t) = \mu_\delta(t)dt + \sigma_\delta dW_\delta(t)$. The first agent-perceived process is

$$d \ln \delta(t) = m_\delta^1(t)dt + \sigma_\delta dW_\delta^1(t),$$

with $m_\delta^1(t)$ following the Gaussian stochastic differential equation (7). This implies that the agent-perceived distribution of $\log(\delta(t))$ is normal with mean $M_\delta(s, t)$ and variance $V_\delta(s, t)$.

- (c) *Joint distribution of $\eta(s)$ and $\delta(s)$:* Because both $\log(\delta(s))$ and $\log(\eta(s))$ are conditionally normal and their dependence is linear, their joint distribution is normal. Let the conditional covariance be $\text{Cov}_{\delta, \eta}(s, t)$, which can easily be obtained as the solution of a deterministic integral, then be

$$\begin{pmatrix} \log \eta(s) \\ \log \delta(s) \end{pmatrix} \sim \mathcal{N} \left(\begin{bmatrix} M_\eta(t, s) \\ M_\delta(t, s) \end{bmatrix}; \begin{bmatrix} V_\eta(t, s) & \text{Cov}_{\delta, \eta}(t, s) \\ \text{Cov}_{\delta, \eta}(t, s) & V_\delta(t, s) \end{bmatrix} \right).$$

The closed-form solutions for V_η , V_δ , and $\text{Cov}_{\delta, \eta}$ are available upon request from the authors. Q.E.D.

Proof of Proposition 3 (Asset Prices):

(a) *Stock price:* From the Euler equation, the stock price $P(t) = \frac{1}{\xi^1(t)} E^1[\int_t^T \xi^1(s) \delta(s) ds]$. Thus,

$$P(t) = \frac{E_t^1 \left[\int_t^T [\delta(s)]^{\gamma-1} (1 + \lambda(s)^{\frac{1}{\gamma-1}})^{1-\gamma} \lambda(s) \delta(s) ds \right]}{[\delta(t)]^{\gamma-1} (1 + \lambda(t)^{\frac{1}{\gamma-1}})^{1-\gamma} \lambda(t)}.$$

Because $\delta(s)$ and $\eta(s)$ are conditionally log normally distributed, it is convenient to define the functions F_η and F_δ ,

$$\begin{aligned} F_\eta(t, s) &= F(t, s, \eta(t), y_1, y_2, Z_\eta) \\ &= (1 + (\lambda(t) e^{M_\eta(s, t) - \sqrt{V_\eta(s, t)} Z_\eta})^{\frac{1}{\gamma-1}})^{1-\gamma} \lambda(t) e^{M_\eta(s, t) - \sqrt{V_\eta(s, t)} Z_\eta} \end{aligned} \quad (\text{A6})$$

$$F_\delta(t, s) = F(t, s, \delta(t), Z_\delta) = \delta(t) e^{M_\delta(s, t) - \sqrt{V_\delta(s, t)} Z_\delta},$$

so that the pricing kernel can be written as $\xi_1(s) = F_\eta(t, s) F_\delta(t, s)^{\gamma-1}$. The numerator of the stock price can be written as

$$\begin{aligned} &E_t^1 \left[\int_t^T F_\eta(t, s) F_\delta(t, s)^\gamma ds \right] \\ &= \int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_\eta(t, s, Z_\eta, Z_\delta) F_\delta(t, s, Z_\eta, Z_\delta)^\gamma \mathcal{N}(Z_\delta, Z_\eta, \rho(t, s)) dZ_\delta dZ_\eta ds, \end{aligned}$$

where $\mathcal{N}(Z_\delta, Z_\eta, \rho(t, s))$ is a bivariate normal probability density function with correlation $\rho(t, s) = \frac{\text{Cov}(t, s)}{\sqrt{V_\delta(t, s)} \sqrt{V_\eta(t, s)}}$. The stock prices can be computed at any desired degree of accuracy by solving for the deterministic integral.

(b) *Option price:* From the Euler conditions, the option price $C(t, H) = \frac{1}{\xi^1(t)} E^1[\xi^1(H) \max(P(H) - K, 0)]$. Thus,

$$C(t, H) = \frac{E^1[F_\eta(t, H) F_\delta(t, H)^{\gamma-1} \max(P(H, \delta(H), \eta(H)) - K, 0)]}{F_\eta(t, t) F_\delta(t, t)^{\gamma-1}}.$$

Because $\delta(t)$ and $\eta(t)$ are jointly log normal, then the numerator can be computed by solving

$$C(t, H) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{\eta}(t, H, Z_{\eta}) F_{\delta}(t, H, Z_{\delta})^{\gamma-1} \\ \times \max(P(H) - K, 0) \mathcal{N}(Z_{\delta}, Z_{\eta}, \rho(H, t)) dZ_{\delta} dZ_{\eta}.$$

(c) *Bond price.* The bond price is given by $B(t, H) = \frac{1}{\xi^1(t)} E_t^1[\xi^1(t, H)]$. Thus,

$$B(t, H) = \frac{1}{F_{\eta}(t, t) F_{\delta}(t, t)^{\gamma-1}} E_t^1[F_{\eta}(t, H) F_{\delta}(t, H)^{\gamma-1}].$$

The expectation part of the product can be calculated using the joint normality of $\delta(t)$ and $\eta(t)$ derived in Lemma 2. Therefore,

$$E_t^1[F_{\eta}(t, H) F_{\delta}(t, H)^{\gamma-1}] \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{\eta}(t, H, Z_{\eta}) F_{\delta}(t, H, Z_{\delta})^{\gamma-1} \mathcal{N}(Z_{\delta}, Z_{\eta}, \rho(s)) dZ_{\delta} dZ_{\eta}.$$

Given the functional form of $F_{\eta}(s, t)$ and $F_{\delta}(s, t)$, the prices of the stock, option, and bond are deterministic integrals and can be computed at any desired level of accuracy using standard numerical integration methods. Q.E.D.

Proof of Proposition 4 (Stock and Option Holdings):

Wealth dynamics and sensitivity to $\delta(t)$ and $\eta(t)$:

The wealth process of the first agent is $X_1(t) = \frac{1}{\xi^1(t)} E^1[\int_t^T \xi^1(s) c_1(s) ds]$. Using the result for the distributions of $\delta(s)$ and $\eta(s)$, the pricing kernel is equal to

$$\frac{\xi^1(s)}{\xi^1(t)} = (e^{M_{\delta}(s,t) - \sqrt{V_{\delta}(s,t)} Z_{\delta}})^{\gamma-1} \left[\frac{1 + \lambda_t e^{\frac{1}{\gamma-1}(M_{\eta}(s,t) - \sqrt{V_{\eta}(s,t)} Z_{\eta})}}{1 + \lambda_t e^{\frac{1}{\gamma-1}}} \right]^{1-\gamma} \times e^{M_{\eta}(s,t) - \sqrt{V_{\eta}(s,t)} Z_{\eta}}.$$

Let $\zeta(t, s; \lambda_t) = \frac{\xi^1(s)}{\xi^1(t)}$, so that

$$\frac{\partial}{\partial \lambda_t} \zeta(s, t; \lambda_t) = e^{M_{\eta}(s,t) - \sqrt{V_{\eta}(s,t)} Z_{\eta}} (e^{M_{\delta}(s,t) - \sqrt{V_{\delta}(s,t)} Z_{\delta}})^{\gamma-1} \\ \times \left[e^{\frac{1}{\gamma-1}(M_{\eta}(s,t) - \sqrt{V_{\eta}(s,t)} Z_{\eta})} - \gamma e^{\frac{1}{\gamma-1}(M_{\eta}(s,t) - \sqrt{V_{\eta}(s,t)} Z_{\eta})} \right. \\ \left. - e^{\frac{1}{\gamma-1}} + \gamma e^{\frac{1}{\gamma-1}} \right] \times A,$$

with

$$A \equiv \left[\left(\frac{\lambda e^{\frac{1}{\gamma-1}(M_\eta(s,t) - \sqrt{V_\eta(s,t)}Z_\eta)} + 1}{\lambda e^{\frac{1}{\gamma-1}} + 1} \right)^\gamma + 2\lambda e^{\frac{1}{\gamma-1}} \left(\frac{\lambda e^{\frac{1}{\gamma-1}(M_\eta(s,t) - \sqrt{V_\eta(s,t)}Z_\eta)} + 1}{\lambda e^{\frac{1}{\gamma-1}} + 1} \right)^\gamma + \lambda^2 e^{\frac{2}{\gamma-1}} \left(\frac{\lambda e^{\frac{1}{\gamma-1}(M_\eta(s,t) - \sqrt{V_\eta(s,t)}Z_\eta)} + 1}{\lambda e^{\frac{1}{\gamma-1}} + 1} \right)^\gamma \right]^{-1}.$$

From Fubini's Theorem, the derivative of the wealth process $X(t)$ with respect to $\delta(t)$ is

$$\frac{\partial X(t)}{\partial \eta(t)} = \frac{y_1}{y_2} E_1 \int_t^T \left[\frac{\partial}{\partial \lambda_t} \zeta(s, t; \lambda_t) \right] \frac{\lambda_t e^{\frac{1}{\gamma-1}(M_\eta(s,t) - \sqrt{V_\eta(s,t)}Z_\eta)}}{1 - \lambda_t e^{\frac{1}{\gamma-1}(M_\eta(s,t) - \sqrt{V_\eta(s,t)}Z_\eta)}} F_\delta(t, s) ds.$$

Similarly, the derivative of the wealth process with respect to $\eta(t)$ is given by

$$\frac{\partial X(t)}{\partial \delta(t)} = E_1 \int_t^T \zeta(s, t; \lambda_t) \frac{\lambda_t e^{\frac{1}{\gamma-1}(M_\eta(s,t) - \sqrt{V_\eta(s,t)}Z_\eta)}}{1 - \lambda_t e^{\frac{1}{\gamma-1}(M_\eta(s,t) - \sqrt{V_\eta(s,t)}Z_\eta)}} e^{M_\delta(s,t) - \sqrt{V_\delta(s,t)}Z_\delta} ds.$$

Asset price sensitivity to $\delta(t)$ and $\eta(t)$:

The stock and option price sensitivities with respect to $\delta(t)$ and $\eta(t)$ are computed from the results of Proposition 3. Because $P(t) = E_t^1[\int_t^T \zeta(s, t; \lambda_t) \delta(s) ds]$, from Fubini's Theorem, the derivative of the wealth process $X(t)$ with respect to $\delta(t)$ is

$$\frac{\partial P(t)}{\partial \eta(t)} = \frac{y_1}{y_2} E_1 \int_t^T \frac{\partial}{\partial \lambda_t} \zeta(s, t; \lambda_t) \delta_t e^{M_\delta(s,t) - \sqrt{V_\delta(s,t)}Z_\delta} ds.$$

Similarly, the derivative of the wealth process with respect to $\delta(t)$ is

$$\frac{\partial P(t)}{\partial \delta(t)} = E_1 \int_t^T \zeta(s, t; \lambda_t) e^{M_\delta(s,t) - \sqrt{V_\delta(s,t)}Z_\delta} ds.$$

Similarly, because the option price is

$$E_t^1[\cdot] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta(s, t; \lambda_t) \max(P(H) - K, 0) N(Z_\delta, Z_\eta, \rho(H, t)) dZ_\delta dZ_\eta,$$

the derivative with respect to $\delta(t)$ and $\eta(t)$ can be computed using Fubini's Theorem. Q.E.D.

Appendix B: Portfolio Choice in the Heston Model

The Heston (1993) model assumes that under the risk-neutral measure the processes for stock returns and volatility are given by

$$\begin{aligned}\frac{dS_t}{S_t} &= (r - \delta)dt + \sqrt{V_t}dW_S(t) \\ dV_t &= k_v(\theta_v - V_t)dt + \sigma_v\sqrt{V_t}dW_V(t).\end{aligned}$$

The correlation coefficient between stock prices and volatility is $\text{Cov}(dW_S(t), dW_V(t)) = \rho dt$. We consider an investor who maximizes the expected utility of his terminal wealth

$$\max_{\pi(t)} E\left(\frac{W^\gamma}{\gamma}\right).$$

As Liu and Pan (2003) show, the solution of this problem is

$$\begin{aligned}\pi_1(t) &= \frac{\lambda_1}{1 - \gamma} - \frac{\lambda_2\rho}{(1 - \gamma)\sqrt{1 - \rho^2}} - \pi_2(t)\frac{(\partial C_t/\partial S_t)S_t}{C_t} \\ \pi_2(t) &= \left[\frac{\lambda_2}{(1 - \gamma)\sigma_v\sqrt{1 - \rho^2}} + H(T - t) \right] \frac{C_t}{(\partial C_t/\partial V_t)},\end{aligned}$$

where C_t is the price of an option calculated as in Heston, $(\partial C_t/\partial S_t)$ and $(\partial C_t/\partial V_t)$ are the sensitivities of the option price to changes in the stock price and volatility, λ_1 and λ_2 are the prices of risk of the stock diffusion and volatility diffusion, and $H(T - t)$ is given by the following functional form

$$\begin{aligned}H(\tau) &= \frac{\exp(k_2\tau)}{2k_2 + (k_1 + k_2)(\exp(k_2\tau) - 1)} \frac{\gamma(\lambda_1 + \lambda_2)}{(1 - \gamma)^2}, \quad k = k_v - \lambda_2 \\ k_1 &= k - \frac{\gamma}{1 - \gamma}(\lambda_1\rho + \lambda_2\sqrt{1 - \rho^2})\sigma, \quad k_2 = \sqrt{k_1^2 - \frac{\gamma(\lambda_1 + \lambda_2)}{(1 - \gamma)^2}\sigma_v^2}.\end{aligned}$$

See Liu and Pan (2003), Liu (2001), and Liu, Longstaff, and Pan (2003) for a detailed derivation of the formulas.

REFERENCES

- Ait-Sahalia, Yacine, and Andrew W. Lo, 1998, Estimation of state-price densities implicit in financial asset prices, *Journal of Finance* 53, 499–547.
- Amin, Kaushik, and Charles M. Lee, 1997, Option trading, price discovery and earnings news dissemination, *Contemporary Accounting Research* 14, 153–192.
- Andersen, Torben G., 1996, Return volatility and trading volume: An information flow interpretation of stochastic volatility, *Journal of Finance* 51, 169–204.
- Back, Kerry, 1993, Asymmetric information and options, *Review of Financial Studies* 6, 435–472.
- Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 1997, Empirical performance of alternative option pricing models, *Journal of Finance* 52, 2003–2049.

- Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 2000, Do call prices and the underlying stock always move in the same direction? *The Review of Financial Studies* 13, 549–584.
- Bakshi, Gurdip, and Nikunj Kapadia, 2003, Delta-hedged gains and the pricing of volatility risk, *Review of Financial Studies* 16, 527–566.
- Bakshi, Gurdip, Nikunj Kapadia, and Dilip Madan, 2003, Stock return characteristics, skew laws, and differential pricing of individual equity options, *Review of Financial Studies* 16, 101–143.
- Basak, Suleyman, 2000, A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous risk, *Journal of Economic Dynamics and Control* 24, 63–95.
- Bates, D., 2000, Post '87 crash fears in the S&P 500 futures option market, *Journal of Econometrics* 94, 181–238.
- Bates, David, 2001, The market for crash risk, *NBER Working paper* 8557.
- Bessembinder, Hendrik, Kalok Chan, and Paul J. Seguin, 1996, An empirical examination of information, differences in opinions, and trading activity, *Journal of Financial Economics* 40, 105–134.
- Biais, Bruno, and Pierre Hillion, 1994, Insider and liquidity trading in stock and options markets, *Review of Financial Studies* 7, 743–780.
- Black, Fischer, and Myron Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–659.
- Brennan, Michael J., 1998, The role of learning in dynamic portfolio decisions, *European Finance Review* 1, 295–306.
- Brennan, Michael J., and H. Henry Cao, 1996, Information, trade and derivative securities, *Review of Financial Studies* 9, 165–208.
- Brennan, Michael J., and R. Solanki, 1981, Optimal portfolio insurance, *Journal of Financial and Quantitative Analysis* 16, 279–300.
- Brennan, Michael J., and Yihong Xia, 2000, Stochastic interest rates and the bond-stock mix, *European Finance Review* 4, 197–210.
- Buraschi, Andrea, and Jens Carsten Jackwerth, 2001, The price of a smile: Hedging and spanning in option markets, *The Review of Financial Studies* 14, 495–527.
- Campbell, S. D., and C. Li, 2001, Option prices with unobserved and regime switching volatility, Working paper, University of Pennsylvania.
- Cao, Charles, Zhiwu Chen, and John Griffin, 2005, The informational content of option volume prior to takeovers, *Journal of Business* 78, 1073–1109.
- Chakravarty, S., H. Gulen, and Stewart Mayhew, 2002, Informed trading in stock and option markets, Working paper, University of Georgia.
- Christensen, B.J., and N.R. Prabhala, 1998, The relation between implied and realized volatility, *Journal of Financial Economics* 3, 125–150.
- Cox, John, and Chi-Fu Huang, 1989, Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory* 49, 33–83.
- Cuoco, Domenico, and Hua He, 1994, Dynamic equilibrium in infinite-dimensional economies with incomplete financial markets, Working paper, University of Pennsylvania.
- David, Alexander, 1997, Fluctuating confidence in stock markets: Implications for returns of volatility, *Journal of Financial and Quantitative Analysis* 32, 427–462.
- David, Alexander, and Pietro Veronesi, 2001, Inflation and earnings uncertainty and the volatility of asset prices: An empirical investigation, Working paper, University of Chicago.
- David, Alexander, and Pietro Veronesi, 2002, Option prices with uncertain fundamentals, Working paper, University of Chicago.
- Derman, Emanuel, and Iraj Kani, 1994, Riding on a smile, *Risk* 7, 139–145.
- Detemple, Jerome, and Shashidhar Murthy, 1994, Intertemporal asset pricing with heterogeneous beliefs, *Journal of Economic Theory* 62, 294–320.
- Detemple, Jerome B., 1986, Asset pricing in a production economy with incomplete information, *Journal of Finance* 41, 383–391.
- Dothan, Michael U., and David Feldman, 1986, Equilibrium interest rates and multiperiod bonds in a partially observable economy, *Journal of Finance* 61, 369–382.
- Duffie, Darrell, J. Pan, and K. Singleton, 2000, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* 68, 1343–1376.

- Dumas, Bernard, Jeff Fleming, and Robert E. Whaley, 1998, Implied volatility functions: Empirical tests, *Journal of Finance* 53, 2059–2106.
- Easley, David, Maureen O'Hara, and P. S. Srinivas, 1998, Option volume and stock prices: Evidence on where informed traders trade, *Journal of Finance* 53, 431–465.
- Figlewski, Stephen, and Gwendolyn P. Webb, 1993, Options, short sales, and market completeness, *Journal of Finance* 48, 761–777.
- Fleming, Jeff, and Robert E. Whaley, 1994, The value of wildcard options, *Journal of Finance* 49, 215–236.
- Franke, Gunter, Richard Stapleton, and Marti Subrahmanyam, 1998, Who buys and who sells options? The role of options in an economy with background risk, *Journal of Economic Theory* 82, 89–109.
- Gallant, Roland, Peter Rossi, and George Tauchen, 1992, Stock prices and volume, *Review of Financial Studies* 5, 199–242.
- Garcia, Rene, Richard Luger, and Éric Renault, 1999, Asymmetric smiles, leverage effects and structural parameters, Working paper, CIRANO.
- Genotte, Gerard, 1986, Optimal portfolio choice under incomplete information, *Journal of Finance* 41, 733–746.
- Grossman, Sanford J., and Zhongquan Zhou, 1996, Equilibrium analysis of portfolio insurance, *Journal of Finance* 51, 1379–1403.
- Guidolin, Massimo, and Allan Timmermann, 2000, Option prices and implied volatility dynamics under Bayesian learning, Working paper, University of California at San Diego.
- Hansen, Lars Peter, and Ravi Jagannathan, 1997, Assessing specification errors in stochastic discount factor models, *Journal of Finance* 52, 557–590.
- Harris, Milton, and Arthur Raviv, 1993, Differences in opinion make a horse race, *Review of Financial Studies* 6, 473–506.
- Heston, Steven L., 1993, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* 6, 327–343.
- Hull, John C., and Alan White, 1987, The pricing of options with stochastic volatilities, *Journal of Finance* 42, 281–300.
- John, Kose, Apoorva Koticha, Ranga Narayanan, and Marti Subramanyam, 2000, Margin rules, informed trading in derivatives and price dynamics, Working paper, New York University.
- Karatzas, Ioannis, John P. Lehoczky, and Stephen E. Shreve, 1987, Optimal portfolio and consumption decisions for a “small investor” on a finite horizon, *SIAM Journal of Control and Optimization* 25, 1557–1586.
- Karatzas, Ioannis, and Steven E. Shreve, 1998, *Methods of Mathematical Finance* (Springer-Verlag, NY).
- Leland, Hayne E., 1980, Who should buy portfolio insurance? *Journal of Finance* 35, 581–594.
- Liu, Jun, 2001, Dynamic portfolio choice and risk aversion, Working paper, UCLA.
- Liu, Jun, Francis Longstaff, and Jun Pan, 2003, Dynamics asset allocation with event risk, *Journal of Finance* 58, 231–259.
- Liu, Jun, and Jun Pan, 2003, Dynamic derivative strategies, *Journal of Financial Economics* 69, 401–430.
- Liptser, Robert S., and Albert N. Shiryaev, 2001, *Statistics of Random Processes* (Springer, NY).
- Madan, D., P. Carr, and E. Chang, 1998, The variance gamma process and option pricing, *European Journal of Finance* 1, 39–55.
- Manaster, Steven, and Richard J. Rendleman, 1982, Option prices as predictors of equilibrium stock prices, *Journal of Finance* 37, 1043–1057.
- Mayhew, Stewart, Atulya Sarin, and Kuldeep Shastri, 1995, The allocation of informed trading across related markets: An analysis of the impact of changes in equity-option margin requirements, *Journal of Finance* 50, 1635–1653.
- Merton, Robert C., 1973, Theory of rational option pricing, *Bell Journal of Economics* 4, 141–83.
- Morris, Stephen, 1995, The common prior assumption in economic theory, *Economics and Philosophy* 11, 227–253.
- Newey, Whitney K., and Kenneth D. West, 1987, A simple positive semi-definite, heteroscedasticity and autocorrelation consistent covariance matrix, *Econometrica* 55, 703–708.

- Pan, J., 2002, The jump-risk premia implicit in options: Evidence from an integrated time-series study, *Journal of Financial Economics* 63, 3–50.
- Pan, Jun, and Allen M. Potesman, 2006, The information in option volume for stock prices, *Review of Financial Studies* (forthcoming).
- Rubinstein, Mark, 1994, Implied binomial trees, *Journal of Finance* 49, 771–818.
- Scheinkman, Jose A., and W. Xiong, 2003, Overconfidence and speculative bubbles, *Journal of Political Economy* 111, 1183–1219.
- Shalen, Catherine T., 1993, Volume, volatility and the dispersion of beliefs, *Review of Financial Studies* 6, 405–434.
- Silverman, B.W., 1986, *Density Estimation* (Chapman and Hall, London).
- Stephan, Jens A., and Robert E. Whaley, 1990, Intraday price change and trading volume relations in the stock and stock option markets, *Journal of Finance* 45, 191–220.
- Varian, Hal R., 1989, Differences of opinion in financial markets, in financial risk: Theory, evidence and implications: Proceedings of the 11th Annual Economic Policy Conference of the Federal Reserve Bank of St. Louis, pp. 3–37.
- Veronesi, Pietro, 1999, Stock market overreaction to bad news in good times: A rational expectations equilibrium model, *Review of Financial Studies* 12, 975–1007.
- Veronesi, Pietro, 2000, How does information quality affect stock returns, *Journal of Finance* 55, 807–837.
- Wiggins, James B., 1987, Option values under stochastic volatility: Theory and empirical estimates, *Journal of Financial Economics* 19, 351–372.
- Xia, Yihong, 2001, Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation, *Journal of Finance* 56, 205–246.
- Zapatero, Fernando, 1998, Effects of financial innovations on market volatility when beliefs are heterogeneous, *Journal of Economic Dynamics and Control* 22, 597–626.

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