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Risk-Neutral Parameter Shifts and Derivatives Pricing in Discrete Time

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ABSTRACT

We obtain a large class of discrete-time risk-neutral valuation relationships, or "preference-free" derivatives pricing models, by imposing a simple restriction on the state-price density process. The risk-neutral stock-return and forward-rate dynamics are obtained by changing only a location parameter, which can be determined independent of the preference and true location parameters. The Gaussian models of Rubinstein (1976), Brennan (1979), and Câmera (2003), and the gamma model of Heston (1993) are all special cases. The model provides simple relationships between expected returns and state-price density parameters analogous to the diffusion case.

RUBINSTEIN (1976) DERIVES THE BLACK–SCHOLES FORMULA in a discrete-time model by imposing a restriction on the joint distribution of the asset price and the representative agent's marginal utility. Similar to the complete-markets diffusion result, the option price is independent of the expected stock return and the agent's preference parameters. As Heston (1993) points out, the noteworthy property is the elimination of three parameters, a stock-price-distribution parameter and two preference parameters (governing subjective time discounting and risk aversion), through the use of stock and bond prices. This so-called risk-neutral valuation relationship (RNVR) has been extended to alternative joint distributions of stock price and marginal utility by Brennan (1979), Câmera (2003), and others. RNVRs are important because they provide conditions in a discrete-time equilibrium setting under which (1) derivative prices are obtained using only parameters that can, in practice, be estimated with precision, and (2) derivative prices provide no information about the preference or expected return parameters that is not already reflected in the stock price. RNVRs resemble the pricing method in the complete-markets diffusion setting, but are obtained through a distributional assumption on marginal utility instead of assumptions of continuous trading and complete markets.

Since RNVRs impose assumptions on the joint distribution of prices and the state-price density (or stochastic discount factor), they are more restrictive than the method of risk-neutral pricing, which basically requires only frictionless markets and the absence of arbitrage. Risk-neutral probabilities

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are obtained by multiplying the true probabilities by the state-price density (which is the marginal rate of substitution of an expected utility maximizer) and scaling by the price of a discount bond. These probabilities represent forward shadow prices of consumption, making asset prices equal to expected payouts discounted at the riskless rate.

This paper shows that most continuous-distribution RNVRs in the literature[1] are special cases of a general model in which investor preferences influence returns only through their effect on a stock-return location parameter. This location parameter can be decomposed into the sum of a risk-free component (the "risk-neutral location parameter") and a risk-premium component, with the latter determined by a preference parameter. Discounting by the state-price density to obtain prices is equivalent to setting the risk-premium to zero and discounting by the riskless rate. Since the risk-neutral location parameter can be solved as a function of the riskless rate (it equates the expected stock return to the riskless rate), derivatives are priced independent of the stock-return location and preference parameters. The same idea applies to our discrete-time version of the Heath–Jarrow–Morton (1992) model. Forward-rate location parameters are given by the sum of risk-neutral and risk-premium components. The risk-aversion component of the state-price density and the risk-premium component of the location parameters both cancel when discounting by the state-price density. The risk-neutral location parameters, and therefore the term-structure derivative prices, can be solved independent of the location and preference parameters. This approach allows us to accommodate time-varying conditional stock-return and forward-rate distributions in a tractable manner.

The basic structure of the model is simple to outline in a one-period setting. We restrict the state-price density so that the risk-neutral distribution of the (possibly multidimensional) underlying state variable is obtained by a shift in the location of the true distribution. That is, the risk-neutral probability of any state z equals the true probability of state $z + \eta$, where η is a (state-independent) constant that embeds agent preferences. In the case of a multivariate-normal state variable, for example, a necessary and sufficient condition for such a shift is a state-price density whose logarithm is affine in the state variable. The shift in the state variable under the risk-neutral distribution induces a state-independent shift in the stock-return location parameter. Analogous to the complete-markets diffusion setting, derivatives are priced by first modifying the stock-return location parameter so that expected stock return equals the riskless rate (since all expected returns equal the riskless rate under the risk-neutral measure), and then computing the expected payout discounted by the riskless rate. We show that in the case of Gaussian Z, the diffusion result can be obtained as the per-period time length goes to zero. In the diffusion limit,

[1] The exceptions are the general model in Heston (1993), which is briefly discussed in Section IV.C, and the model in Madrigal and Smith (1993), who examine risk-neutral measures in a one-period model that are both form invariant (the actual and risk-neutral densities are in the same class) and robust to wealth distribution changes.

Girsanov's Theorem implies that risk-neutral probabilities are necessarily obtained by shifting the true probabilities.

Most RNVR examples in the literature correspond to the special case of a multivariate-normal state variable and a state-price density exponential-affine in that variable. Rubinstein (1976) derives the Black–Scholes formula when the stock-price and state-price density are bivariate lognormal (returns are exponential-affine in the state variable).[2] Brennan (1979) obtains RNVRs in a multiperiod setting under either bivariate lognormality, or bivariate normality of the stock price and the logarithm of the state-price density. Stapleton and Subrahmanyam (1984) generalize these results to multivariate contingent claims, still under the assumptions of multivariate normality or multivariate lognormality. Amin and Ng (1993) obtain an RNVR in a multiperiod model with conditionally bivariate lognormal stock prices and the state-price density. Our stock-derivatives RNVR under normally distributed shocks is similar to theirs, except we allow returns to be any function of the shock (subject to the existence of a unique risk-neutral location parameter), and we allow the parameters to depend on the stock-price (and forward-rate) history. The stock-derivative RNVR in Câmera (2003), obtained in research independent from ours, corresponds to the single-period and single-asset Gaussian case examined here with a strictly monotonic return function.

Another special case is that of an exponential-gamma distributed state variable and a log-gamma distributed state-price density, which is an example in Heston (1993).

Section I presents the stock-derivatives RNVR, including applications with Gaussian and exponential-gamma distributed state variables, and a discussion of the relationship to the diffusion model. Section II presents a discrete-time Heath–Jarrow–Morton model and the RNVR for pricing term-structure derivatives. Section III discusses equilibrium models consistent with a risk-neutral distribution shift. Extensions to subordinated Brownian motion, pure-jump processes, and more general RNVRs are presented in Section IV. Section V concludes the paper. Proofs are contained in the appendices

I. Stock-Derivatives RNVR

We begin with a probability space $(\Omega, \mathcal{F}, P)$, dates $0, 1, \ldots, T$, and a filtration $\{\mathcal{F}_t : t = 0, 1, \ldots, T\}$ to which all stochastic processes introduced in this paper are adapted. The conditional expectation operator $E[\cdot \mid \mathcal{F}_t]$ is abbreviated to E_t throughout (with $E = E_0$). Let $\mathbb{R}_{++}$ denote the strictly positive real line, and for process x, let $\Delta x_t = x_{t+1} - x_t$ denote the forward difference. For any vector v, we let $\|v\| = (v'v)^{1/2}$ denote its norm, and diag(v) the diagonal matrix with the

[2] See also Merton (1973), who notes that the Samuelson and Merton (1969) discrete-time equilibrium option pricing model reduces to the Black-Scholes formula if there is a single constant-relative-risk-aversion investor, and a lognormally distributed stock, which is the only asset in positive net supply.

vector v as its diagonal. The i^{th} row of the matrix σ is denoted σ^{i*}. As usual, statements about equality and uniqueness are in the almost sure sense.

Uncertainty in prices and the state-price density is driven by the d-dimensional state-variable process $Z = (Z^1, \ldots, Z^d)'$. Let $h_t : \Omega \times I\!R^d \to I\!R_{++}$ denote the density function of the shock ΔZ_t conditional on $\mathcal{F}_t$ (think of $h_t(z)$ as a function of both $\mathcal{F}_t$ and the outcome $\Delta Z_t = z$). In addition to one-period riskless borrowing and lending at the short rate r_t over the period $[t, t+1]$, there are n stocks (paying no dividends) with price processes $S = (S^1, \ldots, S^n)'$. Stock returns are assumed to be driven by an affine combination of state-variable shocks:

$$S_{t+1} = \operatorname{diag}(S_t)F_t(\mu_t + \sigma_t \Delta Z_t), \quad t = 0, \ldots, T-1, \tag{1}$$

where $F_t : \Omega \times I\!R^n \to I\!R^n$ is the return function, and the location parameter μ and volatility parameter σ are stochastic processes valued in $I\!R^n$ and $I\!R^{n\times d}$, respectively (that is, μ_t is a length-n vector and σ_t is an $n \times d$ matrix). It is easy to show that equation (1) allows for any time-t conditional marginal distribution for each stock's return; however, the joint distribution of returns and the state-price density is restricted below. Let $\mathcal{H}_t^S$ denote the time-t information set generated by the history of the stock prices.

Markets are assumed to be frictionless and free of arbitrage opportunities, which implies (subject to technicalities) the existence of a state-price density process π. The time-t price, p_t, of a contingent claim paying $K(S_T)$ at time T is therefore $p_t = \pi_t^{-1}E_t(\pi_T K(S_T))$. Since markets in this setting are typically incomplete, the state-price density may not be unique.

The risk-neutral conditional probability density of ΔZ_t is obtained by multiplying the true density by the ratio of the state price density and then dividing by the one-period discount bond price: $\tilde{h}_t(z) = e^{r_t} h_t(z) \frac{\pi_{t+1}}{\pi_t}(z)$. Therefore, $\tilde{h}_t(z)$ equals the time-t shadow forward price (density) of one unit of consumption at time $t+1$ in state z (the state representing the realization of ΔZ_t). The time-t price of a contingent claim paying $K(S_T)$ at T is then $p_t = \tilde{E}_t[\exp(-\sum_{u=t}^{T-1} r_u)K(S_T)]$, where $\tilde{E}$ denotes expectation under the risk-neutral probability measure, $\tilde{P}$.

The key assumption for the RNVRs in Propositions 1 and 2 is the following restriction, which implies that the risk-neutral conditional probability of any state z next period equals the true conditional probability of state $z + \eta_t$, where η_t depends only on time-t information (and not the state).

Assumption 1 (Risk-neutral density shift): *A state-price density process satisfies*

$$\frac{\pi_{t+1}}{\pi_t} = V_t \frac{h_t(\Delta Z_t + \eta_t)}{h_t(\Delta Z_t)}, \quad t = 0, \ldots, T-1, \tag{2}$$

where η and V are stochastic processes valued in $I\!R^d$ and $I\!R_{++}$, respectively.

Since the state-price density must correctly price one-period bonds, we immediately get $V_t = e^{-r_t}$. The risk-neutral density is

$$\tilde{h}_t(z) = e^{r_t} h_t(z) V_t \frac{h_t(z+\eta_t)}{h_t(z)} = h_t(z+\eta_t),$$

or, equivalently, $\Delta\tilde{Z}_t = \Delta Z_t + \eta_t$ has the same distribution under the risk-neutral measure as ΔZ_t has under the original measure. Because the risk-neutral measure change merely shifts the increments of Z by η, the stock-price dynamics under the risk-neutral measure are obtained by a simple shift in the location parameter process μ.

The parameters V and η are related to agent preferences, with V interpreted as a subjective discount factor, since it determines r in equilibrium, and the shift parameter η interpreted as a measure of risk aversion. Idiosyncratic components of Z correspond to zero elements of η, and $\eta = 0$ to risk neutrality. With two further assumptions (Assumptions 2 and 3, below), both parameters, along with the location parameter μ, will be shown to be invisible (using Heston's (1993) terminology), in the sense that they are not needed for derivatives pricing.

The case of Gaussian shocks, examined in Section I.A, includes most RNVRs in the literature and is characterized by the standard multivariate-normal density function

$$h_t(z) = (2\Pi)^{-d/2} \exp\left(-\tfrac{1}{2} z'z\right), \quad z \in I\!R^d. \tag{3}$$

The d elements of ΔZ_t are standard normal, mutually independent, and independent of $\mathcal{F}_t$. One can think of Z as d-dimensional standard Brownian motion sampled at unit time intervals. The state-price density under Assumption 1 is conditionally lognormal:

$$\frac{\pi_{t+1}}{\pi_t} = V_t \exp\left(-\tfrac{1}{2}\eta_t'\eta_t - \eta_t'\Delta Z_t\right).$$

Section I.C examines the case of exponential-gamma distributed shocks.

The next assumption requires the short-rate process and risk-neutral stock-return parameters to depend only on the stock-price history. Under this assumption, the risk-neutral stock-return dynamics and current stock-price history are sufficient to determine the risk-neutral distribution of future stock prices, and therefore the price of any stock derivative. Otherwise, if say σ were to depend directly on Z, the risk-neutral dynamics of σ and therefore the distribution of future prices would depend on η. Note that the invisible parameters V, η, and μ may be adapted to the larger information set.

ASSUMPTION 2: *The processes r, σ, h, and F are adapted to $\mathcal{H}^S$.*

To recover the risk-neutral location parameter from the short-rate and stock-returns equation, we need an invertibility assumption on F. It plays an analogous role to Heston's (1993) nondegeneracy assumption, and requires a unique location parameter process $\tilde{\mu}$ that equates the per-period expected stock returns to the one-period riskless return. If, for example, every component i of F is of the form $F_t^i(\mu_t^i + \sigma_t^{i*}\Delta Z_t)$, with $F_t^i(\cdot)$ being monotonic (which is the typical case in the literature), the assumption is automatically satisfied.

ASSUMPTION 3 (Unique risk-neutral location parameter): *There is a unique solution $\tilde{\mu}_t \in \mathcal{H}_t^S$ to*

$$e^{r_t}\mathbf{1} = E\left[F_t(\tilde{\mu}_t + \sigma_t \Delta Z_t) \,\middle|\, \mathcal{H}_t^S\right], \tag{4}$$

for every t, where $\mathbf{1}$ is a vector of ones.

The restrictive part of the assumption is uniqueness, since Assumption 1, together with the fact that the state-price density must correctly price the stocks, imply existence. The $\mathcal{H}^S$ measurability follows from Assumption 2 (see Appendix A for details). The uniqueness assumption is sufficient, but not always necessary for Proposition 1 to hold. What is necessary is uniqueness of the resulting risk-neutral stock-price distribution.[3]

The main result of the section follows.

PROPOSITION 1: *Suppose stock returns satisfy equation* (1) *and that Assumptions* 1 *to* 3 *hold. Then risk-neutral stock returns satisfy*

$$S_{t+1} = \mathrm{diag}(S_t) F_t(\tilde{\mu}_t + \sigma_t \Delta \tilde{Z}_t), \quad t = 0, \ldots, T-1,$$

where the risk-neutral distribution of $\tilde{Z}$ is the same as the true distribution of Z. Stock derivatives can be priced without knowing the state-price density parameters V or η, or the location parameter μ.

Proof: See Appendix A.

Under the conditions of the proposition, derivative pricing is obtained using the same two-step procedure as in the complete-markets diffusion setting: (1) Change the stock-return location parameter so that expected stock returns equal the riskless rate, and (2) compute the expected derivative payout discounted by the riskless rate process. The parameters V or η, which typically represent unobservable preference parameters, and the location parameter μ, which tends to be more difficult to estimate than σ, do not enter into the pricing formula. Conversely, derivative prices provide no additional information about the preference or location parameters not already impounded in the stock price. The key condition required for the RNVR is Assumption 1. The location shift from the actual to risk-neutral shock distribution implies only a change in the stock-return location parameter under the risk-neutral measure. Since the expected stock return under the risk-neutral measure equals the riskless return,

[3] Assume, for example, a single stock and symmetrically distributed state-variable increments (as in the Gaussian case). Then if the return function is even ($F(x) = F(-x)$ for all x) and $\tilde{\mu}_t$ is a risk-neutral location parameter, then so is $-\tilde{\mu}_t$. But both result in identical risk-neutral return distributions.

the risk-neutral location parameter can be solved as a function of the riskless rate (and stock price history) under the invertibility condition of Assumption 3.

Agent preferences affect asset returns in our model only through a risk premium component of the stock-return location parameter, which is eliminated when discounting by the state price density. The location parameter can be decomposed, using $\Delta\tilde{Z}_t = \Delta Z_t + \eta_t$, as $\mu_t = \tilde{\mu}_t + \sigma_t\eta_t$. The riskless component is $\tilde{\mu}_t$, and the risk-premium component is $\sigma_t\eta_t$; therefore η_t can be interpreted as the market price of risk (per unit of the state variable). Discounting payouts by the state-price density is equivalent to eliminating the risk premium and discounting by the riskless rate. That is, the price of any derivative can be determined as if investors were risk-neutral ($\eta = 0$), in which case the state-price density represents a riskless discount factor, and expected returns all equal the riskless rate.

The similarity to the diffusion setting is not coincidental. We show in Section I.B below that when Z is Gaussian, the state-price density is essentially a discretely sampled continuous-time state-price density with step function (changing at unit intervals) V and η processes. Appendix B shows that the complete-markets diffusion setting is then obtained by letting the per-period time length go to zero.

Allowing the volatility parameter and the return function to depend on the stock-price history allows considerable flexibility in modeling conditional heteroskedasticity of returns. For example, a GARCH-type model for the volatility vector of stock i could take the form

$$\sigma_t^{i*} = \delta^0 + \sigma_{t-1}^{i*}\delta^1 + \cdots + \sigma_{t-p}^{i*}\delta^p + \Delta S'_{t-1}\phi^1 + \cdots + \Delta S'_{t-p}\phi^p,$$

with $\delta^0 \in I\!R^{1\times d}$, $\delta^j \in I\!R^{d\times d}$ and $\phi^j \in I\!R^{n\times d}$, $j = 1, \ldots, p$.

The cases of exponential returns and linear price change functions are particularly tractable, with the risk-neutral location parameter solved in closed form for any state-variable distribution. When the returns of stock i satisfy ${S^i}_{t+1}/S_t^i = \exp(\mu_t^i + \sigma_t^{i*}\Delta Z_t)$, the risk-neutral location parameter is obtained from the moment generating function of the state-variable shock:

$$\tilde{\mu}_t^i = r_t - \ln\left[E_t \exp\left(\sigma_t^{i*}\Delta Z_t\right)\right]. \tag{5}$$

The risk-premium parameter $\sigma_t^{i*}\eta_t$ then has a simple relationship to expected returns:

$$E_t\left(S_{t+1}^i/S_t^i\right) = \exp\left(r_t + \sigma_t^{i*}\eta_t\right).$$

With the linear price-change function $\Delta S_t^i = \mu_t^i + \sigma_t^{i*}\Delta Z_t$, the risk-neutral location parameter is

$$\tilde{\mu}_t^i = (e^{r_t} - 1)S_t^i - \sigma_t^{i*}E_t(\Delta Z_t),$$

and the expected price change is

$$E_t\left(\Delta S_t^i\right) = (e^{r_t} - 1)S_t^i + \sigma_t^{i*}\eta_t.$$

When the state variable is symmetrically distributed ($h_t(z) = h_t(-z)$ for all t and z), additional closed-form solutions are available. For example, if the price-change function is hyperbolic sine, $\Delta S_t^i = \sinh(\mu_t^i + \sigma_t^{i*}\Delta Z_t)$, then

$$\tilde{\mu}_t^i = \operatorname{arcsinh}\left(\frac{(e^{r_t}-1)S_t^i}{E_t \exp\left(\sigma_t^{i*}\Delta Z_t\right)}\right).$$

If the return function is hyperbolic cosine or quadratic, we get closed-form solution pairs, but Proposition 1 still applies to either solution in the case of derivatives on a single stock (see footnote 3).

The following is a multivariate example in which stock returns are linear combinations of exponentials. One interpretation is that each asset is a portfolio of stocks, and each stock has an exponential return function. The risk-neutral location parameters in this model must be solved simultaneously.

EXAMPLE 1: *Linear combinations of exponentials.* Suppose there are $n = d$ stocks with returns given by

$$F_t(\mu_t + \sigma_t \Delta Z_t) = M_t\left[\exp\left(\mu_t^1 + \sigma_t^{1*}\Delta Z_t\right), \ldots, \exp\left(\mu_t^d + \sigma_t^{d*}\Delta Z_t\right)\right]',$$

for every t, where M_t is an $\mathcal{H}_t^S$-measurable $d \times d$ invertible matrix. In the case of Gaussian Z, returns are linear combinations of lognormal random variables. To compute the risk-neutral location parameter, first define the diagonal $d \times d$ matrix Λ_t with ith diagonal element $\Lambda_t^{ii} = E_t \exp(\sigma_t^{i*}\Delta Z_t)$. The risk-neutral location parameters then satisfy

$$\left[e^{\tilde{\mu}_t^1}, \ldots, e^{\tilde{\mu}_t^d}\right]' = e^{r_t}\Lambda_t^{-1}M_t^{-1}\mathbf{1}.$$

Option prices are given by the Black–Scholes model if Z is Gaussian, both the volatility parameter σ and short rate r are constant, and M equals the identity matrix.

Our stock-return model is flexible enough to accommodate discretely distributed returns. In the single stock case, for example, we can generate any marginal discrete return distribution by restricting $F(\cdot)$ to be a monotonic step function. As long as it is not a constant function, the risk-neutral location parameter is unique.

A. Gaussian Applications

Most previous RNVRs are special cases of Proposition 1 with Gaussian Z, in which case stock returns are functions of a conditionally normal variate, and the state-price density under Assumption 1 is conditionally lognormal. The bivariate lognormal RNVRs in Rubinstein (1976), Brennan (1979), Stapleton and Subrahmanyam (1984), and Amin and Ng (1993) for pricing derivatives on stock i correspond to the exponential return function, $S_{t+1}^i/S_t^i = \exp(\mu_t^i + \sigma_t^{i*}\Delta Z_t)$.

Risk-neutral stock returns are conditionally lognormal, but with location parameter (from equation (5)) $\tilde{\mu}_t^i = r_t - \frac{1}{2}\|\sigma_t^{i*}\|^2$. Unlike the earlier papers, Amin and Ng (1993) allow for stochastic state-price density and stock-return parameters. The displaced lognormal RNVR in Câmera (1999) and Câmera and Stapleton (2001) corresponds to $S_{t+1}^i/S_t^i = \phi_t^i + \exp(\mu_t^i + \sigma_t^{i*}\Delta Z_t)$, where $\phi_t^i < e^{r_t}$, and a risk-neutral location parameter $\tilde{\mu}_t^i = \ln(e^{r_t} - \phi_t^i) - \frac{1}{2}\|\sigma_t^{i*}\|^2$. RNVRs for multivariate normality of stock prices and the log state-price density, derived in Brennan (1979) and Stapleton and Subrahmanyam (1984), correspond to $\Delta S_t^i = \mu_t^i + \sigma_t^{i*}\Delta Z_t$.

The RNVR in Câmera (2003), obtained in research independent from ours, corresponds to the single-period and single-asset Gaussian case of our model, with the return function F assumed to be strictly monotonic. The above hyperbolic-sine example is based on theirs.

B. Relationship to the Continuous-Time Brownian Model

This section heuristically derives the diffusion model as a limit of the discrete-time model with Gaussian Z. The details are contained in Appendix B. The main result is that in the continuous-time limit, the state-price density restriction in Assumption 1 becomes void. Continuous trading and complete markets imply the ability to perfectly hedge derivatives, which results in pricing independent of preferences and expected stock returns.

We first change the period length from one to dt and modify the state-price density equation (2) accordingly:

$$\frac{\pi_{t+dt}}{\pi_t} = V_t \frac{h_t(dZ_t + \eta_t\, dt)}{h_t(dZ_t)}, \quad t = 1, \ldots, T.$$

The state variable Z is now standard d-dimensional Brownian motion, and the density function of dZ_t is therefore $h_t(z) = (2\Pi)^{-d/2}\exp(-\frac{z'z}{2dt})$ (recall that each component of dZ_t has variance dt). After substituting the density function and the per-period discount factor, $V_t = e^{-r_t dt}$, we get

$$d\ln(\pi_t) = -\left(r_t + \tfrac{1}{2}\eta_t'\eta_t\right)dt - \eta_t'\, dZ_t. \tag{6}$$

To obtain the stock-price dynamics with period length dt, we consider the case of one stock with exponential return function and scale the location parameter by dt to get $S_{t+dt}/S_t = \exp(\mu_t dt + \sigma_t dZ_t)$; or, equivalently

$$d\ln(S_t) = \mu_t dt + \sigma_t\, dZ_t. \tag{7}$$

In the continuous-time limit, the state-price density necessarily satisfies equation (6), based only on the assumptions of frictionless markets, continuous trading, and uncertainty generated only by the Brownian motion Z. Girsanov's theorem implies that under the risk-neutral measure change, only the Brownian motion drift is altered: The process $d\tilde{Z}_t = dZ_t + \eta_t dt$ is standard Brownian motion under the risk-neutral measure, $\tilde{P}$. For the state-price density to correctly price a stock with the dynamics in equation (7), the risk-neutral drift $\tilde{\mu}_t$ must equate the instantaneous expected stock return and short rate: $\tilde{\mu}_t = r_t - \frac{1}{2}\|\sigma_t\|^2$. Analogous to Assumption 2 in the discrete-time model, the (effective market completeness) assumption in the diffusion model that the stock volatility process is adapted to the stock-price history permits the pricing of any derivative knowing only the risk-neutral stock-price processes.

The discrete-time Gaussian model makes the strong assumption that $\Delta \ln(\pi_t)$ is affine in ΔZ_t. As the time per interval shrinks, the state-price density assumption becomes less restrictive as trading becomes more frequent, and the diffusion model is obtained in the limit.

C. Exponential-Gamma Applications

The *exponential gamma* case is characterized by the density function

$$h_t(z) = \frac{\exp\left[m_t^1 z^1 + \cdots + m_t^d z^d - \left(e^{z^1} + \cdots + e^{z^d}\right)\right]}{\Gamma\left(m_t^1\right)\cdot \cdots \cdot \Gamma\left(m_t^d\right)},$$

where $\Gamma(\cdot)$ is the gamma function and $m_t^1, \ldots, m_t^d$ are strictly positive and deterministic. This implies that ΔZ_t is independent of $\mathcal{F}_t$ and that the exponentials of the state-variable increments, $e^{\Delta Z_t^1}, \ldots, e^{\Delta Z_t^d}$, are independent and gamma-distributed random variables with degree-of-freedom parameters $m_t^1, \ldots, m_t^d$, respectively.[4] Assumption 1 implies that the logarithm of the state-price density ratio is an affine combination of independent gammas:

$$\frac{\pi_{t+1}}{\pi_t} = V_t \exp\left(\sum_{i=1}^{d} m_t^i \eta_t^i + \sum_{i=1}^{d}\left(1 - e^{\eta_t^i}\right)e^{\Delta Z_t^i}\right). \tag{8}$$

The gamma distribution is a rich class, including both the chi-squared and exponential distributions as special cases, and the normal distribution as a limiting case.

Heston (1993) proposes gamma-distributed noise driving stock returns as a parsimonious way to induce the type of skewness and kurtosis in risk-neutral returns that is evident in option prices. He examines a model in which the log return is affine in a one-dimensional gamma-distributed random variable. In our model, this corresponds to a single stock, one-dimensional Z, and

$$\ln(S_{t+1}/S_t) = \phi_t + e^{\mu_t + \Delta Z_t}, \tag{9}$$

where ϕ is some adapted process satisfying $\phi < r$. The change to the risk-neutral measure only affects the "volatility" term e_t^{μ} that scales the gamma-distributed random variable $e^{\Delta Z_t}$. The risk-neutral location parameter is

$$\tilde{\mu}_t = \ln\left[1 - \exp\left(\frac{\phi_t - r_t}{m_t}\right)\right], \tag{10}$$

which yields risk-neutral stock returns independent of V, η, and μ.

[4] That is, the density of $Y = \exp(\Delta Z_t^i)$ is $h(y) = e^{-y} y^{m_t^i - 1}/\Gamma(m_t^i)$, $y > 0$.

A more flexible stock-return model, tractable even with multivariate noise, is the exponential return function

$$S_{t+1}^i/S_t^i = \exp\left(\mu_t^i + \sigma_t^{i*}\Delta Z_t\right) = e^{\mu_t^i}\prod_{j=1}^{d}\left(e^{\Delta Z_t^j}\right)^{\sigma_t^{ij}},$$

where we assume $m_t^j + \sigma_t^{ij} > 0,\ j = 1, \ldots, d$. Returns are driven by products of powers of the independent gamma random variates.[5] In contrast to the returns model in equation (9), the random component of log returns can be both positive and negative. Risk-neutral returns are in the same class, but with a risk-neutral location parameter satisfying

$$e^{\tilde{\mu}_t^i} = e^{r_t}\prod_{j=1}^{d}\frac{\Gamma\left(m_t^j\right)}{\Gamma\left(m_t^j + \sigma_t^{ij}\right)}.$$

II. Term-Structure RNVR

We introduce a discrete-time Heath–Jarrow–Morton term structure model with state-variable shocks having any density function (with support equal to the real line), and forward-rate changes that are any monotonic function of an affine combination of these shocks.[6] The appeal of directly modeling forward-rate dynamics, rather than the short rate, is that we obtain a risk-neutral short-rate process that automatically fits the existing term structure. Similar to the results in Section I, Assumption 1 implies that preferences affect forward rates only through the location parameters, which can be eliminated by comparing riskless and expected discount-bond returns.

Let $P_t(s), s = t+1, t+2, \ldots, T$, denote the time-$t$ price of a unit default-free discount bond maturing at s, and define the time-t forward rate from s to $s+1$ as

$$f_t(s) = \ln[P_t(s)/P_t(s+1)], \quad s = t, \ldots, T-1.$$

Since $P_t(t+1) = e^{-r_t}$, we have $f_t(t) = r_t$. To simplify notation, we let the dot product denote the inner product of two vectors. Assume that forward rates satisfy

$$f_{t+1}(s) = f_t(s) + H_t^s\left(\alpha_t^s + \nu_t^s \cdot \Delta Z_t\right), \quad 0 \le t < s < T, \tag{11}$$

where $H_t^s : \Omega \times I\!R \to I\!R$ is the forward-rate function, and for each s, the location parameter α^s and volatility parameter ν^s are stochastic processes valued in $I\!R$ and $I\!R^d$, respectively. For simplicity, we assume that $F_t(\omega, \cdot)$ is monotonic

[5] Such distributions are contained in the generalized gamma convolution class, members of which are essentially limits of distributions of sums of independent gamma random variables (see Bondesson (1992)).

[6] The suggestion by an anonymous referee to examine the Heath–Jarrow–Morton model is gratefully acknowledged.

(but nonconstant) for all ω and t, though Proposition 2 still holds under the weaker condition that unique risk-neutral forward-rate location parameters exist. We let $\mathcal{H}_t^f$ denote the time-t information set generated by the history of forward rates.

The short-rate process is obtained by writing equation (11) in integrated instead of differenced form, letting $s = t$, and using $f_t(t) = r_t$ to get

$$r_t = f_0(t) + \sum_{u=0}^{t-1} H_u^t(\alpha_u^t + \nu_u^t \cdot \Delta Z_u).$$

The close connection between the stock-price and term-structure RNVRs is seen by using the identities $P_t(s) = e^{-[f_t(t)+f_t(t+1)+\cdots+f_t(s-1)]}$ and $f_t(t) = r_t$ to obtain the bond-price dynamics

$$\frac{P_{t+1}(s)}{P_t(s)} = e^{r_t} \exp\left(-\sum_{u=t+1}^{s-1} H_t^u(\alpha_t^u + \nu_t^u \cdot \Delta Z_t)\right). \tag{12}$$

Similar to the stock-return dynamics in equation (1), bond returns are driven by affine functions of state-variable shocks.

As in Assumption 2, we require the (visible) parameters and functions driving the risk-neutral forward-rates to depend at most on the history of the forward-rate processes. This ensures that the risk-neutral forward-rate dynamics, together with the current forward-rate history, are sufficient to determine the risk-neutral distribution of future forward rates.

ASSUMPTION 2.A: *The processes h, H^s, and $\nu^s, s = 1, \ldots, T$, are adapted to $\mathcal{H}^f$.*

The main difference in the following term-structure RNVR, compared to Proposition 1, is the computation of the risk-neutral location parameters based on the bond-return dynamics (12). At each time t, the risk-neutral location parameters $\{\tilde{\alpha}_t^s; s = t+1, \ldots, T-1\}$ are solved to equate all one-period expected bond returns with the riskless return. But because the location parameter α_t^s affects the time-t expected returns of all bonds maturing before s, the risk-neutral parameters must be solved sequentially, from the earliest to latest maturity.

PROPOSITION 2: *Suppose forward rates satisfy equation* (11), *and Assumptions* 1 *and* 2.A *hold. Then, for each t, the $\mathcal{H}_t^f$-measurable risk-neutral location parameters, $\tilde{\alpha}_t^{t+1}, \ldots, \tilde{\alpha}_t^{T-1}$, can be computed by sequentially solving (starting with $n = t+1$)*

$$1 = E\left(\exp\left[-\sum_{s=t+1}^{n} H_t^s(\tilde{\alpha}_t^s + \nu_t^s \cdot \Delta Z_t)\right] \middle| \mathcal{H}_t^f\right), \quad n = t+1, \ldots, T-1. \tag{13}$$

The risk-neutral forward-rate processes are

$$f_{t+1}(s) = f_t(s) + H_t^s(\tilde{\alpha}_t^s + \nu_t^s \cdot \Delta\tilde{Z}_t), \quad 0 \le t < s < T,$$

and the risk-neutral short-rate process is

$$r_t = f_0(t) + \sum_{u=0}^{t-1} H_u^t(\tilde{\alpha}_u^t + \nu_u^t \cdot \Delta\tilde{Z}_u), \quad t = 1, \ldots, T-1,$$

where the risk-neutral distribution of $\tilde{Z}$ is the same as the true distribution of Z. Term-structure derivatives can be priced without knowing the state-price density parameters V or η, or the location parameters $\alpha^s, s = 2, \ldots, T-1$.

As in Proposition 1, derivative pricing is accomplished in two steps. First replace the forward-rate location parameters with those that equate expected bond returns with the short rate (equation (13)). Then compute the expected derivative payout discounted by the short rate. The risk-neutral location parameters are implicit functions only of the volatility parameters and the observed forward rate history. Pricing is therefore independent of the preference and forward-rate location parameters. The key condition is again Assumption 1, which implies that preferences affect forward rates only through each location parameter α_t^s, which can be decomposed into the sum of a risk-neutral component, $\tilde{\alpha}_t^s$, and a risk-premium component, $\nu_t^s \cdot \eta_t$. The risk-aversion component in the state-price density cancels the risk-premium terms in the location parameter, leaving risk-neutral returns and discounting. Derivatives can therefore be priced as if agents were risk neutral (or $\eta = 0$).

EXAMPLE 2: *Linear forward-rate function.* Letting $H_t^s(x) = x$ for all x, t, and s, equation (12) becomes

$$\frac{P_{t+1}(s)}{P_t(s)} = e^{r_t} \exp\left(\mu_t^s + \sigma_t^s \cdot \Delta Z_t\right), \quad t+2 \le s \le T,$$

where we define $\mu_t^s = -(\alpha_t^{t+1} + \cdots + \alpha_t^{s-1})$ and $\sigma_t^s = -(\nu_t^{t+1} + \cdots + \nu_t^{s-1})$. Modeling in terms of forward rates instead of bond returns is just a matter of parameter redefinition. The risk-neutral location parameters must satisfy (from either equation (13), or equation (5) with $r = 0$)

$$\tilde{\alpha}_t^{t+1} + \cdots + \tilde{\alpha}_t^{s-1} = \ln\left(E_t \exp\left(\sigma_t^s \cdot \Delta Z_t\right)\right), \quad s = t+2, \ldots, T-1,$$

which can be solved sequentially, starting with $s = t+2$, to get an implicit relationship between $\tilde{\alpha}_t^s$ and the volatility parameters $(\nu_t^{t+1}, \ldots, \nu_t^s)$.

In the Gaussian case, discount bond prices are conditionally lognormal, and $\ln(E_t \exp(\sigma_t^s \cdot \Delta Z_t)) = \frac{1}{2}\sigma_t^s \cdot \sigma_t^s$ implies

$$\tilde{\alpha}_t^s = \nu_t^s \cdot \left(\sum_{u=t+1}^{s-1} \nu_t^u + \frac{1}{2}\nu_t^s\right), \tag{14}$$

which is similar (and equal in the diffusion limit) to the drift restriction in Heath, Jarrow, and Morton (1992) derived in a diffusion setting.

In the exponential-gamma case, log returns of discount bond are proportional to products of powers of independent gamma-distributed variates, and the risk-neutral location parameters are[7]

$$\tilde{\alpha}_t^s = \sum_{j=1}^{d} \log\left[\frac{\Gamma\left(m_t^j - \sum_{u=t+1}^{s} \nu_t^{u,j}\right)}{\Gamma\left(m_t^j - \sum_{u=t+1}^{s-1} \nu_t^{u,j}\right)}\right],$$

where $\nu_t^{u,j}$ denotes the jth element of the vector ν_t^u.

The next example specializes the linear forward-rate model to obtain discrete-time versions of the Hull and White (1990) and Vasicek (1977) models that fit the current term structure, but without specifying the distribution of state variable shocks.

EXAMPLE 3: *Discrete-time Hull–White model.* Let $H_t^s(x) = x$ for all x, t, and s, and assume, for each t, that volatilities for forward rates of all maturities are scalar multiples of one another:

$$\nu_t^{t+1} = \Sigma_t, \quad \nu_t^{s+1} = e^{-a_s}\nu_t^s, \quad s = t+1, \ldots, T-1,$$

where Σ is a stochastic process valued in $I\!R^d$ and $a_i, i = 1, \ldots, T-1$ are constant scalars. The impact on time-t forward rates of the shock ΔZ_t therefore declines exponentially for successive maturities. From Proposition 2, the risk-neutral short rate satisfies

$$\Delta r_t = f_0(t+1) - f_0(t) + \delta_t + (e^{-a_t} - 1)(r_t - f_0(t)) + \Sigma_t \cdot \Delta\tilde{Z}_t,$$

where

$$\delta_t = \sum_{u=0}^{t} \tilde{\alpha}_u^{t+1} - e^{-a_t}\sum_{u=0}^{t-1} \tilde{\alpha}_u^t.$$

The short rate incorporates mean reversion (for positive a) and history dependence only through history of the risk-neutral location parameters, which, in turn, are functions of the volatility parameters.

III. Consistency with Equilibrium Models

This section outlines some pure-endowment representative-agent equilibrium models which are consistent with Assumption 1. We begin with time-additive preferences, and then briefly discuss the case of Epstein–Zin (1991) recursive preferences.

[7] Use $\ln(E_t \exp(\sigma_t^s \cdot \Delta Z_t)) = \sum_{j=1}^{d} \log[\Gamma(m_t^j + \sigma_t^{s,j})/\Gamma(m_t^j)]$, where $\sigma_t^{s,j}$ is the jth element of the vector σ_t^s.

If the time-additive utility function

$$E\left(\sum_{t=0}^{T} u(c_t)\prod_{s=0}^{t-1}\mathcal{B}_s\right), \quad \mathcal{B}_s > 0\,\forall s, \tag{15}$$

has an interior optimum $\hat{c}$, then the first-order optimality condition is $\pi_t = u_c(\hat{c}_t)\prod_{s=0}^{t-1}\mathcal{B}_s$. In a pure-trade economy with endowment e, we have $\hat{c} = e$, and therefore Assumption 1 is equivalent to the following consistency condition on the utility function and endowment dynamics:

$$\mathcal{B}_t\frac{u_c(e_{t+1})}{u_c(e_t)} = V_t\frac{h_t(\Delta Z_t + \eta_t)}{h_t(\Delta Z_t)}, \quad t = 1,\ldots,T. \tag{16}$$

In the one-period ($T = 1$) Gaussian model of Câmera (2003), their assumption on the joint distribution of the end-of-period aggregate wealth and stock price is equivalent, in our model, to requiring the marginal utility of terminal wealth, $u_c(e_1)$, to be proportional to $\exp(-\eta_0'\Delta Z_0)$. Equation (16) is then obviously satisfied.

In a dynamic setting, suppose the representative agent's time-additive utility function satisfies

$$u_c(x) = v(x)^{-R}, \qquad R > 0, v'(\cdot) \geq 0, \tag{17}$$

defined for all x such that $v(x) > 0$. For example, $v(x) = x$, for all x, corresponds to power utility with coefficient of relative risk aversion R (log utility is $R = 1$); $v(x) = e^x$ corresponds to exponential utility, with R representing the coefficient of absolute risk aversion. With Gaussian shocks, and an endowment process

$$\Delta\ln(v(e_t)) = a_t + b_t'\Delta Z_t, \tag{18}$$

where a and b are stochastic processes, the consistency condition (16) is satisfied (and Assumption 1 therefore holds) with the parameter assignments

$$\eta_t = Rb_t, \quad V_t = \mathcal{B}_t\exp\left(-Ra_t + \tfrac{1}{2}R^2 b_t'b_t\right). \tag{19}$$

The state-price density parameter η is the product of the risk-aversion coefficient and the consumption-growth volatility parameter, and V is proportional to the subjective discount factor, $\mathcal{B}$. The endowment process is conditionally lognormal in the power utility case (Rubinstein (1976), Brennan (1979), Stapleton and Subrahmanyam (1984), and Amin and Ng (1993)), and conditionally normal in the exponential utility case (Brennan and Stapleton and Subrahmanyam). Rubinstein shows that with power utility, deterministic $\mathcal{B}$, and a deterministic investment opportunity set (meaning deterministic r, μ, σ, F, and h in our setting), the consumption-to-wealth ratio is deterministic; therefore the assumption of a lognormal endowment can be replaced by the assumption of lognormal aggregate wealth.

In the case of exponential-gamma Z, the consistency condition (16) is satisfied in a representative-agent equilibrium with utility satisfying equations (15) and (17), and endowment process[8]

$$\Delta \ln(v(e_t)) = a_t + \sum_{i=1}^{d} b_t^i e^{\Delta Z_t^i}, \quad b_t^i \in (-1/R, 1). \tag{20}$$

From equations (8) and (16), the state–price density parameters are

$$\eta_t^i = \ln\left(1 + R b_t^i\right), \quad i = 1, \ldots, d; \qquad V_t = \mathcal{B}_t e^{-R a_t} \prod_{j=1}^{d} \left(1 + R b_t^j\right)^{-m_t^j}.$$

As in the Gaussian case, η is increasing in the product of the risk aversion coefficient, R, and the endowment volatility parameter, b.

Compensation for risk-bearing in our model is determined by $\sigma\eta$, the inner product of the stock volatility and market price of risk parameters. Assuming a representative-agent equilibrium with utility satisfying equations (15) and (17), Gaussian Z, and endowment equation (18), we obtain

$$\sigma_t \eta_t = R \operatorname{Cov}_t(\sigma_t \Delta Z_t, \Delta \ln[v(e_t)]).$$

In the power-utility case, for example, the right-hand side is the product of the coefficient of relative risk aversion and the conditional covariance between shocks to stock returns and the growth of log endowment.[9]

The state-price density dynamics in Assumption 1 can also be generated in a a representative-agent endowment economy with recursive utility. Epstein and Zin (1991) show that the homothetic recursive utility function[10]

$$U_t = \left[(1-\phi)c_t^{\rho} + \phi\left(E_t U_{t+1}^{\alpha}\right)^{\rho/\alpha}\right]^{1/\rho}, \quad 0 \neq \alpha, \rho < 1, \quad t = 0, 1, 2, \ldots$$

implies a state-price density

$$\pi_t = c_t^{\alpha(\rho-1)/\rho} W_t^{(\alpha/\rho)-1},$$

where c_t and W_t are optimal consumption and wealth. If the investment opportunity set is deterministic, the consumption-to-wealth ratio is deterministic. Therefore, the above assumption of time-additive power utility can be replaced by the assumptions of Epstein–Zin preferences and a deterministic investment opportunity set in either the Gaussian or exponential-gamma models.

[8] The lower and upper bounds on b_t^i insure finite conditional expectations of the state-price density and endowment, respectively.

[9] In the case of exponential-gamma Z and endowment equation (20), we get the approximate relationship

$$\sigma_t \eta_t \approx R \operatorname{Cov}_t(\sigma_t \Delta Z_t, \Delta \ln[v(e_t)])/\bar{m}_t,$$

when $m_t^i = \bar{m}_t$ for all $i = 1, \ldots, d$ (using $\eta \approx Rb$ for small Rb, and $\Delta Z_t \approx e^{\Delta Z_t} - 1$ for small ΔZ_t).

[10] The same results apply in our finite-horizon setting by specifying terminal utility as $U_T = c_T$, with the recursion formula applying up to $T - 1$.

IV. Extensions

This section presents extensions to a discrete-time model with a state-variable process following subordinated Brownian motion, and then a continuous-time pure-jump model. An extension to more general RNVRs is sketched in the last subsection. The jump model is similar to the discrete-time model, except with exponentially-distributed instead of unit-time increments. The subordinated model replaces the unit-time sampling of a state-variable process with random time sampling. We examine only the case of Brownian motion with a gamma subordinator, but RNVRs for other combinations (such as an exponential-gamma process with a gamma subordinator) can be derived using the same approach. In both models we obtain a risk-neutral stock-price process, and therefore derivatives pricing, that is independent both of the stock-return location parameter μ and preference parameters V and η. However, an additional state-price density parameter remains visible and must be solved using a derivative price or equilibrium arguments.

A. RNVRs for Subordinated Brownian Motion

We illustrate the extension to subordinated processes by examining a model of uncertainty similar to the Gaussian model, but with shocks generated by Brownian motion sampled at gamma-distributed time intervals instead of unit time intervals. Let B be d-dimensional standard Brownian motion, and define the one-dimensional discrete-time gamma process $\{G_t; t = 0, 1, \dots\}$ as having independent gamma-distributed increments with the deterministic degree-of-freedom parameter process $m > 0$ (i.e., ΔG_t is gamma-distributed with degree of freedom m_t). We assume G and B are independent. The state-variable process is defined as $Z_t = B(G_t)$. Since $\Delta Z_t = B(G_{t+1}) - B(G_t)$ is equal in distribution to $(\Delta G_t)^{1/2}\Delta B(t)$, its distribution is a mixture of independent normals with a common gamma-distributed variance, ΔG_t.

Praetz (1972) is the first to propose such a mixed distribution for per-period stock returns. Madan and Seneta (1990) introduce a continuous-time version, with Brownian motion and a gamma subordinator, termed the variance-gamma process, to model the logarithm of the stock price (see also Madan, Carr, and Chang (1998)). Option prices in a variance-gamma model are derived in Madan and Milne (1991) in an equilibrium framework that implies (as does ours) a variance-gamma process for $\ln(\pi_t)$. Unlike the above models, ours includes multiple stocks and sources of uncertainty; we allow per-period returns to be any function (subject, as before, to the assumption of a unique risk-neutral location parameter) of the variance-gamma shocks; and we also outline a Heath–Jarrow–Morton model with forward rates driven by the same type of innovations.

Let $h_t(z, g)$ denote the conditional joint density of function $(\Delta Z_t, \Delta G_t)$, which by Bayes' rule can be written $h_t(z,g) = h(z\,|\,g)H(g; m_t)$, where $h(z \mid g) = (2\Pi)^{-d/2}\exp(-\frac{1}{2g}z'z)$ is the conditional density of ΔZ_t given ΔG_t (independent normal with common variance ΔG_t), and $H(g; m_t) = e^{-g}z^{m_t-1}/\Gamma(m_t)$, $g > 0$, is the marginal density ΔG_t (gamma variate with m_t degrees of freedom).

We impose a restriction on the state-price density ratio that allows for two pricing effects: The risk-neutral increments in G are scaled by ψ, and conditional on G, the risk-neutral drift of B is shifted by η per unit time.

ASSUMPTION 4: *A state-price density process satisfies*

$$\frac{\pi_{t+1}}{\pi_t} = V_t \frac{h(\Delta Z_t + \eta_t \Delta G_t \mid \Delta G_t) H(\Delta G_t/\psi_t; m_t)/\psi_t}{h(\Delta Z_t \mid \Delta G_t) H(\Delta G_t; m_t)}, \quad t = 0, \ldots, T-1,$$

where V and ψ are $\mathbb{R}_{++}$-valued stochastic processes, and η is an $\mathbb{R}^d$-valued stochastic process.

Since the risk-neutral density is proportional to the product of the true density and the state-price density ratio, and $V_t = e^{-r_t}$ (since one-period bonds are priced by π), we have

$$\tilde{h}_t(z, g) = h_t(z, g) \frac{h(z + \eta_t g \mid g) H(g/\psi_t; m_t)/\psi_t}{h(z \mid g) H(g; m_t)} = h(z + \eta_t g \mid g) H(g/\psi_t; m_t)/\psi_t.$$

Under the risk-neutral measure, $\Delta \tilde{G}_t = \Delta G_t/\psi_t$ is gamma-distributed with m_t degrees of freedom, and conditional on ΔG_t, $\Delta \tilde{Z}_t = \Delta Z_t + \eta_t \Delta G_t$ is normal with each component having mean zero and variance ΔG_t.

Substituting the density formulas, we see that Assumption 4 is equivalent to the assumption of variance-gamma increments in the log of the state-price density:

$$\Delta \ln(\pi_t) = \mathcal{A}_t + \mathcal{C}_t \Delta G_t - \eta_t' \Delta Z_t,$$

where $\mathcal{A}_t = \ln(V_t) - m_t \ln(\psi_t)$, and $\mathcal{C}_t = 1 - \psi_t^{-1} - \frac{1}{2}\eta_t'\eta_t$.

We interpret V as measuring subjective discounting, η as aversion to state-variable risk conditional on G, and ψ as aversion to G risk. Risk neutrality corresponds to $\eta = 0$ and $\psi = 1$.

An equilibrium model consistent with Assumption 4 is a pure-endowment representative-agent economy with time-additive utility satisfying equations (15) and (17), and an endowment driven by variance-gamma increments:

$$\Delta \ln(v(e_t)) = d_t + a_t \Delta G_t + b_t' \Delta Z_t,$$

where a, b, and d are stochastic processes, and $a_t > \frac{1}{2} R b_t' b_t - R^{-1}$. Assumption 4 then holds with $\eta_t = R b_t$, $\psi_t^{-1} = 1 + R a_t - \frac{1}{2} R^2 b_t' b_t$, and $V_t = \psi_t^m \mathcal{B}_t \exp(-R d_t)$.

Stock returns are assumed to satisfy

$$S_{t+1} = \mathrm{diag}(S_t) F_t(\mu_t \Delta G_t + \sigma_t \Delta Z_t), \quad t = 0, 1, \ldots, T-1, \tag{21}$$

where $F_t : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, and the location parameters μ and σ are stochastic processes valued in $\mathbb{R}^n$ and $\mathbb{R}^{n \times d}$, respectively.

PROPOSITION 3: *Suppose stock returns satisfy equation* (21), *Assumption 4 holds, and, for each t, that r_t, ψ_t, σ_t, h_t, and F_t are $\mathcal{H}_t^S$-measurable and that there is a unique solution $\tilde{\mu}_t \in \mathcal{H}_t^S$ to*

$$e^{r_t}\mathbf{1} = E\big[F_t\big(\tilde{\mu}_t\psi_t\Delta G_t + \sigma_t\psi_t^{1/2}\Delta Z_t\big)\,\big|\,\mathcal{H}_t^S\big]. \tag{22}$$

Then risk-neutral stock returns satisfy

$$S_{t+1} = \text{diag}(S_t)F_t\big(\tilde{\mu}_t\psi_t\Delta\tilde{G}_t + \sigma_t\psi_t^{1/2}\Delta\tilde{Z}_t\big), \quad t = 0, \ldots, T-1,$$

where the risk-neutral joint distribution of $(\tilde{Z}, \tilde{G})$ is the same as the true joint distribution of (Z, G).

Derivatives are priced as if agents were risk-neutral toward shocks in the state variable, but possibly averse to G risk (as measured by ψ). As in the unsubordinated model, the risk-aversion parameter η affects stock returns only through the location parameter, which can be decomposed into the sum of a risk-neutral and risk-premium component: $\mu_t = \tilde{\mu}_t + \sigma_t\eta_t$. In the subordinated model, however, discounting by the state-price density not only cancels the risk-premium term $\sigma\eta$, but also scales the subordinating gamma process. Therefore, ψ enters the risk-neutral stock-return equation directly, as well as indirectly through $\tilde{\mu}_t$ (from equation (22)). Restrictions on the equilibrium model could be imposed so that ψ is a function of observable parameters. For example, if there is no discounting in the utility function, $\mathcal{B} = 1$, and no deterministic growth component in the endowment dynamics (i.e., $d = 0$), then $\psi_t = e^{-r_t/m_t}$. Alternatively, ψ can be solved from an observed derivative price.

A term-structure RNVR can be obtained under the forward-rate dynamics

$$f_{t+1}(s) = f_t(s) + H_t^s\big(\alpha_t^s\Delta G_t + \nu_t^s \cdot \Delta Z_t\big), \quad 0 \le t < s < T.$$

where, as in Section II, $H_t^s : \Omega \times I\!R \to I\!R$ is assumed to be monotonic (but not constant), and for each s, α^s and ν^s are stochastic processes valued in $I\!R$ and $I\!R^d$, respectively. The modification to Proposition 2 is analogous to the stock-derivatives case.

EXAMPLE 4: *Linear forward-rate function.* Letting $H_t^s(x) = x$ for all x, t, and s, then forward-rate changes are conditionally variance-gamma, and discount bond prices satisfy

$$\frac{P_{t+1}(s)}{P_t(s)} = \exp\big(r_t + \mu_t^s\Delta G_t + \sigma_t^s \cdot \Delta Z_t\big), \quad t+2 \le s \le T,$$

where $\mu_t^s = -(\alpha_t^{t+1} + \cdots + \alpha_t^{s-1})$ and $\sigma_t^s = -(\nu_t^{t+1} + \cdots + \nu_t^{s-1})$. It is easy to show that the risk-neutral location parameters have the same solution, equation (14), as in the unsubordinated case in Example 2.

B. RNVRs for Pure-Jump Processes

We consider only the case of stock derivatives and focus on the pure-jump case for simplicity and to emphasize the parallels with the discrete-time model. Unlike the lognormal jump-diffusion models in Merton (1976) and Amin and Ng (1993), jump risk in our model is systematic and can have any distribution.

With the exception of the state-price density, which declines at the discount rate β between jumps, all the processes introduced below are adapted pure-jump processes, with sample paths that are right continuous step functions, changing only at the stopping times, $\tau_1, \tau_2, \ldots$. The stopping times are characterized by the intensity process λ. Since λ is also a step function, then for any $t \in (\tau_j, \tau_{j+1})$, the conditional probability that the next jump occurs in the next dt time interval is $\lambda_{\tau_j} dt$. For any process x in this section, we define the forward difference $\Delta x_{\tau_j} = x_{\tau_{j+1}} - x_{\tau_j}$. Let $h_{\tau_j} : \Omega \times I\!R \to I\!R_{++}$ denote the density function of ΔZ_{τ_j} conditional on $\mathcal{F}_{\tau_j}$.

Stock returns at jumps are assumed to satisfy

$$S_{\tau_{j+1}} = \operatorname{diag}(S_{\tau_j}) F_{\tau_j}\left(\mu_{\tau_j} + \sigma_{\tau_j} \Delta Z_{\tau_j}\right), \tag{23}$$

where $F_{\tau_j} : \Omega \times I\!R^n \to I\!R^n$, the location parameter μ and volatility parameter σ are stochastic processes valued in $I\!R^n$ and $I\!R^{n \times d}$, respectively, and $\tau_0 = 0$. The short rate r is per unit time, so one dollar invested in the riskless account at τ_j grows to $e^{r_{\tau_j} \Delta \tau_j}$ at τ_{j+1}. We assume that $r < \lambda$.

Between jumps, the state-price density is assumed to decline at the rate β (to allow for subjective discounting). We also impose a restriction on its ratio at jumps that results in a risk-neutral shift in the conditional distribution of jumps in the state variable.

ASSUMPTION 5: *A state–price density process satisfies*

$$\frac{\pi_{\tau_{j+1}}}{\pi_{\tau_{j+1}-}} = V_{\tau_j} \frac{h_{\tau_j}(\Delta Z_{\tau_j} + \eta_{\tau_j})}{h_{\tau_j}(\Delta Z_{\tau_j})}, \quad \frac{d\pi_t}{\pi_t} = -\beta dt, \quad t \in (\tau_j, \tau_{j+1}), \quad j = 0, 1, \ldots,$$

where η and V are stochastic processes valued in $I\!R^d$ and $I\!R_{++}$, respectively.

Similar to the Section I model, the conditional risk-neutral density of ΔZ_{τ_j} is $\tilde{h}_{\tau_j}(z) = h_{\tau_j}(z + \eta_{\tau_j})$, where η is again interpreted as measuring aversion to state-variable shocks. Therefore, the risk-neutral conditional distribution of $\Delta \tilde{Z}_{\tau_j} = \Delta Z_{\tau_j} + \eta_{\tau_j}$ is the same as the true conditional distribution of ΔZ_{τ_j}. We interpret V as measuring aversion to jumps themselves (see the equilibrium example below), and β as measuring subjective discounting. If the subjective and riskless discount rates differ, the frequency of shocks is altered under discounting by the state-price density. Under the risk-neutral measure, the magnitude of jumps is increased by η and the per-unit-time probability of a jump is $\tilde{\lambda} = \lambda + \beta - r$ (see the proof of Proposition 4). Risk neutrality corresponds to $\eta = 0$ and $V = 1$ (which together imply $\beta = r$, so that π is a riskless discount factor).

The case of Gaussian jumps, for example, corresponds to the density function in equation (3) and state-price density jumps satisfying

$$\frac{\pi_{\tau_{j+1}}}{\pi_{\tau_{j+1}-}} = V_{\tau_j} \exp\left(-\frac{1}{2}\eta'_{\tau_j}\eta_{\tau_j} - \eta'_{\tau_j}\Delta Z_{\tau_j}\right).$$

An equilibrium model consistent with Assumption 4 is a pure-endowment representative-agent economy with utility function $E\int_0^T e^{-\beta t}u(c_t)\,dt$, with $u_c(\cdot)$ satisfying equation (17), and the pure-jump endowment process $\Delta\ln(v(e_{\tau_j})) = a_{\tau_j} + b'_{\tau_j}\Delta Z_{\tau_j}$, $j = 0, 1, 2, \ldots$, with a and b stochastic processes. Assumption 5 then holds under the parameter assignments in equation (19), with $\mathcal{B}_t = 1$.

The following result is similar to Proposition 1, except that both the location and intensity parameters are changed under the risk-neutral measure, and the equation for determining the risk-neutral location parameter is modified to account for the stochastic time intervals between price changes.

PROPOSITION 4: *Suppose stock returns satisfy equation* (23), *Assumption* 5 *holds, and, for all $t \in [0, T]$, that $r_t, \lambda_t, \beta_t, \sigma_t, h_t$, and F_t are $\mathcal{H}_t^S$-measurable and that there is a unique solution $\tilde{\mu}_t \in \mathcal{H}_t^S$ to*

$$(1 + r_t/\tilde{\lambda}_t)\mathbf{1} = E\left[F_t(\tilde{\mu}_t + \sigma_t\Delta Z_t)\,|\,\mathcal{H}_{t-}^S, \text{jump at } t\right]. \tag{24}$$

Then risk-neutral stock returns satisfy

$$S_{\tau_{j+1}} = \operatorname{diag}(S_{\tau_j})F_{\tau_j}\left(\tilde{\mu}_t + \sigma_{\tau_j}\Delta\tilde{Z}_{\tau_j}\right), \quad j = 0, 1, \ldots,$$

where the conditional (*on $\mathcal{F}_{\tau_j}$*) *distributions of $\Delta\tilde{Z}_{\tau_j}$ under $\tilde{P}$, and ΔZ_{τ_j} under P are the same, but the risk-neutral jump intensity function is $\tilde{\lambda} = \lambda + \beta - r$.*

Proof: See Appendix A.

As in Proposition 3, derivative pricing is risk-neutral with respect to only one of the two sources of risk. Given the subjective discount factor, β, derivatives can be priced as if agents were risk-neutral with respect to jump risk (that is, as if $\eta = 0$), but possibly averse to jumps themselves (that is, V is not necessarily one). The risk-aversion parameter η again affects stock returns only through the stock-return location parameter, which has the decomposition $\mu_t = \tilde{\mu}_t + \sigma_t\eta_t$. The risk-neutral location parameter, $\tilde{\mu}$, equates riskless returns and expected stock returns after changing the intensity process to $\tilde{\lambda}$.[11] Discounting by the state-price density not only cancels the risk-premium term, $\sigma_t\eta_t$, but also changes jump-intensity process through the subjective discount factor β.

[11] This is easier to see by writing equation (24) as

$$r_t\mathbf{1} = \tilde{\lambda}_t\tilde{E}\left[F_t(\tilde{\mu}_t + \sigma_t\Delta\tilde{Z}_t) - 1\,|\,\mathcal{H}_{t-}^S, \text{jump at } t\right].$$

Comparing equations (4) and (24) shows that all the Section I risk-neutral location parameter expressions can be modified to the current jump setting (when the discrete-time stock-price process is replaced by the analogous pure-jump process) by replacing r_t there with $\ln(1 + r_t/\tilde{\lambda}_t)$.

The following example modifies the double exponential Heston (1993) model, discussed in Section I.C, by replacing the discrete-time exponential-gamma state-variable process with a continuous-time exponential-gamma jump process.

EXAMPLE 5: *Pure-jump gamma model.* Let the stock return at jumps satisfy

$$\log(S_{\tau_{j+1}}/S_{\tau_i}) = \phi + \exp\left(\mu + \Delta Z_{\tau_j}\right), \quad j = 1, 2, \ldots.$$

where $\phi \leq 0$. The gamma-distributed random variable $\exp(\Delta Z_{\tau_j})$ is assumed to have $m = 1$ degree of freedom. For simplicity, assume that the intensity, λ, and short rate, r, are strictly positive constants. The total number of jumps in $[0, t]$ is N_t, which is Poisson distributed with mean λt. Therefore, $2e^{-\mu}[\ln(S_t/S_0) - \phi N_t] = 2\sum_{j=1}^{N_t} \exp(\Delta Z_{\tau_j})$ is distributed as noncentral chi-squared with zero degree of freedom and noncentral parameter $2\lambda t$ (see Johnson and Kotz (1970, Chapter 28), and Siegel (1979)). Solving equation (24) for $\tilde{\mu}$ (or, equivalently, replacing e^r with $1 + r/\tilde{\lambda}$ in equation (10) above) we get

$$\tilde{\mu} = \ln\left(\frac{(1 - e^{\phi})(\lambda + \beta) + e^{\phi} r}{\lambda + \beta}\right).$$

When $\phi = 0$, the actual and risk-neutral distributions of $\ln(S_t/S_0)$ are proportional to a noncentral chi-squared random variable.

C. A General RNVR

We sketch a general, though less tractable (especially in the case of a multidimensional case) RNVR under an alternative assumption on the state-price density. Let the conditional density function of the d-dimensional state variable shock ΔZ_t be denoted $h_t(z; \zeta_t)$, where ζ is an $\mathbb{R}^n$-valued stochastic process representing parameters of the distribution. We assume that stock returns satisfy

$$S_{t+1} = \mathrm{diag}(S_t) F_t(\Delta Z_t), \quad t = 0, \ldots, T-1,$$

where $F_t : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, and that the state-price density satisfies

$$\frac{\pi_{t+1}}{\pi_t} = V_t k_t(\Delta Z_t) \frac{h_t(\Delta Z_t; \zeta_t + \eta_t)}{h_t(\Delta Z_t; \zeta_t)},$$

where $k(\cdot)$ is normalized, for convenience, so that $\int_{\mathbb{R}^d} k_t(z) h_t(z; \zeta_t + \eta_t)\, dz = 1$.

Since the state-price density must correctly price the riskless security, we get $V_t = e^{-r_t}$, and the risk-neutral density function is $\tilde{h}_t(z; \zeta_t + \eta_t) = k_t(z) h_t(z; \zeta_t + \eta_t)$. Under a strong invertibility condition, the stock returns equation can be used to solve $\zeta_t + \eta_t$. Assume there is a unique $\tilde{\zeta}_t \in \mathcal{F}_t$ that equates risk-neutral expected stock returns to the riskless return:

$$e^{r_t}\mathbf{1} = \int_{I\!R^d} k_t(z) h_t(z; \tilde{\zeta}_t) F_t(z)\, dz$$

for every t. Then $\tilde{\zeta} = \zeta + \eta$ is the risk-neutral distribution parameter process, and the risk-neutral state-variable and stock-return distributions can be solved without V, η, or ξ. Heston (1993), in a single-stock (and therefore single state-variable, representing the stock price), one-period setting restricts π_{t+1}/π_t to be independent of the distributional parameter ζ_t, which implies that $h_t(z; \xi_t)$ is of the form $a_t b_t(z)^{\xi_t}$. We do not impose this condition in our Assumption 1; therefore, even in the single-stock, one-period setting, the RNVRs do not nest.

In the special case when $k_t(\cdot) \equiv 1$ for all t, the risk-neutral and true state-variable shock distributions are in the same class (a return location parameter shift can be viewed as a special case). Otherwise, we can choose essentially arbitrary marginal distributions for both the state-price density and stock return (although the joint distribution is restricted).

V. Conclusion

This paper presents a general discrete-time model under which derivative securities can be priced independent of preference parameters and a stock-return location parameter. Uncertainty is generated by a state variable with essentially any continuous distribution, and stock returns are modeled within a large class of functions of this state variable. This generality allows, for example, the Black–Scholes model to hold under a large class of distributions of representative-agent marginal utility. Conversely, given any marginal utility distribution in this class, pricing without preference and location parameters is possible for essentially any stock-return distribution.

The key assumption underlying the RNVRs is a restriction on the form of the function linking marginal utility (or, more generally, a state-price density) and the state variable. The restriction implies a simple distributional shift when probabilities are weighted by marginal utilities to obtain shadow prices. Preferences therefore affect returns only through a stock-return location parameter. Derivatives pricing is accomplished by first modifying this location parameter so that the expected stock return equals the riskless rate, and then computing the expected discounted value of the payout. That is, assets are priced as if investors were risk neutral. With a Gaussian state variable, the restriction becomes nonbinding in the diffusion limit, which accounts for the similarity to pricing in the diffusion setting.

In addition to the restriction on the state-price density, we impose an invertibility condition on the stock-return function that allows us to identify a unique risk-neutral location parameter. If more than one location parameter yielded expected returns equal to the riskless rate, we would need additional information (such as option prices or equilibrium restrictions) to identify the solution that gives the correct risk-neutral dynamics. The invertibility condition is weak, however, compared to a more general RNVR (as sketched in Section IV.C), and is satisfied in the single stock case when the return function is monotonic. In fact, the risk-neutral location parameter can be solved explicitly with the return functions typically used in the literature.

Appendix A. Proofs Omitted from the Main Text

Proof of Proposition 1: The state-price density must price one-period unit discount bonds, therefore $e^{-r_t} = E_t(\pi_{t+1}/\pi_t) = V_t$. From $S_t = E_t(S_{t+1}\pi_{t+1})/\pi_t$ and the returns process in equation (1), we have

$$e^{r_t}\mathbf{1} = \tilde{E}\left[F_t(\hat{\mu}_t + \sigma_t \Delta\tilde{Z}_t) \mid \mathcal{F}_t\right] = \tilde{E}\left[F_t(\hat{\mu}_t + \sigma_t \Delta\tilde{Z}_t) \,\middle|\, \mathcal{H}_t^S\right],$$

where $\hat{\mu}_t = \mu_t - \sigma_t \eta_t$, and the second equality follows from the $\mathcal{H}_t^S$-measurability of r_t. The $\mathcal{H}_t^S$-measurability of F_t, h_t, and σ_t imply that there is an $\mathcal{H}_t^S$-measurable modification of $\hat{\mu}_t$ that gives the same conditional expectation. This modification, which we call $\tilde{\mu}_t$, must be the solution of equation (4). (If, for example, F_t is strictly monotonic in each element of its argument, then $\hat{\mu}_t = \tilde{\mu}_t$.) Q.E.D

Proof of Proposition 4: We show first the change in intensity under the risk-neutral measure. The time-τ_j risk-neutral conditional probability that the time to the next jump will exceed s is

$$\tilde{P}_{\tau_j}(\Delta\tau_j > s) = E\left(1_{\{\Delta\tau_j > s\}} \exp(r_{\tau_j} s)\pi_s/\pi_{\tau_j}\right) = P_{\tau_j}(\Delta\tau_j > s) E\left(\exp\left[(r_{\tau_j} - \beta)s\right]\right).$$

Substitute $P_{\tau_j}(\Delta\tau_j > s) = \exp(-\lambda_{\tau_j} s)$ to obtain $\tilde{P}_{\tau_j}(\Delta\tau_j > s) = \exp(-\tilde{\lambda}_{\tau_j} s)$, where $\tilde{\lambda} = \lambda + \beta - r$.

Since investing one dollar at time τ_j yields $\exp(r_{\tau_j}\Delta\tau_j)$ at τ_{j+1}, the state-price density must satisfy

$$1 = E_{\tau_j}\left(\exp\left(r_{\tau_j}\Delta\tau_j\right)\frac{\pi_{\tau_{j+1}}}{\pi_{\tau_j}}\right) = V_{\tau_j} E_{\tau_j}\left(\exp\left[(r_{\tau_j} - \beta)\Delta\tau_j\right]\right) = V_{\tau_j}\frac{\lambda_{\tau_j}}{\tilde{\lambda}_{\tau_j}};$$

therefore $V_{\tau_j} = \tilde{\lambda}_{\tau_j}/\lambda_{\tau_j}$, $j = 1, 2, \ldots$. The risk-neutral conditional density of ΔZ_{τ_j} is

$$\tilde{h}_{\tau_j}(z) = \frac{h_{\tau_j}(z + \eta_{\tau_j})}{h_{\tau_j}(z)} h_{\tau_j}(z) = h_{\tau_j}(z + \eta_{\tau_j}).$$

Equation (24) is derived by considering a one-dollar investment in stock i at time τ_j, which yields the gross return at $F^i_{\tau_j}$ at τ_{j+1}. Therefore

$$\begin{aligned}
\mathbf{1} &= E_{\tau_j}\left[\frac{\pi_{\tau_{j+1}}}{\pi_{\tau_j}} F_{\tau_j}\left(\mu_{\tau_j} + \sigma_{\tau_j}\Delta Z_{\tau_j}\right)\right] \\
&= \tilde{E}_{\tau_j}\left[\exp\left(-r_{\tau_j}\Delta\tau_j\right) F_{\tau_j}\left(\mu_{\tau_j} - \sigma_{\tau_j}\eta_{\tau_j} + \sigma_{\tau_j}\Delta\tilde{Z}_{\tau_j}\right)\right] \\
&= \tilde{E}_{\tau_j}\left[\exp\left(-r_{\tau_j}\Delta\tau_j\right)\right] \tilde{E}_{\tau_j}\left[F_{\tau_j}\left(\mu_{\tau_j} - \sigma_{\tau_j}\eta_{\tau_j} + \sigma_{\tau_j}\Delta\tilde{Z}_{\tau_j}\right)\right].
\end{aligned}$$

The last step follows from the conditional independence of returns and the waiting times between jumps. Now substitute

$$\tilde{E}_{\tau_j}\left[\exp(-r_{\tau_j}\Delta\tau_j)\right] = \frac{\tilde{\lambda}_{\tau_j}}{\tilde{\lambda}_{\tau_j} + r_{\tau_j}}.$$

The proof is completed as in the proof of Proposition 1. Q.E.D.

Appendix B. Convergence to the Continuous-Time Model

This appendix shows that the Gaussian model with an exponential stock-return function converges to the diffusion model as the period length shrinks to zero, and that the state-price density necessarily satisfies the restriction in equation (2) in the limit. The convergence is not surprising, since the discrete-time exponential returns process and state-price density processes can be interpreted as stochastic integrals in a continuous-time setting with simple, or step function, drift and volatility processes. Any continuous-time stochastic integral is, in turn, constructed by taking a sequence of stochastic integrals of simple parameter processes.

The first step is to restate the formulas after changing the period length from 1, as it has been throughout the paper, to $\Delta t = 1/q$, for some positive integer q. The parameter processes corresponding to this partition will be identified by a superscript q. For the purpose of taking the limits below, all the discrete-time processes (identified by superscript q) should be interpreted as step functions, defined at every $t \in [0, T]$, but changing only every Δt. The shocks $\Delta Z_t = Z_{t+\Delta t} - Z_t$ are again normal with zero mean, but the variance is now Δt and the density function is therefore $h_t(z) = (2\Pi)^{-d/2}\exp(-\frac{z'z}{2\Delta t})$.

By Assumption 1, the state-price density satisfies

$$\frac{\pi^q_{t+\delta}}{\pi^q_t} = V_t \frac{h_t\left(\Delta Z_t + \eta^q_t \Delta t\right)}{h_t(\Delta Z_t)}, \quad t = 1, \ldots, T.$$

Substitute $V_t = e^{-r^q_t \Delta t}$ (to correctly price riskless investments) and the density function to get the state-price density equation for the qth partition:

$$\Delta \log\left(\pi^q_t\right) = -\left(r^q_t + \tfrac{1}{2}\eta^{q\prime}_t \eta^q_t\right)\Delta t - \eta^{q\prime}_t \Delta Z_t.$$

Focusing for simplicity on the single stock case, and interpreting the location parameter μ^q as per-unit time, the exponential-form stock-return equation can be written

$$\Delta \ln\left(S^q_t\right) = \mu^q_t \Delta t + \sigma^q_t \Delta Z_t.$$

We now impose conditions so that r^q, η^q, μ^q, and σ^q converge strongly enough to their continuous-time limits r, η, μ, and σ so that S^q and π^q converge. Assume that the processes $r^q, \eta^q, \mu^q, \sigma^q$ are uniformly bounded and that $[r^q - r]^2$,

$[\eta^q - \eta]^2$, $[\mu^q - \mu]^2$ and $[\sigma^q - \sigma]^2$ all converge to zero as q goes to infinity, where $[x]^2 = E(\int_0^T x_s' x_s ds)$ for any possibly vector-valued process x.

Given the convergence of the parameter processes, π^q and S^q converge (in the sense that $[\pi^q - \pi]^2$ and $[S^q - S]^2$ converge to zero) to the stochastic differential equations (see Karatzas and Shreve (1991, Section 3.2))

$$d\ln(\pi_t) = -\left(r_t + \tfrac{1}{2}\eta_t'\eta_t\right)dt - \eta_t'\,dZ_t,$$

$$d\ln(S_t) = \mu_t dt + \sigma_t\,dZ_t.$$

As discussed in Section I.C, if the continuous-time filtration is generated by the Brownian motion Z, the state-price density must necessarily be of this form (after imposing the drift restriction to price short-rate investments) because every adapted process can be represented as a stochastic differential equation with respect to Z. Therefore, Assumption 1 is automatically satisfied.

REFERENCES

Amin, Kaushik I., and Victor K. Ng, 1993, Option valuation with systematic stochastic volatility, *Journal of Finance* 48, 881–910.

Bondesson, Lennart, 1992, *Generalized Gamma Convolutions and Related Classes of Distributions and Densities* (Springer-Verlag, New York).

Brennan, Michael Z., 1979, The pricing of contingent claims in discrete time models, *Journal of Finance* 39, 53–68.

Câmera, António, 1999, An extended set of risk neutral valuation relationships for the pricing of contingent claims, *Review of Derivatives Research* 3, 67–83.

Câmera, António, 2003, A generalization of the Brennan-Rubinstein approach for the pricing of derivatives, *Journal of Finance* 58, 805–819.

Câmera, António, and Richard C. Stapleton, 2001, Forward-price processes and the risk-neutral pricing of options, Working paper, University of Strathclyde.

Epstein, Larry, and Stanley Zin, 1991, Substitution, risk aversion, and the temporal behavior of consumption and asset returns: an empirical analysis, *Journal of Political Economy* 99, 263–286.

Heath, David, Robert Jarrow, and Andrew Morton, 1992, Bond pricing and the term structure of interest rates: A new methodology for contingent claim valuation, *Econometrica* 60, 77–105.

Heston, Steven, 1993, Invisible parameters in option prices, *Journal of Finance* 48, 933–947.

Hull, John, and Alan White, 1990, Pricing interest-rate-derivative securities, *Review of Financial Studies* 3, 573–592.

Johnson, Norman, and Samuel Kotz, 1970, *Continuous Univariate Distributions* (Houghton Mifflin Company, Boston).

Karatzas, Ioannis, and Steven Shreve, 1991, *Brownian Motion and Stochastic Calculus* (Springer-Verlag, New York).

Madan, Dilip B., Peter Carr, and Eric C. Chang, 1998, The variance gamma process and option pricing, Working paper, Robert H. Smith School of Business, University of Maryland.

Madan, Dilip B., and Frank Milne, 1991, Option pricing with V.G. martingale components, *Mathematical Finance* 1, 39–55.

Madan, Dilip B., and Eugene Seneta, 1990, The variance gamma (V.G.) model for share market returns, *Journal of Business* 63, 511–524.

Madrigal, Vincente, and Stephen D. Smith, 1993, Risk-neutral valuation without spanning: The discrete time case, Working paper, Georgia State University College of Business Administration.

Merton, Robert, 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Merton, Robert, 1976, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* 3, 125–144.

Praetz, Peter D., 1972, The distribution of share price changes, *Journal of Business* 45, 49–55.

Rubinstein, Mark, 1976, The valuation of uncertain income streams and the pricing of options, *Bell Journal of Economics* 7, 407–425.

Samuelson, Paul, and Robert Merton, 1969, A complete model of warrant pricing that maximizes utility, *Industrial Management Review* 10, 17–46.

Siegel, Alex F., 1979, The noncentral chi-squared distribution with zero degrees of freedom and testing for uniformity, *Biometrica* 36, 707–719.

Stapleton, Richard C., and Marti G. Subrahmanyam, 1984, The valuation of multivariate contingent claims in discrete time models, *Journal of Finance* 39, 207–228.

Vasicek, Oldrich A., 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics* 5, 177–188.