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Call Options, Points, and Dominance Restrictions on Debt Contracts

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ABSTRACT

We analyze the impact of a contract's length, callability, amortization, and original discount by arbitrage methods. Among instruments that are callable without penalty, longer instruments command a higher interest rate because the borrower possesses the option of repaying relatively more slowly. However, the rate on longer self-amortizing loans cannot be substantially larger than for shorter ones because the payments decrease with contract length. Bounds on the trade-off between points and rate for callable debt are characterized using the trade-off for noncallable debt and the property that the value of the prepayment option increases with the loan's interest rate.

IN RECENT YEARS A WIDE VARIETY of contract terms have been used in the mortgage market and the standard 30-year fixed-rate loan has become relatively less important. In light of the diversity in observed contractual provisions, we examine the relative pricing of different mortgages and other debt instruments. We derive pricing bounds by identifying and eliminating those contractual alternatives that are dominated for every type of borrower. Therefore, the pricing bounds do not require knowledge of the borrower's preferences or private information about his own prepayment likelihoods. Our analysis focuses on the underlying economic intuition, enhancing the understanding of the qualitative behavior of the numerical solutions of parametric valuation models. Arbitrage bounds on the relative pricing of different financial claims yield robust restrictions that do not require an assumed parametric valuation model (in option pricing contexts this is illustrated by Merton (1973), Ross (1978), and, more recently, by Bergman, Grundy, and Wiener (1996)). General results are obtained without imposing assumptions about the prevailing term structure and the stochastic process generating interest rates.

Some of our results exploit the standard characteristic of mortgage loans that borrowers are not subject to a prepayment penalty—that is, the loan is callable at par rather than being subject to a call deferral period or a posi-

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tive call premium. Under the assumption that the loans (or any portion of them) are immediately prepayable at par at the borrower's discretion, longer instruments are less valuable to the lender than shorter ones with the same contractual interest rate because the borrower possesses *the option* of paying down the longer loans more slowly. Therefore, the shorter fixed-rate loan cannot have a higher contract rate than the longer loan with otherwise identical provisions, such as the same points (original discount), for any interest rate process and interest rate conditions. Consequently, the callable yield curve must be (weakly) upward sloping, even if the noncallable yield curve is downward sloping. This conclusion is different from the traditional arbitrage restriction that the value of an American option on a fixed underlying instrument increases with its time to expiration. Here, we compare callable instruments of different maturities so that the maturities of both the underlying noncallable loan and embedded call option are changing. Even if the value of the embedded call option does not increase with maturity, the yield on the callable bond cannot decline and its price cannot increase because borrowers with longer contracts can mimic the exercise strategy of shorter contracts. However, the extent to which the interest rate on the longer instrument can exceed the rate on the shorter loan is limited for self-amortizing loans because the payment on the shorter loan must exceed the payment on the longer instrument. This restriction on the relative payments can be interpreted as a lower bound on the contract rate on the shorter instrument, so that we obtain both lower and upper bounds on its contract rate.

Many of our arguments apply to loans that possess a balloon obligation at maturity—that is, they mature prior to their complete amortization. We derive restrictions on the relative pricing of nonamortizing coupon bonds and amortizing fixed-rate mortgages and among amortizing as well as nonamortizing bonds of various maturities. We use our analysis to obtain restrictions on the relative pricing of adjustable-rate and fixed-rate loans. Some of the pricing results cannot be tightened without stronger assumptions.

Dominance arguments are used to derive restrictions on the trade-off between the points (initial discount) and the periodic loan payment for a given maturity. For both noncallable self-amortizing and nonamortizing loans the trade-off between the points and periodic payments is linear. The amount financed net of points is proportional to the loan payment for noncallable self-amortizing loans of a given maturity. An inequality form of this restriction holds for callable self-amortizing loans as the value of the borrower's call option is increasing in the loan's coupon. This provides a characterization of the points and interest rate trade-off for callable instruments.

Section I explores the effect of repayment options, amortization provisions, and the contract's length on its interest rate. Section II applies the analysis to adjustable-rate loans. Section III examines the relationship between the relative payments of contracts of different lengths and the resulting implications for contract interest rates. Section IV analyzes the trade-off between the interest rate and points (original issue discount) on debt contracts. Concluding comments are offered in Section V.

I. The Repayment Option, Contract Rates, and Length

We examine loan contracts for which the periodic payment is based on amortization in T periods assuming constant payment and rate per period. However, the loan matures in n periods ($n \leq T$), so that the borrower must repay any remaining balance at date n if the loan is not prepaid earlier. By distinguishing between n and T we permit the debt instruments to have a balloon payment at the maturity n, though if $n = T$ the loan is self-amortizing. Similarly, if $T = \infty$, then the loan is nonamortizing. To facilitate comparison of the effect of prepayment options on the contractual interest rate, we assume that the contracts have identical original initial discounts (i.e., "points"). We also assume that the loan contracts are not subject to prepayment penalties—that is, call premia are identically zero.[1] This assumption is central to some of our results because with no call premium the borrower using a longer contract possesses the option of replicating the cash flows of shorter instruments, and also possesses the option of repaying more slowly than the shorter instrument permits. For simplicity, we also assume throughout the analysis that there are no tax effects, and we initially assume that there is no default risk.[2] We do not impose substantive restrictions on the stochastic process generating interest rates, beliefs about future interest rates, or interest rate conditions at the time of contracting. However, we assume that realized interest rates in the capital market are nonnegative (specific implications of this assumption are explored in Black (1995)). Our analysis in this section is also consistent with a borrower's prepayment decision reflecting nonfinancial gains from moving (as in Dunn and McConnell (1981) and Dunn and Spatt (1985)). We permit borrowers to possess private information about their own prepayment likelihoods.[3] Our analysis is also robust to positive refinancing costs because the value of incremental

[1] The presence of due-on-sale provisions in home mortgage contracts is not inconsistent with this assumption. Though Dunn and Spatt (1985) point out that the due-on-sale clause can be interpreted as a stochastic prepayment penalty, the analysis here is robust to the presence of this covenant in the loan contract. We do assume that the instruments being compared have either a similar due-on-sale or assumability provision.

[2] Many of our results can be generalized in such directions. An example that illustrates the robustness of the arbitrage approach to securities with substantial default risk is the analysis in Dammon, Dunn, and Spatt (1993) of RJR Nabisco bonds. A recent treatment of arbitrage and default risk is in Schwartz (1997). Tax effects also can be incorporated from the borrower's perspective by deriving the restrictions for after-tax cash flows. An interesting feature of taxes in mortgage refinancing is that points are deductible only over the full contractual term, which could break our conclusion that the longer instrument must be more valuable for the borrower for contracts with positive points (i.e., loans issued at a discount).

[3] The distribution of the borrower's private information helps determine the set of feasible alternative mortgage contracts (the effect of private information on the borrower's choice of mortgage was first examined by Dunn and Spatt (1988) and Chari and Jagannathan (1989)). The analysis here is robust to the presence of private information and to restrictions on the set of available debt instruments offered in equilibrium. Of course, we assume the coexistence of the contracts undergoing comparison. The absence of dominance implies that each of the alternatives compared is preferred by some type of borrower—that is, no alternative is dominated for all types of borrowers.

prepayment options remains nonnegative, though costly refinancing can reduce their value dramatically. (Dunn and Spatt (1986) analyze the implications of costly refinancing for the valuation of debt contracts and optimal prepayment behavior.) The generality of these assumptions illustrates the robustness of our conclusions.

We now examine the dependence of the loan contract rate on the maturity date, n, and amortization interval, T. Throughout, we defer proofs to the Appendix.

PROPOSITION 1: *The periodic interest rate, R, and periodic payment, P, on a callable loan contract cannot decrease as maturity, n, increases, for a fixed amortization interval, T, where $n \leq T$. Thus, if two loan contracts, $j = 1,2$, have the same T-period amortization interval, and $n_1 > n_2$, then $R_1 \geq R_2$ and $P_1 \geq P_2$.*

An illustration of the proposition is that a mortgage contract with payments based on 30-year amortization and with a balloon obligation in five years cannot have a higher interest rate than a self-amortizing 30-year loan. The assumption that the securities are callable without penalty can be relaxed somewhat for this proposition. Specifically, the proof assumes that the two instruments possess identical nonnegative call premium schedules per dollar prepaid prior to date n_2 and that the longer security is callable without penalty after date n_2. Using an argument similar to Proposition 1, the contract interest rate is also nondecreasing in the amortization interval.

PROPOSITION 2: *The periodic interest rate, R, on a callable loan contract cannot decrease in the loan's amortization interval, T, for a fixed maturity, n, where $n \leq T$. Thus if two loan contracts, $j = 1,2$, have the same maturity, n, and $T_1 > T_2 \geq n$, then $R_1 \geq R_2$.*[4]

Propositions 1 and 2 are robust to the presence of refinancing costs or a borrower's prepayment decision reflecting nonfinancial utility gains from moving, because the borrower using a longer contract still possesses a richer set of options than the borrower using the shorter contract.[5] For example,

[4] In order for the borrower on the longer instrument to be able to exactly replicate the payment pattern on the loan with amortization over a shorter interval, the borrower on the longer contract must be able to prepay some fraction of the outstanding principal each period. This is permitted under mortgage contracts without a prepayment penalty and many callable bonds ("callable in whole or part"). Furthermore, under many conditions the optimal call behavior for an individual borrower is a 0–1 variable (e.g., whether based on nonfinancially motivated utility gains from prepaying or a reduction in the interest cost of the indebtedness) so that preventing fractional prepayments is not very restrictive. Of course, in the presence of significant market imperfections a fractional prepayment can be advantageous.

[5] Consider two loans with the same interest rate. If the borrower on the longer contract chooses to replicate the cash flows of the shorter contract, there are no incremental refinancing costs relative to the shorter contract because the replication results in the same sequence of outstanding principal balances. If the longer contract had a lower interest rate, then the replication of the cash flows of the shorter contract would result in a lower sequence of outstanding balances for the longer contract. So, in contradiction, the longer contract would dominate the shorter one.

refinancing costs are incurred if the borrower chooses to refinance *any* payments, whether scheduled or unscheduled. In contrast, a prepayment penalty (call premium) is imposed only on prepayments and not on scheduled loan payments. Therefore, in the presence of a prepayment penalty, the option to accelerate the payments under the longer contract to mimic the payments under the shorter contract is not costlessly available. As a result, Proposition 2 cannot be extended to identical positive call price schedules prior to the common maturity, n, though Proposition 1 extends to a common call price schedule prior to the earlier maturity n_2.

Propositions 1 and 2 hold in strict form as long as the option to delay repayment possesses some ex ante value. If the term structure were flat and nonstochastic, so that all interest rates were constant through time, then the strict form of our findings would not obtain. Instead, the contract rate on amortizing and nonamortizing instruments of various maturities would be equal.

The results in Propositions 1 and 2 can be applied to various special cases. For example, the contract rate on nonamortizing (coupon) bonds of various maturities can be compared using Proposition 1. The coupon rate (payment) on a nonamortizing, callable coupon bond cannot decrease as the bond's maturity increases. Unlike the yield curve for *noncallable* bonds, Proposition 1 implies that the call option restricts the yield curve for *callable* bonds selling at par to be nondecreasing with the maturity.[6] Hence, even if the noncallable yield curve were inverted, the yield curve for callable bonds would never be downward sloping. In fact, with deterministic interest rates and an inverted noncallable yield curve, longer contracts collapse to one-period instruments due to exercise of the call option, so that the callable yield curve cannot be inverted. With stochastic interest rates and an inverted noncallable yield curve, the yields on callable debt can increase with maturity. However, while the callable yield curve is never inverted, the callable curve can be flatter than the noncallable yield curve. This can occur, for example, when volatility is greater in the near future than in the distant future.[7]

[6] Dybvig, Ingersoll, and Ross (1996) examine by arbitrage arguments the convergence properties of long-term interest rates on perpetual noncallable securities. Related issues are explored in an equilibrium context by Backus, Gregory, and Zin (1989).

[7] We illustrate that the callable yield curve can be flatter than the noncallable curve by an example in which the interest rate volatility occurs only in the initial period. We suppose that the initial period's interest rate is either 0 percent or 100 percent (rates that are equally likely in the "risk-neutral" economy), and that the borrower contracts for a loan at the instant prior to the resolution of uncertainty. The borrower is able to redeem the callable debt as soon as the uncertainty is resolved (i.e., prior to the passage of time). After the first period, the interest rate is 100 percent per period. In this setting we compare the interest rates on nonamortizing callable and noncallable bonds that are either one-period or perpetual bonds. Both the one-period and perpetual callable debt will be immediately redeemed in the low interest rate state, so that the pricing of the debt reflects only the high interest rate state. Both the one-period and perpetual callable loans then have a yield of 100 percent. The yield of the one-period noncallable loan is $33\frac{1}{3}$ percent, and the yield of the perpetual noncallable loan is $66\frac{2}{3}$ percent. In this example the noncallable yield curve is upward sloping, and the callable yield curve is actually flat and therefore less steep than the noncallable yield curve.

Stronger restrictions for nonamortizing callable coupon bonds (beyond the yield curve weakly increasing) cannot be obtained without imposing substantive restrictions on the economic environment. Any nondecreasing set of coupon rates on callable nonamortizing bonds can be generated from an identical noncallable yield curve, since if the noncallable yield curve is deterministic borrowers prepay only to reduce their interest rates. This construction always works because the call option has zero value in an upward-sloping yield curve environment under certainty.

By applying both Propositions 1 and 2 it is also straightforward to show that the contract rate on a self-amortizing (i.e., $T = n$), callable bond cannot decrease as the bond's maturity increases. Again, the borrower would never select the shorter instrument if it possessed a higher contract rate than the longer one because the longer instrument possesses the option of delaying repayments beyond the time mandated by the shorter contract.[8]

Proposition 2 also restricts the relative pricing of amortizing and nonamortizing debt of the same maturity. A callable nonamortizing loan (coupon bond) cannot have a lower contract rate than any (partially or fully) amortizing callable loan of the same maturity, n. A callable self-amortizing loan (some mortgages) cannot have a higher interest rate than any partially amortizing or nonamortizing callable loan of the same maturity, n. By varying T, Proposition 2 implies that the pricing and yields of loans that partially amortize ($n < T < \infty$) must be intermediate between the nonamortizing and self-amortizing contracts. Numerical solutions provided in Dunn and McConnell (1981) illustrate the magnitude of the additional value to the investor (lower yields) in various interest rate states of the self-amortizing callable loan over the nonamortizing callable loan.

At equal contract rates the borrower prefers to issue a nonamortizing callable loan because it possesses the most flexible option to the borrower (principal amortization is not required under the nonamortizing loan).

COROLLARY 1: *A callable nonamortizing loan cannot have a lower contract rate than any other loan of the same maturity (irrespective of option and amortization status).*

The generality of our results is driven by the nature of the call option. We compare the contract rates on instruments of different lengths that cause prices to be the same; however, an alternative interpretation involves comparing differences in prices while holding the coupon rates the same. In particular, comparing the market value of instruments of different lengths (allowing n and T to differ) that are callable at par but possess otherwise identical terms (same contract rate and loan balance), the longer instrument (higher n or T) has a smaller value to the lender and capital market (so at issuance it would possess more points) because the borrower possesses a more flexible prepayment option.

[8] Berk (1999) uses this idea to compare the value of real options of different lengths.

Numerical solutions illustrate the magnitude of the decline in the value of both callable amortizing and nonamortizing securities as a function of the term to maturity in various interest rate states (see the results and discussion in Dunn and McConnell (1981)). A borrower's decision to prepay the longer instrument implies that the same borrower would have prepaid the shorter loan, which possesses a more limited option for the borrower. For example, absent refinancing costs the optimal call boundary is to call the respective instruments when their prices equal the call price of par, a condition first satisfied by the shorter and more valuable debt contract. Since the longer and shorter loans have the same periodic coupon and cash flow, the longer bond must be (weakly) less valuable to investors than the shorter loan at any earlier date (prior to the optimal exercise point) as it is (weakly) less valuable than the shorter loan when it is optimal to call the shorter loan or at the maturity of the shorter loan. The result is analogous to the nondecreasing value of American put and call options on equity in the maturity for dividend-paying stocks, implying that if it is optimal to exercise an option of a specified maturity then it also must be optimal to exercise those options with shorter maturities but otherwise identical terms.

The specific results about callable loans in this section obtain with reverse inequalities for debt instruments that are putable at par, so that the lender can terminate the loan at will.[9] The incremental put option on a longer contract is valuable to the lender. The absence of dominance implies that the rate on longer putable securities cannot exceed those on shorter instruments. However, the rates for putable nonamortizing bonds cannot decline too rapidly with the maturity due to our assumption of nonnegative forward interest rates. If the debt instruments have both American put and call provisions (without premia), then our arguments imply that the contract rate is independent of n and T.[10]

[9] A well-known type of security that is putable (but not callable) is savings bonds, but there are not alternative effective maturities. Though bank deposits of various maturities are often putable, there may be a substantial penalty for redemption prior to maturity. Bank loans are sometimes both putable ("callable" by the lender) and callable (the borrower can prepay). Putable bonds are becoming more widely used in corporate debt as illustrated by Perlmuth (1997).

[10] Under the additional assumption of no refinancing costs it must be strictly optimal for one of the parties to exercise the option to terminate the security, unless it sells at par. Of course, then the assumed variation in the contract length is irrelevant as the instruments reduce to a short-term contract (a discussion of the disadvantages of short-term contracts in mortgage markets is included in Dunn and Spatt (1988)). McConnell and Schwartz (1986) present an analysis of the valuation of debt contracts with both call and put provisions, where the call price schedule exceeds the put price schedule until a few years before maturity and the put option is only exercisable at a discrete set of dates.

Our analysis of the effect of the call option and the qualitative contrast between the impact of the call and put options raises the question of why debt contracts are often callable and rarely putable. While the call and put option can be fairly priced using the tools of modern contingent claims analysis, the different forms for the debt contract lead to different transactions costs. If the contract were not callable, then in light of the secured property backing the loan (necessitated by conventional agency problems) the mortgage borrower could not prepay and transfer the secured property (which would impose a severe agency problem) despite the

The analysis of contract yield, length, and prepayment raises the question of whether the results for the prepayment option also apply to the default option on nonprepayable debt. Merton's (1974) analysis of the risk structure of interest rates points out that the yield on risky single payment debt need not be monotone in the contract maturity. If most of the resolution of uncertainty were prior to the maturity of the shorter of a pair of contracts, then the yield would be lower on the longer contract, because the high default premium required for the early period is "averaged" over a longer interval of time. If this loan were prepayable, then the rate cannot decline as maturity increases because the loan would be prepaid after the interval of high rates and substantial default risk had passed.

To simplify the presentation in our formal analysis of contract yields, we assume that the debt instruments examined are not subject to default risk. Of course, in practice the borrower may find it optimal to default when the collateral value is below the indebtedness. The basic logic underlying our results about the impact of contract length upon the borrower's prepayment option is robust to the presence of default risk. The longer contract, whether measured by the date of the balloon obligation or the length of the amortization interval, cannot have a lower interest rate because no rational borrower would select the shorter contract if it possessed a higher interest rate than the longer contract since the longer contract also provides the borrower with a more flexible combined prepayment and default option.

II. Adjustable-Rate Loans

In this section we apply the analysis concerning callability and debt pricing to adjustable-rate loans. Absent refinancing costs, a "fully-indexed" adjustable-rate loan, which we define as one that resets to sell at par after n periods, is equivalent to a fixed-rate contract with the same amortization length and a maturity equal to the reset period on the adjustable-rate loan. This equivalence allows us to compare the initial interest rate on adjustable-rate loans for different initial reset periods.

presence of a large utility gain from moving. (Dunn and McConnell (1981) model the valuation of GNMAs in the presence of prepayments due to gains from moving and Dunn and Spatt (1985) examine the optimal mortgage contract design and ex post lender strategies in the presence of utility gains from moving.) Similarly, the corporate borrower would not be able to eliminate restrictive covenants that ex post impose high transactional costs (see Kraus (1983)). Therefore, most mortgages and many corporate debt contracts are callable. Notice that if the callability of the mortgage contract were contractually permitted only when the borrower transfers his property and moves, then substantial interest rate declines would induce otherwise inefficient property transactions in order to sustain interest-rate-motivated refinancing. In fact, in a noncompetitive setting with refinancing costs, a putable contract allows the lender to force the borrower to refinance and impose these costs on the borrower. In particular, if the lender could credibly threaten to force the borrower to refinance, then the lender would extract a side payment (a rate concession) from the borrower to avoid the refinancing costs. A detailed analysis of the bargaining power of putable bondholders is in David (1996).

COROLLARY 2: *Absent refinancing costs, the initial rate (and payment) on a fully indexed adjustable-rate contract with a fixed amortization interval cannot decrease as the initial reset period increases.*

For example, Corollary 2 implies that the initial interest rate and payment on a callable fixed-rate instrument must (weakly) exceed the interest rate and payment on a fully indexed adjustable-rate contract with the same amortization interval and with a reset interval shorter than the maturity of the fixed-rate contract. Otherwise, the adjustable-rate contract would be dominated by a strategy of initially financing with the fixed-rate contract, which provides better interest rate protection and a lower initial interest rate.

However, in the presence of refinancing costs, an adjustable-rate contract might be more attractive for the borrower than a fixed-rate balloon contract, because the adjustable-rate contract allows the borrower to avoid the refinancing costs incurred under a fixed-rate balloon contract with the same initial interest rate and maturity equal to the reset interval of the adjustable-rate contract. For example, if refinancing costs are large enough and the term structure for noncallable bonds is downward sloping, then the rate on a fixed-rate contract can be less than the initial rate on a fully indexed adjustable-rate contract.

Even if refinancing is costless we cannot use Proposition 1 to compare fixed-rate loans to adjustable-rate loans unless the interest rate is reset to ensure that the market value equals the par value on the reset date. Typical adjustable-rate loans may not reset to par due to contractual terms such as the loan's interest rate cap, margin, and benchmark index. For example, the initial interest rate on an adjustable instrument can exceed the rate on a fixed-rate contract because the markup over the index at later adjustment dates is low. Alternatively, if the adjustable rate is set in each period (including the initial one) to be a constant markup over a specified benchmark index, then the relationship between the benchmark index and the fixed-rate contract will determine whether in some interest rate states the adjustable rate exceeds the fixed rate.

The arbitrage approach and option arguments can be used to obtain restrictions on adjustable-rate contracts with different caps and floors. Adjustable-rate contracts often contain a periodic cap that limits the size of interest rate increases at each reset date and a lifetime cap that establishes a ceiling on the maximum rate the contract can achieve during its entire lifetime. Holding other loan characteristics constant, borrowers prefer lower caps (either periodic or lifetime) because the resulting path of future contract interest rates is (weakly) lower. Consequently, in order to prevent dominance, a contract with relatively lower caps must possess an initial interest rate that is at least as high. This conclusion is very general and applies directly to caps in observed adjustable-rate loans.

Adjustable-rate contracts also often contain periodic and lifetime floors that limit the maximum periodic and lifetime decreases in the interest rate under the contract. The floors can be costly to the borrower and valuable to the lender because the resulting path of future interest rates is (weakly) higher. Lenders prefer higher to lower floors. So, holding other loan characteristics constant, a contract with a relatively lower floor must have at least as high an initial interest rate. Under somewhat more restrictive conditions than for the analysis above, the periodic and lifetime floors have zero economic value due to the borrower's right to prepay. Specifically, if refinancing is costless and the contractual reset rate is restricted to never providing an interest rate below the market rate, a borrower can replicate the decline in the contract interest rate by refinancing. The restriction on the contractual index structure is necessary to prevent the floor from being valuable due to the possibility of the contractual reset rate being below the market refinancing interest rate. Under these restrictive conditions, we are able to reach several conclusions. First, a tighter collar (a tighter floor and cap) leads to a higher initial interest rate, due to the impact of the cap. Second, a fixed-rate loan can be interpreted as an adjustable-rate loan with a periodic cap and floor equal to the fixed rate (a zero collar). This is an alternative to Corollary 2 to establish that an adjustable-rate loan possesses a (weakly) lower initial interest rate than a fixed-rate loan.

III. Contract Payments and Length

In this section we further examine the impact of the contract's length on its periodic payment and interest rate. As a preliminary to our analysis, note that at each date the outstanding balance on any loan, regardless of option status, is an increasing function of the contract interest rate—illustrating the standard mathematics of amortization. This is a consequence of the observation that at higher contract interest rates relatively less of the periodic payment is allocated to amortization early in the contract, and relatively more is allocated to amortization in the later periods.

LEMMA 1: *The outstanding balance after t periods of a loan (whose initial balance is 1) of fixed amortization interval T,* $1 - [((1+R)^t - 1)/((1+R)^T - 1)]$, *is a strictly increasing function (for* $0 < t < T$ *and T finite) of its contract rate, R.*

In order to obtain further restrictions on contract interest rates, we show that callable and option-free contracts with longer amortization intervals have smaller periodic payments. (The same conclusion applies to putable loans, but the analysis in Section I implies the stronger restriction that longer putable self-amortizing contracts cannot have a higher interest rate.) Rather than requiring the loan to be callable without penalty, we assume that the call premium schedule per dollar of outstanding principal balance is identical prior to the maturity of the shorter loan. For callable debt the proof compares the balance schedules of the loans, though for option-free debt the proof would have been immediate.

PROPOSITION 3: *The periodic payment, P, on a loan contract decreases with an increase in the loan's amortization interval, T, for a loan of maturity, n* ($n \leq T$). *Thus, if two loan contracts,* $j = 1,2$, *have* $n_1 = n_2$ *and* $T_1 > T_2$, *then* $P_1 < P_2$. *Similarly, the periodic payment on the loan contract also decreases in the loan's amortization interval for a self-amortizing loan* ($n = T$). *Thus, if two loan contracts,* $j = 1,2$, *have* $T_1 = n_1 > T_2 = n_2$, *then* $P_1 < P_2$.[11]

From Proposition 3 it follows that the rate on the shorter contract cannot be lower than that which equates the payment on the two contracts. To express the restriction on payments given by Proposition 3 in terms of interest rates, we equate the payment per dollar of initial loan balance on contracts with amortization intervals $T_1 > T_2$ and interest rates R_1 and R_2 to obtain

$$P = \left[\sum_{j=1}^{T_1} \frac{1}{(1+R_1)^j}\right]^{-1} = \left[\sum_{j=1}^{T_2} \frac{1}{(1+R_2)^j}\right]^{-1}. \tag{1}$$

Given a rate on one of the contracts, equation (1) can be used to solve for the bound on the other contract's rate. The results in Propositions 1 and 2 identify lower bounds on the longer contract's rate, relative to the shorter contract's rate. In contrast, the bound identified in equation (1) and Proposition 3 is an upper bound on the longer contract's rate for self-amortizing loans. In contrast to the bounds for self-amortizing debt being described here, for nonamortizing callable debt any increasing contract rate sequence can be supported (see the discussion in Section I). Equation (1) implies that the upper bound on the longer contract's rate increases with the rate on the shorter contract and the amortization interval on the longer contract.

Numerical illustrations of the bounds are shown in Table I. The table provides bounds for contracts with differing amortization intervals but identical option provisions. Table I is constructed by computing the monthly payments on 30-year amortizing loans with annual interest rates on a grid ranging from two percent to 20 percent in increments of two percent. These payments are used in equation (1) to calculate the annual interest rates on a perpetual loan and loans that amortize over 15 and 20 years. The table illustrates that these bounds are surprisingly tight for long instruments when contract interest rates are high because most of the periodic payment is then allocated to interest and relatively little to repay principal. For example, if the 30-year contract rate is 16 percent (in which case less than one percent of the periodic payment is initially available to amortize the principal), then

[11] McConnell and Schallheim (1983) report that for financial leases the equilibrium lease payments decline as the maturity increases due to the depreciation of the underlying asset.

Table I

Payments and Interest Rates for Loans of Various Amortization Intervals

The initial principal balance is $10,000 and the monthly payment is specified in the first column. The remaining columns refer to annual interest rates on amortizing debt of various amortization intervals. The **** entries arise when the monthly payment is insufficient to amortize $10,000 over the indicated amortization interval.

Monthly Payment	Annual Interest Rates for Various Amortization Intervals			
	Perpetual	30-Year	20-Year	15-Year
36.96	4.44	2.00	****	****
47.74	5.73	4.00	1.38	****
59.96	7.19	6.00	3.87	1.03
73.38	8.81	8.00	6.30	3.89
87.76	10.53	10.00	8.65	6.61
102.86	12.34	12.00	10.95	9.25
118.49	14.22	14.00	13.19	11.77
134.48	16.14	16.00	15.38	14.20
150.71	18.09	18.00	17.52	16.54
167.10	20.05	20.00	19.65	18.84

the rate on the perpetuity cannot exceed 16.14 percent and the 20-year contract rate is at least 15.38 percent. In contrast, the bound has little or no bite for short amortization contracts and in low interest rate settings.

IV. Points and Contract Rates

Without imposing parametric assumptions we derive restrictions on the relative contract interest rates and points (original discount) for different debt instruments by using the linearity of the pricing operator (see Cox and Ross (1976)) and assuming the absence of market frictions. We focus initially on noncallable contracts in order to lay the foundation for restrictions on callable debt and to isolate the effects of callability and amortization. We show that for noncallable self-amortizing and nonamortizing loans the trade-off between points and payment is linear. Further, the points are linear in the interest rate for nonamortizing (and one-period) loans and strictly concave in the interest rate for self-amortizing loans.

A noncallable debt instrument i whose maturity is n and amortization interval is T ($T \geq n$) satisfies

$$1 - X_i = \left(\sum_{k=1}^{n} q_k \right) P_i + q_n B_i, \tag{2}$$

where X_i represents the points per dollar of original balance, P_i the corresponding periodic payment, B_i the balloon obligation for loan i (after n payments), and q_k is the value of a dollar of payment at date k. The valuation of the periodic payments and the terminal balloon obligation must equal the net amount financed, $1 - X_i$. Taking the ratio of equation (2) for two different self-amortizing (zero balloon obligation) noncallable loans implies

$$\frac{1-X_i}{P_i} = \frac{1-X_j}{P_j} = \sum_{k=1}^{n} q_k. \tag{3}$$

The amount financed $(1 - X)$ must be proportional to the payment to prevent the dominance of one of the loans, which implies an exact relationship between points and the contract interest rate, after substituting the payment formula, $P(R) = R/(1 - (1 + R)^{-T})$. The only information about the term structure needed to determine the exact trade-off between points and the contract interest rate for self-amortizing noncallable debt is the price of an n-period annuity.[12]

Additional restrictions can be derived by differencing equation (2) for two different loans to obtain

$$X_j - X_i = \left(\sum_{k=1}^{n} q_k\right)(P_i - P_j) + q_n(B_i - B_j). \tag{4}$$

For both self-amortizing and nonamortizing loans, $B_i = B_j$, so that

$$X_j - X_i = \left(\sum_{k=1}^{n} q_k\right)(P_i - P_j),$$

or

$$X_j - X_i = -c(P_j - P_i). \tag{5}$$

Equation (5) describes a linear relation between points and contract payments, where the absolute value of the slope is increasing in n. For nonamortizing loans the periodic payment is identical to the periodic contract rate so that for these loans the trade-off between points and contract rates is also linear.

[12] The increase in points required to reduce the contract interest rate for self-amortizing noncallable instruments increases at a decreasing rate with the contract's maturity. For example, using the relationship derived in equation (3), if a 10 percent loan commanded two points, then a 9.5 percent loan would require 2.26 points if the common maturity were one year, 3.13 points if it were five years, 4.04 points if it were 10 years, 6.1 points if it were 30 years, and 6.9 points if both loans were perpetuities.

For noncallable self-amortizing loans we can combine equations (3) and (5) to obtain

$$\frac{\Delta X}{\Delta P} = \frac{X_j - X_i}{P_j - P_i} = -\frac{1 - X_i}{P_i}. \tag{6}$$

Taking the limit implies

$$X'(P(R)) = -(1 - X)/P(R), \tag{7}$$

so that the chain rule yields

$$\frac{dX(P(R))}{dR} = X'(P(R))P'(R) = -(1 - X)\frac{P'(R)}{P(R)}. \tag{8}$$

Using

$$P(R) = R/[1 - (1 + R)^{-T}] \tag{9}$$

and

$$P'(R) = \frac{[1 - (1 + R)^{-T}] - RT(1 + R)^{-T-1}}{[1 - (1 + R)^{-T}]^2}, \tag{10}$$

we can evaluate the local trade-off between the contract rate and points for any noncallable self-amortizing loan with contract parameters X, R, and T.

The instantaneous trade-off between contract rate and points is illustrated numerically in Table II. The elements of the table are $\frac{1}{12}$ ($|dX/dR|$), where R is the monthly interest rate. For example, in Table II the absolute value of the trade-off for a 10 percent 30-year loan with two points is 8.252. This means that at the margin the points increase by 8.252 points per 100 basis point decrease in the annual contract interest rate. The table's columns show that the trade-off is decreasing in the contract interest rate for fixed values of the other contract parameters. Similarly, the rows in the table show that the trade-off is decreasing in the contract's points for fixed values of the other parameters.

To examine the impact of amortization on the trade-off between points and the contract interest rate we first observe that the periodic payment is convex in the interest rate.

Table II

Absolute Value of Slope of Points—Rate Trade-off for Noncallable Self-Amortizing 30-Year Contracts

The table is based on 30-year noncallable self-amortizing loans. The interpretation of the elements of the table is illustrated by the element corresponding to a 10 percent annual contract rate issued at a discount of five points (a discount of five points means that the borrower receives only $.95 for each dollar of original balance). At the margin (i.e., locally) the annual contract rate decreases by one percent per increase of eight points in the discount. This table does not provide the trade-off for discrete (nonlocal) changes. The trade-off implicit in the table is only a bound for callable loans. Because the elements in the table capture the trade-off between the points and annual interest rate, each element in the table is computed from the monthly interest rate R using the relation

$$\left|\frac{dX(P(R))}{d(12R)}\right| = \frac{1}{12}(1-X)\frac{P'(R)}{P(R)},$$

where

$$P(R) = R/[1-(1+R)^{-T}] \quad \text{and} \quad P'(R) = \frac{1-(1+R)^{-T} - RT(1+R)^{-T-1}}{[1-(1+R)^{-T}]^2}.$$

Annual Interest Rate	Points 0	1	2	3	4	5
2 percent	13.529	13.394	13.259	13.123	12.988	12.853
4 percent	12.076	11.955	11.834	11.713	11.593	11.472
6 percent	10.723	10.616	10.509	10.402	10.294	10.187
8 percent	9.501	9.406	9.311	9.216	9.121	9.026
10 percent	8.421	8.336	8.252	8.168	8.084	8.000
12 percent	7.483	7.409	7.334	7.259	7.184	7.109
14 percent	6.680	6.613	6.547	6.480	6.413	6.346
16 percent	5.996	5.936	5.876	5.816	5.756	5.697
18 percent	5.416	5.362	5.308	5.253	5.199	5.145
20 percent	4.923	4.874	4.824	4.775	4.726	4.677

LEMMA 2: *The periodic payment, $P(R) = R/[1-(1+R)^{-T}]$, is strictly convex in the contract interest rate provided $T > 1$ and T finite.*

For self-amortizing loans the points are linear in the contract payment with negative slope (equation (5)) and the payment is strictly convex in the interest rate, so that the points are strictly concave in the interest rate.

COROLLARY 3: *Consider three noncallable self-amortizing loans of the same maturity with $T > 1$ and T finite, whose initial balance is normalized to one. Let R_j denote the periodic interest rate and X_j the original discount for the jth loan ($j = 1,2,3$). Then, if $R_2 = \lambda R_1 + (1-\lambda)R_3$ for $0 < \lambda < 1$, then $X_2 > \lambda X_1 + (1-\lambda)X_3$; that is, the points per dollar of initial balance are a strictly concave function of the contract interest rate.*

For $T = 1$ the payment satisfies $P(R) = 1 + R$ and for $T = \infty$ (a nonamortizing loan), $P(R) = R$. The points and payment are linear in the interest rate in these cases. In practice, the relationship in Corollary 3 tends to be reversed for callable loans because the impact of the call option typically dominates the effect of the amortization.

The trade-off between contractual points and interest rate that is reflected in equation (3) for noncallable self-amortizing loans can be expressed in inequality form in Proposition 4 for callable self-amortizing debt.

PROPOSITION 4: *If R_1 and R_2 ($R_1 > R_2$) denote the contract interest rates on two callable self-amortizing loans that mature in T periods and P_1 and P_2 ($P_1 > P_2$) denote the corresponding payments, then the corresponding points on the callable instruments, X_1^* and X_2^*, must satisfy*[13]

$$\frac{1 - X_2^*}{P_2} \geq \frac{1 - X_1^*}{P_1} \quad \text{and} \quad X_1^* < X_2^*. \tag{11}$$

Although the points are lower on the high interest rate loan than on the low interest rate loan, they cannot be so much lower that the amount of initial funding $(1 - X^*)$ per dollar of payment is larger for the high interest rate loan.

Substituting the payment formula, $P(R) = R/[1 - (1 + R)^{-T}]$, implies that the trade-off between points and interest rates for callable loans can be written as[14]

$$\frac{1 - X_2^*}{1 - X_1^*} \geq \frac{P_2}{P_1} = \frac{R_2}{R_1} \frac{(1 - (1 + R_1)^{-T})}{(1 - (1 + R_2)^{-T})}. \tag{12}$$

[13] The trade-off between points and interest rate that we characterize is based on newly issued loans. The pricing in the secondary market provides another illustration of the trade-off between price and the interest rate. For callable mortgage-backed securities a 25 basis point increase in coupon often leads to about a one percent decline in price. The observed trade-off of four for one in the secondary market is similar to the observed trade-off at issuance. This trade-off is substantially less than the trade-off for noncallable bonds that is implicit in equations (3) and (8), due to the impact of callability. The dollar spread across coupon differentials is also suggestive of the dollar impact of a change in market interest rates on existing loans. In fact, many mortgages trading near par have durations of approximately four.

[14] An interesting example that illustrates Proposition 4 is the case of a Pittsburgh lender who simultaneously offered a 9.5 percent self-amortizing 15-year contract with 1.75 points and a 9.25 percent self-amortizing 15-year contract with 3.25 points. If we assume *both instruments were noncallable*, self-amortizing, and matured in 15 years (and fixing the parameters 9.25 percent, 9.5 percent, and 1.75), then the points on the 9.25 percent instrument would be 3.1644 (see equation (3)). Hence, for the corresponding callables the condition in Proposition 4 is violated and the 9.25 percent contract is dominated. By selecting the 15-year 9.5 percent contract the borrower's monthly payment per \$10,000 of initial balance would be \$104.42 or \$106.28 per \$10,000 of initial funding under the loan (net of the 1.75 points); the 9.25 percent contract would have a payment of \$102.92 per \$10,000 of initial balance or \$106.38 net of the 3.25

Another interpretation of the points to rate trade-off for callable loans as compared to noncallable loans is that for a given interest rate change the corresponding change in the points is greater for the pair of noncallable loans than for the pair of callable loans. This is a consequence of the value of the call option increasing in the coupon rate.

The trade-off between points and loan payments varies with the specific mix of points and interest rate.[15] We summarize some properties of the balance function in order to establish that points are a strictly convex function of payments for *callable* debt.

LEMMA 3: *The outstanding balance after t periods on a loan whose initial balance is 1, of fixed amortization interval, T,* $1 - [((1 + R)^t - 1)/((1 + R)^T - 1)]$, *is a strictly concave function of its contract rate and payment (for* $0 < t < T$ *and T finite).*

Note that Lemma 3 is actually a generalization of the convexity of the payment function in the contract interest rate (Lemma 2), since the balance after one period per dollar of original balance is $(1 + R) - P(R)$, but Lemma 3 also applies to the balances after each date.

PROPOSITION 5: *Consider three callable loans of the same maturity with an initial balance normalized to 1. Let* P_j *denote the periodic payment on loan j and* X_j^* *denote the initial discount (points) for loan j. If* $P_2 = \lambda P_1 + (1 - \lambda) P_3$ *for* $0 < \lambda < 1$, *then* $X_2^* \leq \lambda X_1^* + (1 - \lambda) X_3^*$; *that is, the points per dollar of initial balance are a convex function of the contract payments.*[16]

We now use this result to examine the shape of the relationship between points and interest rates for callable loans. In the case of nonamortizing loans the contractual payment and rate are identical so that the points are

points. The 9.25 percent is dominated because its normalized payment (and balance) for funding is higher, while its call option is less valuable. Notice that if the contract with the higher rate and lower points were for 30 years instead of 15 years and the previously dominated contract is unaltered (e.g., 15 years), then the call option on the higher rate, longer contract is even more valuable to the borrower so that the lower rate contract is still dominated. This illustrates how restrictions in the paper can be combined.

[15] Although points are typically nonnegative, we are familiar with one lender that has offered a variety of negative points products. A restriction on the magnitude of negative points is that the points for callable debt cannot be a negative amount with an absolute value in excess of the cost of refinancing. If a high interest rate loan had negative points in excess of that level, then the lender would lose money if the borrower refinanced immediately, whether to another such product or to a different lender. In late 1996 and early 1997 a Pittsburgh lender was offering a menu of negative points loans that offered rebates in excess of three percent of the face value of the loan for the highest coupons. This clearly exceeded the refinancing costs of some borrowers with large loans.

[16] Merton (1973) shows that the value of a call option is increasing and convex in the exercise price for equity options. The convexity of the value of the call option in the loan payment is also similar to the result in Merton (1973) that a portfolio of options is at least as valuable as an option on the underlying portfolio of assets. The main complication in Proposition 5 relative to the analysis of equity options is that, except for nonamortizing loans, different balances across contracts must be considered (see the use of Lemma 3 in the proof of Proposition 5).

a convex function of the contractual coupon rates. We demonstrated in Corollary 3 that the points are a strictly concave function of the contractual coupon rates for self-amortizing noncallable loans. On the other hand, the value of the call option is convex in the contract interest rate.[17] For observed levels of volatility the call effect is larger than the amortization effect for callable amortizing loans. Therefore, we typically would expect that points are convex in the contractual interest rates.

V. Concluding Comments

We obtain strong conclusions on the relative pricing of debt contracts under very mild assumptions. These results allow us to understand the economic intuition underlying the pricing of callability, amortization, contract length, and original discount in debt contracts.

Appendix

Proof of Proposition 1: Consider a pair of callable loan contracts with the same amortization interval, T, and balloon payments due at maturities n_1 and n_2, respectively, where $T \geq n_1 > n_2$. Note that the required contract payment increases in the interest rate for a fixed amortization interval since $P(R) = R/(1 - (1 + R)^{-T})$ is increasing in R. If, contrary to the proposition, the longer contract (contract 1) has a lower interest rate (and lower payment), it would be selected by every borrower. The borrower on the longer contract possesses a richer set of options, which includes the possibility of mimicking the payments and prepayment options of the shorter contract. If the borrower on the longer and lower interest rate contract mimics the payments of the shorter contract, then the path of outstanding balances would be lower than under the shorter contract. In order for some borrowers to select the shorter contract, it cannot have a higher interest rate or payment. Q.E.D.

Proof of Proposition 2: We examine a pair of contracts in which the amortization is based on repayment in T_1 and T_2 periods ($T_1 > T_2$). If the respective interest rates satisfy $R_1 < R_2$, then the two contracts cannot coexist because contract 1 is preferred to contract 2 by every borrower. Because contract 1 has a lower interest rate and provides the option of slower repayment than permitted under the higher interest rate contract 2, while contract 1 also possesses all the options enjoyed by the T_2 borrower, contract 1 dominates contract 2. Q.E.D.

[17] The value of the call option can be expressed as a composite function of the loan payment and contract rate, $C(P(R))$. By the chain rule, $dC/dR = C'(P(R))P'(R)$, so that $d^2C/dR^2 = C'(P(R))P''(R) + C''(P(R))(P'(R))^2$. Since the value of the call option is increasing and convex in the payment, and the loan payment is convex in the interest rate (see Lemma 2), it also follows that the value of the call option is convex in the contract interest rate.

Proof of Corollary 1: The most valuable loans for the borrower are those that are callable without penalty. Among these, the nonamortizing loans are most flexible for the borrower. To prevent dominance, the callable nonamortizing loan cannot have a lower contract rate than any other loan of the same maturity. Q.E.D.

Proof of Corollary 2: Both the fully indexed adjustable-rate loan that resets to par in n periods and amortizes over T periods and the balloon obligation loan with maturity n and amortization interval T have the same payments for n periods and the same value at the end of n periods, so they must have identical present values. Therefore, the result is a direct consequence of Proposition 1. Q.E.D.

Proof of Lemma 1: The proof of this lemma is available from the authors upon request.

Proof of Proposition 3: We consider a pair of otherwise identical contracts that can differ in their amortization interval, and, therefore, in their payment and interest rate. For contract i let P_i denote the periodic payment, T_i the amortization interval, and R_i the periodic interest rate. We let $T_1 > T_2$, so that under the contrary supposition $P_1 \geq P_2$, the longer contract has a periodic payment at least as large as the shorter one. The restrictions that $T_1 > T_2$ and $P_1 \geq P_2$ imply $R_1 > R_2$ because the contract interest rate is an increasing function of the coupon payment given the amortization interval and an increasing function of the amortization interval given the coupon payment. Lemma 1 implies that if two contracts have the same amortization interval, then the lower rate contract must have lower outstanding principal balance at each date, except at the loan origination date and at the end of the amortization interval. Under the maintained assumption that the call price schedules per dollar of remaining balance are identical for the two instruments during the interval that both are outstanding, the borrower strictly prefers contract 2 (the shorter contract) because it has lower balances and payments at each date, even if the borrower is constrained to employ the same call policy as required under contract 1. Of course, the contract 2 borrower would do even better by following the optimal call strategy for contract 2. This contradicts the absence of a dominated contract. The logic of the proof applies not only to the cases in which the loans have the same maturity, but also when the loans are self-amortizing with different maturities. Q.E.D.

Proof of Lemma 2: The proof of this lemma is available from the authors upon request.

Proof of Corollary 3: We denote the payment as a function of the contract rate by $P(R)$ and the points as a function of the payment by $X(P(.))$. Differentiating twice yields $d^2X/dR^2 = P''(R)X'(P(R)) + X''(P(R))(P'(R))^2$. Equation (5) implies $X'(P(R)) < 0$ and $X''(P(R)) = 0$, and Lemma 2 implies (for $T > 1$ and T finite) that $P''(R) > 0$. Hence, $d^2X/dR^2 < 0$. Q.E.D.

Proof of Proposition 4: We suppose to the contrary that $[(1 - X_2^*)/P_2] < [(1 - X_1^*)/P_1]$. To show the desired contradiction, we scale contract 2 by P_1/P_2 to have the same payments as contract 1 and show that contract 1 has greater initial funding and lower outstanding balances at all times. Under the contrary hypothesis, the amount of initial funding for contract 1 is greater than the amount of initial funding for P_1/P_2 units of contract 2, $[(P_1/P_2)(1 - X_2^*)] < 1 - X_1^*$. (For noncallable self-amortizing loans, recall that $(1 - X_2)/P_2 = (1 - X_1)/P_1$, the amount of initial funding per dollar of payment is equal for all loans.) The balance on contract 1 is also less than the balance on P_1/P_2 units of contract 2 for $0 < t < T$, $B_1^t < (P_1/P_2)B_2^t$. Since the periodic payments are identical on contract 1 and P_1/P_2 units of contract 2, the inequality is a consequence of the higher interest rate for contract 1 as the balance is the discounted value of the remaining payments at the contract specific interest rate. Having shown that the balances are lower for contract 1 than for P_1/P_2 units of contract 2, the borrower using contract 1 has a lower balance when extinguishing the loan. Contract 1 provides greater initial funding than P_1/P_2 units of contract 2 and has identical periodic payments, but it has lower dollar balances and, therefore, a lower payment to prepay the loan. Borrowers will prefer contract 1 to contract 2, even if the borrowers using contract 1 are constrained to follow the prepayment strategy for contract 2. Of course, the borrowers using contract 1 are even better off if their exercise strategy is not restricted, and, in fact, they find it optimal to prepay at higher interest rates. Hence, contract 2 is dominated by contract 1 from the borrower's perspective, which implies that $(1 - X_2^*)/P_2 \geq (1 - X_1^*)/P_1$. The condition $X_1^* < X_2^*$ also holds, since if $X_1^* \geq X_2^*$, then all borrowers would prefer contract 2 which would offer greater funding, lower periodic payments, and lower balances, due to its lower interest rate per Lemma 1.[18] Q.E.D.

Proof of Lemma 3: The proof of this lemma is available from the authors upon request.

Proof of Proposition 5: In the case of nonamortizing loans the balances are the same across the three loans and the payments equal the interest rates so the result follows directly from the convexity of the call options in the loans. For partially and fully amortizing loans, by Lemma 3 the balance function is strictly concave in the payment. This reinforces the relative unattractiveness of the contract with the intermediate coupon payment that is a consequence of the potential value of making different prepayment decisions on loans with different coupon payments. Given its high balance and less valu-

[18] The result extends to the case in which the contracts have identical refinancing cost functions that increase in the amount to be financed. Since contract 1 has lower dollar balances than P_1/P_2 units of contract 2, the refinancing costs for contract 1 are less than that for contract 2 (this is true for both a proportional refinancing cost component and a fixed cost per contract unit as contract 2 has higher balances and more contractual units). Since the payoff balance for contract 1 is less than for P_1/P_2 units of contract 2, the total payoff cost of contract 1 is less than that of contract 2 and the proof of the proposition extends to refinancing costs.

able call options, the middle coupon loan is dominated for any borrower by a portfolio of the higher and lower payment loans if $(1 - X_2^*) < \lambda(1 - X_1^*) + (1 - \lambda)(1 - X_3^*)$. Therefore, $1 - X_2^* \geq \lambda(1 - X_1^*) + (1 - \lambda)(1 - X_3^*)$. Q.E.D.

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