THE JOURNAL OF FINANCE • VOL. LIV, NO. 2 • APRIL 1999

On the Cross-Sectional Relation between Expected Returns, Betas, and Size

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ABSTRACT

In this paper, I set up scenarios where the mean-variance capital asset pricing model is true and where it is false. Then I investigate whether the coefficients from regressions of population expected excess returns on population betas, and expected excess returns on betas and size, allow us to distinguish between the scenarios. I show that the coefficients from either ordinary least squares or generalized least squares regressions do not allow us to tell whether the model is true or false.

EACH OF THE FOLLOWING FIVE statements has implications for how we might judge whether the Sharpe (1964)–Lintner (1965) mean-variance capital asset pricing model (MV CAPM) is true or false. First, the market portfolio is MV efficient. Second, there is at least one positively weighted efficient portfolio. Third, in the riskless asset version of the model, the market portfolio is the tangency portfolio—it is the point of tangency between a ray emanating from the riskless interest rate and the minimum-variance frontier of risky assets. Fourth, there is a linear relation between the expected returns and market betas of securities, that is, securities plot on the security market line (SML). Fifth, market betas are the only measures of risk needed to explain the cross section of expected returns. If the riskless asset version of the model holds exactly, the fourth and fifth statements have two implications. Either an ordinary least squares (OLS) or a generalized least squares (GLS) regression of population expected excess rates of return on population market portfolio betas will have a zero intercept and a slope equal to the expected excess return on the market.[1] Furthermore, either an OLS or a

* Department of Economics, Faculty of Business Administration, Simon Fraser University. Financial support from the Social Sciences and Humanities Research Council of Canada is greatly appreciated as is the capable assistance of Maciek Kon, John Janmaat, and William Ting. I thank Reo Audette, Avi Bick, George Blazenko, Peter Clarkson, Wayne Ferson, John Heaney, Burton Hollifield, Avi Kamara, Peter Klein, Andy Siegel, Simon Wheatley, and especially John Herzog, Ray Koopman, John Janmaat, two anonymous referees, and the editor René Stulz for extremely helpful comments. But, naturally, I am responsible for both interpretation and errors. The paper was presented at Simon Fraser University, the Pacific Northwest Finance Conference in Seattle, and the Northern Finance Association Meetings in Vancouver. It was originally titled: "Tests of the Capital Asset Pricing Model Based on the Cross-Section of Expected Returns."

[1] An expected excess rate of return is defined as an expected rate of return less the riskless interest rate.

GLS regression of expected excess returns on market portfolio betas and size will have the same intercept and slope for the beta variable as the previous regression and a zero slope for the size variable.

In this paper, I set up scenarios where the CAPM is true and where it is false. When the CAPM is true, all five statements hold exactly. When the CAPM is false, all the statements are almost true—in some scenarios. In other scenarios, some of the statements are almost true—others are grossly violated. I then investigate whether the coefficients from regressions of population expected excess returns on population betas, and expected excess returns on betas and size, allow us to distinguish between the scenarios. I show that the OLS and GLS regression coefficients do not allow us to distinguish between the true and false scenarios.

In the early 1970s, Black, Jensen, and Scholes (1972), Blume and Friend (1973), and Fama and MacBeth (1973) produced the first extensive tests of the model. They focus on the cross-sectional expected return-beta trade-off and the special prediction of the Sharpe–Lintner version of the model that the returns on "zero-beta" or "orthogonal" portfolios (portfolios whose returns are uncorrelated with the returns on the market portfolio) have expected returns equal to the riskless rate of interest. Their findings are well known. The average return-beta plot is almost linear, but the estimated slope of the SML is too flat and the intercept too high. The evidence is interpreted as providing grounds for rejection of the Sharpe–Lintner model and as being consistent with Black's (1972) zero-beta version.

However, two types of difficulties with the tests have been identified. First, there are serious doubts regarding the validity of the tests that focus on either financial or logical rather than statistical or empirical considerations. Roll's (1977) critique is the best known. He argues that the theory is equivalent to the assertion that the market portfolio is MV efficient. Insofar as proxies—that include only a subset of assets—are used for the market portfolio, the CAPM is not being tested. Second, there are several empirical contradictions of the model that have been highlighted in a recent article by Fama and French (1992). They report that size and book-to-market equity combine to capture the cross-sectional variation in average stock returns associated with market beta, size, leverage, book-to-market equity, and earnings-to-price ratios. Arguably the size effect, first documented by Banz (1981) and Reinganum (1981), is the most important, and, therefore, I focus on it exclusively. Fama and French run OLS regressions of: (1) means on betas, calculated relative to a number of proxies for the market portfolio, (2) means on betas and size, and (3) means on size. In the regression of means on betas, the slope is small, positive, and statistically insignificant. In the regression of means on betas and size, the beta coefficient is negative and statistically insignificant, and the size coefficient is negative and statistically significant. In the regression of means on size, the slope is negative and statistically significant. Together these results are taken to be strong evidence against the CAPM.

Papers closely related to this study (Roll and Ross (1994) and Kandel and Stambaugh (1995)), highlight the danger of focusing exclusively on mean-beta space.[2] Roll and Ross derive the set of all possible portfolios that produce a given value for the cross-sectional slope in an OLS regression of population expected returns on population betas. More specifically, they show a market proxy can be nearly MV efficient in mean-variance space and produce a zero slope in an OLS regression of means on that portfolio's betas. Kandel and Stambaugh, also working with population parameters, demonstrate that a market proxy can be almost MV efficient and produce no relationship between means and betas using OLS. Conversely, when the set of securities is repackaged into an alternative set that generates the same portfolio opportunities, there can be a near perfect OLS fit between means and betas calculated relative to a proxy that is grossly inefficient in mean-variance space. See, for example, Figure 1 in Kandel and Stambaugh (1995). More importantly, they show that, in a GLS regression of mean returns on betas, the slope and R^2 are determined uniquely by the mean-variance location of the market index relative to the minimum-variance boundary. In contrast to OLS, GLS gives a zero slope only if the mean return on the market index equals that of the global minimum-variance portfolio. This latter result was derived earlier by Roll (1985)—see his Figure 1. Finally, Kandel and Stambaugh report that their results may be extended to a multifactor model. Once again, repackaging the assets can produce essentially any desired OLS coefficients and R^2.

This paper is similar in spirit to the papers by Roll (1977), Roll and Ross (1994), and Kandel and Stambaugh (1995). However, it differs from the later two in three significant ways. First, the MV CAPM predicts that prices will adjust until the market portfolio is MV efficient. Or, to put it another way, a precondition for the CAPM to hold is that there is at least one positively weighted efficient portfolio. Somewhat surprisingly, neither Roll and Ross nor Kandel and Stambaugh verify whether the minimum-variance frontier contains a positively weighted portfolio. Thus, we cannot be sure whether their results hold if the CAPM is true, that is, if the positively weighted market portfolio is MV efficient. Second, any reasonable proxy portfolio should contain positive weights. Roll and Ross, for example, note that readers of an earlier version of their paper speculated that the central results may be driven by short positions in the market proxies. (See Roll and Ross (1994), Sec. II.B, Plausibility and Short Positions, pp. 111–113.) Roll and Ross are able to construct one example where a proxy portfolio contains positive weights, but the proxies in Stambaugh and Kandel's (1995) Figure 1 do not contain positive weights.[3]

[2] The first version of this paper was completed before I was aware of these papers.

[3] It is clear that the proxy portfolios p and q in Kandel and Stambaugh's (1995) Figure 1 are not positively weighted. First, p and q have higher expected returns than the 10 size portfolios. Second, all the betas with respect to both p and q are less than one. The only way this can happen is if p and q have (some) negative weights.

By way of contrast, I (like Roll) examine two scenarios: one where the CAPM is true, the other where it is false. In the first scenario, the Sharpe–Lintner model holds exactly. In mean-standard deviation space, the positively weighted market portfolio is the MV efficient tangency portfolio. In mean-beta space, securities plot on the SML. In the second scenario, the CAPM is false. In mean-standard deviation space, the position of the now inefficient market portfolio and the tangency portfolio differ—sometimes dramatically—and no positively weighted portfolios plot on the minimum-variance frontier of risky assets. In mean-beta space, securities do not plot on the SML. Second, my examples are consistent with a size effect; that is, low (high) mean assets have high (low) weights in the market portfolio. Third, with one exception, I only employ the apparently innocuous equal-weighted portfolio as a positively weighted proxy for the market portfolio—a proxy commonly employed in empirical work[4] (see, for example, Black et al. (1972), Fama and MacBeth (1973), or Fama and French (1992)).

In these scenarios, I show two main results. First, when the CAPM is true, coefficients of OLS and GLS regressions employing proxy portfolios—that are almost efficient—can incorrectly indicate that the model is false. Second, and perhaps more important, when the CAPM is false, coefficients of OLS, GLS, or both OLS and GLS regressions employing the market portfolio can incorrectly indicate that the model is true.

The paper proceeds as follows. Section I presents the details of the experimental design. Section II records and interprets the results. Section III summarizes the paper.

I. The Experimental Design

A. *Examples where the CAPM is True*

The data set employed in the examples consists of 10 size portfolios compiled from all New York Stock Exchange and American Stock Exchange stocks contained in the Center for Research in Security Prices database. Returns from the 1926 to 1989 period are employed. The covariance matrix is calculated from the historical return data. It serves as the population covariance matrix for the examples. Then, following Best and Grauer (1985), a set of $(\boldsymbol{\Sigma}, \boldsymbol{x_m})$-compatible means—or equivalently SML means—is calculated from

$$\boldsymbol{\mu} = \theta_1 \boldsymbol{\iota} + \theta_2 \boldsymbol{\beta_m}, \tag{1}$$

[4] An equal-weighted portfolio is apparently innocuous in two senses. First, equal-weighted portfolios have positive weights and have been employed extensively in empirical work. Second, the weights in an equal-weighted portfolio are not chosen to produce a specific result, for example, a zero slope in an OLS regression of means on betas. On the other hand, it has never been entirely clear why anyone would choose an equal-weighted portfolio as a proxy for the value-weighted market portfolio.

where $\boldsymbol{\mu}$, $\boldsymbol{\iota}$, and $\mathbf{x}_m$ are n-vectors containing the expected rates of return on the n-assets, ones, and the weights in the market portfolio, respectively; $\boldsymbol{\Sigma}$ is an $n \times n$ positive-definite covariance matrix of asset returns; θ_1 and θ_2 are (positive) scalar constants; and $\boldsymbol{\beta}_m = (\boldsymbol{\Sigma}\mathbf{x}_m)/(\mathbf{x}_m'\boldsymbol{\Sigma}\mathbf{x}_m)$ is an n-vector of market portfolio betas; for example, the jth element of $\boldsymbol{\beta}_m$ is the covariance of the return on security j with the return on the market portfolio divided by the variance of the return on the market portfolio.

The parameters θ_1 and θ_2 are chosen so that the Sharpe–Lintner version of the CAPM holds exactly. θ_1 is chosen so that the zero-beta rate is equal to the risk-free rate of 0.5 percent per month (6 percent per annum), and θ_2 is chosen so that the expected excess return on the MV efficient market portfolio is 0.75 percent per month. This latter value corresponds to the excess return of the Standard & Poor's 500 Index over Treasury bills during the last half century. Thus, the population expected returns in percentage per month are

$$\boldsymbol{\mu} = 0.5\boldsymbol{\iota} + 0.75\boldsymbol{\beta}_m \tag{2}$$

when the hypothesis "the market portfolio is MV efficient" is true.

The CAPM makes no predictions about the magnitude of the weights in the market portfolio, but in order to examine the size effect I assign the market portfolio weights to be consistent with the empirical observation that large (small) firms have low (high) expected returns. I take the average capitalization weights of the decile portfolios over the 1926 to 1992 period to be the weights in the market portfolio. The weights range from 62.29 percent (86.24 percent) in the largest (largest three) capitalization decile(s) to 0.23 percent (1.64 percent) in the smallest (smallest three) capitalization decile(s).[5]

Having set up a world where the Sharpe–Lintner version of the CAPM holds exactly, I then run the following OLS and GLS regressions

$$\mu_j - r = b_0 + b_1 \,\text{beta}_j + \epsilon_j \tag{3}$$

$$\mu_j - r = b_0 + b_1 \,\text{beta}_j + b_2 \,\text{size}_j + \epsilon_j \tag{4}$$

$$\mu_j - r = b_0 + b_1 \,\text{size}_j + \epsilon_j \tag{5}$$

[5] Given a covariance matrix, it is easy to make any portfolio MV efficient. However, in general, the SML means from equation (1) will not be consistent with the scenario of the low-mean asset having the largest weight in the market portfolio. Nonetheless, in the 10 size portfolios universe, the average capitalization weights and the means from equation (1) are exactly consistent with the low-mean, high-weight scenario.

where the size variable is defined to be the market portfolio weights[6] and both market portfolio betas and proxy portfolio betas are employed. (The betas are calculated directly from the covariance matrix and the portfolio weights; that is, $\boldsymbol{\beta}_x = (\boldsymbol{\Sigma}\mathbf{x})/(\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})$, where the portfolio $\mathbf{x}$ may be either the market portfolio or a proxy portfolio.) With market portfolio betas in equation (3), $b_0 = 0$, and $b_1 = \mu_m - r$. In addition, size will have no effect whatsoever. That is, in equation (4) $b_0 = 0$, $b_1 = \mu_m - r$, and $b_2 = 0$. With proxy portfolio betas, it appears to be an open question what the slope and intercept—as well as the interpretation of them—should be. On the one hand, Roll (1977) argues that we can conclude nothing about the CAPM, unless we observe the true market portfolio. On the other hand, Fama (1991) and Fama and French (1992), among others—citing Stambaugh's (1982) empirical evidence that the tests are not sensitive to the proxy for the market—suggest that Roll's critique is overstated. But, in the examples in this paper, we know that the market portfolio is MV efficient. Therefore, any size effect will simply be an artifact arising from the use of proxy portfolio betas. The last regression (5) documents the strength of the relation between expected excess returns and size.

To summarize, in asset universes where the riskless asset version of the CAPM is true, I run OLS and GLS regressions of population expected excess returns on population betas (market portfolio betas and proxy portfolio betas), expected excess returns on betas and size, and expected excess returns on size. In mean-beta space, I plot the data, the OLS regression line, and the SML. (Note that when I use proxy portfolio betas, I still call the line passing through the riskless interest rate and the proxy portfolio the SML. This is a misnomer, but it is consistent with the nomenclature employed in the empirical literature.) In mean-standard deviation space, I plot the data, the minimum-variance frontier of risky assets, and the tangency portfolio. I also keep track of positively weighted minimum-variance portfolios.

B. Examples where the CAPM is False

If the CAPM is true, there is an exact linear trade-off between expected returns and market portfolio betas. If the CAPM is false, securities do not plot on the SML. Therefore, I construct examples where the CAPM is false by perturbing the SML means. But I do so in ways that make the CAPM appear to be true according to the coefficients of the cross-sectional regressions (3) and (4). Roughly speaking, I make the CAPM false by finding a set of perturbations in the SML means that is orthogonal to the betas, the market weights, and a vector of ones.

More specifically, consider the general linear regression model

$$\mathbf{y} = \mathbf{X}\mathbf{B} + \boldsymbol{\epsilon},$$

[6] In most empirical studies the size of a firm is defined as the natural log of its equity where equity is equal to the number of shares times the price per share.

where $\mathbf{y}$ is an n-vector containing n-observations on the dependent variable, $\mathbf{X}$ is an $n \times k$ matrix containing n-observations on the k-independent variables, $\mathbf{B}$ is a k-vector of parameters, and $\boldsymbol{\epsilon}$ is an n-vector of random errors. In an OLS regression, it is assumed that the error terms are independently and identically distributed (i.i.d.) with zero means and covariance matrix $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}$, where $\mathbf{I}$ is an $n \times n$ identity matrix. The OLS estimated coefficients are calculated as

$$\mathbf{b}^{\mathbf{ols}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}. \tag{6}$$

In a regression of expected excess returns on betas, $\mathbf{y}$ contains the expected excess returns, and $\mathbf{X} = [\boldsymbol{\iota}\text{: } \boldsymbol{\beta}]$ is an $n \times 2$ matrix, containing ones in the first column, and either market portfolio betas or proxy portfolio betas in the second column. In the regression of expected excess returns on betas and size, $\mathbf{y}$ again contains the expected excess returns, and $\mathbf{X} = [\boldsymbol{\iota}\text{: } \boldsymbol{\beta}\text{: } \mathbf{x_m}]$ is an $n \times 3$ matrix of ones, betas, and size—market-portfolio weights.

If the Sharpe–Lintner version of the model is true, there is an exact linear relation between expected returns and market portfolio betas. If the CAPM is false, the relation between expected returns and market betas is not linear. However, regressions of population expected excess returns on population market portfolio betas (or proxy portfolio betas) and population expected excess returns on population betas and size can yield exactly the same intercepts and slopes as in the case where the model is true—or the intercepts and slopes can be made to have practically any other values one might wish. To see this, let

$$\mathbf{y} = \boldsymbol{\mu} + \mathbf{e},$$

where $\mathbf{e}$ is an n-vector containing deviations from the security market line means $\boldsymbol{\mu}$; for example, $\mathbf{e} = \boldsymbol{\Delta\mu}$. In this case, the coefficients are

$$\mathbf{b}^{\mathbf{ols}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\mu} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{e} = \mathbf{b}_0^{\mathbf{ols}} + \boldsymbol{\Delta}\mathbf{b}^{\mathbf{ols}}. \tag{7}$$

Clearly, $\mathbf{b}_0^{\mathbf{ols}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\mu}$ gives the coefficients when the CAPM is true, and $\boldsymbol{\Delta}\mathbf{b}^{\mathbf{ols}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{e}$ represents the change in the coefficients. If we set

$$\boldsymbol{\Delta}\mathbf{b}^{\mathbf{ols}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{e} = \mathbf{0}, \tag{8}$$

where $\mathbf{0}$ is a k-vector of zeros, we have exactly the same OLS coefficients as when the CAPM is true. On the other hand, if we set $\boldsymbol{\Delta}\mathbf{b}^{\mathbf{ols}}$ equal to a vector of values other than zeros, we can generate practically any values for the intercept and slope—or intercept and slopes if we include size in the regression.

Instead of restricting the coefficients in equation (8), however, we can simply set $\mathbf{X}'\mathbf{e} = \mathbf{0}$. In the case of an OLS regression of expected excess returns on betas and size the restrictions are

$$\boldsymbol{\iota}'\mathbf{e} = 0, \qquad \boldsymbol{\beta}'\mathbf{e} = 0, \quad \text{and} \quad \mathbf{x}_\mathbf{m}'\mathbf{e} = 0. \tag{9}$$

The advantage of writing the restrictions in this manner is that they provide insights about other variables in addition to the slopes and intercepts of the cross-sectional regressions. For example, the first and third restrictions say that the means of an equal-weighted proxy portfolio and the value-weighted market portfolio are unchanged from their original values.

In a GLS regression, the assumption that the error terms are i.i.d. is relaxed. In this case, the error terms are assumed to have zero means and covariance matrix $\mathbf{\Sigma}$. The GLS coefficients are

$$\mathbf{b}^{\mathbf{gls}} = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{y}.$$

If the CAPM is false, the GLS regressions of population expected excess returns on population betas and expected excess returns on betas and size can again yield exactly the same intercepts and slopes as in the case where the model is true. But the conditions for this to occur are slightly different than in the case of the OLS regressions. To see this, again let $\mathbf{y} = \boldsymbol{\mu} + \mathbf{e}$. Then, the GLS coefficients are

$$\mathbf{b}^{\mathbf{gls}} = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} + (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{e} = \mathbf{b}_0^{\mathbf{gls}} + \mathbf{\Delta b}^{\mathbf{gls}}.$$

As in the case of OLS, $\mathbf{b}_0^{\mathbf{gls}} = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$ gives the coefficients when the CAPM is true, and $\mathbf{\Delta b}^{\mathbf{gls}} = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{e}$ represents the change in the coefficients. As before, if we set

$$\mathbf{\Delta b}^{\mathbf{gls}} = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{e} = \mathbf{0}, \tag{10}$$

we have exactly the same GLS coefficients as when the CAPM is true. Or, more simply, in the case of a regression of expected excess returns on betas and size, we can set $\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{e} = \mathbf{0}$ to yield

$$\boldsymbol{\iota}'\mathbf{\Sigma}^{-1}\mathbf{e} = 0, \qquad \boldsymbol{\beta}'\mathbf{\Sigma}^{-1}\mathbf{e} = 0, \quad \text{and} \quad \mathbf{x}_{\mathbf{m}}'\mathbf{\Sigma}^{-1}\mathbf{e} = 0. \tag{11}$$

Again, two of the three restrictions provide insights about other variables in addition to the coefficients of the cross-sectional regressions. The first restriction says that the mean of the global minimum-variance portfolio of risky assets remains unchanged. The second says that if the equal-weighted proxy portfolio is used to calculate the betas then $\boldsymbol{\beta} = n\mathbf{\Sigma}\boldsymbol{\iota}/(\boldsymbol{\iota}'\mathbf{\Sigma}\boldsymbol{\iota})$ and the GLS restriction $\boldsymbol{\beta}'\mathbf{\Sigma}^{-1}\mathbf{e} = 0$ reduces to the OLS restriction $\boldsymbol{\iota}'\mathbf{e} = 0$. Similarly, if the market portfolio is used to calculate the betas, $\boldsymbol{\beta} = \mathbf{\Sigma}\mathbf{x}_{\mathbf{m}}/(\mathbf{x}_{\mathbf{m}}'\mathbf{\Sigma}\mathbf{x}_{\mathbf{m}})$, and the GLS restriction $\boldsymbol{\beta}'\mathbf{\Sigma}^{-1}\mathbf{e} = 0$ reduces to the OLS restriction $\mathbf{x}_{\mathbf{m}}'\mathbf{e} = 0$.

To summarize, I make the CAPM false by solving for a set of perturbations in the SML means that satisfy some subset of the restrictions in equations (8)–(11)—and, in one case, change the intercept and slope of a regression of expected excess returns on equal-weighted proxy portfolio betas. I then run the OLS and GLS regressions of expected excess returns on population betas, expected excess returns on betas and size, and expected excess returns on size, given in equations (3)–(5). Next, in mean-beta space, I plot the data, the regression line, and the SML. In mean-standard deviation space, I plot the data, the minimum-variance frontier of risky assets, and keep track of the tangency portfolio and positively weighted minimum-variance portfolios.

II. The Results

Space limitations dictate that only a subset of the examples can be reported here. Therefore, I limit discussion to: (1) a 10 size-ranked portfolios universe; (2) market portfolio betas and three instances of proxy portfolio betas; and (3) examples where, with one exception, the tangency portfolio deviates rather dramatically from the market portfolio—when the CAPM is false. However, similar results are generated in 10, 20, and 50 asset universes as well as in a 20 beta-ranked portfolios universe.

The main results are reported in Table I and in Figures 1 to 5. Table I contains the results for OLS and GLS regressions of population expected excess returns on population betas and population expected excess returns on population betas and size. The first panel of Table I and Figures 1 and 2 depict scenarios where the CAPM is true. The final five panels of Table I and Figures 3–5 portray scenarios where the CAPM is false.

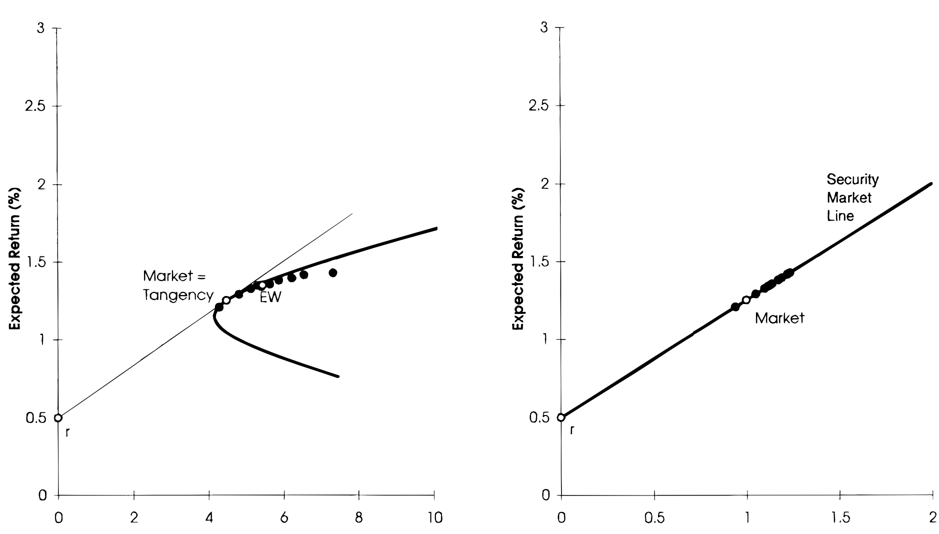

Figure 1. The CAPM is true. The market portfolio is mean-variance efficient and securities plot on the security market line. Coefficients of ordinary least squares and generalized least squares regressions of expected excess returns on market portfolio betas, and expected excess returns on market portfolio betas and size, correctly indicate that the CAPM is true. See Table I, Panel A.

Table I

Results for the Cross-Sectional Regressions

$$\mu_j - r = b_0 + b_1\,\mathrm{beta}_j + \epsilon_j$$

$$\mu_j - r = b_0 + b_1\,\mathrm{beta}_j + b_2\,\mathrm{size}_j + \epsilon_j.$$

In each panel, the risk-free rate is 0.5 percent and the expected return on the market portfolio is 1.25 percent. Thus, when the mean-variance capital asset pricing model (CAPM) is true, $b_0 = 0$ and $b_1 = \mu_m - r = 0.75$ percent in the first regression, and $b_0 = 0$, $b_1 = 0.75$ percent, and $b_2 = 0$ in the second. The first two rows in each panel employ market portfolio betas. Rows three and four in Panels A and F employ equal-weighted proxy portfolio betas. Rows six and seven in Panel A employ a Roll and Ross (1994) proxy portfolio that generates a zero slope in an ordinary least squares (OLS) regression of means on betas. The odd rows in each panel contain the results for OLS and the even rows contain the generalized least squares (GLS) results. In Panel A, the CAPM is true (see Figures 1 and 2). The Roll–Ross proxy portfolio (not shown in the figures) contains negative weights. Its mean equals that of the market portfolio and its standard deviation is only 27 basis points larger. In the remaining panels, the CAPM is false. In Panel B, the tangency portfolio's expected return is 1.35 percent (see Figure 3). In Panel C, the tangency portfolio's expected return is 2.5 percent (see Figure 4). In Panel D, the tangency portfolio's expected return is 26,654 percent. In Panel E, the tangency portfolio's expected return is −21,572 percent (see Figure 5). In Panel F, the tangency portfolio's expected return is 418 percent.

		b_0	b_1	R^2	b_0	b_1	b_2	R^2
				Panel A				
Value-weighted	OLS	0.00	0.75	1.00	0.00	0.75	0.00	1.00
Market portfolio	GLS	0.00	0.75	1.00	0.00	0.75	0.00	1.00
Equal-weighted	OLS	0.46	0.39	0.98	0.53	0.32	−0.07	0.98
Proxy portfolio	GLS	0.36	0.48	0.85	0.42	0.43	−0.04	0.87
Roll–Ross	OLS	0.85	0.00	0.00	1.38	−0.54	−0.35	0.89
Proxy portfolio	GLS	0.30	0.45	0.53	0.48	0.30	−0.09	0.64
				Panel B				
Value-weighted	OLS	0.77	0.00	0.00	1.42	−0.55	−0.29	0.48
Market portfolio	GLS	0.00	0.75	0.51	0.00	0.75	0.00	0.51
				Panel C				
Value-weighted	OLS	0.00	0.75	0.54	0.00	0.75	0.00	0.54
Market portfolio	GLS	0.00	0.75	0.08	0.00	0.75	0.00	0.08
				Panel D				
Value-weighted	OLS	0.00	0.75	0.01	0.00	0.75	0.00	0.01
Market portfolio	GLS	−4.71	5.46	0.05	−11.73	11.44	2.46	0.11
				Panel E				
Value-weighted	OLS	0.00	0.75	0.01	0.00	0.75	0.00	0.01
Market portfolio	GLS	−4.75	5.50	0.05	−11.84	11.54	2.48	0.11
				Panel F				
Value-weighted	OLS	0.00	0.75	0.00	0.00	0.75	0.00	0.00
Market portfolio	GLS	−1.10	1.85	0.00	−1.88	2.52	0.27	0.00
Equal-weighted	OLS	0.59	0.26	0.00	0.97	−0.08	−0.36	0.00
Proxy portfolio	GLS	0.00	0.85	0.00	0.00	0.85	−0.00	0.00

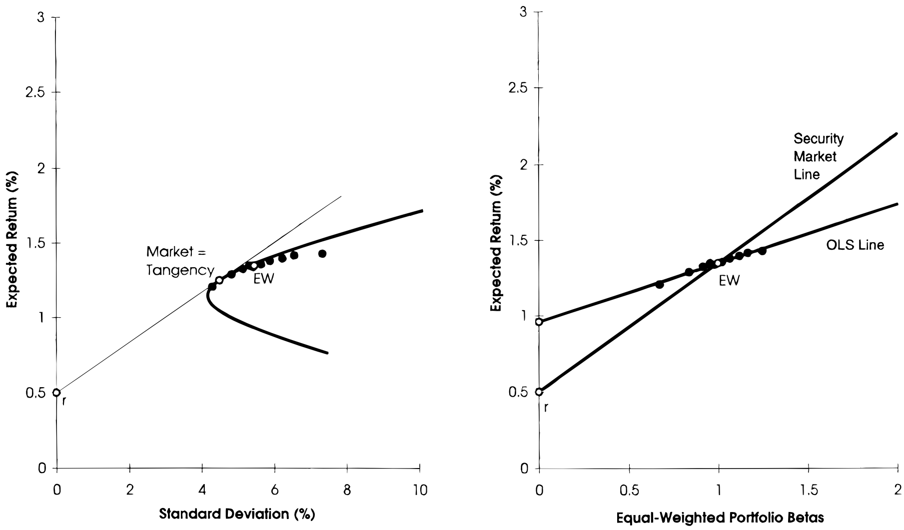

Figure 2. The CAPM is true. The equal-weighted proxy portfolio plots very close to the minimum-variance frontier of risky assets. But, coefficients of ordinary least squares regressions of expected excess returns on equal-weighted proxy portfolio betas, and expected excess returns on equal-weighted proxy portfolio betas and size, incorrectly indicate that the CAPM is false. See Table I, Panel A. Note that, consistent with the empirical literature, the line from the riskless asset through the equal-weighted proxy portfolio is called the security market line. A more accurate label might be the empirical security market line. Also note the difference in the magnitudes of the equal-weighted proxy portfolio betas in this figure and the market portfolio betas in Figure 1.

A. The CAPM Is True

The CAPM is true in a 10 size-ranked portfolios universe shown in Table I, Panel A, and in Figures 1 and 2. The first two rows of Panel A show that with market portfolio betas the coefficients of OLS and GLS regressions are exactly as predicted. That is, in the regression of population expected excess returns on population market portfolio betas, $b_0 = 0$, and $b_1 = \mu_m - r = 0.75$. And, in the regression of expected excess returns on market betas and size, $b_0 = 0$, $b_1 = \mu_m - r = 0.75$, and $b_2 = 0$.[7] Figure 1 contains the corresponding mean-standard deviation and mean-beta plots. The plots are the classic ones envisioned in theory: the market portfolio coincides with the tangency portfolio and securities plot on the SML.

When the CAPM is true, the only way the regression coefficients can lead us to believe that the model is false is if we employ a proxy for the market portfolio. The next two examples illustrate this possibility—one with a pos-

[7] In an OLS (GLS) regression of expected excess returns on size, not reported in Table I, the size coefficient is −0.30 (−0.15). When the CAPM is false, the size coefficient is unchanged if the coefficients in the regression of means on betas and means on betas and size are unchanged, as in, for example, Panels C through F for OLS, and in Panels B and C for GLS. In Panel B, the OLS size coefficient is −0.07. In Panels D and E, the GLS size coefficient is 0.19, and, in Panel F, it is −0.23.

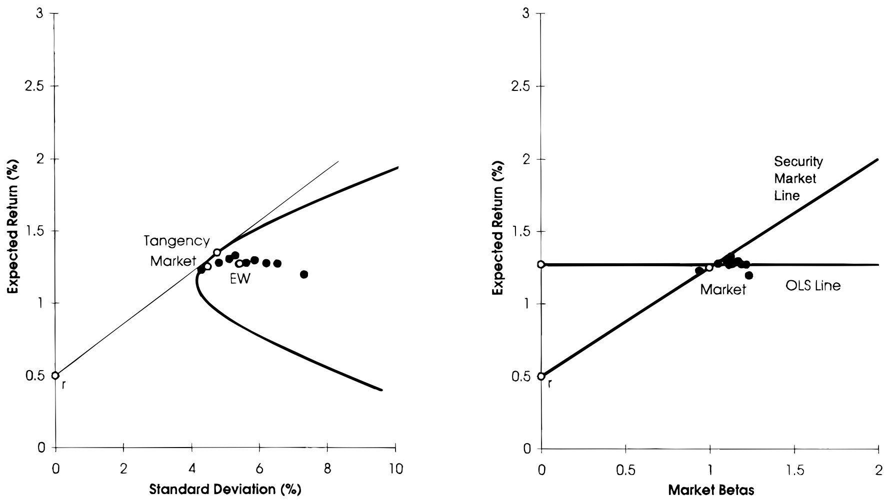

Figure 3. The CAPM is false. Although there are no positively weighted minimum variance portfolios, the market portfolio is almost mean-variance efficient. Yet, in an ordinary least squares (OLS) regression of expected excess returns on market portfolio betas, the slope is zero. And, in an OLS regression of expected excess returns on market portfolio betas and size, both slope coefficients are negative. Thus, the model appears to be dead wrong. On the other hand, coefficients of generalized least squares regressions of expected excess returns on market betas, and expected excess returns on market betas and size, incorrectly indicate that the CAPM is true. See Table I, Panel B.

itively weighted proxy, one without. First, Figure 2 shows that an equal-weighted proxy portfolio plots very close to the minimum-variance frontier of risky assets. Second, Figure 2 and rows three and four of Table I, Panel A, show that OLS and GLS regressions of expected excess returns on equal-weighted proxy portfolio betas are consistent with the well-known result found in Black et al. (1972): the intercept is too large and the slope is too flat. Third, in an OLS and a GLS regression of expected excess returns on equal-weighted portfolio betas and size, the size coefficient is negative. But the size effect is simply an artifact caused by using equal-weighted proxy portfolio betas instead of market portfolio betas.

The noteworthy aspect of this example is that both the MV-efficient market portfolio and the inefficient equal-weighted proxy portfolio contain all positive weights. In the 10 size-ranked portfolios universe, there are no positively weighted portfolios that produce zero slopes in OLS regressions of means on betas.[8] However, if we relax the requirement that the proxy portfolio should contain all positive weights, then a portfolio—with the same mean as the market portfolio and a standard deviation only 27 basis points

[8] In 10, 20, and 50 asset examples not reported in the paper, a weaker statement holds. There are no positively weighted portfolios on the minimum-variance frontier of portfolios that produce a zero slope in an OLS regression of means on betas. I verified this using the method described in Best and Grauer (1992).

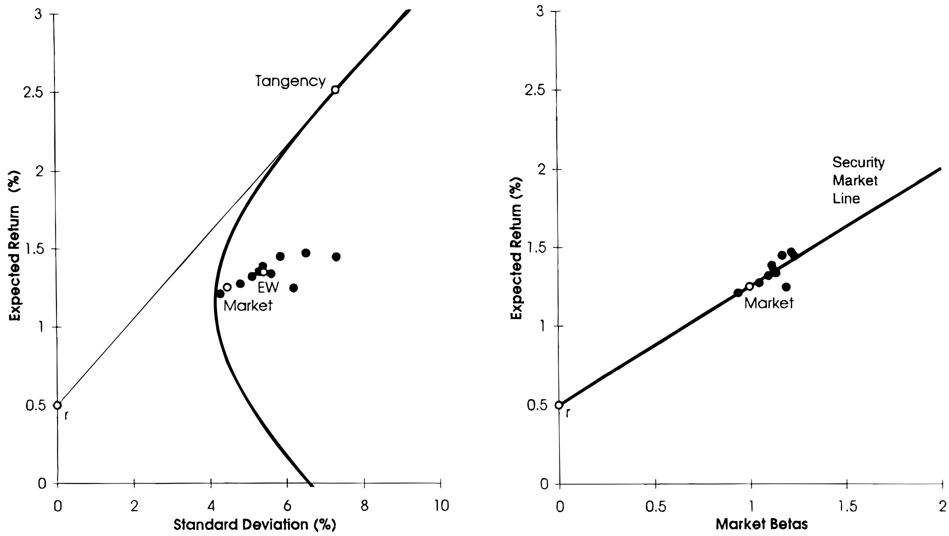

Figure 4. The CAPM is false. The CAPM is clearly false in mean-standard deviation space. The market portfolio is not mean-variance efficient and tangency portfolio's mean is 2.5 percent. However, coefficients of ordinary least squares and generalized least squares regressions of expected excess returns on market portfolio betas, and expected excess returns on market portfolio betas and size, incorrectly indicate that the CAPM is true. See Table I, Panel C.

larger—yields a zero slope in an OLS regression of means on betas. See the fifth and sixth rows of Table I, Panel A. The GLS slope is also too flat. Moreover, in an OLS and a GLS regression of means on betas and size, the size coefficient is negative.

B. The CAPM Is False

In the examples where the CAPM is false, there are no positively weighted minimum-variance portfolios. Nor are there any positively weighted portfolios on the minimum-variance frontier of portfolios that produce zero slopes in OLS regressions of means on betas. In light of this fact and the fact that there are no positively weighted portfolios that produce a zero slope in an OLS regression of means on betas when the CAPM is true, it is surprising to see that the positively weighted market portfolio produces a zero slope in an OLS regression of means on betas in the example shown in Figure 3 and in Table I, Panel B. The model does not look all that bad in mean-standard deviation space. The market portfolio plots very close to the minimum-variance frontier of risky assets and its expected return is only 10 basis less than that of the tangency portfolio. By way of contrast, the OLS and GLS regression coefficients support wildly divergent conclusions. In an OLS regression of expected excess return on market portfolio betas, the slope is zero, and in an OLS regression of expected excess return on market betas and size, both slope coefficients are negative. In other words, the model

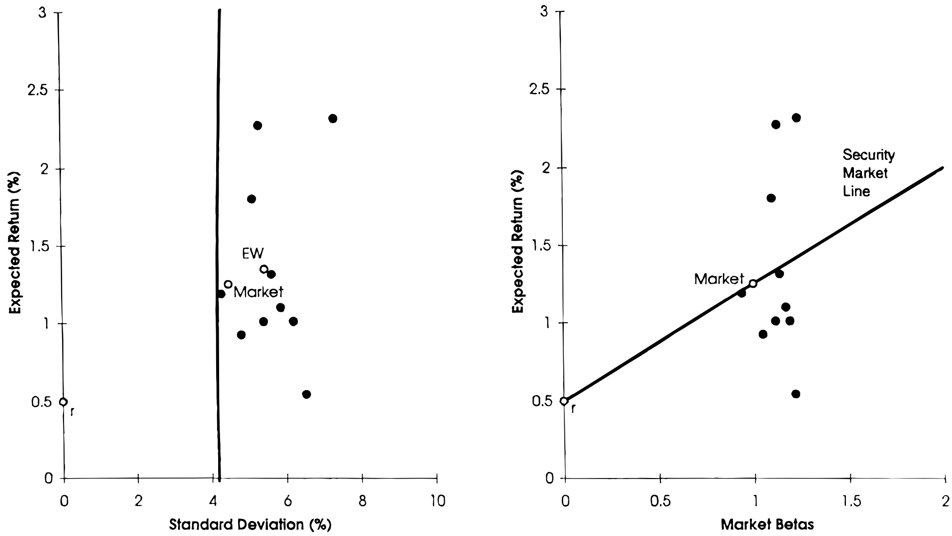

Figure 5. The CAPM is false. The CAPM is clearly false in mean-standard deviation space. The minimum-variance frontier appears to be a vertical line. The tangency portfolio has an expected return of −21,572 percent and standard deviation of 10,157 percent. Yet, coefficients of ordinary least squares regressions of expected excess returns on market portfolio betas, and expected excess returns on market portfolio betas and size, incorrectly indicate that the CAPM is true. See Table I, Panel E.

appears to be dead wrong. On the other hand, the GLS coefficients indicate that the model holds exactly. It is ironic that, although the OLS coefficients may arguably overstate "how wrong the model is," it is the GLS coefficients with market portfolio betas that incorrectly indicate the CAPM is true.

In Figure 4 and in Table I, Panel C, the model is more obviously false in mean-standard deviation space. The tangency portfolio has a 2.5 percent expected rate of return—which is double the inefficient market portfolio's expected rate of return—and the minimum-variance frontier looks appreciably different than it does when the CAPM is true. This occurs even though the expected returns of the 10 size portfolios are little changed, ranging from 1.21 percent to 1.47 percent—compared to a range of 1.21 percent to 1.43 percent when the CAPM is true. On the other hand, all the regression coefficients in Table I, Panel C, are identical to the coefficients in Table I, Panel A, when the CAPM is true. Thus, OLS and GLS regression coefficients with market portfolio betas incorrectly indicate that the CAPM is true.

This is not an isolated case. In examples not recorded in the table or figures, the OLS and GLS regression coefficients are identical to those when the CAPM is true—whether the tangency portfolio's expected rate of return is as low as 1.26 or 1.28 percent or as high as 6.25, 12.5, or even 62.5 percent. Only the R^2 and the 10 size portfolios' expected rates of return differ. In the case where the tangency portfolio's expected rate of return is 1.26 (1.28) percent, two (four) of the tangency weights are negative. But the expected rates of return for the 10 size portfolios are virtually unchanged from the case where the CAPM is true. Furthermore, when the tangency portfolio's expected rate of return is 6.25, 12.5, and 62.5 percent—5, 10, and 50 times the market portfolio's expected rate of return—the 10 size portfolios' expected rates of return are again relatively unchanged (ranging from 1.10 to 1.52 percent, from 0.95 to 1.58 percent, and from 0.35 to 1.85 percent, respectively).

The examples in Table I, Panels D and E, and in Figure 5 are more extreme. The minimum-variance frontier appears to be a vertical line rather than a hyperbola as in Figures 1–4. In Table I, Panel D, the tangency portfolio plots near plus infinity—with an expected rate of return in excess of +26,000 percent and a standard deviation in excess of 12,000 percent. In Table I, Panel E, and in Figure 5, the tangency portfolio plots near minus infinity—with an expected return well below −21,000 percent and a standard deviation in excess of 10,000 percent. (By way of contrast, the expected returns of the 10 size portfolios only range from 0.55 percent to 2.31 percent in both cases.) Yet the coefficients of the OLS regressions of expected excess returns on market portfolio betas, and expected excess returns on market portfolio betas and size, take on exactly the same values as when the CAPM is true. Once more the Sharpe–Lintner version of the CAPM appears to be true when, in fact, in these cases it is again dead wrong. Furthermore, the GLS regression coefficients are almost the same in the two panels even though the tangency portfolio's expected return is +26,654 percent in one case and −21,572 percent in the other.

Table I, Panel F, muddies the waters even further. The CAPM is clearly false—the tangency portfolio's expected return is 418 percent and the expected returns of the 10 size portfolios range from 0.5 percent to 3.72 percent. However, the model appears to be exactly true according to the coefficients of OLS regressions coupled with market portfolio betas, as well as to be almost true in GLS regressions coupled with equal-weighted proxy portfolio betas. This conformity in the regression results employing the market and equal-weighted portfolios in this case is surprising given the complete lack of agreement between the results for the market and equal-weighted portfolios when the CAPM is true.

Clearly, then, the OLS and GLS regression coefficients themselves do not allow us to distinguish between scenarios where the CAPM is true and where it is false. Still, in light of Kandel and Stambaugh's (1995) demonstration that in a GLS regression of means on betas the slope and R^2 are uniquely determined by the mean-variance location of the market index relative to the minimum-variance boundary, a remaining question is: How much better is GLS? The answer is not clear cut. While it provides more information, the GLS R^2 is a somewhat ambiguous measure of efficiency in the presence of a riskless asset. More specifically, the GLS R^2 implicitly compares the positions of the market index, the minimum-variance portfolio with either the same expected return or variance as the market, and the global minimum-variance portfolio. (See Kandel and Stambaugh (1995), pp. 164–167.) However, when there is a riskless asset, we usually compare the positions of the market and tangency portfolios.

The results reported in Table I show that the two measures may give quite different impressions of how inefficient a portfolio is. In the examples, the riskless asset (with a yield of 0.5 percent) and the market portfolio (with an expected return of 1.25 percent and a standard deviation of 4.48 percent) are fixed points. In the GLS regressions of expected excess returns on market portfolio betas, reported in Panels B through F, the R^2 are 0.51, 0.08, 0.05, 0.05, and 0.00, respectively. However, the relative magnitudes of the R^2 are somewhat difficult to reconcile with the expected return of the tangency portfolio, which takes on values of 1.35, 2.5, 26,654, −21,572, and 418 percent in Panels B through F.

III. Summary

For more than 30 years the MV CAPM has formed one of the central paradigms of financial economics, and, for a quarter of a century, the model has been subjected to extensive testing. Yet, for more than 20 years, the tests have been called into question. Roll (1977) shows that the use of a proxy for the market portfolio—that includes only a subset of assets—can make the model appear to be true when it is false, or to be false when it is true. In universes where all the assets are observed, Roll and Ross (1994) demonstrate that a market proxy can be almost MV efficient and produce no relationship between means and betas using OLS. Moreover, Kandel and Stambaugh (1995) show that there can be a near perfect OLS fit between means and betas calculated relative to a proxy that is grossly inefficient. More importantly, they show that, in a GLS regression of mean returns on betas, the slope and R^2 are determined uniquely by the mean-variance location of the market index relative to the minimum-variance boundary. However, neither Roll and Ross nor Kandel and Stambaugh verify whether the minimum-variance frontier contains a positively weighted portfolio. Thus, we cannot be sure whether their results hold if the CAPM is true, that is, if the positively weighted market portfolio is MV efficient. Furthermore, any reasonable proxy portfolio should contain positive weights. Roll and Ross are able to construct one example where a proxy portfolio contains positive weights, but the proxies in Stambaugh and Kandel do not contain positive weights.

My method of analysis differs from that of Roll and Ross (1994) and Kandel and Stambaugh (1995) in two fundamental ways. First, I set up scenarios where the MV CAPM is true and where it is false. When the CAPM is true, all the implications of the model mentioned in the first paragraph of the paper hold exactly. When the model is false, there are no positively weighted minimum-variance portfolios. Nonetheless, the implications of the model are almost true in some scenarios. In other scenarios, some of the implications are almost true but others are not. For example, the positions of the market and tangency portfolios differ dramatically in some cases. Second, in most cases, I do not even employ a proxy for the market. When I do, with one exception I employ a positively weighted proxy.

I then investigate whether the coefficients from OLS and GLS regressions of population expected excess returns on population betas, and expected excess returns on betas and size, allow us to distinguish between the true and false scenarios. I show two main results. First, when the CAPM is true, coefficients of OLS and GLS regressions employing proxy portfolios—that are almost efficient—can incorrectly indicate that the model is false. However, when the CAPM is true in a 10 size portfolios universe, no positively weighted portfolio produces a zero slope in an OLS regression of means on betas. Second, and perhaps more important, when the CAPM is false, coefficients of OLS, GLS, or both OLS and GLS regressions employing market portfolio betas can incorrectly indicate that the model is true even when the market is grossly inefficient. The lack of any clear-cut agreement among the different implications of the CAPM, when it is false, is particularly disturbing. It does not bode well for those seeking to design an unambiguous test of the model.

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