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Do Bonds Span the Fixed Income Markets? Theory and Evidence for Unspanned Stochastic Volatility

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ABSTRACT

Most term structure models assume bond markets are complete, that is, that all fixed income derivatives can be perfectly replicated using solely bonds. However, we find that, in practice, swap rates have limited explanatory power for returns on at-the-money straddles—portfolios mainly exposed to volatility risk. We term this empirical feature "unspanned stochastic volatility" (USV). While USV can be captured within an HJM framework, we demonstrate that bivariate models cannot exhibit USV. We determine necessary and sufficient conditions for trivariate Markov affine systems to exhibit USV. For such USV models, bonds alone may not be sufficient to identify all parameters. Rather, derivatives are needed.

MOST TIME-HOMOGENEOUS MODELS of the term structure predict that bond prices are sufficient to complete the fixed income markets. One implication of this prediction is that fixed income derivatives are redundant securities. Another (related) implication is that bonds can be used to hedge volatility risk. These implications are in contrast to the equity-derivative literature, where it is common to assume that volatility risk cannot be hedged by trading in the underlying stock alone (e.g., Heston (1993)). In such a case, stock options are not redundant securities.[1]

In this paper, we present empirical evidence suggesting that interest rate volatility risk cannot be hedged by a portfolio consisting solely of bonds. Using data on swap rates, caps, and floors for three different currencies, we

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[1] Buraschi and Jackwerth (2001) provide empirical evidence in this direction.

find that there is a rather weak correlation between changes in swap rates and returns on at-the-money straddles.[2] In particular, regression analysis indicates that in some cases as little as 10 percent of straddle returns can be "explained" by changes in the term structure of swap rates. However, the residuals of these regressions are highly cross-correlated across straddle maturities. Indeed, principal components analysis indicates that a single additional state variable can explain more than 85 percent of the remaining variation. These findings strongly suggest that there is at least one state variable which drives innovations in interest rate derivatives, but does not affect innovations in the swap rates (and thus, bond prices) themselves. In other words, these findings suggest that the bond market by itself is incomplete.[3]

We note that it is straightforward to capture this feature, which we term "unspanned stochastic volatility" (USV), by directly specifying the joint dynamics of forward rates (or equivalently, bond prices) and the state variables that drive forward rate volatility (see, e.g., Andreasen, Collin-Dufresne, and Shi (1997), Collin-Dufresne and Goldstein (2001b), and Kimmel (2001a,b)). This approach is analogous to the pricing of equity derivatives (e.g., Heston (1993)) by directly specifying the joint dynamics of a traded asset (i.e., a stock) and its volatility. One disadvantage of such an approach, however, is that in such a framework, bond prices become inputs to the model, rather than predictions of the model. Hence, such an approach provides no testable implications for the cross section of bond prices.

In contrast to modeling forward rate dynamics directly, most models that attempt to investigate the cross-sectional and time-series behavior of bond prices choose a set of latent variables to serve as the state vector, and then define the spot rate as a function of these state variables. The state vector dynamics are typically assumed to be Markov and time homogeneous. Interestingly, most of these models predict that bonds alone are sufficient to complete the fixed income markets. Indeed, most term structure models fall within the so-called affine class of Duffie and Kan (1996, hereafter DK), where bond yields are linear in the *entire set* of state variables. Because all state variables show up in bond prices, these models predict that all sources of risk affecting fixed income derivatives can be completely hedged by a portfolio consisting solely of bonds. For example, the stochastic volatility models of Fong and Vasicek (1991) and Longstaff and Schwartz (1992) generate bond yields that are linear in both the spot rate and the volatility state variables. Hence, these models predict that bonds can be used to hedge volatility risk.

[2] An at-the-money straddle is a portfolio composed of an at-the-money cap and floor. As constructed, this portfolio is hedged against small changes in the interest rate level. Hence, this portfolio is mainly exposed to volatility risk.

[3] A recent paper by Heidari and Wu (2001) confirms our findings. The authors perform an analysis of the factors driving swaption-implied volatilities and document the existence of volatility-specific factors.

Below, we identify a class of time-homogeneous Markov models with a finite state variable representation that provides testable implications for both the time series and cross-sectional behavior of bond prices, yet permits fixed income derivatives to be nonredundant securities. In particular, we identify a class of affine models that can exhibit USV. The affine framework is convenient because it provides closed-form solutions for bond yields that are linear in the state variables. However, we identify parameter restrictions such that bond yields do not depend on the volatility state variable. As such, bonds cannot be used to hedge volatility risk, in turn implying that bonds do not span the fixed income markets.

After providing a formal definition of USV, we show that it is not possible for bivariate Markov affine models to exhibit such behavior, thus ruling out the models of Fong and Vasicek (1991), Longstaff and Schwartz (1992), and Chen and Scott (1993) as potential candidates. More generally, we demonstrate that even nonaffine bivariate models of the short rate cannot generate USV. We then identify necessary and sufficient conditions for a trivariate Markov system to exhibit USV. While such models as Balduzzi, Das, and Foresi (1996), and Chen (1996) cannot satisfy these restrictions, we demonstrate that the maximal $A_1(3)$ model proposed by Dai and Singleton (2000, herafter DS) can exhibit USV.

Focusing on trivariate models, DS analyze the maximal number of parameters that can be identified given a series of bond prices. Below, we argue that their analysis is even more general in that maximality refers to the maximum number of parameters that can be identified given *all* fixed income securities. As we demonstrate, the distinction is important because, for the class of models which exhibit USV, bond prices alone are not sufficient to determine all of the identifiable parameters of the model. Rather, both bonds and fixed income derivatives are needed to identify the system.

The absence of the volatility state variable in bond prices implies that bond innovations are not contemporaneously affected by volatility innovations, and therefore cannot be used to hedge volatility risk instantaneously. Over a longer horizon, however, bond prices *are* affected by changes in volatility. This noncontemporaneous effect begs the question of whether our models, which generate USV in continuous time, will generate USV if data are observed at only discrete time intervals. To investigate whether the proposed class of models can replicate our empirical findings, we simulate a monthly time series of straddle prices and swap rates in both a traditional stochastic volatility model (we use the $A_1(3)$ model of DS (2000)), and in a similar three-factor model which exhibits USV. We then repeat the regression analysis described above for both simulated economies. The results confirm that if a traditional model is used to generate the data sample, then over 95 percent of the variation in straddle prices is explained by changes in swap rates. However, if a USV model is used to generate the data, then only about 25 percent of the variation is explained by changes in the swap rates, similar to the results obtained when actual historical data is used.

To our knowledge, "unspanned stochastic volatility" models were first investigated by Andreasen, Collin-Dufresne, and Shi (1997, hereafter ACS). They note that, while the Heath, Jarrow, and Morton (HJM, 1992) framework uniquely specifies the drift of the forward rates in terms of its volatility structure under the risk-neutral measure, this specification is not sufficient to price interest rate contingent claims if the volatility structure evolves stochastically. Indeed, one also needs to specify the arbitrage-free process for these volatility-specific factors under the risk-neutral measure. Since the HJM restriction does not provide any guidance in that respect, ACS propose that contracts such as futures on yields should be used to identify and calibrate the model.[4]

Below, we do not explicitly consider the "backing-out" of the arbitrage-free dynamics of the volatility-specific state variables. Rather, we take the dynamics under the risk-neutral measure as given, and focus on the pricing implications of such a model for fixed income derivatives. Further, we identify necessary and sufficient conditions for a model to display USV within a time-homogeneous setting. In contrast to ACS, who investigate a two-factor HJM model with a deterministic (Gaussian) volatility structure and price derivatives using a nonrecombining lattice approach, we develop a more general framework. Further, our models possess closed-form solutions for derivatives prices, as shown in, for example, Duffie, Pan, and Singleton (2000, hereafter DPS).

Recently, in independent work, Kimmel (2001a, 2001b) investigates a HJM-type random field model[5] of the term structure where volatility of bond prices is driven by latent variables that possess an affine structure. Further, he identifies the partial differential equation that derivative securities with homogeneous payoff structures satisfy.[6] In his empirical paper, Kimmel (2001b) focuses on the dynamics of bond price volatility and correlation structures, since random field models offer no predictions for the cross section of bond prices. In contrast, we identify a class of affine models that possesses a finite dimensional Markov representation for bond and bond-option prices. While most of our work is restricted to the time-homogenous setting, we also discuss how to simultaneously calibrate the term structures of both interest rates and volatilities.

The rest of the paper is as follows. In Section I, we provide empirical evidence that suggests the bond market is incomplete. In particular, we present evidence that there are sources of risk which drive innovations in straddles but do not drive innovations in the swap rates themselves. In Section II, we identify under what conditions affine models can exhibit USV. In particular, we first show that no two-dimensional system can exhibit USV. We then

[4] Futures on yields have characteristics similar to those of the "log contracts" proposed by Neuberger (1994) to hedge foreign exchange volatility risk. However, futures on yields have the additional feature that permits interest rate level risk to also be hedged.

[5] See, for example, Kennedy (1994, 1997), Goldstein (2000), and Santa-Clara and Sornette (2001).

[6] The partial differential equation he derives is only valid for securities with payoffs that are homogeneous in zero-coupon bond prices.

identify necessary and sufficient restrictions for a three-dimensional affine system to exhibit USV. To demonstrate that the proposed models exhibit USV even over finite time intervals, we repeat the empirical procedure of Section I using data generated from our models, but sampled at only monthly time intervals. In Section III, we show that models exhibiting USV are generated naturally when forward rates, rather than the spot rate and other state variables, are taken as primitives. Some special cases are investigated. We conclude in Section IV.

I. Empirical Support for Unspanned Stochastic Volatility

Empirical evidence suggests that multiple factors are needed to adequately capture the dynamics of the cross section of bond prices.[7] Model-independent factor analysis finds that three factors explain almost all yield curve variation (Litterman and Scheinkman (1991)). Model-dependent investigations within the affine framework (Chen and Scott (1993), DS (2000)) or quadratic models (Ahn, Dittmar, and Gallant (2002)) similarly find that at least three factors are necessary to adequately capture the dynamics of the term structure of interest rates.

Recently, the focus has shifted to estimating the number of factors necessary to price fixed income derivatives. Boudoukh et al. (1997) find that approximately 90 percent of the variation in pricing of mortgage-backed securities can be explained by a few interest-rate-level factors. However, they are unable to explain the final 10 percent. Longstaff, Santa-Clara, and Schwartz (2001a, 2001b) find that, when pricing American-style swaptions, several state variables are needed. They also find that the implied volatility computed from cap prices reflects four state variables. Their findings differ from those of Andersen and Andreasen (2001), who argue that a one-factor model with level dependence of the volatility structure of forward rates might be as effective in fitting and hedging caps and floors. Fan, Gupta, and Ritchken (2001) test various multifactor models of the term structure and find that they systematically misprice caps and floors.

Here we explore a related question. In particular, we investigate how many bonds are needed to hedge interest rate *volatility* risk. Using various proxies for the short rate, previous empirical studies have found that stochastic volatility is a robust feature of short rate dynamics (Brenner, Harjes, and Kroner (1996) and Andersen and Lund (1997)). Here, we examine how much of the variation of returns in straddles (portfolios of at-the-money caps and floors) can be explained by the variations in swap rates. We focus on straddles because they are highly sensitive to bond-price volatility risk.

[7] See, for example, Litterman, Scheinkman, and Weiss (1991), Litterman and Scheinkman (1991), Knez, Litterman, and Scheinkman (1994), Dybvig (1997), Duffie and Singleton (1997), and DS (2000).

A. The Data

We use monthly data on swap rates, caps, and floors for the United States, the United Kingdom, and Japan from Datastream for the period ranging from February 1995 to December 2000. The available swap rate data include maturities of 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 years.[8] The six-month LIBOR rate is used as a proxy for the six-month swap rate.[9] We use the available swap rate data to construct the zero-coupon bond yield curve, which in turn is used to determine the discount factor for the cash-flows in the CAP and FLOOR market. This approach implicitly assumes that (1) the floating leg of the swap contract is valued at par and hence that the quoted swap rate is equivalent to a par-bond rate for an issuer with LIBOR-credit quality,[10] and (2) that CAP, FLOOR, and SWAP markets have "homogeneous" credit quality.[11] We interpolate zero-coupon bond yields with intermediate or nonavailable maturities from the closest available yields to maximize smoothness (i.e., minimize squared curvature) of the obtained yield curves.

The cap and floor data are quoted in terms of implied volatility for at-the-money (ATM) caps and floors. The implied volatilities are obtained using the Black (1976) formula (see Hull (2000, p. 540)). The phrase "at-the-money" implies that both caps and floors have the same strike, which is set to equal the forward swap rate (see Musiela and Rutkowski (1997, p. 393)), implying that caps and floors have the same initial value.[12] Using our computed zero-coupon curve, we transform the implied volatility data into cap and floor prices. Thus, for several different maturities, we obtain a time series of constant maturity, at-the-money cap and floor prices. However, since our goal is to analyze the hedging performance of different term structure models, we need monthly *changes* in prices for a given cap or floor contract. Unfortunately, since the CAP/FLOOR market is mostly a broker/interbank market, we are unable to obtain transaction data on existing cap and floor contracts. We thus resort to interpolation in order to estimate monthly changes in cap and floor prices. The procedure we adopt is the following. In month n we have an implied volatility σ_n for an at-the-money cap with time-to-maturity T and strike K. In month $(n + 1)$ we use data on implied volatility for at-the-money caps for several maturities to interpolate and thus estimate the implied volatility corresponding to a cap with strike K and time-to-maturity

[8] For the United States, we have 1, 2, 3, 4, 5, 7, and 10 for the whole sample, and 6, 8, and 9 years starting February 1997. For Japan, the cap data have a clear reporting flaw in the year 2000, so for Japan, we only used data from February 1995 to December 1999.

[9] We also used an extrapolation of the available swap rates, with negligible impact on the results.

[10] This is a standard textbook assumption, but see Duffie and Singleton (1997) and Collin-Dufresne and Solnik (2001) for a discussion of this assumption.

[11] Using an argument similar to that of Duffie and Singleton (1999), this assumption essentially allows us to use the same instantaneous "default and liquidity risk-adjusted" rate to discount cash flows under the risk-neutral measure.

[12] Indeed the difference between a cap and a floor is by definition equal to a forward swap contract with first payment date equal to the maturity of the shortest caplet/floorlet.

$T - \frac{1}{12}$. We use this estimated implied volatility as an input to the Black formula with strike K and appropriately interpolated forward rates and zero-coupon yields. We thus compute a matrix of one-month changes in prices of at-the-money cap prices. Note that we do not follow a contract over its whole life. Rather, each month we start with ATM cap prices and then use data from the next month to compute one-month price changes. This approach has two advantages. First, this method minimizes the noise introduced by the interpolation procedure used to estimate the implied volatility of the nonobserved cap prices. Second, it considers only portfolios that are "delta neutral" at inception, making them more sensitive to changes in volatility (relative to changes in interest rate levels). We proceed similarly with floors. As a simple consistency check of the interpolation procedure, we test the cap–floor parity condition by computing corresponding monthly changes in the forward swap contract. We find that the interpolated values satisfy ($\text{cap}_{est} - \text{floor}_{est} = \text{forward swap}_{est}$) extremely well.[13]

B. Methodology and Results

We then proceed to analyze monthly returns of straddles for different maturities. Creating straddle portfolios allows us to focus on the presence of unspanned stochastic volatility (USV) because straddles are rather insensitive to (small) changes in the level of interest rates, but are extremely sensitive to changes in volatility.[14] Such an analysis is important for those financial institutions which tend to be delta neutral, that is, those firms which hedge away interest rate risk.

We run separate regressions of changes in straddle prices for maturities of 1, 2, 3, 4, 5, 7, and 10 years on changes in swap rates. Since we would like to obtain an estimate of the "best" possible hedging of volatility risk we can achieve by using swaps, we consider as independent variables as many swap rates as are available. For the U.K. and Japan data, we consider an 11-factor model, and for the U.S., an 8-factor model.[15] The R^2 and adjusted R^2 are reported in Table I.[16] Clearly there is multicollinearity in the independent

[13] We note that, due to interest rate fluctuations, cap and floor prices are no longer equal one month after inception. That is, the forward swap contract does not in general have zero value one month after inception.

[14] Typically a straddle is long an ATM call and ATM put and thus delta neutral and thus sensitive to volatility changes to a first order. We note that caps and floors are actually portfolios of caplets and floorlets each with the same strike. Thus each caplet and floorlet is actually not ATM, and the straddle is not, strictly speaking, delta neutral. However, to a first-order, straddles of caps and floors are still volatility sensitive.

[15] Maturities 0.5 to 11 for the United Kingdom and Japan, maturities 0.5, 1, 2, 3, 4, 7, and 10 for the United States.

[16] One potential criticism with our approach is that, in general, term structure models predict that the factor loadings (i.e., the βs) are interest-level dependent, rather than constant, as we assume. To test this concern, we perform a second regression on the U.S. data where, in addition to the changes in swap levels, we include terms that are products of the level and the change in swap level. If the model that we are testing is seriously misspecified due to the

Table I

R^2 and Adjusted R^2 of the Regression of Straddle Returns

The table gives the R^2 and adjusted R^2 of the regression of straddle returns with maturities {1, 2, 3, 4, 5, 7, 10} years on the changes in swap rates for all available maturities ({0.5, 1, 2, 3, 4, 5, 7, 10} for U.S. data and {0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} for U.K. and Japan data). Although multi-collinearity is evident in the regressors, the R^2 represents an upper bound on the proportion of the variance of straddle returns that can be hedged by trading in swaps.

	U.S. Straddles		U.K. Straddles		Japan Straddles	
Maturity	R^2	Adjusted R^2	R^2	Adjusted R^2	R^2	Adjusted R^2
1	0.215	0.085	0.270	0.134	0.229	0.044
2	0.316	0.202	0.177	0.023	0.239	0.057
3	0.349	0.241	0.155	−0.002	0.305	0.139
4	0.341	0.232	0.162	0.006	0.464	0.336
5	0.385	0.283	0.149	−0.010	0.468	0.340
7	0.439	0.346	0.120	−0.044	0.513	0.396
10	0.478	0.391	0.097	−0.071	0.398	0.254

variables, but the R^2 provides an upper bound to the amount of variation in straddle returns that can be hedged with swaps. The R^2 are quite low, implying that portfolios of bonds (or equivalently, swaps) have very limited ability to hedge volatility risk. We emphasize that these findings are inconsistent with the predictions of the traditional term structure models with stochastic volatility (such as Fong and Vasicek (1991), Longstaff and Schwartz (1992), Chen and Scott (1993), and DK (1996)). Indeed, as we show in Section II.D, replicating the above regression in a simulated traditional affine economy would result in an R^2 well above 90 percent.

After running the multifactor regressions, we estimate (separately for each country) the covariance matrix of the residuals across straddle maturities. We then perform a principal components analysis on each covariance matrix. The corresponding eigenvalues are reported in Table II. We find that the first eigenvalue captures over 80 percent of the remaining variation for each of the three countries.[17] This finding implies that the low R^2 obtained for the regression of straddles on swaps is not due to noisy data but rather to model misspecification. Furthermore, these results suggest that one, or at most two, additional unspanned stochastic volatility state variables are suf-

interest rate level dependence of the βs, then this second regression should pick this up. Instead, the adjusted R^2 of this second regression is almost identical to that obtained for the original regression. This finding is not too surprising, since in the time period we investigate, interest rate levels remained in a fairly narrow range of five to eight percent. We also included other nonlinear terms (squared, cubed, and cross-multiplied changes in swap rates) as independent variables. Similar results are obtained, and thus not reported.

[17] We also perform the same analysis on the correlation matrix of the residuals. Similar results are obtained, and thus are not reported.

Table II

Eigenvalues of Principal Component Decomposition of the Covariance Matrix of Residuals

The table gives eigenvalues of principal component decomposition of the covariance matrix of residuals, ordered by magnitude of the eigenvalue. Note that over 80 percent of the variation is captured by the first principal component for each country.

	U.S. Residuals		U.K. Residuals		Japan Residuals	
Eigenvector	Eigenvalue	% Explained	Eigenvalue	% Explained	Eigenvalue	% Explained
1	0.07184	0.876	0.07751	0.845	0.15700	0.834
2	0.00865	0.105	0.01215	0.132	0.02217	0.118
3	0.00091	0.011	0.00139	0.015	0.00482	0.025
4	0.00035	0.004	0.00038	0.004	0.00235	0.012
5	0.00014	0.002	0.00018	0.002	0.00101	0.005
6	0.00009	0.001	0.00008	0.001	0.00051	0.003
7	0.00002	0.0003	0.00006	0.0006	0.00031	0.0016

ficient to explain almost all of the variation in straddle returns across maturities. That is, the term structure of volatilities is mostly driven by one or two state variables whose dynamics are mostly independent of those factors that drive swap rate innovations.

In Table III, we report the results of regressing the returns of an equally weighted portfolio of straddles on changes in three portfolios of swap rates. These three swap rate portfolios, which capture over 98 percent of the variation of swap rate changes in all three currency markets, are the first three principal components of the swap term structure, and correspond roughly to estimates of changes in level, slope, and curvature of the yield curve. We perform this regression for three reasons. First, by investigating the returns of an equally weighted portfolio of straddles rather than straddles of individual maturities, we provide further evidence that our results are not due to noisy data. Second, by looking at the first three principal components of swap innovations rather than innovations in all available swap maturities, we eliminate multicollinearity. Finally, the factor loadings on these principal components are of interest.

The results confirm our previous findings in that the factors driving the term structure of swap rates can barely explain 20 percent of the variation of straddle returns (in fact, as little as 1.3 percent for the U.K. data[18]). Interestingly, straddle returns appear to be negatively correlated with changes

[18] The reason for the much lower explanatory power of our regressions for U.K. data is that our sample period incorporates the devaluation of the pound in 1998. At that time, interest rates plummeted and implied volatilities on cap and floors spiked. The results are similar if we look at the period prior to the devaluation: It seems that changes in swap rates and straddle prices respond differently to factors driving the devaluation. The period after the devaluation appears to be much more in line with the U.S. and Japan results.

Table III
Coefficient and Adjusted R^2 Obtained Regressing the Return of an Equally Weighted Portfolio of Straddles on Three Portfolios of Swap Rates

The table gives coefficient and adjusted R^2 obtained regressing the return of an equally weighted portfolio of straddles on three portfolios of swap rates replicating the first three factors of a principal analysis of swap rate changes. t-statistics are in parentheses. The same analysis is conducted for each country. The very low R^2 reported for the U.K. data is apparently due to the devaluation of the pound in 1998.

	Fac1	Fac2	Fac3	Adj. R^2
U.S.	−11.05	33.1	38.81	21.3%
	(−3.89)	(2.30)	(1.30)	
U.K.	0.69	15.36	−20.56	1.3%
	(0.22)	(1.70)	(−0.92)	
Japan	−10.65	74.33	−24.83	29.0%
	(−2.48)	(4.40)	(−0.72)	

in level, but positively correlated with changes in slope, at least for the U.S. and Japan markets. The third factor, often associated with curvature, is statistically insignificant in all three markets. The finding that volatility is negatively related to level (holding slope constant) is somewhat counterintuitive from the standard result (e.g., Chan et al. (1992)) that volatility rises with the spot rate level. A possible explanation is that the time series we investigate includes "flight-to-quality" events, where interest rates plummeted while volatility rose.[19]

As an independent check on the proposed interpolation scheme, we also run regressions where the dependent variables are the changes in the implied volatilities of caps and floors, rather than our constructed straddle returns. As noted by Ledoit and Santa-Clara (1999), straddle returns and changes in implied volatilities are likely to have similar information content, since at-the-money options are approximately linear in volatility. Nearly identical results are obtained (and thus not reported), suggesting that the interpolation scheme is not driving our results.[20]

These empirical findings suggest that bonds do not span the fixed income markets. In particular, caps and floors seem to be sensitive to stochastic volatility that cannot be hedged by a position solely in bonds. In the next section, we provide a time-homogeneous framework that is consistent with these findings.

[19] Further, as discussed in note 14, our straddles are not, strictly speaking, delta neutral, which may explain some of these factor loadings.

[20] Further, in a recent paper, Heidari and Wu (2001) perform a similar study on swaption volatilities. They find a slightly higher explanatory power of term structure factors for swaption volatilities. This may partly be due to their running regressions on levels and not changes. However, they also find significant evidence of USV factors in the swaption market.

II. Affine Models of Incomplete Bond Markets

We assume that uncertainty is described by a standard filtered probability space. The innovations that drive fixed income securities are described by a d-dimensional vector of Brownian motions z^Q. The filtration is the natural filtration associated with the Brownian motion. Slightly generalizing equation (4.1) of DK (1996)[21], we consider the class of models whose N-state-variable, d-factor dynamics (with $N \geq d$) possess an affine structure as follows:

$$dX = (aX + b)\,dt + \Sigma\sqrt{v}\,dz^Q, \tag{1}$$

where $a \in R^{N\times N}$, $b \in R^N$, and $\Sigma \in R^{N\times d}$ has rank d. The components of the $(d \times d)$ diagonal matrix v are affine in the state variables X.

A well-known result from linear algebra guarantees that there exists an $(N - d)$-dimensional space, the kernel of the matrix $\Sigma^\top$, such that each vector ψ in this space satisfies $\Sigma^\top\psi = 0$. It is convenient to "rotate" the initial set of state variables $\{X_1,\ldots,X_N\}$ to a new set $\{X_1,\ldots,X_d,X'_{d+1},\ldots,X'_N\}$ so that the last $(N - d)$ of them are defined via[22]

$$X'_i = \sum_{j=1}^{N} \psi_{i,j}X_j \qquad i \in (d + 1,N), \tag{2}$$

or, in matrix notation, $X'_i = \psi_i^\top X$. From their definition, it follows that these state variables are locally deterministic. We thus propose the following.

Definition 1: After suitable rotation, an N-variate, d-factor affine model of the term structure possesses the following properties:

1. There is a set of $N \geq d$ state variables X_i, $i = 1,\ldots N$ that are jointly Markov, where each element of the drift and covariance matrix of the vector process $X(t)$ is affine in the N state variables.
2. The instantaneous risk-free rate is an affine function of these state variables: $r_t = \delta_0 + \sum_{i=1}^{N}\delta_i X_i(t)$, with at least one nonzero δ_i. Furthermore, there is no smaller subset of the N state variables that is both jointly Markov and sufficient to describe the dynamics of r_t.
3. The first d state variables have a diffusion matrix that is full rank. The last $(N - d)$ state variables are locally deterministic.

We now provide a formal definition of incomplete bond markets, consistent with the textbook definition (e.g., Duffie (1996) and Karatzas and Shreve (1998)).

[21] DK only consider the case where the matrix Σ is $N \times N$ and nondegenerate.

[22] Without loss of generality, it is assumed here that the first d state variables of the original set (and thus also the "rotated set") are chosen so that their volatility matrix is full rank, and the ψ_i, $i = d + 1\ldots N$ vectors (with element $\psi_{i,j}$) used for the rotation form a basis of the $(N - d)$-dimensional kernel of $\Sigma^\top$. Below, we will simply refer to this rotated system as X.

Definition 2: Define H as the set of all matrices obtained by stacking any finite collection of bond-price diffusion (row) vectors. Define d_H as the largest rank of any of these matrices. A term structure model generates incomplete bond markets if $d_H < d$.

Within the affine class of models, bond prices can be written in the form

$$P^T(t) = \exp\left(A(T-t) + \sum_{i=1}^{N} B_i(T-t)X_i(t)\right), \tag{3}$$

where $A(\tau), B_1(\tau), \ldots, B_N(\tau)$ are continuous deterministic functions that are solutions to a system of ordinary differential equations (see DK (1996)).

Noting item 3 of definition 1, Itô's lemma implies that incomplete bond markets can also be characterized by the following proposition.

PROPOSITION 1: *An N-variate, d-factor affine model generates incomplete bond markets if and only if there exists a set of parameters $\{\beta_1, \ldots, \beta_d\}$ not all zero such that*

$$\sum_{i=1}^{d} \beta_i B_i(\tau) = 0 \qquad \forall \tau > 0. \tag{4}$$

The number of linearly independent sets of parameter $\{\beta\}$ that satisfy this condition equals the number of state variables that are not spanned by the bond markets, and thus, the number of additional nonbond securities needed to complete the fixed income markets.

The intuition for this result is the following. If equation (4) holds, then without loss of generality, we can take $\beta_d \neq 0$. Then, we can write

$$B_d(\tau) = -\sum_{i=1}^{d-1} \frac{\beta_i}{\beta_d} B_i(\tau). \tag{5}$$

Plugging this into equation (3), we find

$$P^T(t) = \exp\left(A(T-t) + \sum_{i=1}^{d-1} B_i(T-t)\left(X_i(t) - \frac{\beta_i}{\beta_d} X_d(t)\right) + \sum_{i=d+1}^{N} B_i(T-t)X_i(t)\right). \tag{6}$$

It is convenient to change variables from $(X_1, \ldots, X_N)$ to $(Y_1, \ldots, Y_{d-1}, X_d, \ldots, X_N)$, where the $\{Y_i\}$ are defined via

$$Y_i(t) \equiv X_i(t) - \frac{\beta_i}{\beta_d} X_d(t) \qquad \forall i = 1, \ldots d-1. \tag{7}$$

Under this change of variables, bond prices are independent of the state variable X_d:

$$P^T(t) = \exp\left(A(T-t) + \sum_{i=1}^{d-1} B_i(T-t)Y_i(t) + \sum_{i=d+1}^{N} B_i(T-t)X_i(t)\right). \tag{8}$$

Hence, no portfolio that is composed solely of bonds can complete the fixed income markets, because X_d risk cannot be hedged by bonds. More formally, equation (4) implies that the rank of the diffusion matrix of the return on any portfolio comprising bonds of different maturities is less than d.

Below, we determine the necessary conditions for affine models to generate incomplete bond markets. This basically amounts to identifying parameter restrictions on the dynamics of the state variables X_i $(i = 1,\dots,d)$ so that the functions $B_i(\cdot)$ $(i = 1,\dots,d)$ satisfy equation (4). The method we use to identify these parameter restrictions effectively reduces to performing a Taylor series expansion on the functions $B_i(\tau) = \sum_{j=0}^{\infty}(B_i^j(0)/j!)\tau^j$, where $B^j(0)$ refers to the jth time derivative of the function, evaluated at $\tau = 0$. It then follows that equation (4) can be written:

$$0 = \sum_{i=1}^{d} \beta_i \sum_{j=0}^{\infty} \frac{B_i^j(0)}{j!}(\tau)^j \qquad \forall \tau > 0 \tag{9}$$

$$= \sum_{j=0}^{\infty} \frac{\tau^j}{j!}\left(\sum_{i=1}^{d} \beta_i B_i^j(0)\right) \qquad \forall \tau > 0. \tag{10}$$

Equation (10) implies that affine models generate incomplete bond markets if and only if the model satisfies:

$$0 = \sum_{i=1}^{d} \beta_i B_i^j(0) \qquad \forall j = (0,1,\dots,\infty). \tag{11}$$

A. Bivariate Affine USV Models

The bivariate affine models of Fong and Vasicek (1991) and Longstaff and Schwartz (1992) were the first term structure models to incorporate stochastic volatility into a fixed income framework. Hence, it seems natural to investigate under what conditions, if any, bivariate affine term structure models can generate unspanned stochastic volatility. As a point of reference, we note that the model of Heston (1993), which serves as a benchmark for stochastic volatility models of equity options, is a bivariate affine model of equity returns exhibiting USV. Below, however, we demonstrate the following proposition.

PROPOSITION 2: *Bivariate affine models of the term structure cannot generate incomplete bond markets, and thus, cannot exhibit USV.*

Proof: See Appendix A.

The intuition for why bivariate affine models cannot exhibit USV can be provided in terms of duration (P_r) and convexity (P_{rr}). By definition, a bivariate model exhibiting USV would imply bond prices are functions of only the time to maturity and the spot rate, and independent of the spot rate volatility V: $P^T(t,r_t,V_t) = P^T(t,r_t)$. This in turn implies that bond prices must satisfy

$$\frac{1}{2}P_{rr}^T(t,r)\sigma_r^2(t,r,V) + P_r^T(t,r)\mu_r(t,r,V) = rP^T(t,r) - P_t^T(t,r) \quad \forall T. \tag{12}$$

Note that the right-hand side of equation (12) is a function only of r, while the left-hand side is a function of both V and r. Since it is not possible for the ratio of duration and convexity to be constant across maturities, there is no way for the left-hand side to be independent of V, unless the spot rate process itself is one-factor Markov ($\mu_r(t,r,V) = \mu_r(t,r)$, $\sigma_r(t,r,V) = \sigma_r(t,r)$), which also precludes USV. We emphasize that this heuristic argument is not limited to the affine framework.[23] Indeed, we can show more generally the following proposition.

PROPOSITION 3: *Bivariate Markov model of the term structure, affine or otherwise, cannot generate incomplete bond markets, and thus, cannot exhibit USV.*

Proof: See Appendix B.

The implication of Proposition 3 is that at least three state variables are necessary for an affine model to generate incomplete bond markets.

B. Trivariate Affine USV Models

Here we derive necessary conditions for three-dimensional affine models to generate incomplete bond markets. First we prove that one can always conveniently rotate the underlying state variables to make them economically meaningful.

PROPOSITION 4: *Every affine model of three state variables that generates incomplete bond markets can be written so that the three state variables are:*

1. *the spot rate, r,*
2. *the drift of the spot rate, $\mu = (1/dt)E^Q[dr]$, and*
3. *the variance of the spot rate $V = (1/dt)E^Q[(dr)^2]$.*

Proof: See Appendix C.

We emphasize that this proposition is not trivial. First, this proposition implies the following corollary.

[23] We thank Jesper Andreasen for suggesting this.

COROLLARY 1: *Trivariate Gaussian models (such as Langetieg (1980)) cannot generate incomplete bond markets.*

This follows because in a Gaussian framework, the variance of the spot rate is a constant, and thus cannot be a state variable.

Further, Proposition 4 implies that models such as

$$dr_t = \kappa_r(\theta_t - r_t)\,dt + \sigma\sqrt{r_t}\,dz_1^Q \tag{13}$$

$$d\theta_t = \kappa_\theta(\bar{\theta} - \theta_t)\,dt + f\sqrt{r_t}\,dz_1^Q + g\sqrt{V_t}\,dz_2^Q \tag{14}$$

$$dV_t = \kappa_V(\bar{V} - V_t)\,dt + \sigma_V\sqrt{V_t}(\rho\,dz_2^Q + \sqrt{1-\rho^2}\,dz_3^Q) \tag{15}$$

and

$$dr_t = \kappa_r(\theta - r_t)\,dt + \sqrt{V_t}\,dz_1^Q \tag{16}$$

$$dV_t = \kappa_V(\bar{V} - V_t)\,dt + \sigma_V\sqrt{V_t}\,dz_1^Q + \sqrt{\Omega_t}\,dz_2^Q \tag{17}$$

$$d\Omega_t = \kappa_\theta(\bar{\Omega} - \Omega_t)\,dt + \sqrt{\Omega_t}(\rho\,dz_2^Q + \sqrt{1-\rho^2}\,dz_3^Q) \tag{18}$$

cannot generate incomplete bond markets. In the first example, $(1/dt)\mathrm{Var}^Q[dr]$ is not linearly independent of r. In the second example, $(1/dt)\mathrm{E}^Q[dr]$ is not linearly independent of r.

Proposition 4 allows us to limit our search for trivariate models that generate incomplete bond markets to models whose bond prices take the form[24]

$$P^T(s, r_s, \mu_s, V_s) = e^{A(T-s)+B_1(T-s)r_s+B_2(T-s)(\beta_1\mu_s+\beta_2V_s)}. \tag{19}$$

It is convenient to investigate separately those models that have $\beta_2 = 0$, which we refer to as models that exhibit USV, from those models that have $\beta_2 \neq 0$. We examine this second case first. With $\beta_2 \neq 0$, we can change variables from $\{\mu_s, V_s\}$ to $\{\mu_s, x_s \equiv (\beta_1\mu_s + \beta_2V_s)\}$. This implies the bond price can be written

$$P^T(s, r_s, \mu_s, x_s) = P^T(s, r_s, x_s) = e^{A(T-s)+B_1(T-s)r_s+B_2(T-s)x_s}, \tag{20}$$

independent of μ_s. We claim the following.

[24] One may wonder why the term proportional to r is not considered in satisfying equation (4). The answer is that when r is one of the state variables, its coefficient $B_1(T-s)$ is the only one whose first-order Taylor series expansion coefficient is nonzero. The intuition again is that, for small maturities, the bond price goes like $P = e^{-r\tau}$, that is, exponentially linear in τ.

Proposition 5: *There are no trivariate models with state variables* $\{r_t, \mu_t \equiv (1/dt)E^Q[dr_t], x_t\}$ *whose bond prices can be written in the form of equation (20). Hence, all trivariate affine models that generate incomplete bond markets have bond prices of the form*

$$P^T(s, r_s, \mu_s, V_s) = P^T(s, r_s, \mu_s) = e^{A(T-s)+B_1(T-s)r_s+B_2(T-s)\mu_s}. \tag{21}$$

Proof: See Appendix D.

The implication of this proposition is that all trivariate affine models that generate incomplete bond markets are also models that exhibit USV. We emphasize, however, that the former class is larger if we look at models with more than three state variables.

Given Proposition 5, it is convenient to define

$$\frac{1}{dt}E^Q[d\mu] = m_0^\mu + m_r^\mu r + m_\mu^\mu \mu + m_V^\mu V \tag{22}$$

$$\frac{1}{dt}E^Q[(d\mu)^2] = \sigma_0^\mu + \sigma_r^\mu r + \sigma_\mu^\mu \mu + \sigma_V^\mu V \tag{23}$$

$$\frac{1}{dt}E^Q[dr\,d\mu] = c_0^{r,\mu} + c_r^{r,\mu} r + c_\mu^{r,\mu} \mu + c_V^{r,\mu} V. \tag{24}$$

By applying Itô's lemma to equation (21), and then collecting terms of order constant, r, and μ, we find that the time-dependent coefficients are defined through

$$A' = m_0^\mu B_2 + \frac{\sigma_0^\mu}{2} B_2^2 + c_0^{r,\mu} B_1 B_2 \tag{25}$$

$$B_1' = m_r^\mu B_2 + \frac{\sigma_r^\mu}{2} B_2^2 + c_r^{r,\mu} B_1 B_2 - 1 \tag{26}$$

$$B_2' = m_\mu^\mu B_2 + \frac{\sigma_\mu^\mu}{2} B_2^2 + c_\mu^{r,\mu} B_1 B_2 + B_1, \tag{27}$$

and satisfy the boundary conditions

$$A(0) = 0, \qquad B_1(0) = 0, \qquad B_2(0) = 0. \tag{28}$$

Furthermore, by collecting terms of order V, we find that this model exhibits USV if and only if for all dates τ the following condition holds:

$$0 = m_V^{\mu} B_2(\tau) + \frac{\sigma_V^{\mu}}{2} B_2^2(\tau) + c_V^{r,\mu} B_1(\tau) B_2(\tau) + \frac{1}{2} B_1^2(\tau). \tag{29}$$

We claim the following.

PROPOSITION 6: *With the functions $A(\tau)$, $B_1(\tau)$, $B_2(\tau)$ defined implicitly through equations (25)—(27), the necessary and sufficient conditions for the model to exhibit USV (i.e., for equation (29) to hold) are that one of the following two sets of parameter restrictions holds:*

$$\begin{cases} m_r^{\mu} = -(2(c_V^{r,\mu})^2 + c_{\mu}^{r,\mu}) \\ m_{\mu}^{\mu} = 3c_V^{r,\mu} \\ m_V^{\mu} = 1 \\ \sigma_r^{\mu} = -2c_V^{r,\mu}(c_r^{r,\mu} + 2c_{\mu}^{r,\mu} c_V^{r,\mu}) \\ \sigma_{\mu}^{\mu} = 4c_r^{r,\mu} + 6c_{\mu}^{r,\mu} c_V^{r,\mu} \\ \sigma_V^{\mu} = (c_V^{r,\mu})^2 \end{cases} \quad or \quad \begin{cases} m_r^{\mu} = -(2c_{\mu}^{r,\mu} + 2(c_V^{r,\mu})^2 + c_r^{r,\mu}/c_V^{r,\mu}) \\ m_{\mu}^{\mu} = 3c_V^{r,\mu} \\ m_V^{\mu} = 1 \\ \sigma_r^{\mu} = -2c_{\mu}^{r,\mu}(c_{\mu}^{r,\mu} + (c_V^{r,\mu})^2 + c_r^{r,\mu}/c_V^{r,\mu}) \\ \sigma_{\mu}^{\mu} = 4c_V^{r,\mu} c_{\mu}^{r,\mu} + 2c_r^{r,\mu} \\ \sigma_V^{\mu} = c_{\mu}^{r,\mu} + (c_V^{r,\mu})^2 + c_r^{r,\mu}/c_V^{r,\mu} \end{cases} \tag{30}$$

Proof: See Appendix E.

A few points are worth noting. First, the reason that there are two sets of parameter restrictions that generate USV is because equation (29) is a quadratic equation in $B_1(\cdot)$ or $B_2(\cdot)$. This, in turn, generates two possible solutions for $B_1(\cdot)$ in terms of $B_2(\cdot)$. Second, these two sets of restrictions reduce to the same set if and only if $c_V^{r,\mu} \neq 0$ and $c_r^{r,\mu} + c_{\mu}^{r,\mu} c_V^{r,\mu} = 0$. This condition obtains, for example, when the covariance between the short rate and its drift depends only on the volatility, an important special case (basically, the so-called $A_1(3)$ models) which we examine below. Finally, we note that several of the restrictions noted in Proposition 6 occur naturally once we limit the class of models to those which are *admissible*, that is, those which restrict the "square-root" state variables to be nonnegative.

To provide some intuition for the proof of Proposition 6, sufficiency obtains because the right-hand side of equation (29) can be shown to be identically zero when either of the two sets of parameter conditions holds. Necessity obtains because if any one of the conditions is not satisfied, then we can show, by taking repeated time derivatives of the system of ODE's (Ricatti equations) evaluated at $\tau = 0$, that equation (11) cannot hold.

We note that the models of both Chen (1996) and Balduzzi et al. (1996) cannot satisfy these necessary restrictions, and thus cannot display USV. Also, as noted in Corollary 1, the $A_0(3)$ class of models of DS (2000) cannot exhibit USV. However, DS's maximal $A_1(3)$,$A_2(3)$, and $A_3(3)$ models do have the flexibility to exhibit USV.[25]

For simplicity, we only consider the so called $A_1(3)$ family of affine models given by

$$dv = \kappa_v(\hat{v} - v)\,dt + \sigma_v\sqrt{v}\,dz_v^Q \tag{31}$$

$$\begin{aligned} d\theta = {} & [\kappa_\theta(\hat{\theta} - \theta) + \kappa_{\theta r}(\hat{r} - r) + \kappa_{\theta v}(\hat{v} - v)]\,dt \\ & + \sigma_{\theta r}\sqrt{\alpha_r + v}\,dz_r^Q + \sqrt{\sigma_\theta^2 + \beta v}\,dz_\theta^Q + \sigma_{\theta v}\sqrt{v}\,dz_v^Q \end{aligned} \tag{32}$$

$$\begin{aligned} dr = {} & [\kappa_r(\hat{r} - r) + \kappa_{r\theta}(\hat{\theta} - \theta) + \kappa_{rv}(\hat{v} - v)]\,dt \\ & + \sqrt{\alpha_r + v}\,dz_r^Q + \sigma_{r\theta}\sqrt{\sigma_\theta^2 + \beta v}\,dz_\theta^Q + \sigma_{rv}\sqrt{v}\,dz_v^Q, \end{aligned} \tag{33}$$

where z_r,z_θ,z_v are independent standard Brownian motions. These equations correspond to the most general $A_1(3)$ admissible model. Admissibility requires that all processes be well defined and, in the particular case at hand, requires that v be a standard square root process. Intuitively, if any other state variable appears in the drift or diffusion of v, then the positivity of v cannot be guaranteed, because both θ and r can take on both positive and negative values.

We claim the following.

PROPOSITION 7: *Necessary and sufficient parameter restrictions for the $A_1(3)$ model given in equations (31) to (33) to display USV are:*

$$\kappa_{rv}(\kappa_v - \kappa_\theta) + \kappa_{r\theta}\kappa_{\theta v} = 1 + \sigma_{rv}^2 + \sigma_{r\theta}^2\beta \equiv \psi \tag{34}$$

$$9(\kappa_\theta\kappa_r - \kappa_{\theta r}\kappa_{r\theta}) = 2(\kappa_r + \kappa_\theta)^2 \tag{35}$$

$$\frac{\psi}{3}(\kappa_r + \kappa_\theta) = \kappa_r\psi + \kappa_{r\theta}(\sigma_{\theta r} + \beta\sigma_{r\theta} + \sigma_{\theta v}\sigma_{rv}) + \kappa_{rv}\sigma_v\sigma_{rv} \tag{36}$$

$$\begin{aligned} \frac{\psi}{9}(\kappa_r + \kappa_\theta)^2 = {} & (\kappa_r + \kappa_{r\theta}\sigma_{\theta r})^2 + (\kappa_r\sigma_{r\theta} + \kappa_{r\theta})^2\beta \\ & + (\kappa_r\sigma_{rv} + \kappa_{r\theta}\sigma_{\theta v} + \kappa_{rv}\sigma_v)^2. \end{aligned} \tag{37}$$

[25] DS focus on the $A_1(3)$ and $A_2(3)$ models, both of which have three factors and three state variables. These models differ by the number of state variables appearing "under the square root": one for the $A_1(3)$ model and two for the $A_2(3)$ model. Below, we examine in depth the $A_1(3)$ model because DS find that it is somewhat superior at fitting the dynamics of swap rates.

If these restrictions are imposed, then bond prices take the form

$$P^T(t) = \exp\{A(T-t) + B_r(T-t)r_t + B_\theta(T-t)\theta_t + B_v(T-t)v_t\}, \tag{38}$$

where the functions $A(\tau), B_r(\tau), B_\theta(\tau)$ *are*

$$B_r(\tau) = \left(\frac{1}{2(\kappa_r+\kappa_\theta)^2}\right) \times [-9\kappa_\theta + 6(2\kappa_\theta - \kappa_r)e^{-1/3(\kappa_r+\kappa_\theta)\tau} + 3(2\kappa_r - \kappa_\theta)e^{-2/3(\kappa_r+\kappa_\theta)\tau}] \tag{39}$$

$$B_\theta(\tau) = \left(\frac{9\kappa_{r\theta}}{2(\kappa_r+\kappa_\theta)^2}\right)[1 - e^{-1/3(\kappa_r+\kappa_\theta)\tau}]^2 \tag{40}$$

$$\begin{aligned} A(\tau) = \int_0^\tau ds \Bigg[& \frac{\sigma_{\theta r}^2 \alpha_r + \sigma_\theta^2}{2} B_\theta(s)^2 + \frac{\alpha_r + \sigma_{r\theta}^2 \sigma_\theta^2}{2} B_r(s)^2 \\ & + (\sigma_{\theta r}\alpha_r + \sigma_{r\theta}\sigma_\theta^2) B_r(s) B_\theta(s) + B_r(s)(\kappa_r \hat{r} + \kappa_{r\theta}\hat{\theta} + \kappa_{rv}\hat{v}) \\ & + B_\theta(s)(\kappa_\theta \hat{\theta} + \kappa_{\theta r}\hat{r} + \kappa_{\theta v}\hat{v}) + B_v(s)\kappa_v \hat{v} \Bigg]. \end{aligned} \tag{41}$$

USV obtains because

$$B_v(\tau) = \frac{\kappa_{rv}}{\kappa_{r\theta}} B_\theta(\tau) \qquad \forall \tau \geq 0. \tag{42}$$

Proof: To prove that these parameter restrictions are necessary and sufficient, we use Proposition 6, and then change variables from (r,μ,V) to (r,θ,v) given by

$$\mu = \kappa_{r\theta}\hat{\theta} + \kappa_{rv}\hat{v} + \kappa_r \hat{r} - \kappa_r r - \kappa_{r\theta}\theta - \kappa_{rv} v \tag{43}$$

$$V = (\sigma_{r\theta}^2 \sigma_\theta^2 + \alpha_r) + (\sigma_{r\theta}^2 \beta + \sigma_{rv}^2 + 1)v. \tag{44}$$

From DK (1996), we know bond prices take the form of equation (38). This permits us to determine the set of Ricatti equations satisfied by the $A(\cdot)$ and the $B(\cdot)$ functions. In particular, the Ricatti equations for $B_r(\cdot)$ and $B_\theta(\cdot)$ are

$$B_r'(\tau) = -\kappa_r B_r(\tau) - \kappa_{\theta r} B_\theta(\tau) - 1 \tag{45}$$

$$B_\theta'(\tau) = -\kappa_{r\theta} B_r(\tau) - \kappa_\theta B_\theta(\tau), \tag{46}$$

the solutions to which are provided in equations (39) and (40). It is then a matter of straightforward (but tedious) verification that $B_v(\tau) = (\kappa_{rv}/\kappa_{r\theta})B_\theta(\tau)$ solves the ODE if the USV parameter restrictions are satisfied. Q.E.D.

It is interesting to note that by imposing the USV parameter restrictions on the $A_1(3)$ model, we obtain a closed-form solution for the B_v function. This is in contrast to the general $A_1(3)$ model, where B_v does not possess an analytic solution. Furthermore, this closed-form solution possesses the typical "Gaussian" exponential time-decay structure. Clearly though, even under the USV parameter restrictions, this $A_1(3)$ model does not degenerate to a Gaussian model. Thus, we obtain a model with a term structure similar to that of a Gaussian two-factor model, but where the short rate volatility follows an autonomous square root (CIR) process. Further, one can show that for the USV model with $\kappa_{rv} = 0$, the proposition above holds for *any* autonomous one-factor Markov volatility process. In other words, we may have an affine bond price formula for a state vector which is not necessarily affine![26]

C. USV and Maximality

DS identify the maximum number of parameters that can be identified within different classes of three-state variable affine models conditional on observing only bond prices. In particular, they find that the maximal $A_1(3)$ model obtains when the following overidentifying restrictions, $\hat{r} = \kappa_{\theta v} = \kappa_{\theta r} = 0$ and $\kappa_{r\theta} = -\kappa_r$, are applied to equations (31) to (33). In that case, the remaining 14 parameters should be identifiable from bond prices. However, for the $A_1(3)$ model exhibiting USV, it can be seen by looking at the closed-form solution for bond prices that at most 8 parameters (or combinations thereof) are identifiable from the cross section of bond prices.[27] However, the notion of maximality of DS apparently generalizes to the number of state variables and parameters that can be identified by observing panel data on all fixed income securities (i.e., not just bond prices). Indeed, the invariant rotations proposed by DS depend only on the form of the fundamental partial differential equation of fixed income securities, which is independent of the boundary conditions specific to bond prices.

While we have yet to determine the maximum number of parameters which are identifiable for the $A_1(3)$ model exhibiting USV, here we give an example of a model which exhibits USV, guarantees admissibility, and demonstrates that it is not possible to identify all parameters from bond prices alone.

[26] In the particular case where $\kappa_{rv} = 0$ and the USV restrictions (34) to (37) hold, one can show (by substituting the solution into the fundamental PDE) that the bond prices have the form given in equations (38) to (41) with $B_v(\tau) = 0$, $\forall \tau$ even if v follows an arbitrary one factor Markov process $dv_t = \mu_v(v_t, t)\,dt + \sigma_v(v_t, t)\,dz_v^Q$.

[27] Only three parameters are separately identifiable from B_r, B_θ since equation (35) holds, and five parameters are identifiable from A since equation (42) holds.

PROPOSITION 8: *Consider the following model:*

$$dv = (\gamma_v - \kappa_v^Q v)\,dt + \sigma_v \sqrt{v}\,dz_v^Q \tag{47}$$

$$d\theta = \left(\gamma_\theta - 2\kappa_r \theta + \frac{1}{\kappa_r} v\right) dt + \sigma_\theta\, dz_\theta^Q \tag{48}$$

$$dr = \kappa_r(\theta - r)\,dt + \sqrt{\alpha_r + v}\,dz_r^Q + \sigma_{r\theta}\,dz_\theta^Q, \tag{49}$$

with the added restrictions to guarantee that the model is admissible:

$$\gamma_v > 0, \qquad \alpha_r \geq 0. \tag{50}$$

Further assume that the risk premia are such that

$$dz_v^Q = dz_v + \lambda\sqrt{v}\,dt \tag{51}$$

$$dz_\theta^Q = dz_\theta \tag{52}$$

$$dz_r^Q = dz_r \tag{53}$$

then bond prices take the form $((T - t) \equiv \tau)$:

$$P^T(t, r_t, \theta_t) = \exp\{A(\tau) - B_r(\tau) r_t - B_\theta(\tau)\theta_t\}, \tag{54}$$

where

$$B_r(\tau) = \frac{1}{\kappa_r}(1 - e^{-\kappa_r \tau}) \tag{55}$$

$$B_\theta(\tau) = \frac{1}{2\kappa_r}(1 - e^{-\kappa_r \tau})^2 \tag{56}$$

$$\begin{aligned} A(\tau) = \int_0^\tau ds \Bigg[&\frac{\sigma_\theta^2}{2}(B_\theta(s))^2 + \frac{\alpha_r + \sigma_{r\theta}^2}{2}(B_r(s))^2 \\ &+ \sigma_{r\theta}\sigma_\theta B_r(s) B_\theta(s) - \gamma_\theta B_\theta(s) \Bigg]. \end{aligned} \tag{57}$$

Further, κ_v^Q *cannot be identified from bond prices alone. Rather, fixed income derivative prices are needed to identify* κ_v^Q.

Proof: Since the state vector dynamics are affine, we know from DK (1996) that bond prices take the form

$$P^T(t, r_t, \theta_t) = \exp\{A(\tau) - B_r(\tau) r_t - B_\theta(\tau)\theta_t - B_v(\tau) v_t\}, \tag{58}$$

where the bond price satisfies the PDE:

$$\begin{aligned} rP = P_t + P_r[\kappa_r(\theta - r)] + P_\theta\left[\gamma_\theta - 2\kappa_r\theta + \frac{1}{\kappa_r}v\right] + P_v[\gamma_v - \kappa_v^Q v] \\ + \frac{1}{2}P_{rr}[\alpha_r + v + \sigma_{r\theta}^2] + \frac{\sigma_\theta^2}{2}P_{\theta\theta} + \frac{\sigma_v^2}{2}P_{vv}v + P_{r\theta}\sigma_{r\theta}\sigma_\theta. \end{aligned} \tag{59}$$

Collecting terms that are linear in r, θ, v, and constant generate the system of ODE's:

$$B_r'(\tau) = 1 - B_r\kappa_r \tag{60}$$

$$B_\theta'(\tau) = \kappa_r B_r - 2\kappa_r B_\theta \tag{61}$$

$$B_v'(\tau) = \frac{1}{\kappa_r}B_\theta - \kappa_v^Q B_v - \frac{1}{2}B_r^2 - \frac{\sigma_v^2}{2}B_v^2 \tag{62}$$

$$A'(\tau) = -\gamma_\theta B_\theta - \gamma_v B_v + \frac{1}{2}B_r^2[\alpha_r + \sigma_{r\theta}^2] + \frac{1}{2}\sigma_\theta^2 B_\theta^2 + \sigma_{r\theta}\sigma_\theta B_r B_\theta, \tag{63}$$

with the initial conditions $B_r(0) = B_\theta(0) = B_v(0) = A(0) = 0$. The solutions to equations (60) to (61) are those given in equations (55) to (56). Note that these equations satisfy $(1/\kappa_r)B_\theta - \frac{1}{2}B_r^2 = 0$, implying that equation (62) reduces to

$$B_v'(\tau) = -\kappa_v^Q B_v - \frac{\sigma_v^2}{2}B_v^2. \tag{64}$$

Given the initial condition $B_v(0) = 0$, it is clear that $B_v(\tau) = 0$, $\forall\tau$. Hence, this model exhibits USV.

Note that since $B_v(\tau) = 0$, equation (63) reduces to

$$A'(\tau) = -\gamma_\theta B_\theta + \frac{1}{2}B_r^2[\alpha_r + \sigma_{r\theta}^2] + \frac{1}{2}\sigma_\theta^2 B_\theta^2 + \sigma_{r\theta}\sigma_\theta B_r B_\theta, \tag{65}$$

independent of γ_v. More generally, note that bond prices are completely independent of all of the parameters (κ_v^Q, γ_v and σ_v) that drive volatility dynamics. This implies the model can be "extended" to allow for a very simple two-step calibration procedure to fixed income derivatives (such as at-the-money Caps/Floors). First, as in Hull and White (1990), the parameter γ_θ can be made time dependent to fit the term structure of forward rates. Second, some of the parameters $\{\gamma_v, \kappa_v^Q, \sigma_v\}$ can be made time dependent to fit the term structure of volatilities, without affecting the initial calibration of the term structure of forward rates.

Under the historical measure, the volatility dynamics follow

$$\begin{aligned} dv &= (\gamma_v - \kappa_v^Q v)\,dt + \sigma_v\sqrt{v}(dz_v + \lambda\sqrt{v}\,dt) \\ &\equiv (\gamma_v - \kappa_v v)\,dt + \sigma_v\sqrt{v}\,dz_v, \end{aligned} \tag{66}$$

where $\kappa_v \equiv \kappa_v^Q - \sigma_v\lambda$. With this specification of the risk premia, a time series of bond data can identify the state variable v, along with the parameters which show up under the historical measure, namely, κ_v, γ_v, and σ_v. However, given only bond prices, it is not possible to identify κ_v^Q. It is straightforward to demonstrate, though, that κ_v^Q can be identified if other fixed income derivatives are available. Q.E.D.

D. Simulation of USV Model Using Monthly Sampling

The models proposed above generate bond prices that are independent of the current volatility state variable. Hence, these models predict that *instantaneous* bond returns cannot hedge *instantaneous* changes in volatility, and therefore cannot hedge straddles. Note, however, that the empirical support for this class of models comes from data that are sampled monthly. To demonstrate that the proposed class of models is consistent with our empirical findings, we perform the following experiment. We first simulate a time series of monthly swap rate, cap, and floor prices from a particular $A_1(3)$ economy where the parameters governing the state vector do *not* satisfy the USV restrictions.[28] We then regress straddle returns on changes in swap rates. Our results indicate that only three swap rates are necessary for the three-factor model to obtain an R^2 above 90 percent. That is, even with only monthly sampling, and restricting the OLS regression to constant coefficients, the observed straddle returns are almost perfectly explained by changes in swap rates.

We then repeat the same experiment in a similar economy, except this time we adjust the parameter values so that the necessary restrictions for the $A_1(3)$ model to exhibit USV are satisfied. The regression analysis in this simulated economy reveals that, even though data is sampled monthly, one still obtains the "continuous-time" result that (1) only two different swap maturities can be used as regressors, or else the inversion of the covariance matrix becomes nearly singular, and (2) only about 30 percent of the varia-

[28] We chose parameters based on DS (2000, Table II, p. 1964). To compute the cap and floor prices, we use the closed form solution approach proposed by Heston (1993) and extended by DPS.

tion in straddles can be explained by these regressors. These findings imply that our proposed model can generate USV even if data is sampled only monthly.

III. USV within an HJM Framework

Our empirical findings strongly suggest that there are sources of risk that drive innovations in straddle returns, but do not (instantaneously) affect the underlying swap rates. Within a trivariate affine setting, we are able to identify parameter restrictions that generate a class of models consistent with these empirical findings. A potential criticism of this approach, however, is that the imposed "knife-edge" parameterization gives the appearance that the construction of models exhibiting USV is "contrived."

In this section, however, we demonstrate that USV is generated naturally within an HJM framework. Indeed, we demonstrate below that *almost all* HJM stochastic volatility models generate USV. Furthermore, by specializing to models that possess Markov representations, we demonstrate that within an HJM environment, the restrictions found in the previous section arise naturally.

Within the HJM framework, forward rates $f^T(s)$ are taken as inputs. A simple model in this class has the form

$$df^T(s) = \mu_T(s)\,ds + B_T(s,\Sigma_s)\,dz_1^Q(s) \tag{67}$$

$$d\Sigma_s = m_{\Sigma_s}\,ds + \sigma_{\Sigma_s}\,dz_2^Q(s), \tag{68}$$

The drift $\mu_T(s)$ is determined by the volatility structure, as shown by Heath, Jarrow, and Morton (1992). From Itô's lemma and the definition of forward rates, $f^T(s) = -(\partial/\partial T)\log P^T(s)$, we obtain the bond price dynamics

$$\frac{dP_s^T}{P_s^T} = r_s\,ds - B^T(s,\Sigma_s)\,dz_1^Q(s), \tag{69}$$

where we have defined $B^T(s,\Sigma_s) \equiv \int_s^T B_u(s,\Sigma_s)\,du$. We note that either the set equations (67) and (68) or equations (69) and (68), can be used to characterize the system. Thus, from an HJM perspective, we are effectively modeling the dynamics of a set of traded assets (i.e., bonds), and the state variable driving the volatility of these assets.

Define $\rho_{r,V}$ as the correlation between the Brownian motion that generates bond price innovations ($dz_1^Q(s)$) and the Brownian motion that generates volatility ($dz_2^Q(s)$). Note that, excluding the cases $\rho_{r,V} = \pm 1$, *all* HJM stochastic volatility models exhibit USV. In this sense, USV is a very natural phenomenon when one directly models forward rates (or equivalently, bond prices) and their volatility dynamics, rather than modeling spot rate and its volatility dynamics. This result is analogous to USV models of equity prices, such as Heston (1993). That is, when one directly models the dynamics of a traded asset (equity or bond/forward prices) and that asset's volatility, USV is (almost) always generated. In contrast, in a standard affine model, one typically specifies the dynamics of state variables (e.g., the short rate) that are not traded assets. In general, these models will not exhibit USV.[29]

A. A Two-Factor HJM Model Exhibiting USV

Consider the two-factor model

$$df^T(s) = a(s,T)A(s,T)\Omega(s)\,ds + a(s,T)\sqrt{\Omega(s)}\,dz_1^Q(s) \tag{70}$$

$$d\Sigma(s) = \Psi^Q(s)\,ds + b\sqrt{\Omega(s)}(\rho\,dz_1^Q(s) + \sqrt{1-\rho^2}\,dz_2^Q(s)), \tag{71}$$

where $dz_1^Q(s)$ and $dz_2^Q(s)$ are independent Brownian motions, and we have defined

$$\Psi^Q(s) \equiv \psi_0^Q + \psi_1^Q r(s) + \psi_2^Q \Sigma(s) \tag{72}$$

$$\Omega(s) \equiv \omega_0 + \omega_1 r(s) + \omega_2 \Sigma(s) \tag{73}$$

$$A(s,T) \equiv \int_s^T dv\, a(s,v). \tag{74}$$

As first noted by HJM, the drift of the forward rate dynamics under the risk-neutral measure $(a(s,T)A(s,T)\Omega(s))$ is uniquely specified by the volatility structure $(a(s,T)\sqrt{\Omega(s)})$. However, there are no restrictions on $\Psi^Q(s)$, the drift of Σ.[30]

As specified, the model leads to very general dynamics for the term structure of forward rates, and hence also for the risk-free rate. In particular, for arbitrary functions $a(s,T)$, the dynamics of the system $(\{f^T(s)\},\Sigma_s)$ will in

[29] Similarly, if one were to construct a general equilibrium model for (multiple) stock prices starting from some "fundamental" low-dimensional state variable vector (such as in Cox, Ingersoll, and Ross (1985a) or DPS (2000)), the model would necessitate some restrictions on the parameters of the process of the state variables for the stock price processes to exhibit USV. Indeed, in general, all sources of risk would be spanned by the stock prices (as long as the number of state variables is smaller than the number of stock prices).

[30] As noted in Andreasen et al. (1997), the HJM restriction alone does not identify the process of Σ under the risk-neutral measure, since the Girsanov factor associated with z_2 cannot be identified from changes in bond prices alone. To determine the market price of risk associated with volatility-specific risk z_2, either the prices of other interest rate sensitive securities in addition to bond prices must be taken as input to the model, or some equilibrium argument must be made.

general be non-Markov. However, as demonstrated by Cheyette (1995), a Markov representation can be found if we assume the functional form $a(s,T) = a(T)/a(s)$ and $\omega_2 = 0$. Below, we generalize these findings.[31]

B. Markov Representation and Existence

As mentioned previously, for general functions $a(s,T)$, it is not possible to obtain a Markov representation for the model proposed above. Although in a companion paper (Collin-Dufresne and Goldstein (2001b)), we provide closed-form solutions for a large number of derivatives for general functional form of $a(s,T)$, we cannot in general derive simple algorithms to price path-dependent instruments such as American options. In this section, we show that if $a(s,T)$ is modeled as separable, $a(s,T) = a(T)/a(s)$ for some function $a(\cdot)$, then a Markov representation of the model obtains.[32] We claim the following.

PROPOSITION 9: *Assume $a(s,T)$ takes the form $a(s,T) = a(T)/a(s)$. Define*

$$Y(t) = \int_0^t ds\, \Omega(s)\, \frac{a(t)A(t)}{a^2(s)}, \tag{75}$$

where $A(t) = \int_0^t a(s)\, ds$. Then the model proposed in equations (70) to (71) possesses a Markov representation in the three state variables $\{r(t), \Sigma(t), Y(t)\}$. The state vector is affine, and bond prices are exponentially affine functions of the subset $\{r(t), Y(t)\}$ of the state vector. All fixed income derivatives are solutions to a partial differential equation, subject to appropriate boundary conditions.

Proof: Integrating the forward rate dynamics we obtain

$$r(t) = f^t(t) = f^t(0) + Y(t) + X(t), \tag{76}$$

where we have defined

$$X(t) \equiv -\int_0^t ds\, \Omega(s)\left(\frac{a(t)A(s)}{a^2(s)}\right) + \int_0^t dz_1^Q(s)\sqrt{\Omega(s)}\left(\frac{a(t)}{a(s)}\right). \tag{77}$$

[31] When $\omega_2 = 0$, Jeffrey (1995) demonstrates that for the short rate to be one-factor Markov, the functions $a(s,T)$ must satisfy a very specific functional form (his equation (18), p. 631).

[32] For similar separability assumptions made to obtain Markov representation in standard HJM models, see, for example, Carverhill (1994), Cheyette (1995), and Ritchken and Sankarasubramaniam (1995).

Applying Itô's lemma, we obtain the dynamics of $X(t)$, $Y(t)$:

$$dX(t) = \left(-\Omega(t)\frac{A(t)}{a(t)} + \frac{a'(t)}{a(t)}X(t)\right)dt + \sqrt{\Omega(t)}\,dz_1^Q(t), \tag{78}$$

$$dY(t) = \left[\Omega(t)\frac{A(t)}{a(t)} + Y(t)\left(\frac{a'(t)}{a(t)} + \frac{a(t)}{A(t)}\right)\right]dt. \tag{79}$$

Using equation (76), we obtain the dynamics of the short-term rate:

$$dr(t) = \left[\frac{\partial f^t(0)}{\partial t} + Y(t)\left(\frac{a(t)}{A(t)}\right) + (r(t) - f^t(0))\left(\frac{a'(t)}{a(t)}\right)\right]dt + \sqrt{\Omega(t)}\,dz_1^Q(t). \tag{80}$$

Recalling the definition of $\Omega(t)$, it is clear that $\{r(t), Y(t), \Sigma(t)\}$ form a Markov system.

More generally, the forward rates may be written as

$$f^v(t) = f^v(0) + \left(\frac{a(v)}{a(t)}\right)X(t) + \left(\frac{a(v)A(v)}{a(t)A(t)}\right)Y(t). \tag{81}$$

Thus bond prices satisfy

$$\begin{aligned} P^T(t) &= \exp\left[-\int_t^T dv\, f^v(t)\right] && (82)\\ &= \exp\left[-\int_t^T dv\, f^v(0) - M(t,T)X(t) - N(t,T)Y(t)\right] && (83)\\ &= \exp\left[-\int_t^T dv\, f^v(0) - M(t,T)(r(t) - f^t(0) - Y(t)) - N(t,T)Y(t)\right], && (84)\end{aligned}$$

where we have defined

$$M(t,T) \equiv \left(\frac{A(T) - A(t)}{a(t)}\right) \tag{85}$$

$$N(t,T) \equiv \frac{1}{2}\left(\frac{A^2(T) - A^2(t)}{a(t)A(t)}\right). \tag{86}$$

Finally, consider a path-independent European contingent claim that has a payoff at time T that is a function of the entire term structure at time T, that is, $\phi(T) \equiv \phi(T, \{P^v(T)\}_{T \le v \le \bar{T}})$. The price of that security is $\phi(t) = \mathrm{E}^Q[\exp(-\int_t^T r(s)\,ds)\phi(T)|\mathcal{F}_t] = F(t, r(t), \Sigma(t), Y(t))$, where the second equality follows from the Markov property. Moreover, a standard argument (which requires some regularity conditions on F and its derivatives, see Duffie (1996, Appendix E, p. 296)) shows that $\exp(-\int_0^t ds\, r(s))F(t,r(t),\Sigma(t),Y(t))$ is a Martingale and that its drift must vanish, or equivalently

$$\left(\frac{1}{dt}\right)\mathrm{E}_t^Q[dF(t,r_t,\Sigma_t,Y_t)] = r_t F(t,r_t,\Sigma_t,Y_t).$$

Using Itô's lemma we obtain the partial differential equation for the price of the European contingent claim:

$$\begin{aligned} 0 = F_t &+ \Omega(t)\left(\frac{1}{2}F_{rr} + \frac{1}{2}F_{\Sigma\Sigma}b^2 + F_{r\Sigma}\rho b\right) \\ &+ F_Y\left[\Omega(t)\frac{A(t)}{a(t)} + Y(t)\left(\frac{a'(t)}{a(t)} + \frac{a(t)}{A(t)}\right)\right] \\ &+ F_r\left[\frac{\partial f^t(0)}{\partial t} + Y(t)\left(\frac{a(t)}{A(t)}\right) + (r(t) - f^t(0))\left(\frac{a'(t)}{a(t)}\right)\right] \\ &+ F_\Sigma \Psi^Q(t) - rF. \end{aligned} \tag{87}$$

Q.E.D.

Although they do not note the relevance to incomplete bond markets, a similar model appears in de Jong and Santa Clara (1999), where they investigate the special case $a(t) = e^{-\kappa t}$.

This model clearly exhibits unspanned stochastic volatility. In particular, note that bond prices (equation (84)) are exponential-affine functions of r and Y alone, and hence cannot hedge changes in Σ. Since Y is locally deterministic, the innovations of any bond can be hedged by a position in any other bond and the money market fund. However, the set $\{Y, r\}$ are not jointly Markov. As a consequence, the dynamics of bond prices over a finite time period depend on the dynamics of the additional state variable Σ as well.

In general, it is not possible to guarantee that the above stochastic differential equations for r and Σ are well defined. Indeed, for general initial term structures and parameter choices, $\Omega(t)$ may take on negative values.[33] The following lemma demonstrates that there exists a feasible set of parameters such that Ω remains strictly positive (almost surely) and the SDEs are well defined.

For simplicity, we consider the special case $a(t) = e^{-\kappa t}$.

[33] Moreover, the square root diffusion coefficients does not verify the standard Lipschitz conditions at zero, but see Duffie (1996, Appendix E, p. 292) and DK (1996).

PROPOSITION 10: *If the parameters and the initial forward rate curve satisfy*

1. $\kappa \geq 0,\ \omega_1 \geq 0,$
2. $\omega_2 \psi_1^Q - \kappa\omega_1 = \psi_2^Q \omega_1,$
3. $\Omega(0) > 0,$ *and*
4. $\omega_1 f_t(0) + \omega_1 \kappa f^t(0) + \omega_2 \psi_0^Q - \omega_0 \psi_2^Q \geq \frac{1}{2}(\omega_1^2 + \omega_2^2 b^2 + 2\rho b \omega_1 \omega_2),\ \forall t \geq 0,$

then $\Omega_t > 0,\ \forall t \geq 0$ *a.s. and the SDEs for the forward rates* $f^v(t),\ \forall v$ *and the stochastic volatility* $\Sigma(t)$ *are well defined.*

Proof: Note that under condition 1 of the proposition,

$$d\Omega(t) = (\omega_1 f_t(0) + \omega_1 \kappa f^t(0) + \omega_1 Z(t) + \omega_2 \psi_0^Q - \omega_0 \psi_2^Q + \psi_2^Q \Omega(t))\, dt + \beta\sqrt{\Omega(t)}\, dW(t) \tag{88}$$

$$dZ(t) = (\Omega(t) - 2\kappa Z(t))\, dt, \tag{89}$$

where $\beta \equiv \sqrt{\omega_1^2 + \omega_2^2 b^2 + 2\rho b \omega_1 \omega_2}$ and $dW_t = (1/\beta)((\omega_1 + \rho\omega_2 b)\, dz_1(t) + \sqrt{1-\rho^2}\omega_2 b\, dz_2(t))$ is a standard Brownian motion and $Z(0) = 0$. A minor adaptation of the proof of the SDE Theorem in DK (1996) (which extends Feller (1951) to a vector of affine processes) to account for deterministic coefficients in the drift of Ω allows us to conclude that the SDE for forward rates and stochastic volatility state variables are well defined. Q.E.D.

Note that the above proposition puts joint restrictions on both the feasible set of parameters and the initial curve of forward rates. Also note that for this special choice of volatility structure, the model admits a Markov representation of the term structure such that it has an affine structure in the sense of DK (1996) or DPS (2000), but with three distinct features: (1) it is consistent with the initial term structure, (2) it is not time homogeneous, and (3) it results in only a subset of the state variables entering the bond prices exponentially (i.e., the loading of the log-bond price is zero for the state variable Σ). Our approach provides a straightforward and efficient method to construct HJM affine models with unspanned stochastic volatility.[34]

IV. Conclusion

Most time-homogeneous models of the term structure are restrictive in that they assume all sources of risk inherent in the prices of derivative

[34] We note that by appropriately choosing the initial forward curve, this model reduces to a special case of the time-homogeneous affine models presented in the previous section. Indeed, it is a two-factor, three-state variable affine model with USV. The additional locally deterministic state variable is the "lowest cost" to pay in order to obtain a model which exhibits USV.

securities can be completely hedged by a portfolio consisting solely of bonds. Our empirical evidence suggests that this assumption is counterfactual. Indeed, using data from three different countries (the United States, the United Kingdom, and Japan), we find that changes in the term structure of swap rates have very limited explanatory power for returns on at-the-money straddles. We term this feature "unspanned stochastic volatility" (USV). Furthermore, innovations in at-the-money straddle returns are highly correlated. Principal component analysis suggests that a single common factor independent of returns on swap rates explains most of the variation in straddles.

We study conditions under which the affine models can be made consistent with this empirical observation. We find that bivariate Markov models, special cases of which include Fong and Vasicek (1991) and Longstaff and Schwartz (1992), cannot exhibit USV. In other words, two-factor Markov short-rate models necessarily lead to complete bond markets provided sufficient different maturity bonds are traded. Further, we identify necessary and sufficient parameter restrictions for trivariate Markov affine systems to display USV. While such restrictions may appear somewhat contrived, we argue this occurs because the standard affine framework takes as primitives the specification of a low-dimensional Markov vector of state variables which are not traded assets. In contrast, we show that USV occurs naturally when forward rate dynamics (or equivalently, bond price dynamics) are taken as primitives of the model.

Simulated economies of the proposed models suggest that USV can be generated even if data is sampled only monthly. Further, our results suggest that when estimating risk-neutral parameters of a model, it is essential to use as inputs both swaps and fixed income derivative securities. Indeed, it appears that there are some parameters whose estimates have minimal impact on fitting the moments of swap rate data, yet have significant pricing implications for fixed income derivatives such as straddles. Moreover, some risk-neutral parameters are not even, in theory, identifiable given only bond prices, but rather require that fixed income derivatives be observed.

The empirical evidence we provide supporting the concept of USV raises other important testable empirical issues. First, how much additional explanatory power do latent USV state variables possess for explaining the time-series and cross-sectional behavior of bond prices? We note that all previous empirical studies within a time-homogeneous setting implicitly assume that all factors driving the term structure can be inverted from bond prices alone. Second, an implication of USV is that prices of both bonds and fixed income derivatives are needed to determine parameter values. Hence, USV may offer a potential avenue to improve recent attempts at capturing the joint dynamics of the term structure and fixed income derivatives. Finally, USV challenges standard approaches to hedging fixed income derivatives as it requires the use of at least one reference derivative to hedge other fixed income derivatives.

Appendix A: Proof of Proposition 2

By definition, all bivariate affine models can be represented in terms of the spot rate r and some other state variable x, where

$$\begin{bmatrix} dr \\ dx \end{bmatrix} \sim \mathrm{N}\left\{ \begin{bmatrix} (\mu_0^r + \mu_r^r r + \mu_x^r x)\,dt \\ (\mu_0^x + \mu_r^x r + \mu_x^x x)\,dt \end{bmatrix}, \begin{bmatrix} (\sigma_0^r + \sigma_r^r r + \sigma_x^r x)\,dt & (c_0 + c_r r + c_x x)\,dt \\ (c_0 + c_r r + c_x x)\,dt & (\sigma_0^x + \sigma_r^x r + \sigma_x^x x)\,dt \end{bmatrix} \right\}. \tag{A1}$$

That is, the drifts and variances of the two state variables, along with the covariance between the two state variables, are linear in these two state variables.

At date s, the price of a discount bond maturing at date T is defined through

$$P^T(s, r_s, x_s) = \mathrm{E}_s^Q\left[e^{-\int_s^T du\, r_u} \right]. \tag{A2}$$

It is well known that if the system is well defined, then the bond price satisfies the PDE

$$\begin{aligned} 0 = {} & -rP + P_s + P_r(\mu_0^r + \mu_r^r r + \mu_x^r x) + P_x(\mu_0^x + \mu_r^x r + \mu_x^x x) \\ & + \frac{1}{2} P_{rr}(\sigma_0^r + \sigma_r^r r + \sigma_x^r x) + \frac{1}{2} P_{xx}(\sigma_0^x + \sigma_r^x r + \sigma_x^x x) + P_{rx}(c_0 + c_r r + c_x x), \end{aligned} \tag{A3}$$

and that the solution takes the form

$$P(s, r_s, x_s) = e^{A(T-s) + B_1(T-s) r_s + B_2(T-s) x_s}, \tag{A4}$$

with boundary conditions

$$A(0) = 0, \qquad B_1(0) = 0, \qquad B_2(0) = 0. \tag{A5}$$

This implies that the time-dependent coefficients $A(\cdot)$, $B_1(\cdot)$, and $B_2(\cdot)$ satisfy

$$A' = \mu_0^r B_1 + \mu_0^x B_2 + \frac{\sigma_0^r}{2} B_1^2 + \frac{\sigma_0^x}{2} B_2^2 + c_0 B_1 B_2 \tag{A6}$$

$$B_1' = \mu_r^r B_1 + \mu_r^x B_2 + \frac{\sigma_r^r}{2} B_1^2 + \frac{\sigma_r^x}{2} B_2^2 + c_r B_1 B_2 - 1 \tag{A7}$$

$$B_2' = \mu_x^r B_1 + \mu_x^x B_2 + \frac{\sigma_x^r}{2} B_1^2 + \frac{\sigma_x^x}{2} B_2^2 + c_x B_1 B_2. \tag{A8}$$

For this system to generate incomplete bond markets, we have shown in Proposition 2 that there must exist a set of coefficients $\{\beta_1, \beta_2\}$ such that $\beta_1 B_1(\tau) + \beta_2 B_2(\tau) = 0$, with at least one of the coefficients nonzero. However, we see from equations (A7), (A8), and the boundary conditions that $B_1'(0) = -1$, $B_2'(0) = 0$. This implies that (1) $B_1(\tau)$ cannot be identically zero, and that (2) $B_2(\tau)$ cannot be a multiple of $B_1(\tau)$. Therefore, the only possibility for this model to exhibit USV is for $\beta_1 = 0$, $\beta_2 \neq 0$, and $B_2(\tau) = 0$, $\forall \tau$. However, this case is also not possible. Indeed, from equation (A8), the condition $B_2(\tau) = 0$ implies that

$$0 = \mu_x^r B_1(T-t) + \frac{\sigma_x^r}{2} B_1^2(T-t) \qquad \forall T. \tag{A9}$$

Since the dynamics of B_1 is uniquely specified by equation (A7) (with $B_2(\cdot)$ set to zero) and $B_1(\tau)$ is not identically zero, the only way equation (A9) can be satisfied is if $\mu_x^r = 0$, $\sigma_x^r = 0$. In such a case, however, the spot rate process itself is one-factor Markov, implying that all fixed income securities can be expressed as functions of the spot rate only. It may appear at first that equation (A9) can be satisfied by permitting μ_x^r and σ_x^r to take on a particular time dependence. However, this time dependence would only allow a bond with a single maturity T to have its bond price be independent of x: All other maturities U would have $B_2(U-t) \neq 0$. Q.E.D.

Appendix B: Proof of Proposition 3

Consider the following framework:

$$dr = \mu_r(r,x,t)\,dt + \sigma_r(r,x,t)\,dz_1 \tag{B1}$$

$$dx = \mu_x(r,x,t)\,dt + \sigma_x^1(r,x,t)\,dz_1 + \sigma_x^2(r,x,t)\,dz_2, \tag{B2}$$

where z_1, z_2 are two independent Brownian motions under the risk-neutral measure, and $\mu_r, \sigma_r, \mu_x, \sigma_x^1, \sigma_x^2$ are functions satisfying standard regularity conditions for the SDEs to form a well-defined two-factor Markov system (see, e.g., the Appendix of Duffie (1996)). In this case, the diffusion matrix of the state vector,

$$S \equiv \begin{bmatrix} r \\ x \end{bmatrix}$$

is a.e. invertible.

Note that by the Markov property, the price of a zero coupon bond depends only upon the current value of the state variables $P^T(t) = \mathrm{E}[e^{-\int_t^T ds\, r_s} | \mathcal{F}_t] = P^T(t, r_t, x_t)$. Therefore, for bond markets to be incomplete, there must exist coefficients $\{\beta_1, \beta_2\}$, both not zero, such that

$$\beta_1 P_r^T(t,r,x) + \beta_2 P_x^T(t,r,x) = 0 \qquad \forall T, t, r, x. \tag{B3}$$

Here, we have applied Itô's lemma and have used the invertibility of the matrix of the state vector S_t. This is analogous to equation (4) for the affine case.

Since for small maturities (Δ) we have $P^{t+\Delta}(t,r,x) \approx e^{-r\Delta}$ we deduce that (1) there exists $T \geq t$ such that $P_r^T(t,r,x) \neq 0$, and (2) $P_x^T(T,r,x) = 0$. Result (1) implies that to obtain incomplete markets (i.e., for equation (B3) to hold), β_2 cannot be zero. Together with result (2) this implies $\beta_1 = 0$. These results can be proved more rigorously. Write bond price as a Taylor series expansion in time to maturity

$$P^T(t,r_t,x_t) = \sum_{j=0}^{\infty} g_j(t,r_t,x_t)(T-t)^j. \tag{B4}$$

The "final condition" and the definition of the bond price $P^T(t,r_t,x_t) = \mathrm{E}_t^Q[e^{-\int_t^T ds\, r_s}]$ guarantee that $g_0(t,r_t,x_t) = 1$, $g_1(t,r_t,x_t) = -r$. This in turn guarantees that

$$\lim_{T\to t} \frac{P_x^T(t,r_t,x_t)}{P_r^T(t,r_t,x_t)} = 0. \tag{B5}$$

Using this in equation (B3) leads to the result.

Thus, incomplete bond markets obtain only if $P^T(t,r,x) = P^T(t,r)$, $\forall t,T,r,x$. This implies that, if bond markets are incomplete, the fundamental PDE solved by bond prices is

$$\frac{1}{2} P_{rr}^T(t,r)\sigma_r^2(t,r,x) + P_r^T(t,r)\mu_r(t,r,x) = rP^T(t,r) - P_t^T(t,r). \tag{B6}$$

We now demonstrate that this implies that μ, σ must be independent of x and thus that the short rate must be one-factor Markov, in turn implying that bivariate models cannot generate incomplete bond markets.

Note that for any two maturities $\{T_1,T_2\}$ we can write equation (B6) in matrix form as

$$\begin{bmatrix} \frac{1}{2} P_{rr}^{T_1}(t,r) & P_r^{T_1}(t,r) \\ \frac{1}{2} P_{rr}^{T_2}(t,r) & P_r^{T_2}(t,r) \end{bmatrix} \begin{bmatrix} \sigma_r^2(r,x,t) \\ \mu_r(r,x,t) \end{bmatrix} = \begin{bmatrix} rP^{T_1}(t,r) - P_t^{T_1}(t,r) \\ rP^{T_2}(t,r) - P_t^{T_2}(t,r) \end{bmatrix}. \tag{B7}$$

First, suppose there exists T_1, T_2 (possibly dependent on (t,r)), such that the matrix

$$\begin{bmatrix} \frac{1}{2}P_{rr}^{T_1}(t,r) & P_r^{T_1}(t,r) \\ \frac{1}{2}P_{rr}^{T_2}(t,r) & P_r^{T_2}(t,r) \end{bmatrix}$$

is invertible. By premultiplying equation (B7) by this inverted matrix, we see that the right-hand side is a function only of r, implying that both $\mu_r(t,r,x) = \mu_r(t,r)$ and $\sigma_r^2(t,r,x) = \sigma_r^2(t,r)$, which implies that the spot rate is one-factor Markov, in turn implying that bivariate models cannot generate incomplete bond markets.

Second, suppose that it is not possible to find two maturities such that the above matrix is invertible. Then, for all T_1, T_2, the matrix is not full rank and its determinant must be zero:

$$P_{rr}^{T_1}(t,r) \times P_r^{T_2}(t,r) - P_{rr}^{T_2}(t,r) \times P_r^{T_1}(t,r) = 0. \tag{B8}$$

However, we know that for sufficiently small τ, $P(r,\tau) \Rightarrow e^{-r\tau}$, demonstrating that equation (B8) cannot hold in general. (Again, this can be made more rigorous by performing a Taylor series expansion). Hence, our claim follows. Q.E.D.

Appendix C: Proof of Proposition 4

By definition, every affine model of three state variables can be written in terms of the spot rate r and two other (nondegenerate) state variables x and y such that dr, dx, and dy have a drift vector and instantaneous covariance matrix of the form

$$\begin{bmatrix} \frac{1}{dt}\mathrm{E}[dr] \\ \frac{1}{dt}\mathrm{E}[dx] \\ \frac{1}{dt}\mathrm{E}[dy] \end{bmatrix} = \begin{bmatrix} m_0^r + m_r^r r + m_x^r x + m_y^r y \\ m_0^x + m_r^x r + m_x^x x + m_y^x y \\ m_0^y + m_r^y r + m_x^y x + m_y^y y \end{bmatrix} \tag{C1}$$

$$\Sigma^2 = \begin{bmatrix} \sigma_0^r + \sigma_r^r r + \sigma_x^r x + \sigma_y^r y & c_0^{rx} + c_r^{rx} r + c_x^{rx} x + c_y^{rx} y & c_0^{ry} + c_r^{ry} r + c_x^{ry} x + c_y^{ry} y \\ c_0^{rx} + c_r^{rx} r + c_x^{rx} x + c_y^{rx} y & \sigma_0^x + \sigma_r^x r + \sigma_x^x x + \sigma_y^x y & c_0^{xy} + c_r^{xy} r + c_x^{xy} x + c_y^{xy} y \\ c_0^{ry} + c_r^{ry} r + c_x^{ry} x + c_y^{ry} y & c_0^{xy} + c_r^{xy} r + c_x^{xy} x + c_y^{xy} y & \sigma_0^y + \sigma_r^y r + \sigma_x^y x + \sigma_y^y y \end{bmatrix}. \tag{C2}$$

Note that if each of the four coefficients m_x^r, m_y^r, σ_x^r, σ_y^r is zero, then the spot rate process is one-factor Markov. Hence, at least one of these coefficients must be nonzero for the system to display USV. We first consider the case where either σ_x^r or σ_y^r is nonzero, implying that we can take the variance of dr to be a second state variable. We call this case 1. Later, we consider the case where either m_x^r or m_y^r is nonzero, implying that we can take the drift of dr to be a second state variable. We call this case 2.

A. Case 1

It follows from the definition of a trivariate affine USV model that we can describe the system as[35]

$$\begin{bmatrix} \frac{1}{dt}\,\mathrm{E}[dr] \\ \frac{1}{dt}\,\mathrm{E}[dx] \\ \frac{1}{dt}\,\mathrm{E}[dV] \end{bmatrix} = \begin{bmatrix} m_0^r + m_r^r r + m_x^r x + m_V^r V \\ m_0^x + m_r^x r + m_x^x x + m_V^x V \\ m_0^V + m_r^V r + m_x^V x + m_V^V V \end{bmatrix}. \tag{C3}$$

$$\Sigma^2 = \begin{bmatrix} V & c_0^{rx} + c_r^{rx} r + c_x^{rx} x + c_V^{rx} V & c_0^{r,V} + c_r^{r,V} r + c_x^{r,V} x + c_V^{r,V} V \\ c_0^{rx} + c_r^{rx} r + c_x^{rx} x + c_V^{rx} V & \sigma_0^x + \sigma_r^x r + \sigma_x^x x + \sigma_V^x V & c_0^{x,V} + c_r^{x,V} r + c_x^{x,V} x + c_V^{x,V} V \\ c_0^{r,V} + c_r^{r,V} r + c_x^{r,V} x + c_V^{r,V} V & c_0^{x,V} + c_r^{x,V} r + c_x^{x,V} x + c_V^{x,V} V & \sigma_0^V + \sigma_r^V r + \sigma_x^V x + \sigma_V^V V \end{bmatrix}. \tag{C4}$$

Now, for our proposition to be incorrect, it must be that the system can display USV and $m_x^r = 0$, for only then would we not be able to choose $\mathrm{E}[dr]$ as a third state variable to replace x. Further, for the system to display USV, at least one of the parameters $\{m_x^V, \sigma_x^V, c\}$ must be nonzero, or else the system reduces to a bivariate Markov term-structure model in $\{r, V\}$.

To show that imposing the condition $m_x^r = 0$ would preclude USV, we consider two cases. The first case examines whether the bond price can take the form

$$\text{Case 1a:} \quad P(T-t, r_t, x_t) = e^{A(T-t) + B_1(T-t) r_t + B_2(T-t) x_t}. \tag{C5}$$

The second case examines whether the bond price can take the form

$$\text{Case 1b:} \quad P(T-t, r_t, V_t) = e^{A(T-t) + B_1(T-t) r_t + B_3(T-t) V_t}. \tag{C6}$$

[35] Here, the state variable x is not necessarily the same state variable x in the first system, but rather just some arbitrary state variable that is linearly independent of both r and V.

Given that x is an arbitrary state variable, we claim that these two scenarios incorporate all possibilities for equation (4) to hold. The proof is as follows: All affine models have bond prices that can be written in the form

$$P(T-t, r_t, x_t V_t) = e^{A(T-t)+B_1(T-t)r_t+B_2(T-t)x_t+B_3(T-t)V_t}. \tag{C7}$$

Now, it can be shown that $B_1(\tau)$ cannot vanish. Further, it can be shown that $B_1'(0) = -1$, $B_2'(0) = 0$, $B_3'(0) = 0$. The intuition for this result is that, for very short times to maturity τ, the bond price must go like $e^{-r\tau}$. Scenario 1a investigates the possibility $B_3(\tau) = 0$, $\forall \tau$, for then, we set $(\beta_1 = 0, \beta_2 = 0)$ to satisfy equation (4). Analogously, scenario 1b investigates the possibility $B_2(\tau) = 0$, $\forall \tau$, for then, we set $(\beta_1 = 0, \beta_3 = 0)$. Both scenarios include as a special case $B_2(\tau) = 0$, $\forall \tau$ and $B_3(\tau) = 0$, $\forall \tau$. Finally, we claim that the case where none of the $B_i(\tau)$ vanish is also investigated by 1a, because in such a case equation (4) can only hold if $B_2(\tau) = \alpha B_3(\tau)$, for some constant α. But this, in turn, implies that, by change of variables from the arbitrary variable x to another arbitrary $x' \equiv V + \alpha x$, we are back to investigating scenario 1a.

A.1. Case 1a

Assuming that bonds are only functions of $\{r, x\}$ as in equation (C5), and imposing the condition $m_x^r = 0$, the bond price $P(T-t, r_t, x_t)$ satisfies the PDE

$$0 = -rP + P_t + P_r(m_0^r + m_r^r r + m_V^r V) + P_x(m_0^x + m_r^x r + m_x^x x + m_V^x V) + \frac{1}{2} P_{rr} V$$
$$+ \frac{1}{2} P_{xx}(\sigma_0^x + \sigma_r^x r + \sigma_x^x x + \sigma_V^x V) + P_{rx}(c_0^{rx} + c_r^{rx} r + c_x^{rx} x + c_V^{rx} V). \tag{C8}$$

Furthermore, the affine structure of equation (C5) permits us to identify the ODEs satisfied by $B_1(\tau)$ and $B_2(\tau)$ by collecting terms linear in r, x, and V. We find

$$B_1'(\tau) = B_1 m_r^r + B_2 m_r^x + \frac{1}{2} B_2^2 \sigma_r^x + B_1 B_2 c_r^{rx} - 1 \tag{C9}$$

$$B_2'(\tau) = B_2 m_x^x + \frac{1}{2} B_2^2 \sigma_x^x + B_1 B_2 c_x^{rx} \tag{C10}$$

$$0 = B_1 m_V^r + B_2 m_V^x + \frac{1}{2} B_2^2 \sigma_V^x + B_1 B_2 c_V^{rx} + \frac{1}{2} B_1^2, \tag{C11}$$

with the boundary conditions $B_1(0) = 0$, $B_2(0) = 0$. For our proposition to be incorrect, there must exist a solution to equations (C9)–(C11), with at least one of the parameters $\{m_x^V, \sigma_x^V, c\}$ being nonzero.

The proposed method of solution is to note that equations (C9) to (C11) are true for all times to maturity τ, and hence can be differentiated an arbitrarily large number of times. Each differentiation potentially adds another restriction via equation (C11). Effectively, we are performing a Taylor series expansion in time to maturity. For example, the lowest order imposes the restrictions

$$B_1'(0) = -1 \tag{C12}$$

$$B_2'(0) = 0 \tag{C13}$$

$$0 = 0. \tag{C14}$$

Differentiation of equations (C9) to (C11) generates the following system of equations:

$$B_1''(\tau) = B_1' m_r^r + B_2' m_r^x + B_2 B_2' \sigma_r^x + (B_1' B_2 + B_1 B_2') c_r^{rx} \tag{C15}$$

$$B_2''(\tau) = B_2' m_x^x + B_2 B_2' \sigma_x^x + (B_1' B_2 + B_1 B_2') c_x^{rx} \tag{C16}$$

$$0 = B_1' m_V^r + B_2' m_V^x + B_2 B_2' \sigma_V^x + (B_1' B_2 + B_1 B_2') c_V^{rx} + B_1 B_1'. \tag{C17}$$

These equations allow us to identify

$$B_1''(0) = -m_r^r \tag{C18}$$

$$B_2''(0) = 0 \tag{C19}$$

$$0 = -m_V^r. \tag{C20}$$

Hence, for this system to display USV, a necessary condition is that the parameter m_V^r must be set to zero. Continuing in this manner, we find for the next order of differentiation

$$B_1'''(0) = -(m_r^r)^2 \tag{C21}$$

$$B_2'''(0) = 0 \tag{C22}$$

$$0 = 1. \tag{C23}$$

Clearly, equation (C23) demonstrates this model cannot display USV. Q.E.D.

A.2. Case 1b

Assuming that bonds are only functions of $\{r, V\}$ as in equation (C6), and imposing the condition $m_x^r = 0$, the bond price $P(T-t, r_t, V_t)$ satisfies the PDE

$$\begin{aligned} 0 = & -rP + P_t + P_r(m_0^r + m_r^r r + m_V^r V) \\ & + P_V(m_0^V + m_r^V r + m_x^V x + m_V^V V) + \frac{1}{2} P_{rr} V \\ & + \frac{1}{2} P_{VV}(\sigma_0^V + \sigma_r^V r + \sigma_x^V x + \sigma_V^V V) \\ & + P_{rV}(c_0^{r,V} + c_r^{r,V} r + c_x^{r,V} x + c_V^{r,V} V). \end{aligned} \tag{C24}$$

Furthermore, the affine structure of equation (C5) permits us to identify the ODEs satisfied by $B_1(\tau)$ and $B_2(\tau)$ by collecting terms linear in r, x, and V. We find

$$B_1'(\tau) = B_1 m_r^r + B_2 m_r^V + \frac{1}{2} B_2^2 \sigma_r^V + B_1 B_2 c_r^{rV} - 1 \tag{C25}$$

$$0 = B_2 m_x^V + \frac{1}{2} B_2^2 \sigma_x^V + B_1 B_2 c_x^{rV} \tag{C26}$$

$$B_2'(\tau) = B_1 m_V^r + B_2 m_V^V + \frac{1}{2} B_2^2 \sigma_V^V + B_1 B_2 c_V^{rV} + \frac{1}{2} B_1^2, \tag{C27}$$

with the boundary conditions $B_1(0) = 0$, $B_2(0) = 0$. For our proposition to be incorrect, there must exist a solution to equations (C25)–(C27), and at least one of the parameters $\{m_x^V, \sigma_x^V, c_x^{rV}\}$ being nonzero.

Following the same approach as used above, we differentiate the set of equations (C25) to (C27) to see if the model can display USV. The lowest order imposes the restrictions

$$B_1'(0) = -1 \tag{C28}$$

$$0 = 0 \tag{C29}$$

$$B_2'(0) = 0. \tag{C30}$$

However, successive differentiation of equations (C25) to (C27) forces all three coefficients $\{m_x^V, \sigma_x^V, c\}$ to be zero, the proof of which is available upon request. Recall that this implies that the fixed income market is then bivariate Markov, which has been shown previously unable to exhibit USV. Hence, this scenario also cannot display USV. Thus, bond prices of the form in equation (C5) cannot exhibit USV. Q.E.D.

B. Case 2

It follows by definition of a USV model that we can describe the system as

$$\begin{bmatrix} \frac{1}{dt}\,\mathrm{E}[dr] \\ \frac{1}{dt}\,\mathrm{E}[dx] \\ \frac{1}{dt}\,\mathrm{E}[d\mu] \end{bmatrix} = \begin{bmatrix} \mu \\ m_0^x + m_r^x r + m_x^x x + m_\mu^x \mu \\ m_0^\mu + m_r^\mu r + m_x^\mu x + m_\mu^\mu \mu \end{bmatrix} \tag{C31}$$

$$\Sigma^2 = \begin{bmatrix} \sigma_0^r + \sigma_r^r r + \sigma_x^r x + \sigma_\mu^r \mu & c_0^{rx} + c_r^{rx} r + c_x^{rx} x + c_\mu^{rx} \mu & c_0^{r,\mu} + c_r^{r,\mu} r + c_x^{r,\mu} x + c_\mu^{r,\mu} \mu \\ c_0^{rx} + c_r^{rx} r + c_x^{rx} x + c_\mu^{rx} \mu & \sigma_0^x + \sigma_r^x r + \sigma_x^x x + \sigma_\mu^x \mu & c_0^{x,\mu} + c_r^{x,\mu} r + c_x^{x,\mu} x + c_\mu^{x,\mu} \mu \\ c_0^{r,\mu} + c_r^{r,\mu} r + c_x^{r,\mu} x + c_\mu^{r,\mu} \mu & c_0^{x,\mu} + c_r^{x,\mu} r + c_x^{x,\mu} x + c_\mu^{x,\mu} \mu & \sigma_0^\mu + \sigma_r^\mu r + \sigma_x^\mu x + \sigma_\mu^\mu \mu \end{bmatrix}. \tag{C32}$$

Now, for our proposition to be incorrect, it must be that the system can display USV and $\sigma_x^r = 0$, for only then would we not be able to choose $\mathrm{Var}[dr]$ as a third state variable to replace x. Further, for the system to display USV, at least one of the parameters $\{m_x^\mu, \sigma_x^\mu, c_x^{r\mu}\}$ must be nonzero, or else the system reduces to a bivariate Markov term-structure model in $\{r, \mu\}$.

Following the strategy taken above, to show that imposing the condition $\sigma_x^r = 0$ would preclude the model from generating incomplete bond markets, we consider two cases. The first case examines whether the bond price can take the form

$$\text{Case 2a:} \quad P(T-t, r_t, x_t) = e^{A(T-t)+B_1(T-t)r_t+B_2(T-t)x_t}. \tag{C33}$$

The second case examines whether the bond price can take the form

$$\text{Case 2b:} \quad P(T-t, r_t, \mu_t) = e^{A(T-t)+B_1(T-t)r_t+B_2(T-t)\mu_t}. \tag{C34}$$

Following the same argument as in Case 1, given that x is an arbitrary state variable, it follows that these two scenarios incorporate all possibilities for equation (4) to hold. Q.E.D.

B.1. Case 2a

Assuming bond prices are functions of only $\{r, x\}$ as in equation (C33), and imposing the condition $\sigma_x^r = 0$, bond prices $P(T - t, r_t, x_t)$ satisfy the PDE

$$0 = -rP + P_t + P_r(\mu) + P_x(m_0^x + m_r^x r + m_x^x x + m_\mu^x \mu) + \frac{1}{2} P_{rr}(\sigma_0^r + \sigma_r^r r + \sigma_\mu^r \mu)$$
$$+ \frac{1}{2} P_{xx}(\sigma_0^x + \sigma_r^x r + \sigma_x^x x + \sigma_\mu^x \mu) + P_{rx}(c_0^{rx} + c_r^{rx} r + c_x^{rx} x + c_\mu^{rx} \mu). \qquad \text{(C35)}$$

Further assuming that bond prices have an affine structure as in equation (C5), and then collecting terms linear in r, x, and μ, we find that

$$B_1'(\tau) = B_2 m_r^x + \frac{1}{2} B_2^2 \sigma_r^x + B_1 B_2 c_r^{rx} + \frac{1}{2} B_1^2 \sigma_r^r - 1 \qquad \text{(C36)}$$

$$B_2'(\tau) = B_2 m_x^x + \frac{1}{2} B_2^2 \sigma_x^x + B_1 B_2 c_x^{rx} \qquad \text{(C37)}$$

$$0 = B_2 m_\mu^x + \frac{1}{2} B_2^2 \sigma_\mu^x + B_1 B_2 c_\mu^{rx} + \frac{1}{2} \sigma_\mu^r B_1^2 + B_1, \qquad \text{(C38)}$$

with the boundary conditions $B_1(0) = 0$, $B_2(0) = 0$. For our proposition to be incorrect, there must exist solutions to equations (C36) to (C38), with at least one of the parameters $\{m_x^\mu, \sigma_x^\mu, c_x^{r\mu}\}$ not equal to zero.

Following the same approach as used above, we note that equations (C36) to (C38) are true for all times to maturity τ, and hence can be differentiated an arbitrarily large number of times. Each differentiation potentially adds another restriction via equation (C38). For example, the lowest order imposes the restrictions

$$B_1'(0) = -1 \qquad \text{(C39)}$$

$$B_2'(0) = 0 \qquad \text{(C40)}$$

$$0 = 0. \qquad \text{(C41)}$$

Differentiation of equations (C36) to (C38) generates the following system of equations:

$$B_1''(\tau) = B_2' m_r^x + B_2 B_2' \sigma_r^x + (B_1' B_2 + B_1 B_2') c_r^{rx} + B_1 B_1' \sigma_r^r \qquad \text{(C42)}$$

$$B_2''(\tau) = B_2' m_x^x + B_2 B_2' \sigma_x^x + (B_1' B_2 + B_1 B_2') c_x^{rx} \qquad \text{(C43)}$$

$$0 = B_2' m_\mu^x + B_2 B_2' \sigma_\mu^x + (B_1' B_2 + B_1 B_2') c_\mu^{rx} + B_1 B_1' \sigma_\mu^r + B_1'. \qquad \text{(C44)}$$

These equations allow us to identify

$$B_1''(0) = 0 \tag{C45}$$

$$B_2''(0) = 0 \tag{C46}$$

$$0 = -1. \tag{C47}$$

Note that equation (C47) is inconsistent with equation (C39). Hence, this model cannot display USV. Q.E.D.

B.2. Case 2b

Assuming bond prices are functions of only $\{r, \mu\}$ as in equation (C34), and imposing the condition $\sigma_x^r = 0$, bond prices $P(T - t, r_t, \mu_t)$ satisfy

$$\begin{aligned} 0 = {} & -rP + P_t + P_r(\mu) + P_\mu(m_0^\mu + m_r^\mu r + m_x^\mu x + m_\mu^\mu \mu) + \frac{1}{2}P_{rr}(\sigma_0^r + \sigma_r^r r + \sigma_\mu^r \mu) \\ & + \frac{1}{2}P_{\mu\mu}(\sigma_0^\mu + \sigma_r^\mu r + \sigma_x^\mu x + \sigma_\mu^\mu \mu) + P_{r\mu}(c_0^{r,\mu} + c_r^{r,\mu} r + c_x^{r,\mu} x + c_\mu^{r,\mu} \mu). \end{aligned} \tag{C48}$$

Assuming the bond price is of the form in equation (C34), by collecting terms linear in r, x, and μ, we find that

$$B_1'(\tau) = B_2 m_r^\mu + \frac{1}{2}B_2^2\sigma_r^\mu + B_1 B_2 c_r^{r\mu} + \frac{1}{2}B_1^2\sigma_r^r - 1 \tag{C49}$$

$$0 = B_2 m_x^\mu + \frac{1}{2}B_2^2\sigma_x^\mu + B_1 B_2 c_x^{r\mu} \tag{C50}$$

$$B_2'(\tau) = B_2 m_\mu^\mu + \frac{1}{2}B_2^2\sigma_\mu^\mu + B_1 B_2 c_\mu^{r\mu} + \frac{1}{2}B_1^2\sigma_\mu^r + B_1, \tag{C51}$$

with the boundary conditions $B_1(0) = 0$, $B_2(0) = 0$. Below, we prove that these equations are inconsistent, thus proving that this system cannot generate incomplete bond markets.

By setting $\tau = 0$, equations (C49) to (C51) imply

$$B_1'(0) = -1 \tag{C52}$$

$$0 = 0 \tag{C53}$$

$$B_2'(0) = 0. \tag{C54}$$

However, successive differentiation of equations (C49) to (C51) forces all three coefficients $\{m_x^\mu, \sigma_x^\mu, c_x^{r\mu}\}$ to be zero, the proof of which is available upon request. Recall that this implies that the fixed income market is then bivariate Markov, which has been shown previously unable to exhibit USV. Hence, this scenario also cannot display USV. Since both scenarios 2a and 2b cannot exhibit USV, our claim is proved. Q.E.D.

Appendix D: Proof of Proposition

Assuming that bond prices are functions only of time to maturity, the spot rate r, and some generic state variable x as in equation (20), and that the third state variable of the model is $\mu_t \equiv (1/dt)\mathrm{E}^Q[dr_t]$, then bond prices satisfy the partial differential equation

$$\begin{aligned} 0 = &-rP + P_t + P_r(\mu) + P_x(m_0^x + m_r^x r + m_x^x x + m_\mu^x \mu) \\ &+ \frac{1}{2}P_{rr}(\sigma_0^r + \sigma_r^r r + \sigma_x^r x + \sigma_\mu^r \mu) \\ &+ \frac{1}{2}P_{xx}(\sigma_0^x + \sigma_r^x r + \sigma_x^x x + \sigma_\mu^x \mu) + P_{rx}(c_0^{rx} + c_r^{rx} r + c_x^{rx} x + c_\mu^{rx} \mu). \end{aligned} \tag{D1}$$

Further assuming that the bond price is of the form in equation (20), by collecting terms linear in r, x, and μ, we find that

$$B_1'(\tau) = B_2 m_r^x + \frac{1}{2}B_2^2\sigma_r^x + B_1B_2c_r^{rx} + \frac{1}{2}B_1^2\sigma_r^r - 1 \tag{D2}$$

$$B_2'(\tau) = B_2 m_x^x + \frac{1}{2}B_2^2\sigma_x^x + B_1B_2c_x^{rx} + \frac{1}{2}B_1^2\sigma_x^r \tag{D3}$$

$$0 = B_2 m_\mu^x + \frac{1}{2}B_2^2\sigma_\mu^x + B_1B_2c_\mu^{rx} + \frac{1}{2}B_1^2\sigma_\mu^r + B_1, \tag{D4}$$

with the boundary conditions $B_1(0) = 0$, $B_2(0) = 0$. This implies

$$B_1'(0) = -1, \qquad B_2'(0) = 0. \tag{D5}$$

Differentiating equation (D4), and using equation (D5) to evaluate this equation at $\tau = 0$, we find the contradiction

$$0 = -1. \tag{D6}$$

Hence, models such as that proposed in equation (20), where the affine model is trivariate Markov in $\{r, \mu, x\}$, cannot exist. Q.E.D.

Appendix E: Proof of Proposition 6

A. Necessity

Using the initial conditions $B_1(0) = 0, B_2(0) = 0$, equations (26), (27), and (29) imply

$$B_1'(0) = -1 \tag{E1}$$

$$B_2'(0) = 0 \tag{E2}$$

$$0 = 0. \tag{E3}$$

Differentiating equations (26), (27), and (29), and using the conditions in equations (E1) to (E3), we find

$$B_1''(0) = 0 \tag{E4}$$

$$B_2''(0) = -1 \tag{E5}$$

$$0 = 0. \tag{E6}$$

One more differentiation produces

$$B_1'''(0) = -m_r^\mu \tag{E7}$$

$$B_2'''(0) = -m_\mu^\mu \tag{E8}$$

$$0 = -m_V^\mu + 1, \tag{E9}$$

implying that a necessary condition for this model to exhibit USV is $m_V^\mu = 1$, which is one of the conditions for both sets of parameter restrictions given in equation (30). Continuing in this manner, we eventually run into a "branch," where we can choose one of two different conditions to satisfy $0 = \text{RHS}$. After repeating the Taylor series expansion for several more steps along each of these branches, the third equation eventually produces $0 = 0$ for the next $\sim$15 iterations afterwards, strongly suggesting that these necessary restrictions are in fact also sufficient. We now prove this is indeed the case.

B. Sufficiency

Define

$$F(\tau) = m_V^\mu B_2(\tau) + \frac{\sigma_V^\mu}{2} B_2^2(\tau) + c_V^{r,\mu} B_1(\tau) B_2(\tau) + \frac{1}{2} B_1^2(\tau). \tag{E10}$$

which is the right-hand side of equation (29). It is sufficient to show that if either set of parameter restrictions given in equation (30) holds, then $F(\tau) = 0$, $\forall \tau$, if $B_1(\tau), B_2(\tau)$ satisfy the system of ODE given by equations (25, 27). Indeed, in that case, $B_3(\tau) = 0$, $\forall \tau$ and $B_1(\tau), B_2(\tau)$ satisfying the system of ODE given by equations (25, 27) is a solution of the initial system of ODE verified by B_1, B_2, B_3 (given in DK (1996), e.g.). Further, existence and uniqueness of the solution to such a system is proved by Duffie, Filipovic, and Schachermayer (2001).

If either set of parameter restrictions given in equation (30) holds, then substituting into equation (E10), taking derivatives, and substituting expressions for B_1' and B_2' from equations (25, 27), we obtain

$$F'(\tau) = 2(c_V^{r,\mu} + B_2(\tau)(c_r^{r,\mu} + c_\mu^{r,\mu} c_V^{r,\mu}))F(\tau). \tag{E11}$$

The solution to this equation is

$$F(\tau) = F(0) e^{\int_0^\tau ds\, 2(c_V^{r,\mu} + B_2(s)(c_r^{r,\mu} + c_\mu^{r,\mu} c_V^{r,\mu}))}. \tag{E12}$$

Given the initial condition $F(0) = 0$, it is clear that, $F(\tau) = 0$, $\forall \tau$. Q.E.D.

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