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The Limits of Investor Behavior

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ABSTRACT

Many models use noise trader risk and corresponding violations of the Law of One Price to explain pricing anomalies, but include a storage technology in perfectly elastic supply or unlimited asset liability. Storage allows aggregate consumption risk to differ from exogenous fundamental risk, but using aggregate consumption as a factor for asset returns can make noise trader risk superfluous. Using (i) limited asset liability and limited storage withdrawals, or (ii) an endogenous locally riskless interest rate eliminates violations of the Law of One Price. Our main results use only budget equations and market clearing, and require virtually no assumptions about behavior.

THE IDEA THAT REAL-WORLD INVESTORS MAY NOT BE fully rational is an important influence on studies of modern asset pricing. Models of investor behavior naturally emphasize assumptions about investor rationality, but also embedded into these models are budget constraints, market clearing, the absence of single-period arbitrage opportunities, and assumptions about asset liability. We show that these features limit the impact of investor behavior on asset prices except in implausible circumstances. Common assumptions in the literature often thwart these limits and therefore yield misleading economic intuition. Thus, while investor behavior can be important for asset pricing, it is important only in a manner different from that portrayed by models with these assumptions.

We make this point by dissecting the seminal model of De Long et al. (1990) (DSSW). DSSW argue that noise traders can "create their own space," meaning that nonrational investors generate price risk unexplained by classical measures of economic risk, which include endowment and dividend risk. DSSW also argue that noise trader risk can cause violations of the Law of One Price that help explain asset pricing puzzles. However, DSSW's model includes the main assumptions we study, namely, unlimited asset liability and a storage technology in perfectly elastic supply. While these assumptions may seem harmless, we show that they are not. DSSW's explanations of asset pricing puzzles would be invalid given both limited asset liability and limited storage withdrawals,

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or an endogenous locally riskless rate that clears all markets (independent of limits on asset liabilities and storage withdrawals).

Models with unlimited asset liability and a storage technology in perfectly elastic supply are common, perhaps because of their mathematical tractability given other assumptions.[1] Such models include Shleifer and Vishny (1990), Campbell and Kyle (1993), Shleifer and Vishny (1997), Spiegel (1998), Campbell and Cochrane (1999), Campbell (2000), Barberis and Huang (2001), and Kyle and Xiong (2001). Empirical tests of such models include Lee, Sleifer, and Thaler (1991), Campbell and Kyle (1993), Campbell and Cochrane (1999), Gemmill and Thomas (2002), and Mitchell, Pulvino, and Stafford (2002). While we focus on DSSW's model, we provide robust economic explanations of why each of these models can generate deceptive conclusions about investor behavior and asset prices.[2]

Several studies dispute DSSW's conclusions about noise trader risk, including Bhushan, Brown, and Mello (1987), Allen (2001), Brav and Heaton (2002), Constantinides (2002), Kogan et al. (2002), and Ross (2005, ch. 4). These papers focus on whether DSSW's conclusions are robust to different forms of nonrational investor behavior, whether investor behavior is empirically important, or whether noise traders survive in the long run.[3] While all of these studies incorporate specific assumptions about investor behavior, they all share common economic features that limit the properties of asset prices regardless of their specific assumptions.

We argue that many properties of asset prices can be derived without reference to specific assumptions about investor rationality, given minimal and natural assumptions about limited asset liability, market clearing, and limited storage withdrawals. Our paper does not provide a defense for either investor rationality or nonrationality. As such, our paper is in the spirit of Becker (1962), who argues that some important economic theorems (such as downward-sloping demand curves for commodities) are really consequences of general principles (such as budget constraints) and are largely independent of assumptions about investor behavior. Similarly, if one believes that limited asset liability, market clearing, and limited storage withdrawals are reasonable economic assumptions, then one must regard the implied properties of asset prices as inviolable since they are independent of investor rationality.

Our first set of results, given in Section II, questions DSSW's conclusion that noise traders create their own space by causing systematic stock price risk that is unexplained by any fundamental determinant of expected asset

[1] When used with negative exponential utility and normally distributed quantities, for example, the assumptions often permit linear equilibrium quantities.

[2] We do not present results specific to each model. The advantage of DSSW's model is that it is tractable and its assumptions are relatively well specified. An earlier version of this work presents a more general framework, nesting nearly every asset pricing model in the literature. We thank several colleagues for suggesting that we clarify our main message by focusing on the DSSW model.

[3] Bhushan, Brown, and Mello (1987) in particular conclude that it is "difficult, if not impossible," to derive investment or regulatory policy implications from DSSW's model since different noise trader beliefs can produce conflicting conclusions.

returns. DSSW use a specific model to obtain this conclusion: The financial market consists of two assets, a storage technology in perfectly elastic supply and a stock with fixed positive supply. Each period a departing "old" generation with no endowment consumes the payoff of all of its prior financial investments while an incoming "young" generation uses its riskless endowment to buy the old generation's financial assets and consumes nothing. DSSW assume for most of their study that the young generation's total endowment is riskless, and that some of the investors that make up each generation may be nonrational, that is, "noise traders."[4]

DSSW show both that a risky stock price clears the stock market, and the amount of stock price risk varies positively with the proportion of noise traders. DSSW conclude that noise traders create their own space since ". . . none of the variance in the price of [the stock] can be justified by changes in fundamentals: there are no changes in expected dividends in our model or in any fundamental determinant of required returns" (p. 725). This may seem puzzling: How does the young generation afford a risky stock if it invests only a riskless amount? The answer is that the young generation takes appropriate long or short positions in the storage technology to make up the difference between the stock price and its riskless endowment. The young investors' budget equations therefore require that the stock price risk and the risk associated with the amount of storage be equivalent.

Is the risky amount of storage a fundamental determinant of required returns? Yes. Because of the investors' budget constraints, the amount of storage is connected to aggregate consumption, and the Consumption Capital Asset Pricing Model (CCAPM) literature argues that aggregate consumption is a primary determinant of expected asset returns (see Breeden and Litzenberger (1978) and Breeden (1979)). In DSSW's model, the old generation's consumption is aggregate consumption because the young generation does not consume. The old generation's consumption also equals the payoff from its prior investments in financial assets, so aggregate consumption risk is equivalent to both stock price risk and the risk associated with the amount of storage. The budget equations and market clearing for the stock require that these quantities be conditionally perfectly correlated and linear functions of each other, regardless of investor rationality (Proposition 1).

The intuition that noise traders create their own space is misleading for two reasons. First, it ignores aggregate consumption risk as a fundamental determinant of required returns, despite what the CCAPM literature suggests. Rather, it considers only dividend and endowment risk, which are incomplete determinants of expected returns because of the storage technology. Second, DSSW's observation that stock price risk increases with noise trader risk compares economies with varying levels of aggregate consumption, which is

[4] DSSW's specific assumptions about investor nonrationality are unimportant for our purposes. In our Introduction, we describe our results assuming a riskless exogenous endowment and riskless dividends. Just as is the case in DSSW's analysis, this simplifies exposition by highlighting the role of noise trader risk. We describe later, however, that our results extend in a straightforward manner to risky endowments and dividends and a locally riskless rate of interest.

inherently misleading. Indeed, noise traders could not create their own space given the same aggregate consumption for all levels of noise trader risk (Corollary 1); this fact spoils DSSW's explanations for some asset pricing puzzles. While the overlapping generations structure of DSSW's model has special features, our analysis also reveals similar implications for other studies that compare economic quantities across equilibria given a storage technology and varying aggregate consumption.

Our second set of results, given in Section III, examines violations of the Law of One Price, assuming enough investor rationality to permit at least one positive stochastic discount factor that equates the stock price to the expected present value of its dividends over a finite horizon plus its liquidation value. Campbell (2000) calls this the "basic equation of asset pricing." This assumption still allows great flexibility in investor behavior, as it requires only mild assumptions about the monotonicity of some, but not all, investors' preferences for consumption (see Sec. III.A and Ross (2005, ch. 1)). The basic equation of asset pricing is satisfied in DSSW's model (example 1). When the basic equation of asset pricing holds, limited asset liability and limited withdrawals from the storage technology limit the properties of deviations from fundamental value, regardless of what one assumes about investor rationality.

Section III.B shows that a stock price's discount to fundamental value requires unlimited potential liabilities of owning that stock: The present value of those liabilities must exceed the sum of the present value of the stock's dividends and the present value of the exogenous aggregate endowment, provided the latter quantity is finite.[5] This result shows, for example, that DSSW's explanation of the closed-end fund discount puzzle is very fragile: Their explanation requires the closed-end fund to sell at a discount to its fundamental value, which is impossible for exchange-traded closed-end funds because they have nonnegative prices and, hence, no liabilities. While the empirical tests of DSSW's theory by Lee, Sleifer, and Thaler (1991), Gemmill and Thomas (2002), and Mitchell, Pulvino, and Stafford (2002) rely on discounts from fundamental value, they do not address the discrepancy between this impossibility and their empirical data. The limited liability property of empirical asset prices alone negates DSSW's explanation of the closed-end fund puzzle regardless of investor rationality.

Section III.C uses the basic equation of asset pricing to connect deviations from fundamental value to the unreasonably large amounts that the young generation must withdraw from the storage technology. This is linked to the neoclassical intuition that the stock price that has a positive deviation from fundamental value (an asset pricing "bubble") must appropriately "explode" in value through time (see, e.g., Blanchard and Watson (1983) and Santos and Woodford (1997)). If the young generation is to afford the supply of the stock, it must obtain a correspondingly explosive aggregate amount of funds. Even

[5] If these quantities were not finite, then asset prices would deviate from fundamental values even in models with full investor rationality and fully endogenous rates of return (see Shell (1971), Brock (1979), Scheinkman (1988), and Santos and Woodford (1997)).

if the stock price were influenced by nonrational behavior, its explosiveness is a necessary property of a positive deviation from fundamental value, and the explosive amount of funds would be made necessary by the investors' budget equations and market clearing for the stock.

In neoclassical theory, the funds typically come from an infinite-valued aggregate endowment (Shell (1971), Brock (1979), Scheinkman (1988), and Santos and Woodford (1997)). But in DSSW's model, some generation of young investors must make net withdrawals from storage because the aggregate endowment has finite present value. However, the net withdrawals from storage must be explosive enough to match the explosiveness of the positive deviation from fundamental value. In particular, the rate of return on storage, the proxy for the locally riskless rate of return, cannot equate borrowing and lending, for then net withdrawals from storage would always be zero. The net withdrawals cannot be bounded and cannot be asymptotically repaid. More remarkably, the net withdrawals needed to support the stock's premium to fundamental value must sometimes exceed the sum of the present value of the endowments of all future generations and all future stock dividends, or any positive multiple of this present value, which raises the question of how the implicit borrowing could be collateralized in such a model.[6]

It therefore follows that the stock price cannot be less than its fundamental value without an extreme notion of unlimited liability, and it cannot exceed its fundamental value without the unlimited borrowing allowed by a storage technology in perfectly elastic supply. An endogenous riskless rate alone would make impossible any deviation from fundamental value. Valid for virtually any notion of investor rationality, these results suggest that the theoretical underpinnings of DSSW's explanations of asset pricing puzzles are fragile, as are those of models with similar assumptions. The relevance of their explanations to empirically motivated asset pricing puzzles is questionable because real-world asset prices do not possess the features needed to support the explanations.

We now begin our analysis. Section I reviews the basic features of DSSW's model and provides groundwork for our analysis. Our main sections, Sections II and III, present our results about the limits imposed by equilibrium on investor behavior.

I. Review of DSSW's Model

In this section, we review the model of DSSW to remind the reader of its features. In Section I.B, we present DSSW's main conclusions about noise trader risk and equilibrium asset prices. Section I.C builds our framework for studying these conclusions in a manner that deemphasizes the role of investor behavior.

[6] Bhushan, Brown, and Mello (1987) show that position limits on the noise trader's holdings of the risky asset can prevent deviations from fundamental value in DSSW's model. We focus instead on limited liability and limited net withdrawals from storage, and show that these features are necessary and sufficient for preventing such deviations, regardless of investor rationality. Their position limits and assumptions about noise trader beliefs would be superfluous given the limits we study.

A. The Economic Environment

DSSW's model is an overlapping generations (OLG) model similar to that of Samuelson (1958). There are infinitely many trading dates indexed by the finite integers $t = 0, 1, \ldots$. There are two assets. One is available in perfectly elastic supply, has a price equal to one each period, and pays a positive dividend r each period. The other asset is long-lived, has price $P(t)$, and also pays a dividend r each period starting from $t = 1$. We call the first asset the "storage technology" and the second the "stock." The stock's supply is normalized to one unit, and its price is determined endogenously. As DSSW emphasize, if the returns of both assets were equal, then they would be perfect substitutes and the Law of One Price would prevail.

There are infinitely many overlapping generations of investors. Each investor lives for only two periods, arrives with a private endowment of the consumption good, and invests this endowment in one or both of the assets upon arrival. In the following period, the departing investor receives no endowment, entirely consumes the wealth generated by his prior period's investment, and receives utility for doing so. Some investors are rational and correctly anticipate the distribution of future asset prices. Other investors, whom DSSW call "noise traders," do not have rational expectations.

We need no specific assumptions about the individuals' private endowments of the consumption good; however, assumptions about the aggregate endowment are important. We denote the aggregate private endowment of the consumption good by e, which DSSW assume is riskless (p. 708). We make this our default assumption to be consistent with their analysis, but some of our results, particularly those in Section III.C, allow e to be risky. We also assume e is nonnegative.

Most of our results require no specific assumptions about the investors' preferences and rationality. DSSW, however, make the following specific assumption.

ASSUMPTION 1: *DSSW assume that each rational trader chooses a portfolio to maximize expected utility* $-E[\exp(-2\alpha\tilde{c})]$, *where* $\tilde{c}$ *is the terminal consumption,* α *is the coefficient of absolute risk aversion, and E is the rational expectations operator. The noise traders misperceive the expected stock price by an independent and identically distributed normal random variable* $\eta(t)$ *with mean* η^* *and variance* σ_η^2. *The proportion of noise traders is* μ; *the proportion of rational investors is* $1 - \mu$. *The noise traders, who have the mistaken beliefs described previously, maximize expected utility* $-\tilde{E}[\exp(-2\alpha\tilde{c})]$, *where* $\tilde{E}$ *describes the noise traders' expectations.*

We now present terminology that we use to emphasize those of our results that do not depend on Assumption 1 or other assumptions about investor rationality.

DEFINITION 1: *We say a result holds regardless of investor rationality if it holds without reference to assumptions about investor rationality.*

B. DSSW's Stock Price

The form of the stock price that DSSW study is

$$P^{\mathrm{DSSW}}(t) = 1 + \frac{\mu(\eta(t) - \eta^*)}{1+r} + \frac{\mu\eta^*}{r} - \frac{2\alpha\mu^2\sigma_\eta^2}{r(1+r)^2}. \tag{1}$$

Note that (1) requires Assumption 1 and is parameterized by the amount of noise trader risk μ. As DSSW discuss, the first component is the fundamental value of the stock, as we see later. The second is the random component of the stock price, which reflects the uncertainty of the noise traders' beliefs. The third reflects the average beliefs of the noise traders, and the fourth represents compensation for risk caused by the noise traders' beliefs. The last three components become more pronounced when the proportion μ of noise traders increases. DSSW (p. 705) explain the significance:

> *Because the unpredictability of noise traders' future opinions deters arbitrage, prices can diverge significantly from fundamental values even when there is no fundamental risk. Noise traders thus create their own space. All of the main results of our paper come from the observation that arbitrage does not eliminate the effects of noise because noise itself creates risk.*

Two features of (1) are particularly important for our purposes. First, DSSW's stock price is normally distributed and unbounded below when noise traders are present ($\mu > 0$). Second, it equals one whenever noise traders are absent ($\mu = 0$). Together these features suggest that noise traders create their own space since, given (1), the stock would be risky if noise traders were present and would be riskless otherwise. Similarly, (1) suggests that the stock price differs from its fundamental value whenever noise traders are present. Our purpose is to analyze these conclusions.

C. Alternate Stock Price Characterization

The form of DSSW's stock price in equation (1) requires the specific assumptions about investor preferences and rationality given in Assumption 1. We now present an alternate characterization of any stock price that is consistent with the investors' budget equations and market clearing for the stock that is independent of Assumption 1. Our alternate characterization enables us to answer "What if...?" questions about the stock price without specific assumptions about investor rationality. For instance, we ask "What if limited asset liability were a feature of DSSW's model?", and our answer describes the corresponding properties of any stock price given the investors hold the stock's net supply. Being able to answer such "What if . . . ?" questions is important for discovering the limits of investor behavior.

C.1. Budget Constraints and Walras's Law

Our alternate characterization of stock prices uses a version of Walras's Law. The general idea of Walras's Law is that the investors' budget equations and market clearing restrict certain equilibrium aggregate quantities. In our case, Walras's Law restricts the degree to which investor behavior affects equilibrium stock prices. Walras's Law does not depend on investor rationality, because even nonrational investors must choose portfolios that are allowable by their budget set and because the number of shares of stock demanded by all investors must equal the supply. To obtain general intuition, we first present an alternate characterization of the stock prices for any model with a stock and a storage technology; we then refine our characterization by invoking DSSW's assumptions about the OLG structure of their model.

Each investor has a budget equation that determines how much wealth the investor can invest in financial assets. Label this wealth W^i for investor i. Let $\theta^i(t)$ be the number of shares of the stock investor i holds at time t and $\delta^i(t)$ be the amount of wealth invested in the storage technology. By definition,

$$W^i(t) = \theta^i(t)P(t) + \delta^i(t). \tag{2}$$

We can rewrite (2) in terms of both the investor's financial asset holdings in period $t-1$ and current net consumption as

$$W^i(t) = \delta^i(t-1)(1+r) + \theta^i(t-1)(P(t)+r) - (c^i(t) - e^i(t)), \tag{3}$$

where $c^i(t)$ and $e^i(t)$ are investor i's time-t consumption and endowment. This says that the amount investor i invests in financial assets at time t equals the total return on his previous investments at time $t-1$ and his net consumption at time t.

The investors must collectively hold a number of shares of the stock equal to the net supply of the stock, which requires the equality $\sum_i \theta^i(t) = 1$ to hold. By summing (3) over all investors, the aggregate amount invested in financial assets, $W = \sum_i W^i$ satisfies,

$$W(t) = \delta(t) + P(t) = \delta(t-1)(1+r) + P(t) + r - (c(t) - e(t)), \tag{4}$$

where $\delta = \sum_i \delta^i$ is the net investment in the storage technology and c and e are aggregate consumption and endowments, respectively.

We now derive Walras's Law. The model has three markets, namely, the market for the stock, for the storage technology, and for the consumption good. We have already cleared the market for the stock by setting the aggregate demand for the stock equal to the aggregate number of shares. The traditional version of Walras's Law would then say that the remaining markets for storage and consumption either must clear simultaneously or must both fail to clear (Varian (1992)); in this setting, however, what it means for these markets to "clear" is complicated by the assumption that the storage technology has perfectly elastic supply. Our version of Walras's Law says that $\delta(t)$, the net investment in the storage technology, and $\gamma(t) \equiv c(t) - e(t) - r$, the difference between aggregate consumption $c(t)$ and the exogenous supply of consumption $e(t) + r$, are connected by the investors' budget equations.

LEMMA 1 (Walras's Law): *If investors collectively hold the net supply of the stock, that is* $\sum_i \theta^i(t) = 1$ *for all t, then the equality*

$$\delta(t) = \delta(0) - \sum_{j=0}^{t} (1+r)^{t-j} \gamma(j) \tag{5}$$

holds regardless of investor rationality.

Proof: Rewriting (4) yields $W(t) = \delta(t-1)(1+r) + P(t) - \gamma(t)$, which implies that $\delta(t) = W(t) - P(t) = (1+r)\,\delta(t-1) - \gamma(t)$. The result follows by recursion. Q.E.D.

Walras's Law is central to our analysis of DSSW's claims that noise traders create their own space and cause the stock price to deviate from its fundamental value. It says that investors must invest in (withdraw from) the storage technology at time t if the history of aggregate consumption is less than (exceeds) the aggregate exogenous supply of consumption. Similarly, nonzero net investment in storage requires that cumulative aggregate consumption differs from the cumulative aggregate endowment plus the aggregate dividend. In particular, Walras's Law says that requiring the demand for storage to always equal zero (i.e., $\delta(t) \equiv 0$ for all t) is equivalent to requiring aggregate consumption to equal its exogenous supply so that $\gamma(t) \equiv 0$ or, equivalently, $c(t) = e(t) + r$ for all t. Again, these conclusions merely require nonrational investors make portfolio choices that are within their budget and that, collectively, equate the demand and supply of the stock.

C.2. Necessary Property of a DSSW Stock Price

Before we use the OLG structure of DSSW's model, we use Walras's Law to connect the price of the stock to aggregate consumption by using the identity $W(t) = P(t) + \delta(t)$. From (5), we obtain a general characterization of a stock price consistent with the investors' budget equations and stock market clearing, which is given by

$$P(t) = W(t) + \sum_{j=0}^{t} (1+r)^{t-j} \gamma(j) - \delta(0). \tag{6}$$

If net investments in the storage technology were not permitted ($\delta(t) \equiv 0$), then the stock price would equal the investors' wealth invested in financial assets. When $\delta(t)$ is allowed to be nonzero, the stock price can be higher or lower than the wealth invested in financial assets, but the investors must withdraw a net amount from storage ($\delta(t) < 0$) if $P(t) > W(t)$ or invest a net amount in storage if $P(t) < W(t)$. For general models, equation (6) shows that the history of aggregate consumption must be consistent with the current amount of net withdrawals from or investments in storage.

DSSW's OLG assumptions refine matters: The young generation consumes nothing and invests its entire endowment in financial assets, so $W^i(t) = e^i(t)$ for a young investor i. For an old investor i, $W^i(t) = 0$ since the old investor receives no endowment ($e^i(t) = 0$) and consumes his entire financial wealth ($c^i(t) = \delta^i(t-1)(1+r) + \theta^i(t-1)(P(t)+r)$). Summing over all investors yields $W(t) = e(t)$, so (6) becomes

$$P(t) = e(t) - \delta(t) = e(t) + \sum_{j=0}^{t}(1+r)^{t-j}\gamma(j) - \delta(0). \tag{7}$$

We derive (7) by applying DSSW's OLG assumptions to (6), but one may verify (7) specifically for DSSW's stock price (1) given the investors' optimal consumption plans under Assumption 1.

We use the term "DSSW-style model" to refer to models with similar properties.

DEFINITION 2 (DSSW-style models): *We use "DSSW-style model" to refer to an economic model in which the stock price satisfies (7), which would follow from assumptions that would imply $W(t) = e(t)$.*

We now have sufficient structure in place to dissect DSSW's model.

II. Noise Trader Risk and Fundamental Risk

Noise traders seemingly create their own space by creating price risk in excess of that implied by fundamental risk. As DSSW (p. 712) state,

> *... both sophisticated investors and noise traders always hold portfolios that possess the same amount of fundamental risk: zero. Any intuition to the effect that investors in the risky asset "ought" to receive higher expected returns because they perform the valuable social function of risk bearing neglects to consider that the noise traders' speculation is the only source of risk. For the economy as a whole, there is no risk to be borne.*

By assumption, the aggregate endowment e and the net stock dividend r are riskless in DSSW's model. The amount of wealth W that investors invest in the financial assets is riskless, since $W(t) = e(t)$. DSSW therefore argue that the fact that the stock price is risky indicates that noise traders create their own space because there is no apparent fundamental risk in the economy.

Is there really no risk to be borne by the economy as a whole? No, the representation of the stock price in (7) says that stock price risk is channeled through the investors' budget equations and the market clearing requirement for the stock into equivalent aggregate consumption risk. If even nonrational investors make portfolio choices that are within their budget and that, collectively, equate the demand and supply of the stock, then the stock price and the net investment in storage are conditionally perfectly correlated with aggregate consumption since one is a linear function of the other in a DSSW-style model, as we now show.

PROPOSITION 1: *Any stock price for a DSSW-style model can be written as*

$$P(t) = e(t) - \delta(t) = c(t) - \delta(t-1)(1+r) - r, \tag{8}$$

regardless of investor rationality. Given that $e(t)$ is riskless, stock price risk is fully described by $\delta(t)$, the net investment in storage. The stock price $P(t)$ is also a linear function of aggregate consumption $c(t)$ with time $(t-1)$-measurable coefficients, so its risk is equivalently described by the distribution of future aggregate consumption.

Proof: Both expressions use the property that $W(t) = e(t)$ in a DSSW-style model. The first follows from the definition of δ, and the second follows from (4). Q.E.D.

The CCAPM literature argues that aggregate consumption is often the primary determinant of stock returns. Breeden (1979, p. 279) notes that "... wealth is not a sufficient statistic for the marginal utility of a dollar—consumption is." Breeden and Litzenberger (1978, sec. V) and Breeden (1979, sec. 5) emphasize that an asset's value is ultimately determined by the investors' marginal utilities for consumption, which are more generally determined by aggregate consumption.[7] In a DSSW-style model, stock price risk and aggregate consumption risk are identical. DSSW's claim that "none of the variance in the price of [the stock] can be justified by changes in fundamentals [because] there are no changes in expected dividends in our model or in any fundamental determinant of required returns" (p. 725) is incorrect: The stock price risk is fully captured by aggregate consumption risk, simply because investors make portfolio choices that are within their budget and that, collectively, equate the demand and supply of the stock.

The stock price is risky only because investors use the storage technology to consume more or less than the exogenous endowment of consumption.

COROLLARY 1: *When the net investment in the storage technology is always identically zero ($\delta(t) \equiv 0$ for all t) or is riskless in a DSSW-style model, aggregate consumption and the stock price must both be riskless, and noise traders cannot create their own space.*

Although the amount of wealth invested in the financial assets each period is riskless (since $W(t) = e(t)$), the returns to the stock are risky because each generation makes withdrawals from or investments in the storage technology to make up the difference between the stock price and their riskless endowment. When noise traders are absent in DSSW's model, it so happens that these withdrawals or investments are deterministic, and the stock price is consequently riskless. When noise traders are present, however, these withdrawals and investments are perfectly correlated with noise trader beliefs, giving the

[7] For single-period models, Mossin (1973) demonstrates that aggregate wealth is a sufficient statistic of fundamental risk. Breeden and Litzenberger (1978) point out that this does not generalize to multiperiod models except in special cases.

impression that noise traders create their own space. Comparing apples to apples, either by considering aggregate consumption as a fundamental source of risk or by fixing the level of aggregate consumption across equilibria, would make noise trader risk superfluous.

This fundamentally alters some of DSSW's explanations of asset pricing puzzles. Noise traders do not create price risk unexplained by classic measures of risk. Moreover, DSSW's attempt to compare equilibria with the "same amount of fundamental risk: zero" masks the varying consumption risk allowed by the storage technology. Other models make similar comparisons. Campbell and Kyle (1993), Spiegel (1998), Campbell and Cochrane (1999), Barberis and Huang (2001), and Kyle and Xiong (2001) present similar conclusions about excess volatility and contagion using varying aggregate consumption and a storage technology in perfectly elastic supply. For these models, Walras's Law (Lemma 1) and (6) show that this can create additional price risk that is not fully captured by endowment and dividend risk. It is not clear whether excess volatility, excess risk premia, and financial contagion in these models reflect varying aggregate consumption risk, varying noise trader risk, or both.[8]

Remark 1: Our results do not depend on a fixed rate of storage or riskless stock dividends; indeed, making net investment in storage identically equal to zero might require an endogenous and locally riskless rate of return on storage. We make these assumptions only to highlight the source of risk in the economy, much as DSSW do (DSSW attribute this risk to noise traders, while we attribute it to risky aggregate consumption). Denote a locally riskless storage return by $r(t)$ and potentially risky stock dividends by $d(t)$. Redefining $\gamma(t)$ as $c(t) - d(t) - e(t)$, Walras's Law implies that $\delta(t) = -\sum_{j=0}^{t} f(j,t)\gamma(j)$, where $f(j,t) = \Pi_{l=j+1}^{t}(1 + r(l))$ if $j < t$ and $f(t,t) = 1$. Given $W(t) = e(t)$, our results suitably modified would still say that (i) noise traders could not create their own space by creating price risk unexplained by aggregate consumption risk, and (ii) given no net investment in storage, stock price risk would be identical to aggregate endowment risk.

III. Stock Price Misvaluation

DSSW explain several asset pricing puzzles based on the deviation of the stock price from its fundamental value whenever noise traders are present.

[8] Some of these models would require additional assumptions to trace the sources of risk and include other features that make them seemingly intractable for this purpose. Although Campbell and Kyle (1993) and Kyle and Xiong (2001) are continuous-time models, our intuition extends to them. The continuous-time budget equation is $dW^i(t) = r(W^i(t) - \theta^i(t)P(t))\,dt + \theta^i(t)\,dP(t) + \theta^i(t)\,dD(t) - dC^i(t)$. Denote the supply of the stock by $\bar{\theta}(t)$ to emphasize that it may be stochastic (as the authors above assume). Define $\delta(t) = W(t) - \bar{\theta}(t)P(t)$ and $\gamma(t) = C(t) - \bar{\theta}(0)D(0) - \int_0^t \bar{\theta}(s)\,dD(s)$. Summing each budget equation over the $I(t)$ individuals, we find that $d\delta(t) = r\delta(t)\,dt - d\gamma(t)$. Thus, we obtain $\delta(t) = \delta(0) - \int_0^t e^{r(t-s)}\,d\gamma(s)$, which is the continuous-time analog of (5). As above, the excess volatility of $\bar{\theta}(t)P(t)$ may reflect the term $\int_0^t e^{r(t-s)}\,d\gamma(s)$.

In the context of DSSW's model, our main conclusion of this section is that this misvaluation of the stock price would be completely eliminated given one of the following two features: (1) limited asset liability and limited net withdrawals from the storage technology, or (2) an endogenous locally riskless rate of interest that equates withdrawals from and investments in storage. Thus, DSSW's explanations of asset pricing puzzles would be invalidated by either feature.

Using the context of the DSSW model identifies the specific links among limited asset liability, net withdrawals from storage, and an endogenous riskless rate. However, the economic intuition applies to virtually any economic model regardless of investor rationality. To make this clear, we first review the idea of deviations from fundamental value in Section III.A to identify the assumptions about investor rationality that it requires. Section III.B explains how certain properties of asset prices, such as limited liability, generally restrict stock price misvaluation. Section III.C uses the OLG structure of DSSW's model to connect stock price misvaluation to assumptions about the storage technology. Section III.D generalizes our results to multiple risky assets.

A. The Economics of Fundamental Value

DSSW state that the fundamental value of the stock's dividends equals one for any amount of noise trader risk, and that the stock price deviates from its fundamental value whenever noise traders are present. Using DSSW's assumption about noise trader beliefs (Assumption 1), we immediately see that this follows because

$$P^{\mathrm{DSSW}}(t) - 1 = \frac{\mu(\eta(t) - \eta^*)}{1+r} + \frac{\mu\eta^*}{r} - \frac{2\alpha\mu^2\sigma_\eta^2}{r(1+r)^2}, \tag{9}$$

which is normally distributed when $\mu > 0$. Our main purpose in this section is first to show that the fundamental value of the stock in DSSW's model does not depend on their specific assumption about investor nonrationality (it equals one regardless of investor rationality), and second to characterize deviations from fundamental value in a manner that is robust for any assumption about investor rationality.

Fundamental value is connected to what Campbell (2000, p. 1517) calls the "basic equation[s] of asset pricing," which in our case are given by

$$P(t) = E_t[\rho_{t,t+1}(P(t+1) + r)] \tag{10}$$

and

$$1 = E_t[\rho_{t,t+1}(1 + r)]. \tag{11}$$

The existence of the strictly positive random variables $\rho_{t,t+1}$ in equations (10) and (11) is essentially equivalent to the absence of single-period arbitrage strategies, that is, strategies that require no investment at time t and offer a chance of profit with no chance of loss at time $t+1$.[9] Many random variables $\rho_{t,t+1}$ may satisfy (10) and (11), but this is not a concern because our results hold for any of them.

The random variable $\rho_{t,t+1}$ can sometimes be given by investors' marginal utilities, which is the case in DSSW's model.

Example 1: For the DSSW model, one such random variable $\rho_{t,t+1}$ is given by the marginal utilities $u'(c^i(t+1))/E_t[u'(c^i(t+1))]$, where $u(x) = -\exp(-2\alpha x)$ is the utility function of the rational investors, $u'(x)$ denotes marginal utility, and $c^i(t+1)$ is the consumption at time $t+1$ of the old generation of rational investors born at time t.

Define a stochastic discount factor ρ by $\rho(t) \equiv \Pi_{s=0}^{t-1}\rho_{s,s+1}$. Using iterated expectations, we rewrite the basic equations of asset pricing as

$$P(t) = E_t\left[\sum_{j=t+1}^{T}\frac{\rho(j)}{\rho(t)}r + \frac{\rho(T)}{\rho(t)}P(T)\right] \tag{12}$$

and

$$1 = E_t\left[\frac{\rho(T)}{\rho(t)}(1+r)^{T-t}\right]. \tag{13}$$

Combining equations (12) and (13) produces

$$P(t) = \sum_{j=t+1}^{T}\frac{r}{(1+r)^{j-t}} + E_t\left[\frac{\rho(T)}{\rho(t)}P(T)\right]. \tag{14}$$

The fundamental value of an asset is often described as the present value of the asset's future dividends. Because DSSW's stock has riskless dividends, the calculation of fundamental value as a present value is straightforward. Given any stochastic discount factor ρ, the present value is

$$E_t\left[\sum_{j=t+1}^{\infty}\frac{\rho(j)}{\rho(t)}r\right] = \sum_{j=t+1}^{\infty}\frac{r}{(1+r)^{j-t}} = 1. \tag{15}$$

The basic equations of asset pricing require that the stock's fundamental value equals one for any stochastic discount factor ρ, and regardless of investor rationality.

However, the economic importance of fundamental value is its relation to violations of the Law of One Price. A deviation of the stock price from its fundamental value indicates that there is a portfolio that replicates the stock's

[9] The relations among the absence of arbitrage, the existence of stochastic discount factors, and optimal strategies have been documented in results collectively known as the "Fundamental Theorem of Asset Pricing." The general relations involve complications that we do not need to address (see Dybvig and Ross (1987), Harrison and Kreps (1979), Kreps (1981), and Loewenstein and Willard (2000a) for further discussion). Our analysis uses only the absence of single-period arbitrage opportunities. DSSW's "limits of arbitrage" have no bearing on our analysis since it uses a different notion of arbitrage that risks realizing losses over a single period but profits over a sufficiently long horizon.

dividends at a different price. In DSSW's model, the one unit of the stock and one unit of the storage technology both provide a dividend stream of r each period, and the Law of One Price suggests that their prices should be equal. Yet, in DSSW's model, the stock price P^{DSSW} differs from the cost of one unit of the storage technology (the cost is one, consistent with (15)) when noise traders are present. Hence, deviations from fundamental value and violations of the Law of One Price are fundamentally linked.[10]

Deviations from fundamental value are easy to see given (9): If noise traders are present ($\mu > 0$), then (9) is normally distributed and no stochastic discount factor ρ can equate the stock's price and its fundamental value. If noise traders are absent ($\mu = 0$), every ρ equates the stock price and its fundamental value. For our purposes, a representation of a deviation from fundamental value that does not depend on Assumption 1 is more useful. Our next lemma provides this representation.

LEMMA 2: *The stock price deviates from its fundamental value if and only if*

$$\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}P(T)\right] \tag{16}$$

does not equal zero.

Proof: This follows directly from equations (14) and (15). Q.E.D.

This classical property of asset pricing bubbles appears in studies by Tirole (1982), Blanchard and Watson (1983), Santos and Woodford (1997), Loewenstein and Willard (2000b), and others. The representation in (16) is valid for any assumptions about investor behavior that are consistent with the basic equations of asset pricing, including DSSW's specific assumption about investor behavior. Indeed, (16) requires only the absence of arbitrage opportunities that profit during some investor's lifetime, so the range of assumptions about investor rationality allowed by (16) is very broad.

B. Price Bounds and Deviations from Fundamental Value

DSSW's stock price is normally distributed when noise traders are present. Normally distributed asset prices also appear in other popular models, including Campbell and Kyle (1993), Campbell and Cochrane (1999), and Kyle and Xiong (2001). Prices with this property have no upper or lower bounds. In this

[10] The assumption that dividends are riskless merely simplifies the calculation of fundamental value. In particular, the present value of the stock's dividends equals one for any stochastic discount factor in DSSW's model. Our results in this section accommodate the more robust interpretation of fundamental value as the lowest-cost limited liability portfolio that (super)replicates an asset's dividends. If the stock price equals the present value of its dividends for some stochastic discount factor, then it also equals the lowest cost of a limited liability portfolio that replicates its dividends (see, e.g., Santos and Woodford (1997)).

section, we study how this affects DSSW's conclusions. Section III.B.2 uses our results to show that DSSW's explanation of the closed-end fund discount puzzle is not valid for funds having limited liability.

B.1. Bounds Limiting Deviations from Fundamental Value

The central idea of this section is as follows: Equation (16) expresses deviations from fundamental value as a failure of the limit

$$\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}P(T)\right] \tag{17}$$

to converge to zero. Properties of the stock price $P(T)$, particularly upper or lower bounds, directly affect the limit in (17) and thereby limit the properties of deviations from fundamental value. One does not need to refer to assumptions about investor rationality to obtain these limits on deviations from fundamental value.

Our first result in this section shows that assets with bounded prices cannot deviate from their fundamental values.

PROPOSITION 2: *If the stock price P is bounded above and below across all states of nature, then P must equal the fundamental value of the stock's dividends relative to any stochastic discount factor ρ, regardless of investor rationality.*

Proof: Suppose $|P(t)| \leq M$ for all t, where M is a nonnegative constant. Using Jensen's Inequality, we obtain $0 \leq \lim_{T\to\infty}|E_t[\rho(T)P(T)]| \leq \lim_{T\to\infty}E_t[\rho(T)|P(T)|] \leq \lim_{T\to\infty}M(1+r)^{-T} = 0$, where we recall that the discount rate r is positive. Q.E.D.

Given a positive riskless rate of interest, it is necessary for the stock price to be unbounded if it deviates from its fundamental value.[11] It is sometimes difficult to ascertain whether real-world asset prices have upper bounds, but many types of assets undoubtedly have lower bounds, particularly assets with limited liability such as those traded on public exchanges. Investors in exchange-traded stocks and mutual funds cannot lose more than their initial investment in an asset.

We now show that a mild notion of limited liability prevents the stock from being underpriced relative to its fundamental value.

PROPOSITION 3: *Define $P^-(t) = \max\{-P(t), 0\}$. Let $\Gamma(t)$ be any stream of consumption having finite present value; that is, $E_t[\sum_{j=t+1}^{\infty} \frac{\rho(j)}{\rho(t)}\Gamma(j)] < \infty$. If for all T,*

[11] Proposition 2 seemingly contradicts an unpublished example by DSSW (see DSSW's footnote 4; the working paper De Long et al. (1987, appendix II) contains the example). DSSW do not present their example prices directly; they instead present them as the limit of equilibrium prices for economies that are arbitrage free. However, the limit of these prices contains an arbitrage and cannot be equilibrium prices in the corresponding limit economy. Thus DSSW's example is not arbitrage free and does not contradict our Proposition 2.

$$E_t\left[\frac{\rho(T)}{\rho(t)}P^-(T)\right] \le E_t\left[\sum_{j=T+1}^{\infty}\frac{\rho(j)}{\rho(t)}\Gamma(j)\right], \tag{18}$$

then P cannot be less than the fundamental value of its dividends relative to any stochastic discount factor ρ. In particular, if P is bounded below and r is positive, then P cannot be less than the fundamental value of its dividends relative to any stochastic discount factor ρ.[12] *Both conclusions hold regardless of investor rationality.*

Proof: Condition (18) implies that $\lim_{T\to\infty} E_t[\rho(T)P(T)] \ge 0$, from which the first claim follows. We now prove the second claim. Let M be a finite positive constant so that $P(t) \ge -M$. By (12), we see that $E_t[\rho(T)P(T)] \ge -E_t[\rho(T)M]$, and the statement follows from noting that $E_t[\rho(T)M] = M(1+r)^{-T} \to 0$ if r is positive. Q.E.D.

The stock price cannot be less than its fundamental values unless the liabilities of the stock dominate every finite-valued stream of consumption. In particular, there is no finite-valued collateral that the investors could post to ensure that they would honor the stock's future liabilities.[13] Assets that have strict limited liability, in that the purchaser of the asset cannot lose more than the initial investment, cannot be underpriced relative to their fundamental values since the absence of single-period arbitrage opportunities implies the assets have nonnegative prices.[14] In the next section, we use Proposition 3 to show that DSSW's noise-trader explanation of the closed-end fund puzzle is inconsistent with limited liability independent of assumptions about investor rationality.

Remark 2: All of our results in this section extend to a risky dividend $d(t)$, which DSSW study in their Section III.B, and to a possibly stochastic riskless one-period rate of return $r(t)$ chosen to clear the riskless bond market (also see Remark 1). The basic equations of asset pricing would become

$$P(t) = E_t\left[\sum_{j=t+1}^{T}\frac{\rho(j)}{\rho(t)}d(j) + \frac{\rho(T)}{\rho(t)}P(T)\right] \tag{19}$$

and

$$1 = E_t\left[\frac{\rho(T)}{\rho(t)}\Pi_{j=t}^{T}(1+r(j))\right] \tag{20}$$

[12] Although we assume the riskless rate is fixed to be consistent with DSSW's interpretation of the riskless asset as a storage technology, this statement can be extended to a locally riskless rate of interest bounded away from zero.

[13] Unbounded negative prices are potentially inconsistent with moral hazard. In DSSW's model, for example, an old generation with negative financial wealth would prefer to default on its trading obligations since consuming nothing provides more utility.

[14] This is connected to the so-called "impossibility of negative bubbles," a common feature of models with rational investors (Diba and Grossman (1988) and Santos and Woodford (1997)).

(compare to (12) and (13)). Even with a risky dividend and a locally riskless rate of interest, our results connect the properties of P to deviations from fundamental value, which are still described by the asymptotic properties of $E_t[\rho(T)P(T)]$. In particular, Propositions 2 and 3 would not change.

B.2. Application to Closed-End Fund Discounts

DSSW provide a leading explanation of closed-end fund discounts, an explanation whose intuition is used in empirical studies such as Lee, Sleifer, and Thaler (1991), Gemmill and Thomas (2002), and Mitchell, Pulvino, and Stafford (2002), which use data from exchange-traded closed-end funds that have strict limited liability. We now show, however, that DSSW's theory cannot explain the discount of any closed-end fund with limited asset liability, particularly those traded on public exchanges. We also provide a necessary condition to explain the closed-end fund puzzle, independent of whether investors are rational or nonrational.

To explain the closed-end fund puzzle using noise trader risk, DSSW interpret the stock as a closed-end fund, and the storage technology as the fund's underlying assets. Under this interpretation, the fund pays the same dividend as its underlying assets. DSSW's stock price (which here is the closed-end fund) is normally distributed, and can be underpriced relative to its fundamental value (which here is the value of its underlying assets, or its net asset value) since

$$\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}P(T)\right] = \frac{\mu(\eta(t)-\eta^*)}{1+r} + \frac{\mu\eta^*}{r} - \frac{2\alpha\mu^2\sigma_\eta^2}{r(1+r)^2}. \tag{21}$$

Consequently, in DSSW's model the closed-end fund's price is at a premium or a discount to its net asset value depending on whether (21) is positive or negative. DSSW conclude that noise trader risk can explain the closed-end fund puzzle:

> *In most cases, fundamental value is difficult to measure ... But the fundamental value of a closed-end fund is easily assessed: the fund pays dividends equal to the sum of the dividends paid by the stocks in its portfolio and so should sell for the market price of its portfolio. Yet closed-end funds sell and have sold at large and substantially fluctuating discounts ... The concept of noise trader risk can explain...the persistent and variable discounts on closed-end funds ...* (p. 728)

However, DSSW's explanation of the closed-end fund puzzle conflicts with the fact that exchange-traded closed-end funds have no liabilities and, hence, nonnegative prices. Proposition 3 shows that such closed-end funds cannot sell at a discount to their fundamental value. This result requires only that the basic equations of asset pricing hold (or, equivalently, the absence of single-period arbitrage opportunities), and is inviolable for any additional assumption about investor rationality.

Empirical evidence strongly suggests, however, that exchange-traded closed-end funds do sell at a discount to their net asset values. Reconciling this with

Proposition 3 requires that the net asset value of a closed-end fund differs from its fundamental value.

COROLLARY 2: *If a limited liability closed-end fund sells at a discount to its net asset value, then its net asset value must be greater than the fundamental value of the fund, regardless of investor rationality.*

Proof: If at time t the price of the fund must be at least the fund's fundamental value, then the net asset value of the fund can be greater than the fund's price only if the net asset value exceeds the fund's fundamental value. Q.E.D.

Any explanation of closed-end fund discounts for exchange-traded funds must provide a reason for the fundamental value of the fund to be less than its net asset value, regardless of whether investors are assumed to be rational or nonrational. Such reasons include those in which the dividends of the underlying assets are reduced as they pass through the fund, perhaps because of management fees, transactions costs, agency costs, or taxes (Lee, Sleifer, and Thaler (1991) and Ross (2005, ch. 4) discuss these costs) since these costs lower the fund's fundamental value relative to the fundamental value of its holdings. Even in the absence of such costs, another explanation might be that the fund's underlying assets are priced above their fundamental values, while the fund itself is priced at (or perhaps slightly above) its fundamental value.

DSSW's theory does not explain the closed-end fund puzzle both because noise trader risk does not affect the fund's dividends and because the net asset value is assumed to be the fundamental value of the fund. Thus, arguments by Lee, Sleifer, and Thaler (1991) and others that management fees, agency costs, and other costs inadequately explain the time series properties of fund discounts do not support DSSW's theory. While investor rationality might ultimately be important for explaining the puzzle, the reason must be consistent with limited asset liability.

C. Limited Withdrawals from Storage and Asset Prices

We now show that DSSW's explanations of asset pricing puzzles critically depend on the ability of investors to withdraw unreasonably large amounts of consumption from storage. To show this, we use only the basic equations of asset pricing and the requirement that investors collectively demand the net supply of the stock. We need no further assumptions about investor preferences.

The fundamental intuition behind this conclusion is that positive deviations from fundamental value, or what one might call asset pricing bubbles, require the stock to appropriately explode in value. Whether investors are rational or nonrational, the explosiveness in the stock price is identified with explosive investor wealth through the investors' budget equations and the market clearing requirement for the stock: Investors must collectively hold one share of the stock, so they must collectively be able to afford it. As we see in Section II, the investors' budget equations dictate that investors make net withdrawals from the storage technology when the stock price exceeds their endowments. Consequently, an explosive stock price implies an explosive amount of net withdrawals from storage, so explosive in fact that riskless borrowing and lending do not cancel and the borrowing cannot be collateralized.

Recall that in a DSSW-style model, we may write $P(T) = e(T) - \delta(T)$, where $e(T)$ is the aggregate endowment of the young generation at time T and $\delta(T)$ is their net investment in storage (Section I.C). As $e(T)$ represents the aggregate amount of investable wealth in DSSW's model, the young generation of investors must make a net withdrawal from storage ($\delta(T) < 0$) if $P(T) > e(T)$. A deviation from fundamental value is linked to net investments in storage via the relation

$$\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}P(T)\right] = \lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}(e(T)-\delta(T))\right]. \tag{22}$$

Thus, the asymptotic properties of the value of $e(T) - \delta(T)$ determine the properties of the stock price's deviation from fundamental value.

Equation (22) is valid for any assumption about investor rationality in a DSSW-style model. Given DSSW's specific assumption about noise trader beliefs (Assumption 1) and their additional assumption that e is riskless and appropriately finite, equation (22) becomes

$$\begin{aligned}\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}P^{\mathrm{DSSW}}(T)\right] &= -\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}\delta(T)\right] \\ &= \frac{\mu(\eta(t)-\eta^*)}{1+r} + \frac{\mu\eta^*}{r} - \frac{2\alpha\mu^2\sigma_\eta^2}{r(1+r)^2}.\end{aligned} \tag{23}$$

(We are more specific about "appropriately finite" in a moment.) Thus, in DSSW's model, the asymptotic value of net investment in the storage technology is normally distributed. Note, however, that $\delta(\cdot)$ itself must have a positive probability of becoming very positive and very negative infinitely many times in the future to avoid being "killed off" by the discounting implicit in $\rho(\cdot)/\rho(t)$.

Below is the general result given the aggregate endowment e has finite value.[15] Unless stated otherwise, we allow e to be risky for the remainder of our paper.

LEMMA 3: *Suppose that the value of the exogenous aggregate endowment is finite, meaning that for any stochastic discount factor* ρ,

[15] The assumption that the value of the aggregate endowment is finite forces deviations from fundamental value to appear in the properties of the asymptotic present value of $\delta(T)$, rather than of $e(T)$. For example, even if the exogenous endowment were the only source of risk (i.e., if $\delta(t) \equiv 0$), the stock price would deviate from its fundamental value if and only if the value of the aggregate endowment were infinite since

$$\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}P(T)\right] = \lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}e(T)\right] \neq 0,$$

a neoclassical equivalence studied by Shell (1971), Santos and Woodford (1997), and others. For a DSSW-style model, the value of the aggregate endowment is finite whenever the asymptotic growth rate of e is less than r. Also note that if e were riskless, then the deviations from fundamental value caused by e would be riskless and uncorrelated with noise trader risk.

$$E_t\left[\sum_{j=t+1}^{\infty}\frac{\rho(j)}{\rho(t)}e(j)\right]<\infty \qquad \textit{almost surely}. \tag{24}$$

Then the condition

$$\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}\delta(T)\right]=0 \qquad \textit{almost surely} \tag{25}$$

implies that the stock price for the DSSW-style model must equal the fundamental value of its dividends, regardless of investor rationality.

Proof: A finite value of the aggregate endowment implies $\lim_{T\to\infty}E_t \times [\rho(T)e(T)] = 0$. The result follows immediately from (23). Q.E.D.

Regardless of investor rationality, the stock price cannot deviate from its fundamental value if riskless borrowing and lending are equal.

PROPOSITION 4: *Suppose that the value of the exogenous aggregate endowment is finite. If the locally riskless rate of interest equates withdrawals from and investment in storage so that $\delta(t) \equiv 0$ holds, then any stock price for the DSSW-style model must equal the fundamental value of its dividends, regardless of investor rationality.*[16]

Proposition 4 says that none of DSSW's explanations of asset pricing puzzles could be obtained if the storage technology were interpreted as a riskless investment opportunity in zero net supply. It also says that their explanations could not be obtained if their economy were free of all fundamental risk, including aggregate consumption risk (Section II). It also raises questions about the robustness of explanations of asset pricing puzzles that appear in other models that utilize storage technologies or linear production technologies, such as Campbell and Kyle (1993), Spiegel (1998), Campbell and Cochrane (1999), and Kyle and Xiong (2001).

Proposition 4 is valid for assets that have unlimited liability. As we see in Section III.B.1, limited liability prevents an asset from selling at a discount to its fundamental value. By itself, this limits the properties of investment in the storage technology, as our next result shows.

LEMMA 4: *Suppose that the value of the exogenous aggregate endowment is finite, and that the future liabilities of owning the stock price have finite present value, in the sense that (18) holds. Because the stock price cannot be less than its fundamental value, the condition*

$$\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}\delta(T)\right]\le 0 \qquad \textit{almost surely} \tag{26}$$

[16] See Remark 2 for the appropriate changes in notation needed for the basic equations of asset pricing to accommodate a locally riskless rate $r(t)$.

must hold for every stochastic discount factor ρ regardless of investor rationality.

Lemma 4 says that given limited liability and a finite-valued aggregate endowment, the stock's price deviates from its fundamental value if and only if the asymptotic present value of the net investment in storage is negative: The investors must asymptotically withdraw large and unbounded amounts from storage. Consequently, if

$$\lim_{T\to\infty} E_t\left[\frac{\rho(T)}{\rho(t)}\delta(T)\right] \geq 0 \qquad \text{almost surely} \tag{27}$$

holds for at least one stochastic discount factor ρ, then the stock must equal its fundamental value. Condition (27) is automatic if the market for riskless borrowing and lending clears so that $\delta(t) \equiv 0$, a requirement of many OLG models. It is also automatic if investors must asymptotically repay their withdrawals from consumption, something that might be required to rule out Ponzi schemes. Bounded net withdrawals ($\delta \geq -M$ for a positive constant M) or net withdrawals that grow asymptotically slower than the per-period cost of storage $r(\delta(t) > -M(1+g)^{-t}$ for all t with $0 \leq g < r)$ would also imply (27).[17]

The economic intuition for why the withdrawal limit (27) prevents the stock price from exceeding its fundamental value is related to the classical idea that the stock price must be appropriately explosive if it does. The stock price cannot exceed its fundamental value if

$$(\forall\, T > t)E_t\left[\frac{\rho(T)}{\rho(t)}P(T)\right] \leq E_t\left[\sum_{j=T}^{\infty}\frac{\rho(j)}{\rho(t)}\Gamma(j)\right] \tag{28}$$

holds for at least one stochastic discount factor ρ and at least one nonnegative adapted consumption stream Γ with finite value; that is, $E_t[\sum_{j=T}^{\infty}\frac{\rho(j)}{\rho(t)}\Gamma(j)] < \infty$.[18] Given that the value of the aggregate endowment e is finite, this says that asymptotically the present value of the stock price must exceed the value of the entire future exogenous supply of consumption since the right-hand side of (28) would have finite value if we take $\Gamma(t) = e(t) + r$. In a sense, the stock price must grow larger than the economy itself if it has a positive asset pricing bubble, independent of whether investors are rational or nonrational.

Given a finite-valued endowment e, clearing the market for riskless borrowing and lending prevents the stock price from exceeding its fundamental value because investors simply would not have access to enough wealth to afford it

[17] Like Cox, Ingersoll, and Ross (1985) do for investments in linear production technologies, we simply assume that the equilibrium properties of net withdrawals satisfy certain limits without constraining any investor's portfolio decision. The no-arbitrage conditions (12) and (13) do not change, and the stock's fundamental value remains unambiguous (Section III.A). See Campbell and Cochrane (1999) for the equivalence between economies with a perfectly inelastic storage technology and those with linear production technologies.

[18] This follows from taking limits of both sides of (28) as $T \to \infty$ and noting that the right-hand side must converge to zero since Γ is nonnegative.

when it becomes very expensive.[19] In DSSW's model, however, there are no withdrawal limits, so investors acquire the wealth they need by withdrawing from storage. But to keep up with the stock price, the investors must withdraw enormous amounts; indeed, the stock price must equal its fundamental value if

$$(\forall\, T > t) - E_t\left[\frac{\rho(T)}{\rho(t)}\min\{\delta(T), 0\}\right] \le E_t\left[\sum_{j=T}^{\infty}\frac{\rho(j)}{\rho(t)}\Gamma(j)\right] \tag{29}$$

holds for at least one stochastic discount factor ρ and at least one nonnegative adapted consumption stream Γ with finite value. Again, if we take $\Gamma(t) = e(t) + r$, or for that matter any positive multiple of this quantity, the right-hand side of (29) would be finite, and a limited liability stock's price would exceed its fundamental value only if the investors at time t were able to withdraw more than any multiple of the present value of the endowments of all future generations and all future stock dividends. Again, what one assumes about investor rationality is irrelevant for this conclusion.

Our next result summarizes these ideas.

PROPOSITION 5: *Assume that the value of the exogenous aggregate endowment is finite, and assume that net withdrawals from storage are limited, in the sense that (27) or (29) holds. Also assume that the stock is a limited liability asset in the sense that (18) holds. Then any stock price for the DSSW-style model must equal the fundamental value of its dividends, regardless of investor rationality.*

In DSSW's model, negative deviations from fundamental value require unlimited asset liability. Positive deviations require unlimited and unreasonably large net withdrawals from the storage technology. Deviations from fundamental value are inconsistent with an endogenous locally riskless rate of interest that equates borrowing and lending. This is true regardless of what one assumes about investor rationality.

D. Multiple Assets

Our preceding analysis maintains DSSW's assumption that there is one risky asset and a storage technology in perfectly elastic supply. We now restate our results for multiple risky assets. Our main point that DSSW's explanations of asset pricing puzzles cannot be supported given the more realistic assumptions about limited asset liability and limited storage withdrawals remain the same, but some of the details are more subtle. We omit proofs in this section unless they differ significantly from their analogs in the preceding sections.

Suppose there are K long-lived stocks available at each time t. Index the stocks by $k = 1, \ldots, K$, and assume that the fixed supply of a given stock k

[19] See Shell (1971), Brock (1979), Scheinkman (1988), and Santos and Woodford (1997), who examine deviations from fundamental values when investors must acquire the necessary funds from an infinite-valued aggregate endowment.

equals $\bar{\theta}^k$. We assume $\bar{\theta}^k$ is nonnegative; if it is positive, the stock is in positive net supply, and if it is zero, it is in zero net supply. Denote the supply of all of the stocks by the K-dimensional vector $\bar{\theta} = [\bar{\theta}^1, \ldots, \bar{\theta}^K]$. We denote the exdividend price of stock k at time t by $P^k(t)$ and the dividend it pays at time t by $d^k(t)$, which may be risky. Let $P(t)$ and $d(t)$ be corresponding K-dimensional vectors denoting the time-t exdividend stock prices and dividends. We assume that $d(t)$ is nonnegative at each time t, but allow $P(t)$ to be negative unless we state otherwise.

We retain all of our other definitions and assumptions. Given multiple assets, the basic equations of asset pricing become

$$(\forall k = 1, \ldots, K), \qquad P^k(t) = E_t\left[\sum_{j=t+1}^{T} \frac{\rho(j)}{\rho(t)} d^k(j) + \frac{\rho(T)}{\rho(t)} P^k(T)\right] \tag{30}$$

and

$$1 = E_t\left[\frac{\rho(T)}{\rho(t)} \Pi_{j=t}^{T}(1 + r(j))\right], \tag{31}$$

where ρ is a stochastic discount factor. The locally riskless rate $r(t)$ may be stochastic and potentially endogenous. The fundamental value of stock k relative to ρ remains

$$E_t\left[\sum_{j=t+1}^{\infty} \frac{\rho(j)}{\rho(t)} d^k(j)\right]. \tag{32}$$

We compare the value of the market portfolio, given by $\bar{\theta} P(t)$, to the fundamental value of the market, given by

$$E_t\left[\sum_{j=t+1}^{\infty} \frac{\rho(j)}{\rho(t)} \bar{\theta} d(j)\right]. \tag{33}$$

Note that assets in zero net supply ($\bar{\theta}^k = 0$) contribute neither to the value of the market portfolio nor to the fundamental value of the market.

As before, we use budget equations and market clearing in the stock to relate deviations of stock market prices to their fundamental values. Here is the analog of (7) appropriate for the multiple stock model:

LEMMA 5: *The value of the market portfolio for a DSSW-style model must satisfy*

$$\bar{\theta} P(t) = W(t) - \delta(t) = e(t) - \delta(t) \tag{34}$$

regardless of investor rationality.

The difference between the case with multiple assets from before is that (34) implies only that the net amount that is not invested in all stocks must be invested in storage (one cannot use (34) to describe what is invested in storage given what is invested in only one of many stocks). Given a stochastic discount factor ρ, we find that

$$\bar{\theta}P(t) = E_t\left[\sum_{j=t+1}^{T}\frac{\rho(j)}{\rho(t)}\bar{\theta}d(j) + \frac{\rho(T)}{\rho(t)}(e(T) - \delta(T))\right], \tag{35}$$

which follows from summing (30) over the stocks and applying Lemma 5. Using (35), we can connect limits to storage withdrawals to deviations from fundamental values. Our next result identifies when the market portfolio can be overpriced relative to its fundamental value.

PROPOSITION 6: *Assume that the value of the aggregate endowment is finite, and assume that net withdrawals from storage are limited, in the sense that (25), (27), or (29) holds. Then the value of the market portfolio for a DSSW-style model cannot exceed the fundamental value of the market, regardless of investor rationality.*

Proposition 6 says that if investor withdrawals are appropriately limited, then

$$\bar{\theta}P(t) \leq E_t\left[\sum_{j=t+1}^{\infty}\frac{\rho(j)}{\rho(t)}\bar{\theta}d(j)\right]. \tag{36}$$

In contrast to the one-stock case (Proposition 5), limited withdrawals from storage alone do not necessarily prevent any individual stock from exceeding its fundamental value, even if the market for the storage technology clears. While the investors on aggregate still need a large amount of funds to purchase an overpriced asset in positive net supply, it is not necessary for them to withdraw from storage since another positive net supply stock may be correspondingly underpriced. Put differently, a positive bubble on one stock may be offset by a negative bubble on another, possibly allowing investors to maintain limited storage withdrawals by simultaneously holding stocks with offsetting bubbles.

However, the feature of limited liability supply assets rules out negative bubbles, and together with limited withdrawals, prevents deviations from fundamental value altogether for those assets that contribute to the value of the market portfolio.

PROPOSITION 7: *Assume that the value of the exogenous aggregate endowment is finite, and assume that net withdrawals from the storage technology are limited, in the sense that (25), (27), or (29) holds. Also suppose that the prices of the positive net supply assets have limited liability, in the sense that their prices are nonnegative or that (18) holds for each asset. Then, the price of any positive net supply asset in a DSSW-style model must equal the fundamental value of that asset's dividends, regardless of investor rationality.*[20]

[20] For our purpose, it is enough to show that the stock price equals its fundamental value for one stochastic discount factor. See footnote 10 for more details.

Proof: Limited withdrawals require that $\bar{\theta}P(t)$ be no greater than the fundamental value of market portfolio (see (36)). Limited liability requires P^k to be no less than the fundamental value of stock k's dividends. As the fundamental value of the market portfolio is the sum of each stock's fundamental value weighted by $\bar{\theta}^k$, the price of stock k cannot exceed its fundamental value if $\bar{\theta}^k$ is positive. Q.E.D.

We again find that limited storage withdrawals and limited liability together prevent deviations from fundamental value, as do an endogenous locally riskless rate and limited liability. One difference from the one-asset case is that an endogenous locally riskless rate of interest by itself may not rule out deviations from fundamental value for individual assets, although it ensures that the market portfolio is correctly priced. A second difference is that these assumptions rule out deviations from fundamental value only for stocks in positive net supply. Assets in zero net supply do not contribute to aggregate investable wealth, so mispricing on these assets does not cause the distortions that are inconsistent with an equilibrium (see, e.g., Santos and Woodford (1997) and Loewenstein and Willard (2000b)). The more realistic assumptions of limited liability and limited net withdrawals from storage continue to invalidate DSSW's noise trader explanations of asset pricing puzzles.

IV. Conclusion

The conclusions of this paper are built on the idea that certain economic principles limit the properties of asset prices independent of investor behavior, and that the limits implied by limited asset liability, market clearing, and limited withdrawals from the storage technology have been inadequately appreciated. Models that deviate from these assumptions risk offering misleading economic insights, no matter how tantalizing such insights may seem. As we show, DSSW's assumptions about investor opportunity sets and budget constraints are much more critical for their explanations of asset pricing puzzles than are their assumptions about investor rationality.

DSSW's conclusions are at odds with the broader economic intuition developed from the assumptions of limited asset liability, market clearing, and limited withdrawals from the storage technology. DSSW's explanation of why expected stock returns potentially include unique compensation for noise trader risk ignores the fundamental importance of aggregate consumption risk discussed by Breeden and Litzenberger (1978). Their explanation of closed-end fund discounts fails for funds that have limited liability, a result related to the "impossibility of negative bubbles" that commonly appears in neoclassical asset pricing (Diba and Grossman (1988)). Their explanations of other asset pricing puzzles fail unless the economy as a whole borrows with unfettered abandon from a mysterious source of consumption and never repays what it owes—an intuition linked to the known explosive nature of asset pricing bubbles (Shell (1971), Brock (1979), and Scheinkman (1988)). DSSW's explanations are incompatible with an endogenous locally riskless rate that equates borrowing and lending. These issues—not the issue of investor rationality—determine the relevance of DSSW's explanations of real-world asset pricing puzzles.

Our economic intuition extends well beyond the DSSW model. It extends to long-lived investors, risky endowments, and risky dividends. We do not emphasize informational asymmetries and nonrational updating rules, but our proofs indicate that adding these features is unlikely to change the nature of our results. The economic intuition of our results is longstanding, clear, and compatible with more general neoclassical models of full rationality that are built on the same behavior-independent principles that we study. Simply put, investor behavior can be important for equilibrium asset prices, but only within certain limits that apply universally to all assumptions about investor behavior.

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