The American Put Option and Its Critical Stock Price

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ABSTRACT

We derive an expression for the critical stock price for the American put. We start by expressing the put price as an integral involving first-passage probabilities. This approach yields intuition for Merton's result for the perpetual put. We then consider the finite-lived case. Using (1) the fact that the put value ceases to depend on time when the critical stock price is reached and (2) the result that an American put equals a European put plus an early-exercise premium, we derive the critical stock price. We approximate the critical-stock-price function to compute accurate put prices.

KIM (1990), JACKA (1991), AND CARR, JARROW, AND MYNENI (1992) showed that the American put price is equal to the corresponding European put price plus an integral representing the early-exercise premium. However, to use their approach one needs to know the critical stock price, which may be defined in three equivalent ways.

1. It is the value of the stock price at which one is indifferent between exercising and not exercising the put.
2. It is the highest value of the stock price for which the value of the put is equal to the exercise price less the stock price.
3. It is the highest value of the stock price at which the put value does not depend on time to maturity.

The second definition has ordinarily been used to find the critical stock price, but it requires solving an integral equation numerically. If it is optimal to exercise the put immediately, then its value cannot depend on how much time remains to maturity. Hence, we have the third definition, which, we believe, is new to the literature. We show that the third definition leads to an analytic expression for the critical stock price. Our result is important not only because it provides a better way to get American put prices but also

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because it provides a better understanding of the early-exercise boundary, which is important in its own right. For example, one empirical implication of our equation is that early exercises of in-the-money puts should increase dramatically as expiration nears. Furthermore, our approach can be used for other derivative securities where early exercise is a real possibility. Of course, it can also be applied to corporate securities, real options, insurance, banking products, and other contracts that are modeled as options. Finally, there is pedagogical value in having a simple equation for the critical stock price.

The problem of valuing the American put is clearly important. Over 20 papers have been published on this topic. The most recent papers (Carr and Faguet (1994), Broadie and Detemple (1996), Huang, Subrahmanyam, and Yu (1996), Ho, Stapleton, and Subrahmanyam (1997), and Ju (1998)) provide a good overview of the literature. Because of the inherent difficulty of the problem, solutions have tended to be mathematically complex and challenging to compute. An illustrative example is the Geske and Johnson (1984) expression for the value of the American put, which is written as an infinite series of multivariate normal terms.

Simple expressions for the American put price exist only for a few special cases. When the interest rate is zero, no interest is lost by waiting until maturity to exercise, so the American put is equivalent to a European put, which has a simple expression. When the put is perpetual, the expiration date always remains infinitely far into the future, so the put price does not depend explicitly on time. Merton (1973) used this fact to convert the partial differential equation derived by Black and Scholes (1973) into an ordinary differential equation. His solution is simple, but it bears no resemblance to either the expression of Geske and Johnson (1984) or the expression for the zero-interest-rate case. Moreover, there is no obvious intuition for the stock price entering the equation as an inverse power law. Finally, Merton's equation lacks any obvious connection to the normal probability distributions that arise naturally in other option-pricing equations.

In this paper we derive results that establish links among alternative approaches to valuing the American put, and we develop simple expressions for the critical stock price in the finite-lived case. We start by writing the American put price as an integral involving the first-passage probability of the stock price hitting the critical stock price. Using the "first-passage approach" we provide alternate derivations of expressions for put prices in the perpetual-put case. We are then able to provide intuition for Merton's result and show its connection to the general option-pricing literature.

Next, we consider the finite-lived case. The first-passage approach is used to get a simple functional form that can be used to *approximate* critical stock prices. We then use the fact that the critical stock price for the American put can be found by equating to zero the derivative of the American put price with respect to time to maturity. This yields a critical-stock-price expression that is *exact* (and approximations that can be used to evaluate it more easily). Intuitively, this method finds the critical stock price for a put by equating the interest-rate effect to the volatility effect. For calls these two effects ordinarily reinforce rather than offset each other, so there is no early exercise unless there is a dividend, in which case the equations of Roll (1977) and Geske (1979) (see also Whaley (1981) and Geske (1981)) can be used to value the calls. However, for currency options and options on futures the interest-rate effect can offset the volatility effect for calls because the effective interest rate can be negative or the continuous dividend yield can be large enough. Our approach could be used for these cases and also for various exotic options, such as barrier options and Asian options.

Given our analytic expressions for the critical stock price, we can find the American put price by using either the first-passage probability approach or the approach derived in Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992). Lacking an exact expression for the first-passage probability for the general case, we use the latter approach to value puts.

The paper proceeds as follows. Section I uses the first-passage approach to derive the perpetual-put value. In Section II we derive exact and approximate expressions for the critical stock price for the finite-lived case. Section III contains numerical results demonstrating the behavior and accuracy of our critical-stock-price expressions. These include comparisons of American put values computed by various competing methods. Section IV is a summary.

I. The First-Passage Approach and the Perpetual-Put Case

The American put will be exercised as soon as the stock price hits the critical stock price. Thus, the put price can be expressed as the expectation of the discounted value of the exercise price less the critical stock price. We will use this approach to derive analytic expressions for the price of a perpetual put.

To calculate the expectation we need the probability of exercise at any given time. The first-passage probability is exactly what we need, for the option can only be exercised once, and it is optimal to exercise the first time the critical-stock-price boundary is reached. We use in one case an expression provided by Feller (1971) and in another an expression by Revuz and Yor (1991). The integrals we obtain are complicated but can be reduced by using a Laplace transform listed in Abramowitz and Stegun (1972). A very similar approach is used in Ingersoll (1987) and in Kim (1990) to value the rebate portion of a perpetual down-and-out option and a perpetual American call, respectively.

In general, when the stock price and the critical stock price have no discontinuities, we have

$$P = \max_{S_C} \int_0^T e^{-rt}(X - S_c) f\, dt, \qquad (S > S_c), \tag{1}$$

where P, r, T, X, and S_c are the American put price, risk-free rate, time to maturity, exercise price, and critical stock price, respectively. Let S be the current stock price (at time $t = 0$). The first factor in the integral is the discount factor, the second is the payoff when exercise occurs, and the third, f, is the first-passage probability, that is, the probability that the stock price will decline from S to S_c for the first time at time t. The maximization is taken over all possible functions, $S_c(\tau)$, where $\tau \equiv T - t$. If $S \leq S_c(T)$ we simply have $P = X - S$.

In general, S_c is a decreasing function of τ and may not be a known function. Also, f in general depends on S, S_c, and t. However, in the case where $T = \infty$, S_c is a constant and can be found from the first-order condition. Feller (1971) provides a formula for the first-passage probability for this case. First, define

$$Z = \frac{1}{\sigma}\left[\log S/S_t + \left(r - \frac{1}{2}\sigma^2\right)t\right], \tag{2}$$

where S_t is the value of the stock at time t and σ is the standard deviation of the rate of return on the stock. Under the standard assumptions, Z is normal with a zero mean and a variance of t. Second, define

$$a = \frac{1}{\sigma}\left[\log S/S_c + \left(r - \frac{1}{2}\sigma^2\right)t\right]. \tag{3}$$

Intuitively, a can be thought of as a standardized measure of how far the put is from early exercise. First consider the case $\gamma \equiv 2r/\sigma^2 = 1$ so that a is constant (as in equation (7) below). Equation (1) therefore requires an expression for the first-passage time of a (standardized) Brownian motion to the "position" a. Feller (1971, p. 171) discusses this case; he argues that the distribution obeys a fundamental stability property and concludes that the expression for the first-passage probability is given by his equation (2.5) ($P(T_a < t) = 2[1 - N(a/\sqrt{t})]$). He notes that his equation (2.5) is equivalent in distribution to his equation (2.1) ($F(x) = 2[1 - N(1/\sqrt{x})]$) with density

$$f(x)dx = \frac{1}{\sqrt{2\pi x^3}} e^{-1/(2x)}dx. \tag{4}$$

Computing the American put value requires the appropriate density obtained through the transformation $x = t/a^2$, so that

$$f(t)dt = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/(2t)}dt, \tag{5}$$

in equation (1). The constant in equation (5) is chosen so that $f(t)$ is normalized; that is, the integral of $f(t)$ from zero to infinity is one. Because S_c is a constant in this case,

$$P = \frac{(X - S_c)}{\sqrt{2\pi}} \int_0^\infty \frac{ae^{-rt}e^{-a^2/(2t)}}{t^{3/2}} dt, \tag{6}$$

where it is understood that S_c is the value yielding the maximum P; that is, we suppress the max notation. The integral in equation (6) can be done using the Laplace transform table of Abramowitz and Stegun (1972, p. 1026). When $\gamma = 1$,

$$a = \frac{1}{\sigma} \log S/S_c, \tag{7}$$

a constant. The result for the Laplace transform is

$$\int_0^{\infty} \frac{k}{2\sqrt{\pi t^3}} e^{-k^2/(4t)} e^{-st} dt = e^{-k\sqrt{s}}, \qquad (k > 0). \tag{8}$$

Laplace transforms are often useful in finance, as the exponential factor containing the s term can be interpreted as a continuous discount factor. In general, s is just the independent variable in the Laplace transform, but in finance it is the discount rate.

Letting $s = r$ and $k^2 = 2a^2$, we obtain

$$P = (X - S_c)\exp[-(\sqrt{2r}/\sigma)\log S/S_c]. \tag{9}$$

Simplifying, we obtain

$$P = (X - S_c)S_c/S. \tag{10}$$

Merton obtained the same result by noting that the Black and Scholes (1973) partial differential equation

$$\frac{\partial P}{\partial t} = rP - rS\frac{\partial P}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} \tag{11}$$

simplifies to an ordinary differential equation when $T = \infty$ and the partial with respect to time vanishes. When $\gamma = 1$, one can show by substituting equation (10) into

$$rP - rS\frac{dP}{dS} - \frac{1}{2}\sigma^2 S^2 \frac{d^2 P}{dS^2} = 0 \tag{12}$$

that equation (10) satisfies equation (12).

Our derivation provides some insight into Merton's result. Although no normal distributions appear in the final result, equation (10), normality is assumed to get the first-passage probability given in equation (5). The second factor in equation (10) can be identified from equation (1) as the expected discount factor. Alternatively, we may write equation (10) as

$$P = X(S_c/S) - S(S_c/S)^2. \tag{13}$$

The normal density function does not appear explicitly in equation (10) or (13), but it is implicit in S_c. Now, $\partial P/\partial S_c = X/S - 2S_c/S = 0$, as $S_c = X/2$ for this case. Because, from equation (10), $\partial P/\partial S = -(X - S_c)S_c/S^2$, we have

$$\frac{\partial P}{\partial S} = -(S_c/S)^2. \tag{14}$$

Thus, we may interpret equation (13) by saying the American put price equals the exercise price times the expected discount factor plus the stock price times the (negative) hedge ratio. This is very similar to the European put, where

$$p = Xe^{-rT}N(-d_2) - SN(-d_1) \tag{15}$$

and d_1 and d_2 are as defined in Black and Scholes (1973). Because we can interpret equation (15) by saying the put price equals the discounted exercise price times the risk-neutral probability that the put will be exercised plus the stock price times the hedge ratio, it is clear that equations (13) and (15) are actually very similar. Thus, Merton's formula is not merely a mathematical result but has a simple intuitive interpretation.

We can also derive expressions for the case γ not equal to one, but some care is required, for in this case a, defined in equation (3), is now a function of time. Making the transformation from x to t then yields a very complicated expression. However, if we neglect terms that are small when γ is close to one, then we can use equation (7) instead of equation (3) for the first factor in the integrand in equation (6). After some calculations (given in the Appendix), we get

$$P = (X - S_c)\left(\frac{S_c}{S}\right)^{\gamma}, \tag{16}$$

which is Merton's result. Neither equation (10) nor equation (16) contains the normal density function, but the normal density is used in deriving equation (5), and thus it is implicit in S_c and P.

Note that equation (16) is the exact result. Therefore, it must be the case that the density used in equation (A1) (see the Appendix),

$$f(t) = \frac{\log S/S_c e^{-a^2/(2t)}}{\sqrt{2\pi}\sigma t^{3/2}}, \tag{17}$$

with a defined in equation (3), is the exact expression for the density when T approaches infinity. Note that, when $\gamma = 1$, equations (17) and (5) are the same and equation (16) reduces to equation (10). Comparing equations (A1) and (16), we see that for $r = 0$ the density integrates to one, as required.

In fact, although this case may be interpreted as the first passage of stock price (with γ not required to be one) to the constant critical stock price S_c, transformations (2) and (3) render it equivalent to the first passage of a (standard) Brownian motion to a *linear* boundary, which has been considered in the literature on Brownian motion (see Revuz and Yor (1991)).

As in Merton's approach, our alternative derivations of equations (13) and (16) benefited from mathematical simplifications that are possible when time to expiration is infinite. We consider the more difficult case of finite-lived puts in the next section.

II. The Critical-Stock-Price Function for Finite T

When $T < \infty$, the critical-stock-price function is no longer a constant, and the first-passage approach requires considering the passage of a Brownian motion to a nonlinear (curved) boundary. A monograph by Lerche (1986) (see also Durbin (1992) and references therein) addresses this more difficult case. However, the general expression giving the exact $f(t)$ takes the form of an integral equation in which $f(t)$ is only defined implicitly, and convenient closed-form solutions are available only for certain specialized cases. A major purpose of Lerche (1986) is to characterize conditions under which first-passage probabilities may be estimated using the so-called tangent approximation:

$$f = \frac{\log \dfrac{S}{S_c} + \dfrac{\dot{S}_c}{S_c} t}{\sqrt{2\pi}\sigma t^{3/2}} e^{-a^2/(2t)}, \tag{18}$$

where the dot indicates the time derivative and S_c and a are general functions of t. Intuitively, the second term in the numerator is the elasticity of the critical stock price with respect to time. If the elasticity—the curvature—is high, S is likely to hit S_c, and so the probability density is high. Note that equation (18) reduces to equation (17) when T is infinite, as $\dot{S}_c$ goes to zero. The tangent approximation works well in those cases where the boundary (the critical-stock-price function in our case) does not have too much curvature. In fact, if S_c is assumed to have a simple exponential form (as in Omberg (1987) and Ju (1998)) then equation (18) provides the *exact* first-passage probability density, and equation (1) can be reduced to expressions involving cumulative normal distributions.

In the Appendix we derive an expression for S_c by starting with the tangent approximation. The result we obtain,

$$S_c/X = \frac{\gamma}{1+\gamma} + \frac{e^{-\beta_2\tau - \alpha_1\sqrt{\tau}}}{1+\gamma}, \tag{19}$$

where α_1 and β_2 are defined in the Appendix, is *not* a simple exponential form and has too much curvature for the tangent approximation to yield accurate put values. Thus, this result must only be an approximation for S_c. Next, we derive the exact result (equations (22) and (23)), which is essentially of the same form as equation (19).

We can get an exact expression by noting that $S = S_c$ when

$$\frac{\partial P}{\partial \tau} = 0 \tag{20}$$

Intuitively, when the stock price falls to S_c, it is optimal to exercise immediately; hence, it does not matter how much time is left to maturity. The value of the put is $X - S$, which does not depend on τ. Alternatively, equation (20) describes the point where the interest effect is exactly offset by the volatility effect.

To the best of our knowledge, this equation has not been used before. Traditionally, S_c has been found as the solution to the integral equation $P = X - S_c$; however, this approach does not yield an analytic solution. To get an analytic expression for this derivative, we use the equation obtained by Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992) and used in Huang, Subrahmanyam, and Yu (1996):

$$P = p + \int_0^T rXe^{-rt}N(-d_2(S,S_c,t))\,dt, \tag{21}$$

where

$$d_2(S,S_c,t) = \frac{\log S/S_c + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}.$$

Equation (21) states that the American put is equivalent to the corresponding European put plus an early exercise premium, which is the value of the right to exchange the European put for the American. In the Appendix we show that

$$\frac{S_c}{X} = e^{-(r+(1/2)\sigma^2)\tau - g\sigma\sqrt{\tau}}, \tag{22}$$

where

$$g = \pm\sqrt{2\log\frac{\sigma^2}{\dfrac{2r}{\sqrt{\alpha}}\,x\log xe^{-\alpha(r+(1/2)\sigma^2)^2\tau/(2\sigma^2)}}}, \tag{23}$$

α is defined in the Appendix, and $x = X/S_c$. The function g typically has a value of about 1.5. Intuitively, it measures approximately how many standard deviations the stock price must be in the money before the put will be exercised. Note that S_c/X only depends on two dimensionless quantities, namely, $\sigma^2\tau$, a measure of the time to maturity, and γ, a measure of the ratio of the interest-rate effect to the volatility effect.

Because we have so far made no approximations, equation (22), together with equation (23), is *exact for all* τ. However, we must estimate a value for α, and solving for S_c (or, equivalently, g) requires an iterative procedure because it is implicitly defined. Moreover, it might appear that equation (22) has the wrong asymptote for large τ; that is, it appears to asymptote to zero instead of $\gamma X/(1 + \gamma)$, as required. But, we can show that the asymptote is in fact correct, and we can develop good approximations for both g and α.

For small τ we must choose the positive root in equation (23) (see equation (A4) in the Appendix), but g decreases with τ and eventually becomes zero. For even larger τ, g becomes negative; that is, we choose the negative root in equation (23). For equation (22) to have the correct asymptote (i.e., the same as in equations (A3) and (19)), we must have, for very large τ,

$$g = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{1+\gamma}{\gamma} - \left(r + \frac{1}{2}\sigma^2\right)\tau\right]. \tag{24}$$

We need to know which sign to choose in equation (23); that is, we need to know when g is zero. We can get a fairly good estimate of τ_0, the point at which g passes through zero, by setting $g = 0$ in equation (24). We obtain

$$\tau_0 \approx \frac{\log\dfrac{1+\gamma}{\gamma}}{r + \dfrac{1}{2}\sigma^2}. \tag{25}$$

Equation (25) gives values that are slightly too high, as S_c will be slightly above the asymptote at τ_0, whereas equation (24) assumes S_c equals the asymptote. The exact expression is found by letting $g = 0$ in equation (23), that is, letting the argument of the logarithm equal one. We get

$$\tau_0 = \frac{2\log\dfrac{2\sqrt{\alpha}}{\gamma(1+\gamma)\sigma^2\tau_0}}{\sigma^2(1+\gamma)\left[1 - \dfrac{\alpha}{4}(1+\gamma)\right]}. \tag{26}$$

Equation (26) must be solved iteratively.

In the Appendix we obtain an expression for α:

$$\alpha = 1 - \frac{A}{1 + \dfrac{(1+\gamma)^2}{4}\gamma^2\tau}, \tag{27}$$

where

$$A = \frac{1}{2}\left(\frac{\gamma}{1+\gamma}\right)^2. \tag{28}$$

The expression for A is an approximation, albeit a very good one.

Finally, we derive (in the Appendix) the following approximation for g for small τ:

$$g = \sqrt{\log \frac{\sigma^2}{4er^2\tau/\alpha}}. \tag{29}$$

For typical puts g is between one and two, which means early exercise occurs when the stock price is a couple of standard deviations in the money. At this point the volatility effect is small, because a deep-in-the-money put has lost much of its option (i.e., limited-liability) character. Hence, the interest-rate effect can equal the volatility effect. Note that equation (29) depends on r and σ in the proper way. For example, when $r = 0$, there is no reason to exercise early, no matter how low the stock price, so S_c should be zero. In fact, $r = 0$ in equation (29) gives $g = \infty$; hence, $S_c = 0$, as required.

Many approaches for valuing the American put use the European put as a starting point. For small r and τ the American and European puts are almost the same, but as τ becomes large, the two diverge, and this often results in large errors for these approaches. Our approach will suffer from this same problem if we use equation (29) but not if we use equation (23). In fact, the only source of error (apart from numerical problems) from using equations (21), (22), (23), (27), and (28) is the approximate nature of equation (28), and this error disappears as τ becomes large. In the next section we will show that using equations (21), (22), and (23) gives accurate American put prices. We also use equation (29) and an approximation to equation (21) to get a very fast (but less accurate) method for computing American put prices.

III. Numerical Evaluation

In this section we evaluate the results of the previous section for purposes of computing critical stock prices and put values. As usual, there is a tradeoff between accuracy and speed, which we explore below. We first consider

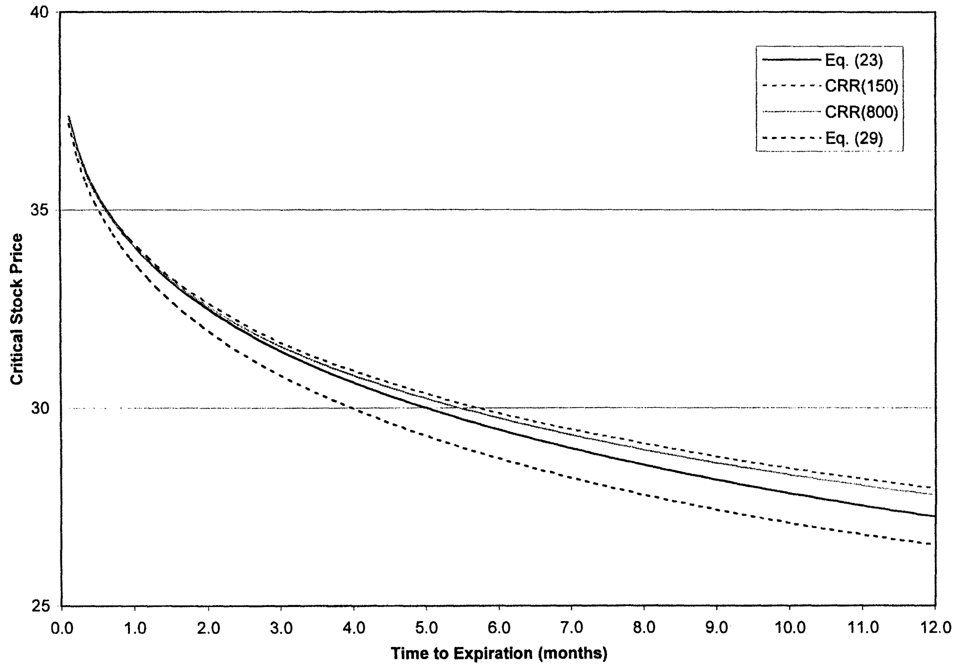

Figure 1. Critical-stock-price functions (r = 0.0488, σ = 0.3, X = \$40). This figure gives plots of the critical stock price versus time to maturity for the case $r = 0.0488$ (continuous rate), $\sigma = 0.3$, and $X = \$40$. The solid line is from equations (22) and (23). The lowest (dotted) line is from equations (22) and (29). The highest (short-dashed) line is found using the Cox et al. (1979) (CRR) method with 150 time steps, and the (long-dashed) line just below it is found using the CRR method with 800 time steps. (The discrete rate $r = 0.05$ was used for the CRR methods.)

the behavior of two S_c functions from the previous section by providing comparisons to critical-stock-price values from alternative methods. We then examine the accuracy of computed put values (assuming that errors from all other sources are small) to provide an indirect test of the S_c functions. Finally, we compare alternative procedures for valuing American puts.

We first compare plots of four S_c functions for a specific case that extends well-known examples from Cox and Rubinstein (1985, p. 249): $r = 0.05$, $\sigma = 0.3$, and $X = \$40$ over the range $\tau = 0$ to 12 months (see Figure 1). We use two functions from the previous section. One is equation (22) combined with equation (23), and the other is equation (22) combined with equation (29), where equations (27) and (28) are used to compute α for both functions. The former requires solution of an implicit equation, whereas the latter does not, and, a priori, the former approach ought to have smaller errors. To provide a comparison, two more plots were obtained by using the approach of finding the largest value of S for which $P = X - S$, where P is computed using the Cox , Ross, and Rubinstein (1979) method (a.k.a. the "binomial method"). Plots were obtained using 150 and 800 time steps.

All four curves are qualitatively similar and illustrate the general behavior of the critical-stock-price function. For intermediate τ, S_c grows exponentially, but for very small τ, S_c grows very fast, approaching X with an ever-increasing slope. This means that exercises will tend to occur near, but not at, maturity. As always, exercise occurs when the volatility effect is no longer larger than the interest-rate effect. Crudely speaking, the interest-rate effect goes as $r\tau$, whereas the volatility effect goes as $\sigma\sqrt{\tau}$. That is, the interest lost by waiting until maturity decreases linearly as τ approaches zero, but the volatility effect decreases much faster: the amount by which the stock must go into the money to be several standard deviations below the exercise price is decreasing as the square root of τ, and so the volatility effect, which decreases as the stock gets deeper into the money (because the asymmetry of gains and losses due to the limited-liability nature of the put decreases), decreases rapidly. Hence, the critical stock price rises rapidly near maturity. This behavior of S_c gives rise to an easily tested prediction: Early exercises of in-the-money puts should increase dramatically as expiration nears. If one has the data on exercises and stock prices, one can also test directly the functional form we derived for S_c.

In an unjustly neglected paper, Omberg (1987) gives qualitatively similar results for the critical stock price. Omberg used an exponential function for the critical stock price, as it was the only function consistent with the first-passage probability density he was using. However, he found that his results were better if he let the exponential function hit $\tau = 0$ below X and added a discontinuous jump to X. In our case, we have *derived* the functional form, and, though the derivative becomes infinite at $\tau = 0$, the function has no discontinuity.

Figure 1 reveals some differences among the curves. For $\tau < 2$ months, three of the curves agree rather closely. For $\tau > 2$ months the envelope of curves is bounded above by the Cox et al. (1979) (150) curve (the CRR curve) and below by the equation (29) curve. The equation (23) curve is intermediate, lying below both the CRR curves for $\tau > 2$ months, with divergence increasing in τ. Note, though, that the CRR curves move *toward* the equation (23) curve as the number of steps increases.

Detailed numerical values for three specific cases (examples from Cox and Rubinstein (1985, p. 249)) are given in Table I. Table I includes values from Cox and Rubinstein (1985) and from the Cox et al. (1979) method with 2,000 steps. The Cox and Rubinstein values are all too high, and the value of 36 would seem to be in error. The Cox et al. values all decrease as the steps increase. There are "crossovers" between our curves and the CRR curves, with equation (29) giving the biggest differences.

An alternative approach to evaluating the accuracy of the critical stock prices is to compare the accuracy of put values obtained using procedures that directly incorporate the critical-stock-price function. For example, either equation (1) or equation (21) can be used to compute put values, and if the numerical integration and all other calculations are performed accurately, then any remaining errors would presumably be due to inaccuracies in the critical-stock-price functions.

Table I

Comparison of Selected $S_c(\tau)$ Values.

This table compares values for the critical stock price determined from equation (22) combined with equation (23), and also with equation (29), to those in Cox and Rubinstein (1985) and to values using the Cox et al. (1979) method with 150, 800, and 2,000 time steps. The continuous interest rate, r, is 0.0488, and the corresponding discrete rate of 0.05 is used for the Cox and Rubinstein and Cox et al. methods. The exercise price is $40 in all cases. The time to maturity, T, is in months. The standard deviation of the rate of return on the stock is denoted by σ.

Method	$\sigma = 0.2$, $T = 9$	$\sigma = 0.3$, $T = 1$	$\sigma = 0.3$, $T = 9$
Cox and Rubinstein (1985)	36.00	34.28	28.80
Cox et al. (1979) (150 steps)	33.06	34.17	28.75
Cox et al. (1979) (800 steps)	32.95	34.11	28.59
Cox et al. (1979) (2,000 steps)	32.91	34.09	28.55
Equations (22) and (23)	32.59	34.11	28.17
Equations (22) and (29)	33.78	33.69	27.42

We investigated a variety of ways to use equation (1) for valuing finite-lived puts. Difficulties arise because, in addition to a critical-stock-price function, equation (1) also requires an expression for $f(t)$, and, as previously noted, an exact expression is not readily available. One obvious possibility is to use the tangent approximation from equation (18). We determined that this approach does not have the desired accuracy for our purposes. When time to expiration is large, the tangent approximation can be quite good. Indeed, this follows from our results in Section I, because it is clear that the boundary function $a(t)$ will be nearly linear over this range, and the tangent approximation will be very close to the true density function. However, when time to expiration is quite short, the boundary function $a(t)$ can exhibit a relatively high degree of curvature. The degree of difficulty this creates will depend on the particular features of the put being evaluated.

The practical alternative is equation (21), which only requires numerical integration of dimension two. Furthermore, adopting this approach places us within the same framework used by Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992) and Huang, Subrahmanyam, and Yu (1996), with the latter providing a natural basis for comparison of numerical results.

Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992) developed equation (21) as part of an overall procedure in which S_c must be determined first. However, actually computing S_c was regarded to be burdensome, requiring the solution of an integral equation. Huang, Subrahmanyam, and Yu (1996) reduced computation time by using a sequence of approximations to S_c, combined with Richardson extrapolation, to compute put values (as in Geske and Johnson (1984) and Bunch and Johnson (1992)) and also their associated derivatives. The Huang, Subrahmanyam, and Yu (1996) procedure is equivalent to the Geske and Johnson method with equation (21) in place of the infinite series of multivariate normal integrals. Because our S_c's are analytic expressions, we can use either numerical integration of equation (21) or extrapolation to compute put values.

Table II

American Put Values

This table provides American put values computed seven different ways: the binomial method with 150 and 10,000 time steps; numerical integration using equations (21), (22), and (23); integration using equation (29) instead of equation (23); the Huang, Subrahmanyam, and Yu (1996) four-point extrapolation using equations (23) and (29); and the Huang, Subrahmanyam, and Yu (1996) recursive four-point extrapolation. The interest rate is 0.0488 (0.05) in continuous (discrete) terms, and the stock price is $40. The exercise price, standard deviation, and time to maturity in months are denoted as X, σ, and T, respectively. Values for the Geske and Johnson (1984), accelerated binomial, finite-difference, and recursive four-point methods are those reported in Table 1 of Huang, Subrahmanyam, and Yu (1996).

	$\sigma = 0.2$			$\sigma = 0.3$			$\sigma = 0.4$		
Method	$T = 1$	$T = 4$	$T = 7$	$T = 1$	$T = 4$	$T = 7$	$T = 1$	$T = 4$	$T = 7$
Panel A: $X = 35$									
Binomial, $N = 150$	0.0061	0.1995	0.4340	0.0775	0.6993	1.2239	0.2454	1.3505	2.1602
Binomial, $N = 10{,}000$	0.0062	0.2004	0.4328	0.0774	0.6975	1.2198	0.2466	1.3460	2.1549
Numerical integration using equation (23)	0.0062	0.2005	0.4332	0.0774	0.6978	1.2203	0.2466	1.3465	2.1555
Integration using equation (29)	0.0062	0.2002	0.4326	0.0774	0.6966	1.2169	0.2465	1.3442	2.1494
Four-point extrapolation using equation (23)	0.0062	0.1999	0.4313	0.0773	0.6964	1.2172	0.2464	1.3445	2.1518
Four-point extrapolation using equation (29)	0.0062	0.2006	0.4361	0.0773	0.6961	1.2173	0.2463	1.3425	2.1457
Huang, Subrahmanyam, and Yu (1996) four-point extrapolation	0.0062	0.2004	0.4337	0.0775	0.6973	1.2233	0.2467	1.3468	2.1603
Panel B: $X = 40$									
Binomial, $N = 150$	0.8512	1.5783	1.9886	1.3083	2.4799	3.1665	1.7658	3.3836	4.3480
Binomial, $N = 10{,}000$	0.8522	1.5798	1.9904	1.3099	2.4825	3.1696	1.7681	3.3874	4.3526
Numerical integration using equation (23)	0.8528	1.5813	1.9915	1.3104	2.4837	3.1705	1.7686	3.3886	4.3539
Integration using equation (29)	0.8518	1.5788	1.9914	1.3092	2.4776	3.1591	1.7672	3.3811	4.3380
Four-point extrapolation using equation (23)	0.8523	1.5807	1.9905	1.3096	2.4825	3.1695	1.7677	3.3868	4.3523
Four-point extrapolation using equation (29)	0.8526	1.5893	2.0172	1.3091	2.4816	3.1697	1.7667	3.3801	4.3376
Huang, Subrahmanyam, and Yu (1996) four-point extrapolation	0.8543	1.5873	1.9987	1.3116	2.4919	3.1842	1.7694	3.3970	4.3699
Panel C: $X = 45$									
Binomial, $N = 150$	5.0000	5.0886	5.2677	5.0600	5.7065	6.2448	5.2875	6.5103	7.3897
Binomial, $N = 10{,}000$	5.0000	5.0883	5.2670	5.0597	5.7056	6.2436	5.2868	6.5099	7.3830
Numerical integration using equation (23)	5.0020	5.0911	5.2647	5.0621	5.7077	6.2438	5.2885	6.5120	7.3846
Integration using equation (29)	4.9983	5.0895	5.2859	5.0533	5.6886	6.2151	5.2819	6.4927	7.3507
Four-point extrapolation using equation (23)	5.0014	5.0926	5.2607	5.0615	5.7031	6.2369	5.2876	6.5112	7.3829
Four-point extrapolation using equation (29)	5.0022	5.1447	5.4048	5.0584	5.7014	6.2394	5.2841	6.4982	7.3581
Huang, Subrahmanyam, and Yu (1996) four-point extrapolation	5.0020	5.0594	5.2631	5.0604	5.6970	6.2303	5.2853	6.5128	7.3865

In what follows, we compute values that can be directly compared to those used by Huang, Subrahmanyam, and Yu (1996, Table 1, p. 292). Table II contains computed prices for a set of 27 American puts. The benchmark for accurate put values is the binomial method with 10,000 steps (reported in Huang, Subrahmanyam, and Yu (1996, Table 1, column 4) and the second row of each panel of our Table II). Rows 1 and 7 contain results from the binomial method with 150 steps and the Huang, Subrahmanyam, and Yu (recursive four-point extrapolation) values, respectively. Rows 3 through 6 contain values obtained through the four possible combinations of equations (23) or (29) with numerical integration or four-point extrapolation. We compute our own results for the binomial-150 method to provide numerical benchmarks to allow comparisons to Table 1 of Huang, Subrahmanyam, and Yu (1996) (through appropriate standardization).[1] Table III reports accuracy and CPU-time measurements for the methods from Table II, as well as for the Geske and Johnson (1984), accelerated binomial, and finite-difference methods. (The results from the latter methods are taken from Huang, Subrahmanyam, and Yu (1996), which also gives the full set of put values for these methods.)

First, consider the accuracy of methods based on our critical-stock-price functions. The method using equation (23) with numerical integration produces accurate put values, with root-mean-square error (RMSE) of 0.13 cents. The largest absolute error in the set of 27 puts is 28/100 of a cent. The "RMSE-ratio" = RMSE (method being evaluated)/RMSE (binomial-150) is 0.50, indicating that the method has "twice the accuracy" of the binomial-150 method. In fact, the two methods based on equation (23) are the two with the smallest errors. Methods based on equation (29) have less error than the accelerated binomial and finite-difference methods but have larger error than the Geske and Johnson (1984) and recursive four-point methods.

We next compare the methods on the "accuracy/speed" trade-off. Equation (23) using the numerical integration method (row 6) has the least error but is slower than all the others except the finite-difference method. However, we see that extrapolation can be used to speed up the method substantially without losing too much accuracy. Equation (23) using the four-point extrapolation method (row 7) is faster *and* has less error than all the remaining methods (excluding equation (23)/integration). In particular, the method is both "twice as fast" and "twice as accurate" (according to these measures) as the recursive four-point method.

The methods based on equation (29) are faster but have more error, as expected. For example, equation (29) with four-point extrapolation is extremely fast (nearly 100 times faster than the binomial-150 method and 10 times faster than the recursive four-point) but has a much larger overall error (RMSE ratio = 11.34). However, it is worth noting that the method is

[1] Numerical results were obtained using Fortran 90 on a DEC Alpha workstation. Equation (23) was solved using a safeguarded Newton method. Numerical integration was performed using the IMSL subroutine DQDAGS. Four-point extrapolation to approximate equation (21) was performed as described in Huang, Subrahmanyam, and Yu (1996).

Table III

RMSE and CPU Time for American Put Prices and Hedge Ratios

Using the 27 put values from Table II and the hedge ratios from Table IV, Panel A gives the root-mean-square error (RMSE) over the 27 puts, the ratio of RMSE for a particular method to the RMSE for the binomial method with 150 time steps, the total CPU time to compute all 27 puts (in seconds), and the ratio of CPU time for a particular method to the CPU time for the binomial-150 method. Panel B provides delta results corresponding to the price results in Panel A and Table II. Values for the Geske and Johnson (1984), accelerated binomial, finite-difference, and recursive four-point methods are those reported in Tables 1 and 2 of Huang, Subrahmanyam, and Yu (1996). (CPU times taken from Tables 1 and 2 of Huang, Subrahmanyam, and Yu (1996) have been adjusted using binomial-150 time as a benchmark.)

Panel A: Put Values

Method	RMSE	RMSE Ratio	CPU Time	CPU Ratio
Binomial-150	2.6380E-03	1.00	1.03E-01	1.00
Geske and Johnson (1984)	5.3383E-03	2.02	—	—
Accelerated binomial	3.5302E-02	13.38	6.85E-02	0.67
Finite difference	4.1041E-02	15.56	5.65E-01	5.50
Recursive four-point	6.7047E-03	2.54	1.17E-02	0.11
Equation (23)/integration	1.3235E-03	0.50	1.75E-01	1.71
Equation (23)/four-point	2.2850E-03	0.87	6.58E-03	0.06
Equation (29)/integration	1.1050E-02	4.19	4.67E-02	0.45
Equation (29)/four-point	2.9920E-02	11.34	1.21E-03	0.01

Panel B: Hedge Ratios

Method	RMSE	RMSE Ratio		CPU Ratio
Binomial-150	1.0593E-03	1.00		1.00
Geske and Johnson (1984)	2.5289E-03	2.39		—
Accelerated binomial	4.2777E-02	40.38		0.66
Finite difference	4.2216E-02	39.85		1.83
Recursive four-point	5.1702E-03	4.88		0.04
Equation (23)/four-point	1.9262E-03	1.82		0.02

still quite accurate for many of the puts in the table and most of the error is associated with two of the more difficult deep-in-the-money puts. Simple decision rules could be developed that would use this very fast approach for all but the most difficult of American puts.

Equation (23) using the four-point extrapolation has an additional attractive feature, in that it may also be used in a straightforward way to compute various risk-management parameters (e.g., delta, gamma, vega, theta) using the method described in Huang, Subrahmanyam, and Yu (1996): We use their method except that we use equations (22) and (23) to get the critical-stock-price function. As an example, we include the results for the hedge ratios (deltas) corresponding to those in Huang, Subrahmanyam, and Yu (1996, Table 2). See Panel B of Table III and Table IV for delta results corresponding to the price results in Table II and Panel A of Table III. The hedge ratios from the equation (23)/four-point method (row 3 of each panel of Table IV) have slightly larger errors than those from the binomial-150

Table IV

Hedge Ratio (Delta) Values

This table provides hedge ratios (deltas) computed four different ways: the binomial method with 150 and 10,000 time steps; the Huang, Subrahmanyam, and Yu (1996) recursive four-point extrapolation; and the Huang, Subrahmanyam, and Yu (1996) four-point extrapolation using equation (23). The interest rate is 0.0488 (0.05) in continuous (discrete) terms; the stock price is 40; and X, σ, and T are the exercise price, standard deviation, and time to maturity in months, respectively. Values for the recursive four-point method are those reported in Table 2 of Huang, Subrahmanyam, and Yu (1996).

	$\sigma = 0.2$			$\sigma = 0.3$			$\sigma = 0.4$		
Method	$T = 1$	$T = 4$	$T = 7$	$T = 1$	$T = 4$	$T = 7$	$T = 1$	$T = 4$	$T=7$
Panel A: $X = 35$									
Binomial, $N = 150$	−0.0081	−0.0911	−0.1355	−0.0526	−0.1758	−0.2144	−0.1073	−0.2277	−0.2557
Binomial, $N = 10{,}000$	−0.0080	−0.0901	−0.1338	−0.0516	−0.1741	−0.2126	−0.1062	−0.2260	−0.2539
Four-point extrapolation using equation (23)	−0.0080	−0.0898	−0.1335	−0.0515	−0.1738	−0.2125	−0.1062	−0.2257	−0.2539
Huang, Subrahmanyam, and Yu (1996) four-point extrapolation	−0.0080	−0.0900	−0.1351	−0.0516	−0.1744	−0.2145	−0.1062	−0.2265	−0.2558
Panel B: $X = 40$									
Binomial, $N = 150$	−0.4698	−0.4444	−0.4299	−0.4699	−0.4429	−0.4268	−0.4673	−0.4370	−0.4186
Binomial, $N = 10{,}000$	−0.4693	−0.4435	−0.4287	−0.4695	−0.4420	−0.4256	−0.4668	−0.4360	−0.4173
Numerical integration using equation (23)	−0.4703	−0.4442	−0.4280	−0.4699	−0.4429	−0.4262	−0.4670	−0.4368	−0.4182
Huang, Subrahmanyam, and Yu (1996) four-point extrapolation	−0.4712	−0.4416	−0.4228	−0.4710	−0.4429	−0.4243	−0.4679	−0.4376	−0.4177
Panel C: $X = 45$									
Binomial, $N = 150$	−1.0000	−0.8803	−0.7944	−0.9220	−0.7260	−0.6520	−0.8350	−0.6474	−0.5819
Binomial, $N = 10{,}000$	−1.0000	−0.8812	−0.7948	−0.9233	−0.7266	−0.6520	−0.8363	−0.6475	−0.5819
Numerical integration using equation (23)	−1.0004	−0.8878	−0.8002	−0.9261	−0.7248	−0.6490	−0.8357	−0.6468	−0.5806
Huang, Subrahmanyam, and Yu (1996) four-point extrapolation	−0.9993	−0.8964	−0.8112	−0.9293	−0.7229	−0.6459	−0.8341	−0.6424	−0.5753

method (row 1) but are still twice as good as those from the recursive four-point method (row 4). However, the method is now even faster relative to the binomial-150 method because the equations are analytic.

IV. Discussion

We have used the first-passage probability approach and the formula from Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992) to derive exact expressions for the critical-stock-price function and for the American put price for the perpetual and finite-lived cases. For small times to maturity we have derived a simple approximation for the critical-stock-price function that can be used with the extrapolation method of Huang, Subrahmanyam, and Yu (1996) to yield a method for valuing puts that is very fast and has small errors.

A key element of our derivation is our use of the fact that exercise occurs when the interest-rate effect is exactly offset by the volatility effect. Analytically, this means that the derivative of the put price with respect to time to maturity is zero at the point of early exercise. Our fundamental result, equation (22), yields easily computed functional forms with important implications; for example, exercises of in-the-money puts should increase dramatically as maturity nears. Although this behavior might be generally expected given the collective experience of researchers, behavior near maturity has generally been crudely approximated by exponential forms or by *ad hoc* jump corrections. Furthermore, the availability of our functional forms makes it easier to test a variety of implications using empirical data.

Our method can be applied to currency options and options on futures; to exotic options, such as barrier options and Asian options; and to corporate securities and other contracts modeled as options. Dividends can be handled as in Geske and Johnson (1984), or, in the case of index options, the continuous-dividend assumption can be used.

For pedagogical purposes, equations (21) and (22) can be used to explain the American put. One can either add equation (29) or just note that g is typically about 1.5.

Extending our results could be a fruitful area for future research. First, critical-stock-price functions could be derived for other derivative securities. Second, a better approximation might be found to replace equation (29). Third, alternative first-passage probability equations might be developed that would lead to additional insights and better computational procedures.

Appendix

In this Appendix we present details of the derivations omitted in the text.

Deriving equation (16):

$$P = (X - S_c)\left(\frac{S_c}{S}\right)^{\gamma}.$$

Equation (6) becomes

$$P = \frac{X - S_c}{\sqrt{2\pi}} \exp\left[-\left(r - \frac{1}{2}\sigma^2\right)\log(S/S_c)/\sigma^2\right] \int_0^{\infty} \frac{\log S/S_c}{\sigma t^{3/2}} e^{-st} e^{-k^2/4t} dt, \tag{A1}$$

where

$$s = r + \frac{(r - \frac{1}{2}\sigma^2)^2}{2\sigma^2} = \frac{(r + \frac{1}{2}\sigma^2)^2}{2\sigma^2} \quad \text{and} \quad k^2 = \frac{2(\log S/S_c)^2}{\sigma^2}.$$

Simplifying, using the definite integral from equation (8), and simplifying again, we get equation (16).

Deriving equation (19):

$$S_c/X = \frac{\gamma}{1+\gamma} + \frac{e^{-\beta_2\tau - \alpha_1\sqrt{\tau}}}{1+\gamma}.$$

We solve for S_c by using the tangent approximation for f and substituting equation (1) into equation (11) to obtain a second-order ordinary differential equation for S_c:

$$t\frac{\ddot{S}_c}{S_c} + (\beta_1 - 1)t\left(\frac{\dot{S}_c}{S_c}\right)^2 + \left[\beta_1 \log\frac{S}{S_c} + \beta_2 t\right]\frac{\dot{S}_c}{S_c} + \beta_2 \log\frac{S}{S_c} = 0, \tag{A2}$$

where

$$\beta_1 = \frac{a}{\sigma t} - \frac{S_c}{X - S_c},$$

and

$$\beta_2 = \frac{\sigma a}{2t} + \frac{a^2}{2t^2} - r - \frac{3}{2t} - \frac{ar}{\sigma t}.$$

When t is large and S is chosen to be close to S_c, β_2 reduces to

$$\beta_2 = -\frac{(r + \frac{1}{2}\sigma^2)^2}{2\sigma^2} \equiv -\beta,$$

a constant (cf. the expression for s below equation (A1)).[2] For $\tau \equiv T - t$ very large, S_c must approach $\gamma X/(1+\gamma)$, and $\dot{S}_c$ becomes small. Neglecting small terms in equation (A2) gives

$$\ddot{S}_c + \beta_2 \dot{S}_c = 0.$$

The solution is

$$S_c/X = \frac{\gamma}{1+\gamma} + ce^{-\beta\tau}, \tag{A3}$$

where c is a positive constant. If $\tau \to 0$, then $S_c \to$ X and $\beta_1 \to -\infty$, and equation (A2) reduces to

$$\ddot{S}_c = \frac{\dot{S}_c^2}{X - S_c}$$

which is solved by

$$S_c/X = e^{-\alpha_1\sqrt{\tau}} \tag{A4}$$

for any $\alpha_1 > 0$. In general, this approach indicates that the solution should have the form given in equation (19).

Deriving equations (22) and (23):

$$\frac{S_c}{X} = e^{-(r+(1/2)\sigma^2)\tau - g\sigma\sqrt{\tau}},$$

$$g = \pm\sqrt{2\log\frac{\sigma^2}{\frac{2r}{\sqrt{\alpha}}x\log xe^{-\alpha(r+(1/2)\sigma^2)^2\tau/(2\sigma^2)}}}.$$

Taking the derivative of equation (21) with respect to τ, setting it equal to zero, canceling terms, using the identity $SN'(d_1) = Xe^{-rt}N'(d_2)$ and setting $S = S_c$, we get

$$-\int_0^T \frac{rX}{\sigma\sqrt{t}} e^{-rt} N'(d_2(S_c, S_c, t)) \frac{\dot{S}_c}{S_c} dt + \frac{\sigma}{2\sqrt{T}} S_c N'(d_1) = 0.$$

[2] The critical stock price should not depend on S. This gives us some latitude, and so we choose S such that the final expression for S_c will not depend on S.

The first term can be interpreted as the negative of the interest-rate effect and the second as the volatility effect. Exercise does not ordinarily occur as soon as the put gets into the money, for the volatility effect exceeds the interest-rate effect. As the put gets deeper into the money, the put begins to lose its option character. It begins to resemble a forward contract, and the volatility effect declines. Exercise occurs when these two effects become equal. Solving for $N'(d_1)$ and then for d_1 we get

$$S_c/X = e^{-(r+(1/2)\sigma^2)\tau - g\sigma\sqrt{\tau}}, \tag{A5}$$

where

$$g = \pm\sqrt{\log\frac{\sigma^4}{8\pi r^2 J^2}}, \tag{A6}$$

and

$$J = \frac{X}{S_c}\int_0^T \frac{e^{-rt}}{\sqrt{\dfrac{t}{T}}} N'(d_2(S_c, S_c, t))\,\frac{\dot{S}_c}{S_c}\,dt. \tag{A7}$$

To evaluate the integral, we pull all the factors but the last one out of the integral and evaluate them at αT, where $0 \le \alpha \le 1$. The remaining expression can be integrated to get equations (22) and (23).

Deriving equations (27) and (28):

$$\alpha = 1 - \frac{A}{1 + \dfrac{(1+\gamma)^2}{4}\sigma^2\tau},$$

$$A = \frac{1}{2}\left(\frac{\gamma}{1+\gamma}\right)^2.$$

For large τ equations (23) and (24) both imply that g goes as $\sqrt{\tau}$. Hence, for equations (23) and (24) to agree, at very large τ we must have $\alpha = 1$. We can derive an approximation for α by equating the squares of the right-hand sides of equations (23) and (24). We obtain

$$(\alpha - 1)\frac{(1+\gamma)^2}{4}\sigma^2\tau + \log\alpha - \frac{L^2}{\sigma^2\tau} + (1+\gamma)L - \log(1+\gamma)^2 - \log L^2 = 0 \tag{A8}$$

where x is taken to be its asymptotic value, $(1 + \gamma)/\gamma$, and $L \equiv \log[(1 + \gamma)/\gamma]$. For large τ the first term dominates, and so α must approach one. But then $\log \alpha$ approaches $\alpha - 1$, so we can solve equation (A8) to get

$$\alpha = 1 - \frac{A}{1 + \frac{(1+\gamma)^2}{4}\sigma^2\tau}, \tag{A9}$$

where

$$A \equiv (1+\gamma)L - \log(1+\gamma)^2 - \log L^2 - \frac{L^2}{\sigma^2\tau}.$$

When τ is not large we can derive an approximation for α by equating the right-hand sides of equations (25) and (26). We obtain

$$\alpha = \frac{4}{1+\gamma}\left[1 - \frac{1}{L}\log\frac{\sqrt{\alpha}}{\gamma L}\right]. \tag{A10}$$

Equation (A10) must be solved iteratively. A good initial value for the iteration (and a good approximation for α) is obtained by using equation (A9) with

$$A = \frac{1}{2}\left(\frac{\gamma}{1+\gamma}\right)^2, \tag{A11}$$

which is equation (28).

The intuition for this approximation is as follows. For very small time intervals, the integral in J (see equation (A7)) can be approximated by pulling factors out of the integral and evaluating them at the midpoint (i.e., $\alpha = 0.5$), provided the factor remaining in the integral is well behaved. However, $\dot{S}_c$ approaches infinity as $\tau \to 0$. For very small γ, $\dot{S}_c$ strongly resembles a delta function, in which case the integrand should be valued at the endpoint (i.e., $\alpha = 1$).

Deriving equation (29):

$$g = \sqrt{\log\frac{\sigma^2}{4er^2\tau/\alpha}}.$$

For very small τ we can use the approximation

$$x \log x = x - 1$$

and equation (22) to reduce equation (23) to

$$g = \sqrt{\log \frac{\sigma^2}{4r^2\tau g^2/\alpha}}. \tag{A12}$$

Using

$$\log g^2 = 1 - \frac{1}{g^2} + R,$$

where R is the remainder term, we can rewrite equation (A12) as a quadratic equation for g^2, which is solved to yield

$$g = \sqrt{b + \sqrt{b^2 + 1}},$$

where

$$b = \frac{1}{2} \log \frac{\sigma^2}{4e^{1+R}r^2\tau/\alpha}.$$

For small τ, $R > 0$ and $b > 1$. If we ignore R and the constant in the inner radical, the errors are offsetting, and we get an approximation that requires no iteration:

$$g = \sqrt{\log \frac{\sigma^2}{4er^2\tau/\alpha}}. \tag{A13}$$

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