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The Stochastic Volatility of Short-Term Interest Rates: Some International Evidence

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ABSTRACT

This paper estimates a stochastic volatility model of short-term riskless interest rate dynamics. Estimated interest rate dynamics are broadly similar across a number of countries and reliable evidence of stochastic volatility is found throughout. In contrast to stock returns, interest rate volatility exhibits faster mean-reverting behavior and innovations in interest rate volatility are negligibly correlated with innovations in interest rates. The less persistent behavior of interest rate volatility reflects the fact that interest rate dynamics are impacted by transient economic shocks such as central bank announcements and other macroeconomic news.

THE PAST DECADE HAS WITNESSED UNPRECEDENTED GROWTH in the market for fixed-income derivatives, domestically as well as internationally. In light of this growth, recent research has attempted to empirically characterize the dynamics of short-term riskless interest rates necessary for the valuation of many of these securities. With few exceptions much of this evidence has been concerned with the behavior of U.S. interest rates. The purpose of this paper is to investigate the dynamics of short-term interest rates across a number of countries.

A particularly noteworthy feature, at least of the U.S. evidence, is that the volatility of short-term interest rates is itself volatile. The stochastic nature of volatility is an important property of interest rate dynamics which we wish to focus on in our comparison. To do so practically, this paper implements a computationally efficient and statistically reliable method for estimating stochastic volatility models in finance. We apply this method to Chan et al.'s (1992) specification of short-term riskless interest rate dynamics augmented with stochastic volatility.

For a clean international comparison, we use proxies for short-term riskless interest rates drawn from the *same* market, the London interbank market, over the *same* sample period. Unlike Nowman (1997), we find that the

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dynamics of one-month Euro-dollar, Euro-mark, Euro-sterling, and Euro-yen rates are broadly similar and are all characterized by stochastic interest rate volatility. In contrast to estimated stock return dynamics, stochastic interest rate volatility is distinguished by a faster mean-reverting behavior. Furthermore, innovations in interest rates are only negligibly correlated with innovations in interest rate volatility, which also differs from the evidence on stock returns where a reliably negative relation prevails between returns and volatility. Finally, for all of the sampled series we confirm Chan et al.'s (1992) claim that interest rate volatility is sensitive to the level of interest rates, though this relation is difficult to estimate precisely.

Our analysis provides additional economic insight into the persistence of financial market volatility. The differences documented between the stochastic volatility dynamics of short-term interest rates and stock returns suggest that different economic shocks impact these assets and that return shocks vary in persistence depending on their source. For example, the faster mean-reverting behavior of stochastic interest rate volatility is consistent with short-term interest rates being impacted by transient and less persistent economic shocks. We find that these shocks are dominated by central bank announcements of base rate changes in the Japanese and U.K. data and by macroeconomic announcements in the U.S. data. Short-term interest rates respond significantly yet quickly to these unanticipated announcements. This conclusion is similar to the findings of Jones, Lamont, and Lumsdaine (1998) who document that volatility shocks to U.S. Treasury bonds arising from scheduled macroeconomic announcements are not persistent at all. The transient nature of these economic shocks has direct implications for modeling the conditional heteroskedasticity of short-term interest rates. For an EGARCH specification we find that an extremely fat-tailed t-distributed innovations process is needed to accommodate these transient shocks and so make the EGARCH model statistically indistinguishable from the stochastic volatility model. Taken together, our results point to the central role played by underlying economic shocks in determining the dynamics of financial market volatility.

The plan of this paper is as follows. Section I summarizes the short-term interest rate series. Section II puts forward the stochastic volatility model of short-term riskless interest rate dynamics and investigates its statistical estimation. We use a filtering framework in which volatility is treated as a latent variable. Since volatility enters the interest rate model in a multiplicative fashion, we explicitly take into account the resultant nonnormality of the measurement error which in quasi-maximum likelihood (QML) estimation is treated as if it were normal. Section III presents our empirical results. Section IV compares our empirical results to those obtained for an EGARCH specification of interest rate volatility and documents the nature of the economic shocks impacting short-term riskless interest rate dynamics. Section V concludes the paper.

I. Data Description

The short-term riskless rate of interest is proxied by a one-month interest rate. We use a variety of one-month rates. Duffee (1996) argues that because of the increased idiosyncratic variation of U.S. Treasury yields since the early 1980s, we should not rely exclusively on Treasury yields in calibrating models of short-term interest rate dynamics.

One-month London Euro-currency rates: We provide an international comparison of short-term interest rate dynamics by using a number of one-month rates drawn from the London Euro-currency market. Each of these rates is a London interbank rate denominated in a given currency. In particular, we use weekly observations of one-month Euro-dollar, Euro-mark, Euro-sterling, and Euro-yen (middle) rates taken from Datastream[1] over the sample period January 1986 to December 1995, giving 522 observations for each series.

One-month U.K. interbank rates: Following Nowman (1997), the short-term riskless rate of interest in the U.K. is proxied by the one-month U.K. interbank (middle) rate. Weekly observations over the sample period January 1975 to September 1995, for a total of 1079 observations, are also taken from Datastream.[2] This sample period corresponds roughly to Nowman's sample period of March 1975 to March 1995.

One-month U.S. Treasury bill yields: One-month U.S. Treasury bill yields (the average of the bid and ask) are taken from the risk-free rate file of the Center for Research in Security Prices (CRSP). The sample period is June 1964 to December 1989, for a total of 307 observations, and corresponds to the sample period used by Chan et al. (1992).[3]

II. A Stochastic Volatility Model of Short-Term Riskless Interest Rates

It is by now acknowledged that the volatility of short-term interest rate changes is itself volatile. The stochastic nature of interest rate volatility is suggested by Dybvig (1995) and incorporated by Longstaff and Schwartz (1992) in their two-factor equilibrium term structure model. Empirical verification of stochastic interest rate volatility for the case of U.S. Treasury bill yields is provided by Andersen and Lund (1997), and in discrete-time GARCH extensions of Chan et al. (1992) by Brenner, Harjes, and Kroner (1996) and Koedijk et al. (1997).

[1] Datastream files ECUS$1M, ECWGM1M, ECUK£1M, and ECJAP1M, respectively. We use Wednesday-to-Wednesday observations.

[2] Datastream file LDNIB1M. We use Wednesday-to-Wednesday observations. These data are unavailable prior to 1975.

[3] Chan et al. proxy the short-term riskless rate of interest by the one-month Treasury yield (the average of the bid and ask) reported in CRSP's Fama 12-month Treasury bill term structure file. See Duffee (1996) for a discussion of the problems associated with this and other one-month Treasury yield series as proxies for the instantaneous default-free rate.

We incorporate stochastic interest rate volatility into Chan et al.'s proposed discrete-time model[4] of short-term riskless interest rate dynamics:

$$r_t = (a + br_{t-1}) + \sigma_{t-1} r_{t-1}^{\gamma} \epsilon_{1,t} \tag{1}$$

$$\ln \sigma_t^2 - \mu = \beta(\ln \sigma_{t-1}^2 - \mu) + \xi \epsilon_{2,t} \tag{2}$$

with $\epsilon_{1,t}$ and $\epsilon_{2,t}$ being i.i.d. standard normals. The parameters a and b characterize the interest rate's linear drift component[5] and the parameter γ allows the volatility of r to depend on its level. Unlike Chan et al., the assumed unobservable interest rate volatility, σ_t, evolves stochastically. In particular, the dynamics of $\ln \sigma_t^2$ are given by an $AR(1)$ specification with the speed of adjustment to its unconditional mean μ governed by the parameter β and its volatility characterized by the parameter ξ.

We estimate the drift parameters a and b by OLS; these estimators are consistent even when volatility is stochastic. Letting $x_t \equiv \ln \sigma_t^2$ and $res_t \equiv r_t - (\hat{a} + \hat{b} r_{t-1})$, we take the logarithm of the squared observations and consider the following equivalent discrete-time specification:

$$y_t \equiv \ln(res_t^2) = x_{t-1} + 2\gamma \ln(r_{t-1}) + \ln(\epsilon_{1,t}^2) \tag{3}$$

$$x_t - \mu = \beta(x_{t-1} - \mu) + \xi \epsilon_{2,t}. \tag{4}$$

By squaring observations, however, we lose all information about dependence between ϵ_1 and ϵ_2. This entails no loss of statistical efficiency if these error terms are uncorrelated, and indeed our empirical analysis confirms that the correlation coefficient equals approximately zero across our sam-

[4] Dassios (1993) establishes that the density of the discretized variance process converges to the density of the continuous-time variance process at rate h, where h denotes the equally spaced interval between observations. In the case of deterministic interest rate volatility, Nowman (1997) presents evidence that the bias arising from using a discrete-time approximation is very small if interest rate data are sampled sufficiently often. Shoji and Ozaki (1997) also come to the same conclusion when investigating the small sample performance of various procedures for estimating continuous-time stochastic processes from discrete observations.

[5] Although identifying the appropriate drift specification is beyond the scope of this article, we did attempt to assess whether our stochastic volatility estimates are sensitive to the assumed drift specification. Our evidence suggests that these estimates are robust to different drift specifications. For example, Aït-Sahalia (1996) suggests, at least in the deterministic volatility case, that a nonlinear drift of the form $a + br_t + cr_t^2 + d/r_t$ is an empirically more appropriate specification. Assuming this nonlinear drift, we obtain the following stochastic volatility parameter estimates (with asymptotic standard errors in parentheses) for the geometrically scaled U.K. interbank rate data: $\mu = -7.948$ (0.137), $\beta = 0.712$ (0.046), $\xi = 1.230$ (0.099), and $\gamma = 0.568$ (0.212). Notice that there is little difference between these estimates and the corresponding estimates in Table I based on the linear drift specification.

pled interest rate series.[6] That is, upward shocks to interest rates are as likely as downward shocks to increase interest rate volatility. This is in contrast to stock returns where the empirical evidence is consistent with a reliably negative relation since downward shocks to stock returns tend to be associated with increased stock volatility (see, e.g., Nelson (1991)).

Expressions (3) and (4) give a linear time series model but with non-Gaussian errors since $\ln(\epsilon_1^2)$ is $\log \chi^2$ distributed under the maintained assumption that ϵ_1 is normally distributed. Treating $\ln(\epsilon_1^2)$ as if it were normally distributed, we can straightforwardly apply the Kalman filter to estimate the parameters of this model using quasi-maximum likelihood (QML) (Harvey, Ruiz, and Shephard (1994)). However, this approximate filter is adversely affected by outlying observations in the $\log \chi^2$ distribution's skewed left tail which are highly irregular under the normal approximation. Kim, Shephard, and Chib (1998) approximate the $\log \chi^2$ density by a mixture of seven normal densities, but the approximation is computed once and does not rely on the data. Alternatively, to improve statistical performance while maintaining the tractability of QML, we now turn our attention to an alternative filtering procedure that *explicitly* incorporates this nonnormality.

A. Practical Integration-Based Filtering of Non-Gaussian Time Series

Given a set of T observations $y_1, \ldots, y_T$, the likelihood of the parameter vector $\psi \equiv \{\mu, \beta, \xi, \gamma\}$ can be expressed as

$$\mathcal{L}(\psi) = \prod_{t=1}^{T} p(y_t | Y_{t-1}), \tag{5}$$

where $Y_t \equiv \{y_t, y_{t-1}, \ldots, y_1\}$ denotes the history of the observable through time t and $p(y_t|Y_{t-1})$ denotes the conditional density of y_t given the history of the process through time $t-1$.

This likelihood function may be evaluated using an iterative filtering procedure. Given a prior on the state $p(x_{t-1}|Y_{t-1})$, there are three stages to this procedure: a projection to obtain $p(x_t|Y_{t-1})$, followed by an integration to calculate the conditional likelihood, and finally an updating to obtain $p(x_t|Y_t)$. Proceeding sequentially in this manner we may calculate $\mathcal{L}$. The maximum likelihood estimator $\hat{\psi}$ is found by maximizing this function with respect to ψ.

[6] To account for this correlation a computationally intensive nonlinear filter using numerical double integration is required; see, for example Kitagawa (1987). We implement this nonlinear filter for our sampled interest rate series and obtain the following estimates of this correlation coefficient ρ (with asymptotic standard errors in parentheses): $\hat{\rho} = 0.163$ (0.082) for the Euro-mark series, $\hat{\rho} = 0.224$ (0.091) for the Euro-yen series, $\hat{\rho} = -0.029$ (0.103) for the Euro-sterling series, $\hat{\rho} = 0.109$ (0.086) for the Euro-dollar series, and $\hat{\rho} = 0.043$ (0.062) for the U.K. interbank series. Convergence is not achieved for the monthly U.S. Treasury yield series, perhaps because of the relatively small number of observations.

For a linear state space model with normal errors the Kalman filter may be applied and the first two moments of $p(x_t|Y_t)$ are obtained from the first two moments of $p(x_{t-1}|Y_{t-1})$ by simple matrix calculations (see Harvey (1989)). This simple sequential scheme is, however, not applicable to non-Gaussian time series models such as ours.

Rather than assuming normal measurement errors, as is the case with QML, we follow Frühwirth-Schnatter (1994) and approximate the prior density by a normal density with the same first and second moments as the prior. With this assumption,[7] the filter is then implemented in a similar fashion to the Gaussian case. The projection, based on a normal prior, preserves normality and so can be implemented analytically. The evaluation of the conditional likelihood requires numerical integration as the measurement error is $\log \chi^2$ distributed, but the approximation can be made highly accurate in our case as the integration is single dimensional. Computation of the first and second moments of the updated prior each requires an additional single-dimensional numerical integration and so the iterative scheme may be continued. Because the numerical integration is of the same dimension as the observation vector, this integration-based procedure provides a practical method of filtering univariate or multivariate financial time series of moderate dimension even for a very high-dimensional state vector.[8]

In Appendix A the sampling properties of the non-Gaussian estimation procedure are compared with those of Jacquier, Polson, and Rossi's (1994) Bayesian estimation procedure and with those of Sandmann and Koopman's

[7] The filter approximates the posterior $p(x_t|Y_t)$ by a normal density. If the posterior is close to a normal density this error is small. The experimental results of Fahrmeir (1992) indicate that with increasing t the posterior tends to be normal even in cases where the observation density is extremely nonnormal. Additionally, we "warm up" the filter for 12 observations to improve this accuracy.

[8] For example, following Balduzzi et al. (1997), our estimation procedure can also accommodate a stochastic mean rate, θ_t:

$$r_t = (1 - b)\theta_{t-1} + br_{t-1} + \exp(\tfrac{1}{2}x_{t-1})r_{t-1}^{\gamma}\epsilon_{1,t}$$

$$x_t - \mu_x = \beta_x(x_{t-1} - \mu_x) + \xi_x\epsilon_{2,t}$$

$$\theta_t - \mu_\theta = \beta_\theta(\theta_{t-1} - \mu_\theta) + \xi_\theta\epsilon_{3,t}$$

with $\epsilon_{1,t} \sim$ i.i.d. $N(0,1)$ and $\epsilon_{2,t}, \epsilon_{3,t} \sim BVN(0,\Sigma)$. Notice that the stochastic mean rate enters the specification in a linear fashion and so all necessary means and covariances can be obtained analytically without recourse to additional numerical integration. Also, this specification does not preclude the possibility of negative mean rates θ_t. Using geometrically scaled U.S. Treasury bill yields, whose dynamics over our sample period are sufficiently away from a unit root (Ball and Torous (1996)) to allow meaningful inference, we obtain the following results (with asymptotic standard errors in parentheses): $b = 0.850$ (0.073), $\gamma = 0.671$ (0.246), $\mu_x = -4.783$ (0.309), $\beta_x = 0.905$ (0.048), $\xi_x = 0.428$ (0.111), $\beta_\theta = 0.965$ (0.028), $\mu_\theta = 1.038$ (0.154), and $\xi_\theta = 0.081$ (0.028). Though these results are consistent with the presence of a stochastic mean rate in U.S. Treasury bill yields, the estimates of the parameters characterizing stochastic volatility $(\gamma, \mu_x, \beta_x, \xi_x)$ are once again similar to the corresponding estimates in Table I.

(1996) Monte Carlo–based estimation procedure for a generic stochastic volatility model. The non-Gaussian estimator is as efficient as these alternative estimation procedures across all the posited parameter values and exhibits similar properties.

III. Estimation and Empirical Results

The stochastic volatility model of short-term riskless interest rate dynamics, expressions (3) and (4), is estimated using one-month London Euro-currency rates, one-month U.K. interbank rates, as well as the one-month U.S. Treasury bill yields. The sampling properties of the non-Gaussian estimator applied to the stochastic volatility models of short-term riskless interest rate dynamics are explored in Appendix B. As detailed in Appendix C, we geometrically scale all interest rate data to minimize the collinearity between parameters. Table I presents the results.

We see clear evidence across all interest rate series of mean-reverting ($\hat{\beta} < 1$) stochastic volatility ($\hat{\xi} > 0$). Compared to stock returns (see, e.g., Jacquier et al. (1994)), stochastic interest rate volatility is less persistent. That is, stochastic interest rate volatility reverts more quickly to its long-term mean as evidenced by the relatively low β estimates. Though interest rate volatility would seem to be sensitive to the level of interest rates ($\hat{\gamma} > 0$), this relation is difficult to measure accurately.

The results of a number of diagnostic tests based on standardized one-step-ahead predictive residuals, also tabulated in Table I, indicate that the stochastic volatility model performs quite well across the sampled one-month interest rate series. Ljung–Box (1978) statistics testing for the presence of serial correlation up to lag k (k = 1, 2, 5, and 10) in the standardized predictive residuals and the squared standardized predictive residuals ($Q(k)$ and $Q^2(k)$, respectively) do not indicate any first- or second-order serial dependence except for the U.K. one-month interbank rate series where we have evidence of first-order serial dependence at longer lags ($k > 2$). Also the excess kurtosis of the standardized predictive residuals is minimal throughout.

One-month London Euro-currencies: The estimation results based on one-month Euro-dollar, Euro-mark, Euro-sterling, and Euro-yen rates are broadly consistent with one another. For each of these interest rate series we have reliable evidence of mean-reverting stochastic interest rate volatility but we cannot statistically distinguish between the corresponding estimates of the stochastic volatility's speed of adjustment, β, or between the corresponding estimates of its volatility, ξ. As expected, there are also no statistically significant differences in the estimated relation between short-term interest rate volatility and the level of interest rates across these series owing to the relative imprecision of the γ estimates. For example, we cannot statistically reject that $\gamma = 0.5$ (Cox, Ingersoll, and Ross (1985)) characterizes the dynamics of each of the sampled Euro-currency rates.

Table I

Parameter Estimates of the Stochastic Volatility Model of Short-Term Riskless Interest Rate Dynamics

We estimate the parameters of the stochastic volatility model of short-term riskless interest rate dynamics:

$$\ln(res_{t+1}^2) = x_t + 2\gamma \ln(r_t) + \ln(\epsilon_{1,t+1}^2)$$

$$x_{t+1} - \mu = \beta(x_t - \mu) + \xi\epsilon_{2,t+1},$$

where $x_t \equiv \ln \sigma_t^2$ is the logarithm of the unobserved volatility σ_t^2 and is assumed to follow an autoregressive specification, $res_{t+1} \equiv \Delta r_{t+1} - a - br_t$ and the parameters a and b are estimated by OLS. The model is cast in state-space form and the resultant likelihood function is evaluated using an integration-based filter which assumes that $\ln(\epsilon_{1,t+1}^2)$ is log χ^2 distributed and the prior on the state is normally distributed. The first 12 observations of the samples are used to initialize the filtering procedure. Asymptotic standard errors of the parameter estimates are in parentheses. The short-term riskless interest rate r_t is proxied by a number of one-month interest rates. Weekly observations on the London one-month Eurocurrency rates, middle rates, for the sample period 1985 to 1995 are obtained from Datastream. Weekly observations on the one-month U.K. interbank rate, middle rate, for the sample period 1975 to 1995 are also obtained from Datastream. Monthly observations on the one-month U.S. Treasury bill yield over the sample period used by Chan et al. (1992), June 1964 to December 1989, are obtained from the CRSP risk-free rate file. All interest rate data are geometrically scaled. Ljung–Box (1978) statistics testing for the presence of serial correlation up to lag k ($k = 1, 2, 5$, and 10) in the standardized predictive residuals and the squared standardized predictive residuals ($Q(k)$ and $Q^2(k)$, respectively) are also presented with p values in parentheses. The excess kurtosis of the model's standardized predictive residuals is denoted by Kurt.

Interest Rate	a	b	μ	β	ξ	γ	$Q(1)$	$Q(2)$	$Q(5)$	$Q(10)$	$Q^2(1)$	$Q^2(2)$	$Q^2(5)$	$Q^2(10)$	Kurt
Euro currencies															
German mark	0.003	0.997	−8.169	0.759	1.033	0.425	1.752	2.969	6.107	11.428	0.228	0.867	2.664	5.348	−0.165
	(0.004)	(0.003)	(0.211)	(0.052)	(0.140)	(0.299)	(0.186)	(0.226)	(0.296)	(0.325)	(0.633)	(0.648)	(0.752)	(0.867)	
Japanese yen	−0.002	0.998	−7.197	0.623	1.201	0.310	2.208	2.296	4.977	16.715	1.217	1.691	2.439	6.213	−0.153
	(0.005)	(0.004)	(0.165)	(0.080)	(0.126)	(0.127)	(0.137)	(0.317)	(0.419)	(0.081)	(0.270)	(0.429)	(0.786)	(0.797)	
U.K. sterling	0.003	0.996	−8.414	0.759	0.944	0.398	1.127	1.519	4.409	16.017	0.642	1.247	4.039	5.916	0.615
	(0.004)	(0.003)	(0.192)	(0.068)	(0.128)	(0.274)	(0.289)	(0.468)	(0.492)	(0.100)	(0.423)	(0.536)	(0.544)	(0.822)	
U.S. dollar	0.005	0.994	−8.011	0.562	1.280	0.754	2.612	2.7032	6.530	8.125	2.422	2.538	5.475	10.444	−0.041
	(0.004)	(0.004)	(0.163)	(0.108)	(0.174)	(0.214)	(0.106)	(0.259)	(0.258)	(0.247)	(0.120)	(0.281)	(0.361)	(0.402)	
U.K. interbank	0.017	0.983	−7.722	0.689	1.102	0.449	2.524	4.425	41.521	69.871	1.860	3.315	8.249	9.444	0.463
	(0.004)	(0.004)	(0.124)	(0.059)	(0.102)	(0.182)	(0.112)	(0.109)	(0.001)	(0.001)	(0.173)	(0.191)	(0.143)	(0.491)	
U.S. Treasury bill	0.063	0.943	−4.563	0.928	0.318	0.651	0.803	0.824	1.773	13.327	1.176	1.576	4.678	7.099	0.035
	(0.022)	(0.019)	(0.281)	(0.043)	(0.094)	(0.255)	(0.307)	(0.662)	(0.880)	(0.206)	(0.278)	(0.455)	(0.456)	(0.716)	

One-month U.K. interbank rates: Nowman (1997) concludes when using monthly observations of the one-month U.K. interbank rate that U.K. interest rate volatility is not sensitive to the level of interest rates. When we use monthly observations of these rates and allow volatility to evolve stochastically we arrive at the same conclusion. However, from Table I we see that when we sample these rates on a weekly basis, thereby increasing the number of observations over the sample period, we estimate $\hat{\gamma} = 0.449$ and conclude that U.K. interest rate data are sensitive to the level of interest rates.

This comparison suggests that it is difficult to precisely estimate γ and only with a large amount of data, more than 1000 weekly observations in our case, can accurate estimates be obtained.[9] Accordingly, Nowman's (1997) results should not be interpreted as evidence that U.K. interest rate volatility is not sensitive to the level of interest rates but rather that monthly observations over a 20-year sample period are insufficient to accurately characterize this relation.

One-month U.S. Treasury bill yields: Once stochastic volatility is incorporated into the Chan et al. framework we still have evidence that the volatility of U.S. Treasury yields is sensitive to the level of yields but no longer to the extent estimated by Chan et al.; for example, using the non-Gaussian procedure we estimate $\hat{\gamma} = 0.651$.[10] The estimated stochastic volatility process for U.S. Treasury yields differs, at least for this sample period, from that estimated for the other interest rate series—being more persistent and so reverting much more slowly to its long-term mean.[11]

IV. Comparison with EGARCH and the Empirical Results

ARCH models have been extensively relied upon to characterize volatility in a variety of financial markets. For example, Longstaff and Schwartz (1992), among others, have modeled short-term interest rate volatility using a GARCH(1,1) specification. A comparison against the stochastic volatility model provides further insights into short-term interest rate dynamics.

[9] For example, Andersen and Lund (1997) use more than 2100 weekly observations on three-month U.S. Treasury yields from January 1954 through April 1995 to estimate $\hat{\gamma} = 0.544$ with $se(\hat{\gamma}) = 0.084$. This precision, however, is at the expense of assuming stationarity in the model parameters over an extended sample period of more than 40 years.

[10] Similarly, Brenner et al. (1996) in their discrete-time GARCH extension of Chan et al. find that $\hat{\gamma} = 0.54$ when they use monthly returns of 30-day U.S. Treasury bills over the January 1960 to December 1993 sample period. In contrast, Koedijk et al. (1997) assume a nonlinear drift specification in their discrete-time extension of Chan et al. and find that $\hat{\gamma} = 1.24$ when they use monthly returns of one-month U.S. Treasury bills over the January 1968 to July 1996 sample period.

[11] Andersen and Lund (1997) also find that the estimated volatility process for U.S. Treasury yields reverts slowly to its long-term mean. This result may reflect the fact that their sample period, like ours, encompasses the period from October 1979 to September 1982 in which the Federal Reserve Board adopted monetary targets that resulted in increased interest rate volatility throughout this period.

For illustrative purposes, the discrete-time version of Chan et al.'s model of short-term riskless interest rate dynamics is now augmented by an EGARCH(1,1) (Nelson (1991)) specification for volatility:

$$res_t = \exp(\tfrac{1}{2}x_{t-1})r_{t-1}^{\gamma}\epsilon_t \tag{6}$$

$$x_t - \mu = \beta(x_{t-1} - \mu) + g_0\epsilon_{t-1} + g_1(|\epsilon_{t-1}| - E|\epsilon_{t-1}|). \tag{7}$$

As before, $res_t \equiv r_t - (\hat{a} + \hat{b}r_{t-1})$. We assume $\epsilon_t \sim$ i.i.d. $N(0,1)$, or alternatively, $\epsilon_t \sim$ i.i.d. t_ν. Unlike the stochastic volatility model, there is now only one source of uncertainty, ϵ_t, and the stochastic nature of volatility is introduced by modeling volatility as a function of past realizations of this error.

Unfortunately, as noted by Kim et al. (1998), a standard likelihood ratio test cannot be used to compare the stochastic volatility model (expressions (3) and (4)) to the EGARCH model (expressions (6) and (7)) since the models are not nested. To make comparisons, we follow Naik and Lee (1994) and use Vuong's (1989) modified likelihood ratio test for comparing nonnested hypotheses.[12] Under the null hypothesis that the competing models are statistically indistinguishable, the Vuong statistic is standard normally distributed.

From Table II we see that the Vuong test rejects the EGARCH model with normally distributed innovations in favor of the stochastic volatility model for all but one of the interest rate series (U.S. Treasury yields).[13]

Focusing our attention on the EGARCH model with t-distributed innovations, ARCH effects in (log) interest rate volatility are evident in Table II for each of the sampled one-month interest rate series. Applying the diagnostic tests to the EGARCH model's standardized residuals reveals that, like the stochastic volatility model, the EGARCH model performs quite well. Only for the U.K. one-month interbank rate series do we have any evidence of model misspecification as significant first-order serial dependence is present in the corresponding standardized residuals at all lags. Also the excess kurtosis of the standardized residuals is once again minimal throughout.

However, the estimated degrees of freedom, $\hat{\nu}$, associated with the t-distributed innovations are approximately two for all but one of the sampled interest rate series (U.S. Treasury yields),[14] as compared to approximately five to 10 for stock returns (Engle and Bollerslev (1986)). With such t-distributed innovations, the Vuong test cannot statistically distinguish be-

[12] Vuong's (1989) test statistic is a likelihood ratio statistic adjusted by the standard deviation of the difference in maximized log-likelihood functions under the competing models.

[13] Our conclusions are not altered if another EGARCH specification with normal innovations (e.g., EGARCH(2,2)) is assumed. Because of these rejections, Table II does not report the parameter results nor the results of the diagnostic tests applied to the EGARCH model with normal innovations.

[14] Similarly, using monthly returns on 30-day U.S. Treasury bills over the January 1960 to December 1993 sample period, Brenner et al. (1996) estimate $\hat{\nu} = 9.48$ in a discrete-time GARCH extension of the CKLS model assuming t-distributed innovations.

Table II

Parameter Estimates of the EGARCH-*t* Model of Short-Term Riskless Interest Rate Dynamics

We estimate the parameters of the EGARCH model of short-term riskless interest rate dynamics:

$$res_t = \exp(\tfrac{1}{2}x_t)r_t^{\gamma}\epsilon_t$$

$$x_t - \mu = \beta(x_{t-1} - \mu) + g_0\epsilon_{t-1} + g_1(|\epsilon_{t-1}| - E|\epsilon_{t-1}|),$$

where $res_t \equiv \Delta r_t - (\hat{a} + \hat{b}r_{t-1})$ and the parameters a and b are estimated by OLS. We assume t-distributed innovations with ν degrees of freedom. Asymptotic standard errors of the parameter estimates are in parentheses. The short-term riskless interest rate is proxied by a number of one-month interest rates. Weekly observations on the London one-month Euro-currency rates, middle rates, for the sample period 1985 to 1995 are obtained from Datastream. Weekly observations on the one-month U.K. interbank rate, middle rate, for the sample period 1975 to 1995 are also obtained from Datastream. Monthly observations on the one-month U.S. Treasury bill yield over the sample period used by Chan et al. (1992), June 1964 to December 1989, are obtained from the CRSP risk-free rate file. All interest rate data are geometrically scaled. Ljung–Box (1978) statistics testing for the presence of serial correlation up to lag k ($k = 1, 2, 5,$ and 10) in the standardized predictive residuals and the squared standardized predictive residuals ($Q(k)$ and $Q^2(k)$, respectively) are also presented with p values in parentheses. The excess kurtosis of the model's standardized predictive residuals is denoted by Kurt. The value of the Vuong (1989) statistic for testing the EGARCH-t model against the stochastic volatility model is denoted by V_t; $V_{\mathcal{N}}$ is the value of the Vuong (1989) statistic for testing the EGARCH model with normal innovations against the stochastic volatility model.

Interest Rate	$V_{\mathcal{N}}$	μ	β	g_0	g_1	γ	ν
Euro currencies							
German mark	3.32*	−9.077	0.813	0.086	0.390	0.069	2.822
		(0.264)	(0.050)	(0.041)	(0.062)	(0.285)	(0.416)
Japanese yen	5.16*	−8.492	0.854	0.132	0.210	0.319	2.007
		(0.375)	(0.056)	(0.038)	(0.049)	(0.145)	(0.328)
U.K. sterling	4.37*	−9.881	0.902	−0.002	0.181	0.249	2.241
		(0.397)	(0.037)	(0.027)	(0.037)	(0.314)	(0.273)
U.S. dollar	3.89*	−8.998	0.825	0.050	0.164	0.762	2.074
		(0.242)	(0.071)	(0.036)	(0.052)	(0.220)	(0.286)
U.K. interbank	3.14*	−9.035	0.873	−0.036	0.174	0.519	2.098
		(0.204)	(0.029)	(0.023)	(0.030)	(0.228)	(0.176)
U.S. Treasury bill	1.39	−5.067	0.960	0.183	0.206	−0.448	6.598
		(0.543)	(0.016)	(0.129)	(0.068)	(0.779)	(2.361)

Interest Rate	V_t	$Q(1)$	$Q(2)$	$Q(5)$	$Q(10)$	$Q^2(1)$	$Q^2(2)$	$Q^2(5)$	$Q^2(10)$	Kurt
Euro currencies										
German mark	0.105	1.284	2.132	4.780	10.554	0.079	0.169	2.168	5.004	−0.061
		(0.257)	(0.344)	(0.443)	(0.393)	(0.778)	(0.919)	(0.826)	(0.891)	
Japanese yen	0.815	3.065	3.117	5.787	17.244	0.323	0.323	0.457	2.289	−0.244
		(0.080)	(0.210)	(0.328)	(0.069)	(0.570)	(0.849)	(0.994)	(0.994)	
U.K. sterling	−1.66	2.382	2.570	5.436	17.791	0.513	0.605	3.006	4.755	0.056
		(0.123)	(0.277)	(0.365)	(0.059)	(0.474)	(0.739)	(0.699)	(0.907)	
U.S. dollar	−0.684	3.184	3.191	6.792	12.691	0.390	0.859	1.993	6.149	−0.214
		(0.074)	(0.203)	(0.237)	(0.241)	(0.533)	(0.651)	(0.850)	(0.803)	
U.K. interbank	−1.09	4.213	7.498	44.274	76.106	2.342	2.694	4.517	6.284	0.121
		(0.040)	(0.024)	(0.001)	(0.001)	(0.126)	(0.260)	(0.478)	(0.791)	
U.S. Treasury bill	0.24	1.010	1.020	1.884	12.718	2.456	2.862	7.351	9.134	0.052
		(0.315)	(0.600)	(0.865)	(0.240)	(0.117)	(0.239)	(0.196)	(0.519)	

* The Vuong statistic is significant at the 5 percent level.

tween the stochastic volatility model and the EGARCH model. But for two (or fewer) degrees of freedom the variance of a t-distributed random variable is infinite (and all higher order moments do not exist), making statistical inference potentially more difficult. For example, Phillips (1990) shows for a particular class of infinite variance errors that t-ratio statistics and other self-normalized sums in these errors are bimodally distributed in asymptotic as well as finite samples. This suggests that in a Monte Carlo analysis with infinite variance driving processes standard asymptotic measures of precision based on the normal distribution will be incorrect.

To better understand these estimation results, recall that ARCH models are predicated by the persistent nature of changing volatility. That is, periods of high and low volatility tend to cluster together (Engle (1982)). If short-term interest rates are impacted by more transient economic shocks then ARCH models may provide a less accurate description of their volatility dynamics because past errors do not accurately forecast volatility. Fat-tailed t-distributed innovations remedy this deficiency by better accommodating these transient economic shocks. The empirical adequacy of the stochastic volatility model stems from the fact that it is also characterized by fat-tailed changes in interest rates since by construction, unlike the EGARCH model, it simultaneously incorporates interest rate and variance innovations to yield a variance mixture.

These results are illustrated in Figure 1 where we plot the Euro-mark interest rate series and display for each sample point t the difference between the maximized log-likelihood values, based on data through $t - 1$, of the stochastic volatility and EGARCH models evaluated at their respective sample-wide maximum likelihood estimates. Intuitively, a positive difference at t is consistent with the stochastic volatility model being better able to explain the data at t.

Notice in Panel A of Figure 1 that for most of the sample points the stochastic volatility model's improvement over the EGARCH model with normal innovations is minimal. However, there are a small number of very brief episodes where substantial improvement is evident. We also identify in Panel A specific events, if any, associated with each of the five largest discrepancies between the likelihoods of the stochastic volatility model and the EGARCH model assuming normal innovations. In particular, we searched the *Financial Times* in the vicinity of these dates for mention of any noteworthy events that may have affected German credit markets. A Bundesbank announcement in December 1990 resulted in the largest discrepancy, but other economic shocks also are important. These include the ERM crisis (September 1992), announcement of an OPEC production agreement (November 1988), and disclosure of dialogue with East Germany on monetary union (February 1990). In the remaining case (December 1986), no specific economic event could be found.

Apart from the ERM crisis (September 1992), the corresponding discrepancies between the stochastic volatility model and the EGARCH model with normal innovations for the Euro-pound interest rate series are dominated by

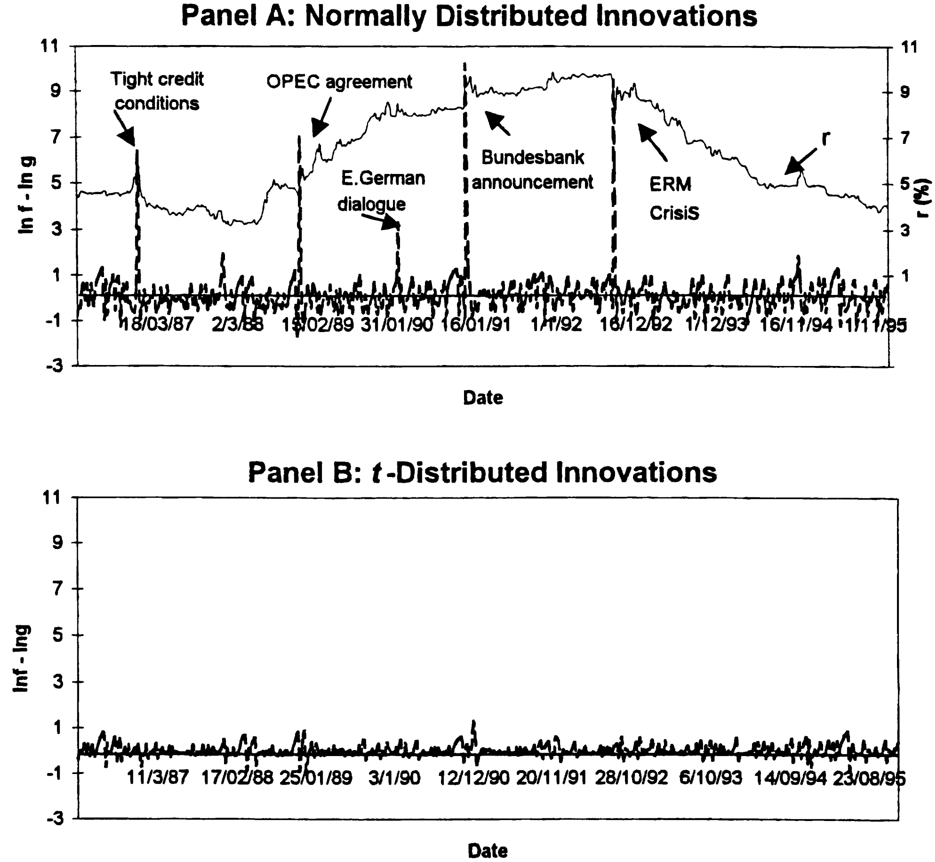

Figure 1. A comparison of the stochastic volatility model with the EGARCH model. We plot the Euro-mark interest rate series (in percentage points) and display for each sample point t the difference between the maximized log-likelihood values, based on data through $t-1$, of the stochastic volatility and EGARCH models evaluated at their respective sample-wide maximum likelihood estimates. Panel A displays the results assuming an EGARCH model with normal innovations and Panel B displays the results assuming an EGARCH model with t-distributed innovations. A positive (negative) difference at t is consistent with the stochastic volatility model being better (less) able to explain the data at t.

announcements of base rate changes by the Bank of England—both increases (May 1989, August 1987, November 1988) and decreases (October 1990). Central bank actions also dominate the Euro-yen interest rate results: discount rate increases (October 1989) and decreases (January 1990, April 1995), open market operations (October 1988), and pledges by the Bank of Japan to lower rates (April 1995).

In contrast, the largest discrepancies for the Euro-dollar interest rate series result from surprises in the macroeconomic activity data that are released. Examples include unexpectedly strong economic growth (November

1988), revisions of past growth rates (December 1992), and increases in consumer confidence (November 1993). These results complement those of Edison (1996) and Fleming and Remolona (1997), among others, who demonstrate at daily and intraday frequencies, respectively, that both short-term and long-term U.S. Treasury rates in the post-October 1979 sample period respond systematically and quickly to surprises in the release of U.S. macroeconomic activity data.

Alternatively, we can interpret these events as giving rise to the outliers that are captured by assuming t-distributed innovations in the EGARCH model. That such a low number of degrees of freedom is needed for the t-distribution reflects the fact that these outliers are minimal for a great majority of the time but are substantial otherwise. The resultant improvement in the fit of the EGARCH model for the Euro-mark interest rate series is evident in Panel B of Figure 1 where we see that the difference between the maximized log-likelihood values of the stochastic volatility model and the EGARCH model with t-distributed innovations is negligible across all of the sample points. However, with such low degrees of freedom the EGARCH model with t-distributed innovations is less stable than the stochastic volatility model. Using the Euro-mark series, we follow Andersen and Lund (1997) and fix the parameter values at the point estimates of the EGARCH-t model and the stochastic volatility model and construct a 100,000 long sample of realizations from each of these models. The samples are then broken down into consecutive subsamples of 1000 observations and the properties of these subsamples are analyzed. The simulated EGARCH-t model displays erratic behavior: the overall maximum simulated interest rate over the subsamples is in excess of 1250 percent and 10 of the 100 subsample maxima are in excess of 100 percent. In contrast, for the simulated stochastic volatility model the overall maximum is never in excess of 15 percent.

V. Conclusions

The volatility of short-term interest rates is stochastic. This conclusion holds for all of the one-month interest rate series examined. Models of short-term riskless interest rate dynamics that assume deterministic volatility are misspecified. Ignoring stochastic volatility results in systematic errors in the valuation and hedging of fixed-income derivative securities.

Stochastic interest rate volatility differs from stochastic stock return volatility. Most significantly, interest rate volatility is characterized by a faster mean-reverting behavior reflecting the impact of transient economic shocks such as central bank announcements of base rate changes. Future work explicitly incorporating the implications of central bank actions (Balduzzi, Bertola, and Foresi (1997)) and monetary shocks in general on the behavior of interest rates holds the promise of enhancing our understanding of short-term interest rate dynamics and ultimately of improving the pricing of fixed-income derivative securities. In general, economic shocks play a central role in determining the dynamics of financial market volatility, and the differential persistence of volatility that we observe across financial markets depends critically on the source of their economic shocks.

Appendix A. Sampling Experiments for the Generic Stochastic Volatility Model

We provide evidence comparing the sampling properties of the non-Gaussian estimator to alternative procedures for estimating stochastic volatility models recently proposed by Jacquier et al. (1994) and Sandmann and Koopman (1996). We use the following generic model, which has been used previously to benchmark statistical performance in the stochastic volatility literature:

$$y_t = \sqrt{h_{t-1}} u_t \tag{A1}$$

$$\ln h_t = \alpha + \delta \ln h_{t-1} + \sigma_v v_t, \qquad u_t, v_t \sim \text{i.i.d. } N(0,1). \tag{A2}$$

We make our results directly comparable by calibrating this model precisely as in Jacquier et al. (1994) and Sandmann and Koopman (1996). Parameter values α, δ, and σ_v are chosen to investigate the sensitivity of the estimation methods (i) to the size of the stochastic volatility component (the squared coefficient of variation $\text{var}(h)/E(h)^2 = 10$, 1.0, 0.1), and (ii) to the persistence of the stochastic component ($\delta = 0.9$, 0.95, 0.98), while fixing $E(h) = 0.0009$, or approximately a 20 percent annualized standard deviation for weekly data. See Jacquier et al. (1994) for a discussion of the relevance of these parameter values in the stochastic volatility literature. Time series of length $n = 500$ are simulated and sampling distributions are based on 500 replications.

The results are tabulated in Table AI. For each combination of parameter values the corresponding sampling results of Jacquier et al. (1994) and Sandmann and Koopman (1996) are reproduced in the first and second rows, respectively; the third row gives the sampling results for the non-Gaussian estimator. The non-Gaussian estimator is as efficient as the Jacquier et al. (1994) and the Sandmann and Koopman (1996) estimation procedures across all the posited parameter values and exhibits similar biases. In particular, the sampling performance of each of the competing estimation procedures noticeably deteriorates as the size of the stochastic component (measured by $\text{var}(h)/E(h)^2$) decreases. For example, the stochastic volatility's speed of adjustment parameter δ is estimated with increasing downward bias as well as decreasing precision as $\text{var}(h)/E(h)^2$ is decreased. In contrast, varying the persistence of the stochastic component (measured by δ) appears to have little effect on the sampling performance of the estimation procedures.

Table AI

A Comparison of the Sampling Properties of the Non-Gaussian Estimation Procedure for the Generic Stochastic Volatility Model

We compare the sampling properties of the non-Gaussian estimation procedure with Jacquier et al.'s (1994) Bayesian estimation procedure and with Sandmann and Koopman's (1996) Monte Carlo–based estimation procedure for the generic stochastic volatility model:

$$y_t = \sqrt{h_{t-1}u_t}$$

$$\ln h_t = \alpha + \delta \ln h_{t-1} + \sigma_v v_t .$$

We choose values of the model parameters α, δ, and σ_v to investigate the sensitivity of these estimation procedures to the size of the stochastic volatility component by varying the squared coefficient of variation, $\mathrm{var}(h)/E(h)^2$, and to the persistence of the stochastic component by varying δ. All parameter value combinations are consistent with $E(h) = 0.009$ or approximately a 20 percent annualized standard deviation in weekly data. Time series of length $n = 500$ are simulated and the resultant sampling distributions are based on 500 replications. We report the average estimated parameter values and corresponding root mean squared errors (in parentheses). The results for the Bayesian estimator are reproduced from Jacquier et al. (1994, Table 7); the results for the Monte Carlo–based estimator are adapted from Sandmann and Koopman (1996, Table 2).

	α	δ	σ_v	α	δ	σ_v	α	δ	σ_v
$\mathrm{Var}(h)/E[h]^2 = 10$	−0.821	0.9	0.675	−0.4106	0.95	0.4835	−0.1642	0.98	0.308
Bayes	−0.68	0.92	0.56	−0.46	0.94	0.46	−0.19	0.98	0.35
	(0.22)	(0.03)	(0.12)	(0.16)	(0.02)	(0.06)	(0.08)	(0.01)	(0.06)
Monte Carlo	−0.66	0.91	0.62	−0.36	0.95	0.46	−0.17	0.98	0.30
	(0.25)	(0.03)	(0.15)	(0.15)	(0.02)	(0.06)	(0.11)	(0.02)	(0.05)
Nonnormal	−0.91	0.89	0.69	−0.49	0.94	0.49	−0.26	0.97	0.32
	(0.27)	(0.03)	(0.08)	(0.19)	(0.02)	(0.07)	(0.16)	(0.02)	(0.05)
$\mathrm{Var}(h)/E[h]^2 = 1$	−0.736	0.9	0.363	−0.368	0.95	0.26	−0.1472	0.98	0.166
Bayes	−0.87	0.88	0.35	−0.56	0.92	0.28	−0.22	0.97	0.23
	(0.34)	(0.05)	(0.07)	(0.34)	(0.05)	(0.07)	(0.14)	(0.02)	(0.08)
Monte Carlo	−0.60	0.90	0.34	−0.33	0.95	0.25	−0.16	0.97	0.16
	(0.30)	(0.04)	(0.08)	(0.19)	(0.03)	(0.06)	(0.16)	(0.03)	(0.05)
Nonnormal	−0.87	0.88	0.37	−0.49	0.93	0.27	−0.27	0.96	0.18
	(0.37)	(0.05)	(0.08)	(0.26)	(0.03)	(0.06)	(0.14)	(0.03)	(0.05)
$\mathrm{Var}(h)/E[h]^2 = 0.1$	−0.706	0.9	0.135	−0.353	0.95	0.0964	−0.141	0.98	0.0614
Bayes	−1.54	0.78	0.15	−1.12	0.84	0.12	−0.66	0.91	0.14
	(1.35)	(0.19)	(0.08)	(1.15)	(0.16)	(0.07)	(0.83)	(0.12)	(0.10)
Monte Carlo	−0.88	0.85	0.10	−0.64	0.89	0.09	−0.45	0.92	0.07
	(1.52)	(0.22)	(0.11)	(1.37)	(0.24)	(0.08)	(1.15)	(0.20)	(0.06)
Nonnormal	−1.22	0.83	0.16	−0.79	0.89	0.12	−0.42	0.94	0.08
	(1.31)	(0.18)	(0.11)	(1.00)	(0.15)	(0.09)	(0.81)	(0.12)	(0.07)

Appendix B. Sampling Experiments for the Stochastic Volatility Model of Interest Rate Dynamics

We investigate the sampling properties of the non-Gaussian estimator when applied to the stochastic volatility model of short-term riskless interest rate dynamics.

The first sampling experiment assumes parameter values broadly consistent with those characterizing the U.S. Treasury bill yield results based on monthly observations ($a = 0.06$, $b = 0.92$, $\mu = -6.0$, $\beta = 0.93$, $\xi = 0.32$, and $\gamma = 0.65$). Interest rate and volatility series each of length $n = 500$ observations are simulated. After geometrically scaling these interest rate data, the parameters a and b are estimated using OLS. The stochastic volatility interest rate model, expressions (3) and (4), is then estimated using the non-Gaussian procedure. The experiment is repeated 500 times and the sampling distributions of the estimators of μ, β, ξ, and γ are tabulated in Panel A of Table BI.

On the whole, the parameters appear to be estimated unbiasedly and precisely. However, as in the case of the generic stochastic volatility model (see Table AI), the stochastic volatility's speed of adjustment parameter, β, is estimated with a downward bias.

To further investigate the robustness of the non-Gaussian estimator, Panel B of Table BI presents the sampling results when we assume parameter values broadly consistent with those characterizing the Euro-dollar rate results based on weekly observations ($a = 0.005$, $b = 0.99$, $\mu = -8.0$, $\beta = 0.5$, $\xi = 1.2$, and $\gamma = 0.75$). Once again the parameters appear to be estimated precisely.

Appendix C. Geometric Scaling

Empirically characterizing the relation between the volatility of short-term interest rate changes and the level of rates is made difficult by the collinearity between the parameters of interest. For example, because of the collinearity between γ and σ, Gray (1996) fixes $\gamma = 0.5$ since the "interpretation of the individual parameter estimates (is) questionable at best"(p. 36). Rather than simply assuming a particular relation between the volatility and level of interest rates, this appendix shows how the geometric scaling of interest rate data minimizes this collinearity.

We motivate geometric scaling by couching our analysis in the deterministic volatility case

$$r_t = a + br_{t-1} + \sigma r_{t-1}^{\gamma}\epsilon_t, \qquad \epsilon_t \sim \text{i.i.d. } N(0,1). \tag{C1}$$

Since the parameters a and b do not affect the collinearity between γ and σ, we assume they are known or have been consistently estimated previously. As a result, consider

$$res_t = \sigma r_{t-1}^{\gamma}\epsilon_t, \tag{C2}$$

Table BI

Sampling Properties of the Non-Gaussian Estimation Procedure: Stochastic Volatility Model of Interest Rate Dynamics

This table summarizes the sampling properties of the non-Gaussian estimation procedure when applied to the stochastic volatility model of interest rate dynamics. We simulate interest rate and volatility series each of length $n = 500$ observations according to

$$\Delta r_t = (a + br_{t-1})\Delta t + \sigma_t r_{t-1}^{\gamma}\sqrt{\Delta t}\epsilon_{1,t}$$

$$\ln \sigma_t^2 - \mu = \beta(\ln \sigma_{t-1}^2 - \mu)\Delta t + \xi\sqrt{\Delta t}\epsilon_{2,t}.$$

In Panel A we use parameter values consistent with our estimation results for monthly U.S. Treasury bill data: $a = 0.06$, $b = 0.94$, $\mu = -4.5$, $\beta = 0.93$, $\xi = 0.32$, and $\gamma = 0.65$. In Panel B we use parameter values consistent with our estimation results for weekly Euro-dollar data: $a = 0.005$, $b = 0.99$, $\mu = -9.0$, $\beta = 0.50$, $\xi = 1.2$, and $\gamma = 0.75$. After geometrically scaling the interest rate data, the parameters a and b are estimated by OLS. The remaining parameters of the stochastic volatility model are estimated using the non-Gaussian procedure. The experiment is repeated 500 times and the sampling distributions of the estimators are summarized by the sample mean, the sample standard deviation, the sample skewness, the sample excess kurtosis, and selected percentiles.

Assumed Parameter Value	*Mean*	*Std*	*Skew*	*Kurt*	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Panel A: Assuming Representative Monthly Parameters									
$a = 0.06$	0.077	0.024	0.716	1.310	0.050	0.060	0.075	0.091	0.109
$b = 0.94$	0.927	0.024	−0.717	1.362	0.896	0.913	0.929	0.943	0.954
$\mu = -4.5$	−4.456	0.247	−0.062	−0.159	−4.788	−4.623	−4.459	−4.282	−4.123
$\beta = 0.93$	0.907	0.047	−1.610	3.294	0.852	0.887	0.918	0.938	0.953
$\xi = 0.32$	0.332	0.078	0.811	1.867	0.243	0.275	0.323	0.374	0.427
$\gamma = 0.65$	0.654	0.177	0.088	0.104	0.433	0.533	0.652	0.766	0.878
Panel B: Assuming Representative Weekly Parameters									
$a = 0.005$	0.014	0.008	1.413	3.045	0.005	0.008	0.012	0.018	0.025
$b = 0.99$	0.985	0.008	−1.433	3.071	0.973	0.981	0.986	0.990	0.993
$\mu = -9.0$	−9.190	1.637	0.065	1.392	−11.227	−10.149	−9.206	−8.234	−7.160
$\beta = 0.50$	0.473	0.099	−0.480	0.573	0.342	0.416	0.479	0.538	0.592
$\xi = 1.20$	1.179	0.118	−0.287	0.101	1.028	1.100	1.183	1.263	1.327
$\gamma = 0.75$	0.757	0.264	−0.011	1.183	0.434	0.598	0.753	0.915	1.086

or, equivalently,

$$\ln(res_t^2) = \ln\sigma^2 + 2\gamma\ln r_{t-1} + \ln\epsilon_t^2. \tag{C3}$$

The geometric mean of $\{r_1,\ldots,r_N\}$ is $\dot{r} = (\prod_{t=1}^{N} r_i)^{1/N}$. If we scale or divide the interest rate data by $\dot{r}$ then, since

$$\frac{1}{N}\sum_t \ln(res_t^2) = \frac{1}{N}\sum_t \ln\sigma^2 + 2\gamma\frac{1}{N}\sum_t \ln r_{t-1} + \frac{1}{N}\sum_t \ln\epsilon_t^2, \tag{C4}$$

it follows that $\ln\sigma^2$ is consistently estimated by

$$\frac{1}{N}\sum_t \ln(res_t^2) + 1.27, \tag{C5}$$

where we use the fact that for geometrically scaled interest rate data $\sum_t \ln r_{t-1} = 0$ and the mean of $\log\chi^2_{(1)}$ is -1.270 (Abramowitz and Stegun (1970)). In other words, *regardless* of the value of γ, we obtain a consistent estimate of $\ln\sigma^2$. Or, in the case of stochastic volatility, we obtain a consistent estimate of μ.

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