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An Empirical Comparison of Forward-Rate and Spot-Rate Models for Valuing Interest-Rate Options

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ABSTRACT

Our main goal is to investigate the question of which interest-rate options valuation models are better suited to support the management of interest-rate risk. We use the German market to test seven spot-rate and forward-rate models with one and two factors for interest-rate warrants for the period from 1990 to 1993. We identify a one-factor forward-rate model and two spot-rate models with two factors that are not significantly outperformed by any of the other four models. Further rankings are possible if additional criteria are applied.

A VALUATION MODEL FOR INTEREST-RATE derivatives represents the core of any system designed to measure, control, and supervise interest-rate risk. This is true regardless of whether a value-at-risk methodology, sensitivity analysis, stress test, or scenario technique is applied. Unfortunately, there is no empirical evidence that evaluates the performance of the most popular competing pricing models using the same data from a risk management perspective. This paper provides such empirical evidence using data from the German market for interest-rate warrants for the period from 1990 to 1993.

The more recent valuation models are dominated by two groups of models, forward-rate and spot-rate models. The approach of the first group, pioneered by Ho and Lee (1986) and Heath, Jarrow, and Morton (HJM) (1992), directly uses the arbitrage-free dynamics of the entire zero bond price curve or, equivalently, the term structure of forward rates to price interest-rate derivatives. We refer to this approach as the forward-rate (or HJM) model. The approach of the second group (e.g., see Hull and White (1990, 1993), Black, Derman, and Toy (1990), Black and Karasinski (1991), Jamshidian (1991), and Sandmann and Sondermann (1993)) is based on the dynamics of

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the instantaneous spot interest rate. The second group's papers follow a suggestion by Cox, Ingersoll, and Ross (1985) and fit the endogenous term structure (and volatility structure) of interest rates to the observed term structure. This fitting is achieved through time-dependent parameters of the stochastic factor processes. Both approaches that model the stochastic behavior of the term structure of interest rates, as well as the subsequent valuation of derivatives, are closely related to each other. In fact, for some of the models, a mathematical equivalence is easily established.

In this comprehensive empirical study we test one-factor and two-factor spot-rate and forward-rate models. The paper's main goal is to clarify the question of whether spot- or forward-rate models are better suited to support the measurement, control, and supervision of interest-rate risk. Existing work provides no answer to this important question. Though the two classes of models are close to each other from a theoretical point of view, they can exhibit very different behaviors in application. Furthermore, it is not at all obvious whether two-factor models outperform one-factor models in each class.

The models we test have the following basic structures. The one-factor spot-rate model is characterized by a mean-reverting drift and a diffusion coefficient with constant elasticity. Within the class of two-factor spot-rate models we consider two variants: In the first model, the factors are identified with a long rate and the difference between this long rate and a short rate. In the second model, the factors we consider are the short rate and its volatility. These two-factor models represent generalizations of models developed by Schaefer and Schwartz (1984) and Longstaff and Schwartz (1992).

We test two forward-rate models with one factor. The first model represents the continuous-time version of Ho and Lee's (1986) model with Gaussian forward rates and constant (absolute) volatility. The second model has a linear proportional volatility—that is, the proportional volatility depends solely on time-to-maturity of forward rates. We also test two forward-rate models with two factors. We determine the volatility functions of both models by principal component analyses. The volatilities, therefore, depend on empirically specified factor loadings and these factors' volatilities. In one model, forward-rate volatilities are independent of forward rates. In the other, they are proportionally dependent on these rates.

Assessing the applicability of the different models within a risk management system requires two important decisions about the test methodology. First, valuation models within risk management systems must be capable of predicting future option prices if they are to correctly measure risk exposure. This capability is best evaluated by the ex ante predictability of a model. Therefore, we use the valuation quality of a model, not its ability to identify mispriced options, as the most important assessment criterion. This implies that this study is not a test of an option market's efficiency.

Second, we estimate all parameters, including the volatility, from time series. There are two reasons to estimate parameters historically rather than implicitly. We cannot prematurely rule out that model-dependent parameters, such as implied volatilities, favor one model versus the other. More important, testing valuation models with implied parameters only represents a "local test," in the sense that the current option price is used to value the same option one period later. Therefore, "local tests" consider only small deviations from observed prices of derivatives. Tests on the basis of historically estimated parameters are "global," in the sense that they do not use information from derivatives markets.

In addition to the valuation quality, there are other important criteria for assessing a valuation model for interest-rate options. These refer to the difficulties in estimating the model parameters, in fitting the model to the current term and volatility structures, in computing the option values numerically, and to the stability of the model's performance over different time periods.

We test the seven valuation models for interest-rate warrants from the German market for the period from 1990 to 1993. In contrast to standardized options traded on the German Futures and Options Exchange (DTB) in Frankfurt, interest-rate warrants are options issued by banks. Underlying these warrants are German government bonds, which represent the most liquid market segment in the German bond market. The market for interest-rate warrants started in 1989 and is now more liquid than the market of standardized options on the BUND-Future that are traded on the DTB. We also selected this market segment of German interest-rate options because warrants show a much wider variety of terms than do standardized options. There are warrants of both the European and American types whose maturities range up to 2.9 years, compared with nine months for the standardized futures options on the BUND-Future.

Very few papers study the empirical performance of models for the valuation of interest-rate options. Dietrich-Campbell and Schwartz (1986) value interest-rate options on U.S. government bonds and treasury bills, using the two-factor Brennan and Schwartz (1982) model. Bühler and Schulze (1995), Flesaker (1993), and Amin and Morton (1994) present empirical studies of the HJM model. The study by Bühler and Schulze analyzes the optimal call policy of callable bonds that are issued by German public authorities. Flesaker, as well as Amin and Morton, presents results for Eurodollar futures options. However, none of these studies compares spot-rate and forward-rate models.

The paper is organized as follows. In Section I we describe our selection of the empirically tested models on the basis of an extensive data analysis and the decisions made in the different implementation steps. Section II presents relevant information on the German bond and interest-rate options market. In Section III we describe the design of the empirical study and provide estimation results for the input data of the different models. The valuation results, including a multivariate error analysis and a paired comparison of the seven models, appear in Section IV. We summarize our final assessment of the models and present conclusions in Section V.

I. Selected Valuation Models and Their Implementation

In this section, we describe our selection of the empirically tested spot-rate and forward-rate models. We also specify the formal structure of the models and describe the decisions made in the different implementation steps.

A. Data Analysis and Selection of Models

The first step in testing interest-rate options valuation models is to preselect the basic features of the models. Preselection refers to the number of factors driving the term structure of interest rates and the functional form of the stochastic processes for these factors. To do this, we perform an extensive data analysis of the factor structure of zero bond yields and the behavior of individual yields in the German bond market. We then summarize our findings, which are the basis of the models discussed in Section I.B. The details are presented in Bühler et al. (1996).

We first apply principal component analyses to determine the number of factors. Because of the high degree of correlation among yields with different maturities, two factors explain more than 95 percent of the variation in the term structure of interest rates. These findings are stable over different time periods with varying lengths during 1970 to 1993. Litterman and Scheinkman (1991) report similar results for the U.S. market. On the basis of these results, we only consider one- and two-factor models. One-factor models should be understood as a reference case against which we measure the improvement of introducing a second factor. Of course, to understand whether two factors are sufficient, the whole study should be carried out for three factors.

A.1. One-Factor Models

The distinguishing feature of a one-factor forward-rate model is the functional form of the forward-rate volatility. Amin and Morton (1994) test different parsimonious (one and two parameter) parameterizations and find that the number of parameters has a stronger effect on the behavior of the model than does the form of the models used. Two-parameter models tend to fit prices better. Indeed, the model with the best fit in-sample and out-of-sample is the two-parameter model with a linear proportional volatility function. However, the one-parameter models result in implied parameter estimates that are more stable, and the models earn larger and more consistent profits from their perceived mispricings. Amin and Morton conclude that the model with constant volatility (the absolute model) seems to be preferable among the one-parameter models.[1]

In light of these findings, we test two one-factor HJM models, a one-parameter and a two-parameter model. The one-parameter model we choose for our investigation is the one with constant volatility, which is in fact the continuous-time version of the Ho and Lee model. The two-parameter model is the one with linear proportional volatility.

One-factor spot-rate models start with a specification of the process of the short rate, r. In line with studies for other markets, we find that an increase (decrease) in short rates is more likely than a decrease (increase) if the values of the rates are historically low (high). Additionally, large interest-rate movements take place in periods of high interest rates, and moderate

[1] Amin and Morton (1994) test the HJM models using Eurodollar futures and options. Some of their results might stem from the short maturity of the options considered.

movements are observed in low-rate periods.[2] Therefore, we use the standard model for the short rate, a mean-reverting process with an instantaneous volatility shown as σr^{ϵ}. Contrary to the results of Chan et al. (1992), who report a value of 1.5 for the exponent ϵ, we find values between 0.5 and 1.[3] These estimates result in a unique solution of the stochastic differential equation of r.

A.2. Two-Factor Models

For a two-factor forward-rate model, two volatility functions—one for each factor—must be specified. We consider two different structural specifications. In the first case, we assume that both volatility functions are independent of the forward rate's level. In the second case, the two volatility functions are proportional to the forward rates. We empirically determine the precise functional form of the volatility functions in both two-factor models from principal component analyses.

Our choice of the state variables for the two two-factor spot-rate models we investigate is motivated by two empirical findings. First, principal component analyses in combination with regression analyses reveal that the first component can be identified with the level of the yield curve; the second is closely related to the spread between the long and the short rate.[4] We take these findings as a guideline for the construction of our first two-factor model, which uses both a long rate and the spread between the same long rate and the short rate as factors. The basic idea for this line of approach goes back to Brennan and Schwartz (1979) and Schaefer and Schwartz (1984).

The second two-factor model is based on the observation that the short-rate volatility exhibits typical volatility clusterings. Therefore, a model with stochastic volatility of the short rate could be an appropriate description of the data. This model represents a generalization of the Longstaff and Schwartz (1992) approach.

Both two-factor spot-rate models are special cases of the affine class of term structure models. (See Duffie and Kan (1996), pp. 383–391.) Although they are mathematically equivalent, empirically, they can behave very differently.

B. Review of the Models and Basic Implementation Steps

B.1. The Forward-Rate Models

Here, we briefly review the HJM approach and the concrete implementation realized in this study. Rather than discussing the approach in general terms, we concentrate on the simplest derivation for the forward-rate models that we use in our empirical investigation.

[2] See Chan et al. (1992) for the U.S. market, Barone at al. (1991) for the Italian market, and Walter (1996) for the German market.

[3] See Uhrig and Walter (1996). The details of this estimation procedure are discussed later.

[4] These results are also found for other markets. For example, see Litterman and Scheinkman (1991) for the U.S. market, Rebonato (1996) for the British market, and Bühler and Zimmermann (1996) for a recent study of the Swiss market.

Table I

Forward-Rate Models under Consideration

This table summarizes the parametric specification of the volatility functions. $\sigma(t,T,f)$ denotes the volatility function for the one-factor models. $\sigma_1(t,T,f)$ and $\sigma_2(t,T,f)$ represent the two volatility functions for the two-factor models. f denotes the instantaneous forward rate at date t for instantaneous borrowing or lending at date T ($T \geq t$). σ, σ_0, and σ_1 are positive parameters. $\sigma_1(t,T)$ and $\sigma_2(t,T)$ are functions of t and T, which are to be empirically determined. In order to avoid an explosion of the forward-rate processes in finite time, the proportional volatility is capped by a large positive number M.

Panel A. One-Factor Models	
$\sigma(t,T,f) \equiv \sigma$	Absolute I
$\sigma(t,T,f) = (\sigma_0 + \sigma_1(T-t))\min(f,M)$	Linear proportional
Panel B. Two-Factor Models	
$\sigma_1(t,T,f) = \sigma_1(t,T)$ $\sigma_2(t,T,f) = \sigma_2(t,T)$	Absolute II
$\sigma_1(t,T,f) = \sigma_1(t,T)\min(f,M)$ $\sigma_2(t,T,f) = \sigma_2(t,T)\min(f,M)$	Proportional II

The fundamental building block of this approach is the whole instantaneous forward-rate curve. HJM start with a fixed number of unspecified factors that drive the dynamics of these forward rates:[5]

$$df(t,T) = \alpha(t,T,\cdot)\,dt + \sum_{i=1}^{2} \sigma_i(t,T,f)\,dz_i(t), \tag{1}$$

where $f(t,T)$ denotes the instantaneous forward interest rate at date t for borrowing or lending at date T ($T \geq t$), $z_1(t)$ and $z_2(t)$ denote independent one-dimensional Brownian motions, and $\alpha(t,T,\cdot)$ and $\sigma_i(t,T,f)$ are the drift and the volatility coefficients of the forward rate of maturity T. As HJM show, when a number of regularity conditions and a standard no-arbitrage condition are satisfied, then the drift of the forward rates under the risk-neutral measure is uniquely determined by the volatility functions $\sigma_i(t,T,f)$:

$$\alpha(t,T,\cdot) = \sum_{i=1}^{2} \sigma_i(t,T,f) \int_t^T \sigma_i(t,s,f)\,ds. \tag{2}$$

As we noted earlier, we focus on four specifications of this approach: two one-factor models and two two-factor models. The parametric specification of the volatility functions is shown in Table I.

[5] In the following, we concentrate on two factors. In general, drift and volatilities can depend on the path of the forward-rate curve. See Heath et al. (1992), p. 80.

The first step in implementing any particular valuation model for interest-rate derivatives is to estimate the current yield curve, which is used in the forward-rate models in the form of the current forward-rate curve $f(0,T)$. We discuss the term structure estimation procedure in detail in Section III.

In the second step, we estimate the volatility parameters. As discussed earlier, we estimate these parameters from time-series observations of forward rates. We obtain the volatility parameters of the two one-factor models directly from forward-rate changes and relative forward-rate changes, respectively.

By means of principal component analyses, we determine the volatilities of the forward rates $f(t,T)$ in the two-factor models by the factor loadings and the volatilities of the two independent factors.

In the third step, we compute the option prices. In this step, we discretize equation (1) under the risk-neutral measure by building a binomial tree model. This tree is nonrecombining for the Linear Proportional and the Proportional II forward-rate models. Supporting the findings of Amin and Morton (1994), we find that seven time steps are sufficient to achieve accurate option prices. A seven-step binomial nonrecombining tree contains 254 (128 final) nodes for the one-factor models and 3,279 (2,187 final) nodes for the two-factor models. As usual, in the backward induction procedure we consider the premature exercise feature of American options, taking at each node the maximum of the intrinsic value and the value of the option if not exercised.

B.2. The Spot-Rate Models

Unlike the forward-rate models that define the stochastic behavior of the term structure relative to the observable current term structure, spot-rate models must be adapted to the current interest rates and volatilities. Following a suggestion made by Cox et al. (1985), we achieve this adaptation through time-dependent parameters of the stochastic factor processes. These time-dependent parameters are determined in such a way that both the endogenous term and volatility structures fit with the observable one. In principle, this requires an inversion of the valuation formula.

In our approach, we base the inversion of the spot-rate models on the fundamental valuation equation for derivatives as derived in the general equilibrium setting by Cox et al. (1985).[6] If the value of a derivative with payoff at time T is a sufficiently smooth function $F(x_1,x_2,t,T)$ of time and two independent diffusion processes x_1, x_2 with drift and diffusion functions $\mu_i(x_i,t)$, $\sigma_i(x_i,t)$, $i = 1,2$, then F satisfies the following partial differential equation in the no-exercise region:

$$F_t + F_{x_1}[\mu_1 - \Theta_1\sigma_1] + F_{x_2}[\mu_2 - \Theta_2\sigma_2] + F_{x_1x_1}\tfrac{1}{2}\sigma_1^2 + F_{x_2x_2}\tfrac{1}{2}\sigma_2^2 = rF, \qquad (3)$$

[6] See Cox et al. (1985), p. 388. For an exposition of the Hull and White approach, refer to Hull and White (1990). A discrete variant is described in Hull and White (1993). For a presentation of the inversion problem in the framework of equivalent martingales, see Schmidt (1994), p. 15.

where r represents the instantaneous risk-free rate, and $\Theta_1(x_1, x_2, t)$ and $\Theta_2(x_1, x_2, t)$ are the market prices of risk for the two factors. It follows from no-arbitrage arguments that Θ_1 and Θ_2 are real-valued functions of only the state variables and time. The subscripts of F denote partial derivatives. We obtain the values for contingent claims by solving this parabolic partial differential equation subject to appropriate initial and boundary conditions.

In this study, we derive the three spot-rate models tested from equation (3) by specifying the nature of the stochastic processes driving the factors, the functional form of the market prices of risk, and the relation between the factors and the instantaneous spot rate, r. Table II summarizes the assumptions underlying the one-factor model and the models with two factors.

B.2.1. One-Factor Spot-Rate Model. The first model is a one-factor interest-rate model, in which we assume that the dynamics of the short-term interest rate $r(t)$ exhibit mean-reversion and that the diffusion coefficient depends on the level of the short rate. κ, γ, σ, and ϵ are positive constants.

The basic idea behind Hull and White's (1990) procedure is to allow for time-dependent parameters in the risk-neutralized process of r, in order to match the solution of equation (3), in the case of zero bonds, to an exogenously given term structure of zero bond prices. If the elasticity parameter ϵ is positive, this calibration must be carried out numerically. Generally, any of the parameters can be selected as a time-dependent function. However, for several economic and technical reasons, we select the market price of risk as a time-dependent function.[7] In principle, if a second parameter is assumed to depend on time, the model can also be fitted to an exogenously given current volatility structure. However, this procedure has an important drawback: this second time-dependent parameter results in unstable and partially unrealistic future endogenous volatility structures. (Hull and White (1993, 1996) report similar results.)

In light of these findings, we do not calibrate the model to the whole current volatility structure with a second time-dependent parameter. Instead, we fit only two points of the endogenous volatility structure to the observable one, the volatilities of the short and the long rates. Volatilities of intermediate rates are interpolated by the model.

The advantage of this parsimonious fitting procedure is that the model results in stable future volatility structures. Technically, we achieve the two-point calibration of the model to the current volatility structure by the (constant) mean-reversion parameter κ, which determines the transmission of the instantaneous interest rate's volatility to the volatilities of long rates (cf. Uhrig and Walter (1996), pp. 87–88).

[7] The arguments refer to the economic interpretation of the calibration process, the existence of a solution for the fitting function, the consequences for the endogenous volatility structure, and the relation between risk-adjusted and original measure. Cf. Heath et al. (1992), pp.96–97, and Uhrig and Walter (1996), pp. 84–85.

Table II

Spot-Rate Models under Consideration

This table describes the basic structure of the stochastic processes behind the respective factors, the functional form of the market prices of factor risk, and the relation between the factors and the instantaneous spot rate, r. The one-factor model is driven by a one-dimensional Brownian motion, z. z_1 and z_2 denote the independent one-dimensional Brownian motions driving the two-factor models. The market price of interest-rate risk is denoted by Θ. Θ_1 and Θ_2 denote the market prices of factor risk for the factors of the two-factor models. κ, κ_x, and κ_y are positive mean-reversion parameters. The positive parameters γ, γ_s, γ_x, and γ_y denote the long-term stationary means of the respective factors. σ, ϵ, σ_l, σ_s, σ_x, and σ_y are positive parameters of the volatility functions. $\lambda(t)$ denotes a time-dependent function. ϕ is a constant parameter.

Factor	Nature of Processes	Factor Risk	Short Rate
Short-rate r	$dr = \kappa(\gamma - r)dt + \sigma r^{\epsilon} dz$	$\Theta = \lambda(t)$	Given
Long-rate l	$dl = \sigma_l \sqrt{l} dz_1$	$\Theta_1 = \phi\sqrt{l}$	
Spread s between long and short rate	$ds = (\gamma_s - s)dt + \sigma_s dz_2$	$\Theta_2 = \lambda(t)$	$r = l - s$
Two unspecified factors x, y, linearly related	$dx = \kappa_x(\gamma_x - x)\,dt + \sigma_x\sqrt{x}\,dz_1$	$\Theta_1 = 0$	
to the short rate and its volatility	$dy = \kappa_y(\gamma_y - y)\,dt + \sigma_y\sqrt{y}\,dz_2$	$\Theta_2 = \lambda(t)\sqrt{y}$	$r = x + y$

The model's implementation requires four steps. First, we determine the current yield curve of zero bonds from prices of coupon bonds. Second, we use an Euler-discretization to estimate the drift and volatility parameters of the short-rate process. We obtain the maximum likelihood estimates from time-series observations of the one-month money market rate. Third, we achieve the fitting to the current yield curve and to the volatility of the long rate with maturity $\bar{T} = 9$ years by using a numerical algorithm that simultaneously determines the time-dependent function $\lambda(t)$ and the parameter κ. This requires the solution of the partial differential equation (3) with one state variable $x_1 = r$ under the following conditions:

i. maturity condition for zero bonds: $F(r,T,T) \equiv 1$

ii. fitting condition for endogenous zero bond prices to observed prices $\hat{F}(T)$ conditional on the current instantaneous rate, $r(0)$:

$$F(r(0),0,T) = \hat{F}(T) \qquad (0 < T \leq \bar{T}) \tag{4}$$

iii. fitting condition for the volatility of the long rate:

$$-\frac{1}{T}\,\frac{F_r(r(0),0,\bar{T})}{F(r(0),0,\bar{T})} = \hat{v}. \tag{5}$$

The left-hand side of equation (5) represents the ratio of the endogenous yield volatility of a zero bond with maturity $\bar{T}$ and the short-rate volatility $\sigma r(0)^{\epsilon}$. The right-hand side denotes the ratio of the historical observed volatility of the zero bond yield with maturity $\bar{T}$ and the short-rate volatility. We use the inverted implicit finite difference method introduced by Uhrig and Walter (1996) to solve equations (3)–(5).[8] In the fourth and final step, we determine the values of the interest-rate warrants by solving equation (3) together with the appropriate boundary conditions for calls and puts by a fully implicit Crank–Nicholson method. We use a time step Δt of one day for the time variable and a grid size of 1/30 for the transformed state variable $z = 1/(1 + r/r(0))$. This change of the state variable, proposed by Brennan and Schwartz (1979), has the advantage that the original state space $[0,\infty)$ is transformed into the bounded interval $[0,1]$, and that the boundary conditions are easier to handle. For $z = 0$ ($r = \infty$) we use the fact that the values of bonds and European options are zero, and for $z = 1$ ($r = 0$) we exploit numerically the special structure of the transformed partial differential equation (3). We take into account the early exercise possibilities for American options in the recursive backward procedure.

B.2.2. Two-Factor Model with Long Rate and Spread. The choice of the state variables in the second model is motivated by the findings in our data analysis process, and by the fact that the correlation among the changes of

[8] Hull and White (1993) solve equations (3)–(5) on the basis of a trinomial tree. This method can be considered as an explicit scheme. Cf. Brennan and Schwartz (1978), p. 464.

these two factors is less than 0.23. These empirical observations suggest that a two-factor model that uses a long-term rate l and the spread s between the long-term rate l and the short-term rate r as independent stochastic factors can plausibly describe the yield curve dynamics.

An examination of the time-series behavior of long rates in the German bond market reveals that there is only a slight mean-reversion tendency in long rates. The maximum likelihood estimate of the discrete version of the process $dl = \kappa_l(\gamma_l - l)dt + \sigma_l l^{\epsilon_l} dz_1$ for the mean-reversion parameter κ_l is not significantly different from zero on the 1 percent level. As a proxy for the long rate, we use the yield to maturity of a nine-year zero bond.

To exclude, additionally, negative long rates l and to keep the model analytically tractable, we model the long rate as a martingale with a square root representation $\sigma_l \sqrt{l}$ of the diffusion coefficient.[9] The spread process is assumed to follow an Ornstein–Uhlenbeck process, in line with the observation that this process can take on both positive and negative values. The parameters σ_l, γ_s, and σ_s are positive and constant.

We calibrate the model to the current term structure of interest rates by means of a time-dependent market price of spread risk. Because of the separability of variables and the choice of an Ornstein–Uhlenbeck process for the spread, we can solve this problem analytically. Moreover, this spread process ensures that the endogenous term structure of interest rates can be adapted to every observed one. This is not generally true for two-factor models, which use nonnegative state variables.

We use the market price of long-term interest-rate risk to overcome a problem that is typical for two-factor models in which both factors are interest rates (see also Duffie and Kan (1996), p. 383). The state variable l is labeled "long rate," but it does not have this property, since the price of a zero bond depends on both state variables s and l. Therefore, a zero bond with a maturity corresponding to l has a yield to maturity that does not rely only on l. This internal inconsistency can be considerably reduced by an appropriate choice of the market price of risk θ_1 of the long rate.[10]

Again, the implementation of the model consists of four steps. The first step coincides with the first step of the one-factor model. The second step is reduced to an estimation of the two volatilities of the long rate and the spread. These two parameters also reflect the information about the volatility structure. In the third step, the endogenous zero bond prices are analytically fitted to the observed prices by exploiting the separability of the solution $F(l,s,t,T) = G(l,t,T)H(s,t,T)$. Finally, we compute the option values by means of the alternating direction implicit method.[11]

[9] However, this modeling has the drawback that for this process, $l = 0$ is an absorbing barrier.

[10] θ_1 is chosen such that $-(1/T)\ln(F(l,s,t,t + T) = l$ holds for the specific time-to-maturity of $T = 10$ years.

[11] In contrast to the valuation of zero coupon bonds within the two-factor models, the valuation of options on coupon bonds makes it necessary to solve a partial differential equation in two state variables, because the terminal condition cannot be separated.

The grid sizes for the time and state variables are fixed identically to that of the one-factor model. Because the state variable s can take on both positive and negative values, we choose a special treatment. In the numerical procedure, we restrict this state variable on the interval $(-\infty, l(0)]$. We choose the upper boundary $l(0)$ to ensure nonnegative short rates for the current level of l. By an appropriate change of the state variable, we transform the original state space $(-\infty, l(0)]$ into the state space $[0,1]$. For $s = -\infty$, we use the fact that the values of bonds and European options are zero. For $s = l(0)$ we impose a boundary condition, setting the second derivative F_{ss} equal to zero.

B.2.3. Two-Factor Model with the Short Rate and Its Volatility. From our data analysis step, we know that the short rates exhibit volatility clusters. This behavior can be modeled approximately by stochastic volatility. Since interest-rate volatility is a key variable in option pricing, a promising approach might be to add to the one-factor model the level of interest-rate volatility as a second state variable.

The resulting model represents a generalized version of the general equilibrium model proposed by Longstaff and Schwartz (1992). Longstaff and Schwartz base their model on assumptions about the stochastic evolution of two abstract independent factors x and y, described in Table II, in which κ_x, γ_x, σ_x, κ_y, γ_y, and σ_y are (positive) parameters. The short-term rate r and its instantaneous variance V are determined endogenously as part of the equilibrium:

$$r = x + y \tag{6}$$

$$V = \sigma_x^2 x + \sigma_y^2 y. \tag{7}$$

Using this system of linear equations, we can represent the fundamental valuation equation (3) for interest-rate derivatives in terms of the observable state variables r and V.

To achieve consistency with the current term structure, we generalize the model by allowing for a time-dependent risk parameter. Because of the separability of the partial differential equation in the state variables x and y, we can reduce the adaptation of the endogenous to the exogenous term structure of interest rates to the adaptation problem within the one-factor Cox et al. (1985) model.

Again, the implementation procedure requires four steps. As in the other models, the first step involves the estimation of the current term structure of interest rates. Compared with the other two spot-rate models, the estimation of the parameters in the second step is more complex, because the volatility of the short-term rate is not directly observable. Following Longstaff and Schwartz (1993), we use a two-phase approach. In the first phase, we estimate the volatility of the short-term rate by using a generalized autoregressive conditional heteroskedastic (GARCH) model. In the second phase, we estimate the parameters describing the movement of the short-term interest rate and its volatility. To estimate these parameters, we equate the first two moments of the long-run stationary unconditional distribution of r and V with their historical counterparts.

In addition to these four equations, we obtain two further conditions by choosing the volatility parameters σ_x^2 and σ_y^2 as the minimum and the maximum of the ratio $V(t)/r(t)$, respectively. By using these six conditions, we calculate the six parameters of the model by solving a nonlinear system of six equations.

The calibration of the model to the current term structure in the third step follows the same procedure as the one-factor model. Again, we apply the inverted implicit difference method. The computation of the option prices in the fourth step is comparable to the procedure in the other two-factor spot-rate model.

II. The German Fixed-Income Market

The German bond market is the third largest in the world. At year-end 1995, the nominal value of outstanding publicly issued bonds totaled more than three trillion deutsch marks (DM).

Traditionally, the bank bond sector is the largest component. However, bonds issued by the federal government are the most liquid. Typically, so-called *Bundesanleihen* (BUNDs) are issued with an initial maturity of ten years and *Bundesobligationen* (BOBLs) with an initial maturity of five years. BUNDs are termed as long term, and BOBLs as medium term. Various interest-rate derivatives have been launched in the last seven years, of which the BUND-Future (futures on ten-year government bonds) at the Deutsche Terminbörse is the most popular.

Three types of interest-rate options trade in Germany: options on the BUND-Future and on the BOBL-Future (futures on five-year government bonds), interest-rate warrants, and over-the-counter (OTC) interest-rate options of all types.

Because the market for options on futures is not particularly liquid and sufficiently long-term time-series data are not available for OTC-options, our empirical study is based on the valuation of interest-rate warrants. Underlying these warrants are German government bonds. Most of them are the American type, with maturities of up to three years. Therefore, they represent a more diverse sample than would standardized options traded at options exchanges.

A. Government Bonds

Most German government bond issues are straight bonds with a fixed coupon size and one coupon payment per year. These bonds build a homogeneous market segment in bankruptcy risk, liquidity, and taxes. A subsample of these bonds represents the underlyings of the interest-rate warrants considered in our study. In the period from 1990 to 1993, the coupons of these bonds varied between 5 percent and 10.75 percent. The initial maturity of these bonds was between five and ten years. This maturity structure implies that the longest maturity of actively traded BUNDs lies between nine and ten years. To avoid an extrapolation of the term structure of interest rates beyond the maturity of traded bonds, the *long rate* is defined as the nine-year rate.

BUNDs and BOBLs are listed at each of the eight exchanges in Germany. At every exchange, trading is organized as a call-auction market with a single market-clearing price at noon each day. This auction price is set so that the turnover is maximized. A small subsample of the government bonds also trades in continuous auctions at the Frankfurt Stock Exchange, and on XETRA, the computerized trading system. The bond prices used in our study come from the daily noon auction process carried out at the Frankfurt Stock Exchange, which is the largest in Germany. The data are available from the German Financial Data Base Mannheim/Karlsruhe.[12]

B. Interest-Rate Warrants

German interest-rate warrants began trading at the end of 1989. These instruments are issued by banks and, as in the bond market, exchange trading takes place in daily noon auctions, with one single market-clearing price each day. In this study, we use all call and put options listed in the two market segments, *Amtlicher Handel* and *Geregelter Markt*, of the Frankfurt Stock Exchange. The sample period covers the period from January 1990 through November 1993. During this period, nineteen different calls and fourteen different puts traded on thirteen different German government bonds. Ten of the thirteen underlying bonds were BUNDs, the remaining three were BOBLs.

During the sample period, the time-to-maturity of the bonds ranged from 6.9 to 9.1 years for the long-term bonds, and from 3.4 to 3.8 years for the medium-term bonds. The average time-to-maturity for the options was 0.85 years, with a maximum of 2.91 years. With the exception of three European interest-rate warrants, the options under consideration were American-type options. We use weekly observations. The total number of option prices we collected amounts to 1,751. A detailed description of the interest-rate warrants' terms, including the average number of daily trades and the average daily turnover, appears in Table AI in the Appendix.

C. Money Market Rates

Bid and offer rates in the German money market are available for one day, as well as one month, and two, three, six, twelve, and twenty-four months. Because the daily rate fluctuates strongly and the level and changes of this

[12] This database was established under the Empirical Capital Market Research Program, which is supported by the German National Science Foundation.

rate are only loosely related to other short-, medium-, and long-term rates, the daily rate cannot reasonably be used to explain the evolution of the whole term structure of interest rates. However, these restrictions do not hold to the same extent for the second shortest rate, the monthly rate. This rate is therefore selected as "the short rate" for the empirical part of our study.[13] In addition, we use German money market rates with a time-to-maturity of up to six months to support the yield curve estimation for short maturities.

III. Design of the Study and Estimation Results

A. Methodology

The quality of different valuation models can be assessed by at least two well-known methodologies. The first compares out-of-sample differences between model and market prices; the second checks whether observed differences can be exploited by a dynamic replication strategy.

In this study, we apply the first strategy. The reason for this is that our test is not directed toward the efficiency of the German market for interest-rate warrants or toward the applicability of the models considered in a trading environment. Instead, we address the problem of which of the models are best suited to measure the exposure to interest-rate risk. This goal of the study, together with additional arguments elaborated earlier, results in our decision to estimate the parameters of the stochastic factors from time series. As the valuation quality of the different models is unavoidably assessed together with the estimation procedure of the input data, we use the same raw data and the same statistical methodology throughout:

i. For all models, we base the estimation of the term structures of interest rates on an identical set of German government bonds. The forward rates for the HJM models are determined from these term structures.
ii. We estimate all time-independent parameters of the models by the maximum likelihood method.
iii. Concerning the length of the historical time series, we distinguish between structural and volatility parameters. We estimate the structural parameters (those parameters that are relevant for the basic structure of the model) by using an estimation period of at least 20 years. The volatility parameters must reflect the current market information. Therefore, we estimate them by using only observations of the previous nine to twelve months. Figure 1 illustrates this procedure.

[13] Pearson and Sun (1994), p. 1285, point out that the identification of the instantaneous rate r with the one-month money market rate introduces measurement errors because the instantaneous interest rate does not depend on the market prices of risk, but the one-month rate does. This effect could be avoided by using state variables that can be observed. Since, contrary to the study of Pearson and Sun, we fit the model to the current term structure, we avoid this error. The choice of the one-month rate does, however, affect the volatility estimates.

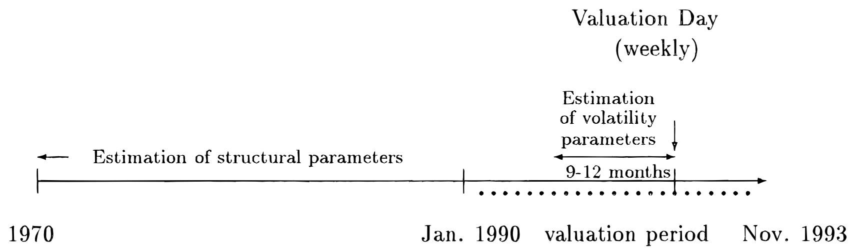

Figure 1. Design of the study. This figure shows the basic design of our study. The valuation period from January 5, 1990, through November 16, 1993, consists of 204 weeks. One day (Friday) is taken from each of these weeks as a valuation day. For each of these valuation days, we carry out the following steps for each of the seven models: (i) estimation of the current term structure of interest rates, (ii) estimation of the structural and volatility parameters, (iii) calibration of the spot-rate models to the current term and volatility structures of interest rates, and (iv) valuation of all interest-rate warrants traded on the current day. Concerning the length of the historical time series, we distinguish between structural and volatility parameters. We estimate the structural parameters (those parameters that are relevant for the basic structure of the model) by using an estimation period of at least twenty years. The volatility parameters must reflect the current market information. Therefore, we estimate them by using observations of the previous nine to twelve months.

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1. estimation of the current term structure of interest rates
2. estimation of the structural and volatility parameters
3. calibration of the spot-rate models to the current term and volatility structures of interest rates
4. valuation of all interest-rate warrants traded on the current day.

B. Estimation of the Current Term Structure of Interest Rates

We estimate the term structure of interest rates from the homogeneous market segment of government bonds. Since the German government does not issue zero coupon bonds, and a stripping possibility comparable to the U.S. STRIPS program did not exist in Germany until July 1997, the term structure of interest rates can only be determined from traded coupon bonds. To estimate the current term structure of interest rates, we use all the straight bonds issued by the German government that have a time-to-maturity from six months to ten years. We exclude bonds with a maturity less than six months, because this market segment has a lower liquidity and its transaction costs influence short-term yields more than long-term yields. Instead, we construct synthetic short-term bonds to reflect the prevailing money market rates.

Table III

Summary Statistics of the Estimation Quality of Term Structures of Interest Rates

This table shows summary statistics of the deviations between the theoretical and market bond prices. The sample period is January 1990 to November 1993. The observations are weekly. The deviations are measured in DM per DM 100 nominal value.

Time Period	1990–1993	1990	1991	1992	1993
Valuation days	204	52	52	52	48
Total number of observations	20308	5392	5310	5052	4554
Average number of bonds per valuation day	99.55	103.69	102.12	97.15	94.88
Mean absolute deviation	0.1477	0.1330	0.1556	0.1854	0.1138
Standard deviation of absolute deviations	0.1658	0.1409	0.1862	0.1900	0.1252

The estimation procedure for the term structure of interest rates affects the valuation of interest-rate options in two important ways: First, we value the bond underlying the option using the term structure of interest rates. Second, in models of the HJM type, we estimate the volatility of the forward rates from a time series of the term structures of interest rates. In the spot-rate models, the estimated term structure directly affects the time-dependent market price of risk $\lambda(t)$.

These two effects result in different requirements on the term structure estimates that are not in line with each other: The first effect leads to the recommendation to implement an estimation procedure that minimizes the deviations between observed and theoretical prices of the underlyings. The reason behind this is that deviations are directly transferred to differences between the observed and theoretical option values. Consequently, these differences should not be attributed to the valuation model. However, such an estimation fully transfers noise of coupon bond data to the term structure of interest rates, the noise in which results in irregular time-dependent market prices of risk in the spot-rate models and leads to unreasonably high volatility estimates of forward rates. In both cases, the ex ante predictability of the valuation models turns out to be very low. Therefore, a balance between accuracy and smoothness of the term structure of interest rates must be determined.

We achieve this compromise in two steps. In the first step, for every cash flow date of one of the bonds in the sample, we determine a discount factor by using a quadratic linear programming approach. This results in discrete term structure estimates with the highest possible accuracy in explaining observed bond prices. In the second step, we smooth out this discrete term structure by using cubic splines with ten nodes. This smoothing procedure increases the mean absolute deviation in the sample period from DM 0.071 to DM 0.148 per DM 100 nominal value. Table III shows some summary statistics of the deviations between the theoretical and the market prices of the bonds.

C. Parameter Estimates

Due to four complications, parameter estimates across models are not directly comparable. First, we use different factors (forward rates, spot rates, volatility) as basic variables. Second, even if we concentrate on models with the same factors, drift and volatility functions differ in form. Third, the number of different parameters that we need to estimate varies substantially across models. The one-factor forward-rate model with absolute volatility requires the estimation of one single parameter. In contrast, six parameters must be estimated for the spot-rate model with stochastic volatility. The fourth complication is that all forward-rate models and the two-factor spot-rate model with long rate and spread as factors have only volatility parameters, but the two remaing spot-rate models depend on both structural and volatility parameters.[14]

C.1. Estimation of Structural Parameters

In the one-factor spot-rate model, we interpret two parameters, the long-term mean γ and the elasticity parameter ϵ, as structural parameters. We estimate these parameters by using the discrete process obtained by applying the Euler scheme. We calculate both parameters for each valuation day from the time series of the one-month money market rates, starting in January 1970 and running up to (but excluding) the current valuation day.[15] The maximum likelihood estimates of γ vary between 0.062 and 0.067 and those of ϵ between 0.77 and 0.90.

The stationary mean γ_s of the spread process is the only candidate for classification as a structural parameter in the two-factor spot-rate model with long rate and spread. However, there is no need to estimate this parameter separately in the risk-neutral process, because it appears only in combination with the market price of spread risk. Since we use the latter parameter to calibrate the model to the initial term structure of interest rates, we can use an arbitrary value for γ_s.

The first requirement for a historical parameter estimation for the two-factor spot-rate model with stochastic volatility is a time series of volatilities of the short rate. The relation between the short-term interest rate and its volatility is interpreted as a structural link and is described as in Longstaff and Schwartz (1992) by the following GARCH model:

$$r_t - r_{t-1} = b_0 + b_1 r_{t-1} + b_2 V_t + \epsilon_t, \tag{8}$$

$$\epsilon_t \sim N(0, V_t), \quad \text{and} \tag{9}$$

$$V_t = a_0 + a_1 r_{t-1} + a_2 V_{t-1} + a_3 \epsilon_{t-1}^2, \tag{10}$$

[14] The reason some models do not comprise structural parameters is that they are not formulated in the most general form. So, in developing these models, we have used the available degrees of freedom to give these models more structure.

[15] Contrary to the U.S. time series of interest rates, we have no evidence of structural shifts in the time series of interest rates at the end of the 1970s in Germany.

where $b_i, i = 0,\ldots,2$ and $a_j, j = 0,\ldots,3$ denote the parameters of the GARCH model. The maximum likelihood estimates of this GARCH model for weekly observations of one-month money market rates from January 1970 to December 1989 appear in Table AII in the Appendix. We use the estimated GARCH model (8)–(10) to determine the volatility of the short rate for each valuation day from January 1990 to November 1993.[16] The second set of structural parameters in the two-factor model are the first two moments, $E(r_\infty)$ and $\mathrm{Var}(r_\infty)$, of the long-run stationary unconditional distribution of the short rate. As with the structural parameters in the one-factor model, we obtain the two moments from weekly observations of one-month money market rates from January 1970 up to (but not including) the current valuation day. The estimates for the first moment $E(r_\infty)$ vary between 0.064 and 0.068. The estimates for the variance $\mathrm{Var}(r_\infty)$ are close to 0.0007 for the entire valuation period.

C.2. Estimation of Volatility Parameters

The second type of parameters are related to interest-rate volatilities. Empirical results for a wide variety of markets show that volatilities vary significantly over time. Thus, we estimate the volatility parameters from short time series with a lengths of nine to twelve months.

As we have noted for the HJM models, only volatility parameters need to be estimated. We base all estimates on time series of weekly changes in instantaneous forward rates $f(t,T)$ with a fixed maturity date T. The time series covers a period of nine months. For the one-factor models, we use these forward-rate changes $f(t,T) - f(t - \Delta t, T)$ directly to estimate the volatility parameters σ and σ_0, σ_1, respectively. For the two-factor models, on every valuation date t, we conduct a principal component analysis to extract the first two principal components (factors) that explain the comovement of the forward-rate changes. The volatilities of these two factors and the corresponding factor loadings determine the two required volatility functions $\sigma_1(t,T)$ and $\sigma_2(t,T)$ of Table I. We summarize the results of the parameter estimation for the HJM models in Table AIII in the Appendix.

The spot-rate models require the estimation of widely differing volatility parameters. Therefore, we give a brief description of the estimation procedure and the obtained results for each of the models. For the one-factor model, we need the volatility σ of the short-rate process and the volatility of a long-rate process. As described earlier, we use the latter estimate to determine implicitly the mean-reversion parameter κ so that the endogenous volatility of the long rate equals the estimated volatility. As a by-product, this implicit estimation of κ allows us to cope with the problem that the maximum likelihood estimates of κ prove to be considerably upward biased. We estimate the two volatility parameters from changes of the one-month money market rates and the nine-year zero bond yields derived from the

[16] For a detailed description of the estimation procedure and the parameter estimates, their standard errors, and the log likelihood function value for the GARCH model, see Uhrig (1996).

term structures of interest rates, respectively. Again, for each valuation day, we base our estimates on the weekly observations of the previous nine months. Panel A of Table AIV in the Appendix presents summary statistics for the two parameters.

The two-factor model with the long-term rate and spread requires the volatilities of these two factors. Once again, we determine the parameters from the weekly changes of the nine-year yield as well as the changes of the spread, using observations of the nine months preceding the valuation day. We summarize the estimates in Panel B, Table AIV, in the Appendix.

For the two-factor model with stochastic volatility, we need to estimate the first two moments of the volatility's long-run stationary unconditional distribution, $E(V_\infty)$ and $\text{Var}(V_\infty)$, and the maximum and minimum values for the ratio $V(t)/r(t)$. We determine these four parameters from the weekly changes of estimated volatilities and the weekly changes of the one-month money market rate, based on our observations of the preceding twelve months. The summary statistics of these results appear in Panel C, Table AIV, in the Appendix. We use these results and the estimates of the structural parameters $E(r_\infty)$, $\text{Var}(r_\infty)$ to obtain the six parameters, κ_x, γ_x, σ_x, κ_y, γ_y, and σ_y, that describe the dynamics of the two unspecified factors, x and y. As in Longstaff and Schwartz (1993), we compute the parameters from the nonlinear system of equations represented in Table AIV in the Appendix. Panel C, Table AIV, in the Appendix also displays the estimated values for these six parameters.

The volatility estimates we obtain for the seven models are not directly comparable. To facilitate the comparison of the results across models, we compute the volatilities for two selected rates from these estimates. More precisely, for the forward-rate models, we determine the implied instantaneous standard deviation of changes in instantaneous spot rates $f(t,t)$, and also in instantaneous forward rates $f(t,t+9)$ that mature nine years from the current day. Accordingly, for the spot-rate models, we compute the instantaneous standard deviation of changes in instantaneous spot rates $r(t)$ as well as in the instantaneous nine-year zero yields $r(9)$ implied by the models.

To compute these volatilities for the forward-rate one-factor models, we only need the estimated volatility parameters and, for the linear proportional model, the forward rates $f(t,t)$ and $f(t,t+9)$ for each day. For the two-factor forward-rate models, we also require the corresponding factor loadings.

The computation of the long rate's volatilities is much more cumbersome for those two spot-rate models in which l is not a factor of the model. The models' endogenous volatility depends, in general, not only on the constant parameters of the models and the current value of the nine-year yield, but also on the time-dependent market price of risk, which itself is influenced through the calibration process by the entire term structure of interest rates on the day under consideration.

In Tables IV and V we report the estimation results for the spot-rate and forward-rate models, respectively. Before discussing the results, we point out that the estimates for the nine-year forward-rate volatilities reported in

Table IV

Estimated (Absolute) Volatilities of the Spot and Nine-Year Forward Rate for the Forward-Rate Models

This table presents the estimated (absolute) volatilities of the spot and nine-year forward rate for the forward-rate models. Absolute I is the one-factor model with constant volatility. Linear proportional denotes the one-factor model with linear proportional volatility. Absolute II is the two-factor model in which both volatility functions are independent of the level of the forward rate. In the two-factor model Proportional II, both volatility functions are proportional to the level of the forward rate. The parameters σ_{spot} and σ_9 are the (absolute) volatilities of the instantaneous spot rate $r(t) = f(t,t)$ and the instantaneous nine-year forward rate $f(t,t+9)$ implied by the different models. If multiplied by 100, they represent volatilities in percent p.a. The parameters are estimated for 204 Fridays within the valuation period from January 5, 1990, through November 16, 1993.

Model	Volatilities	Mean	Minimum	Maximum	Standard Deviation
Absolute I	σ_{spot}	0.0128	0.0097	0.0159	0.0017
	σ_9	0.0128	0.0097	0.0159	0.0017
	σ_9/σ_{spot}	1	1	1	0
Linear proportional	σ_{spot}	0.0093	0.0047	0.0160	0.0026
	σ_9	0.0174	0.0110	0.0278	0.0032
	σ_9/σ_{spot}	2.0341	0.9110	4.1094	0.7133
Absolute II	σ_{spot}	0.0055	0.0010	0.0156	0.0033
	σ_9	0.0116	0.0059	0.0232	0.0051
	σ_9/σ_{spot}	2.9870	0.4181	9.9396	2.1322
Proportional II	σ_{spot}	0.0061	0.0013	0.0176	0.0037
	σ_9	0.0105	0.0052	0.0247	0.0050
	σ_9/σ_{spot}	2.5168	0.3081	8.9511	1.8922

Table IV are not comparable to the estimates for the nine-year zero-rate volatilities of Table V. The nine-year forward-rate volatility refers to an *instantaneous* forward rate maturing nine years from the current day. The latter refers to a rate for a period that is nine years long. Only the instantaneous volatility of the spot rate $r(t) = f(t,t)$ is comparable across all models.

If σ_{spot} and σ_9 are multiplied by 100, we obtain the (absolute) volatilities in percentage per annum (p.a.). A division of σ_{spot}, as estimated for the absolute model by an average short rate of 0.088 in the valuation period, results in an approximation for the mean relative volatility of about 15 percent p.a.

On average, the volatilities of the forward rates increase with maturity. The only exception is the absolute one-factor model in which constant volatilities are assumed. The (constant) volatility of this model is approximately equal to the arithmetic mean of σ_{spot} and σ_9 in the linear proportional model.

Additionally, we find that volatilities are very similar across the two-factor HJM models. In comparison to the one-factor models, both the mean volatility of the spot and the nine-year forward rate are lower. However, the standard deviations are higher for the two-factor models. This bias of the

Table V

Estimated (Absolute) Volatilities of the Spot and Nine-Year Zero Bond Yield for the Spot-Rate Models

This table presents the estimated (absolute) volatilities of the spot and nine-year zero bond yield for the spot-rate models. Short rate is the one-factor model. Long rate and spread denotes the two-factor model with the long rate and the spread as stochastic factors. The stochastic volatility model is denoted as Short rate and volatility. The parameters σ_{spot} and σ_9 are the absolute volatilities of the spot rate $r(t)$ and the nine-year zero bond yield l implied by the different models. If multiplied by 100, they represent volatilities in percent p.a. The parameters are estimated for 204 Fridays within the valuation period from January 5, 1990, through November 16, 1993.

Model	Volatilities	Mean	Minimum	Maximum	Standard Deviation
Short rate	σ_{spot}	0.0149	0.0090	0.0212	0.0031
	σ_9	0.0103	0.0060	0.0190	0.0038
	σ_9/σ_{spot}	0.6840	0.4236	1.0580	0.1450
Long rate and spread	σ_{spot}	0.0146	0.0094	0.0214	0.0032
	σ_9	0.0075	0.0044	0.0130	0.0025
	σ_9/σ_{spot}	0.5028	0.3739	0.6354	0.0682
Short rate and volatility	σ_{spot}	0.0185	0.0127	0.0351	0.0038
	σ_9	0.0090	0.0047	0.0112	0.0013
	σ_9/σ_{spot}	0.5112	0.1352	0.7288	0.1316

estimates relative to the one-factor models and the high variability of those estimates might be one reason for the surprising valuation results that we present in Section IV.

Implied volatility functions obtained from cap data often exhibit a humped volatility structure (see, e.g., Amin and Morton (1994), p. 160, and Hull and White (1996), p. 33). However, this structure is not reflected in historical forward-rate changes. To the contrary, we find that for the majority of valuation days in our research period, historical volatilities of forward-rate changes increase with maturity. One reason for this result could be that term structures estimated by cubic spline techniques result in highly volatile long-term forward rates. Furthermore, we note that implied forward-rate volatilities are typically calculated using Black's (1976) model. They therefore represent volatilities of relative forward-rate changes.

The volatilities of the spot rates decrease with time-to-maturity for all spot-rate models. On average, the volatility of the instantaneous spot rate is twice as high as the volatility of the nine-year rate. The volatilities for the short rate are very similar for the one-factor model and the two-factor model with long rate and spread, even though they are determined very differently.

The short-rate volatility is highest for the two-factor model with stochastic volatility. This difference can be explained by the fact that for this model, the short-rate volatility is not a model parameter, but a state variable. The realizations of this state variable are determined by a GARCH procedure; the volatility estimates for the other two models are smoothed by taking an average of squared differences in the time series of short rates.

The endogenous volatility of the nine-year zero bond yield differs considerably across the three models. The mean volatility of the nine-year rate in the generalized version of the Longstaff and Schwartz (1992) model lies between the estimates of the other two models. This represents a positive result insofar as only the short-rate volatility is given exogenously and the model is not calibrated to the current volatility structure.

The differences between the nine-year volatilities for the one-factor model and the two-factor model with the long rate and spread as factors happen because we choose the mean-reversion parameter κ in the one-factor model so that the model-endogenous volatility structure σ_9/σ_{spot} fits the empirically determined volatility structure. Because the historical and model endogenous volatilities σr^{ϵ} for the short-rate are different,[17] historical and endogenous nine-year volatilities must also differ.

A comparison of the short-rate volatilities in Tables IV and V shows that, on average, the forward-rate models result in lower short-rate volatilities than do the spot-rate models. The reason for this difference is that the HJM volatility functions are not only estimated using short-rate changes, but also must reflect the behavior of all other forward rates.

A cautious comparison of the parameter estimates of the one-factor HJM models in Table AIII with the results by Amin and Morton (1994) shows that for the absolute model, the mean estimates do not differ greatly. On the other hand, the standard deviation of our historical estimate is lower than their implied values. For the linear proportional model, Amin and Morton report a negative effect ($\sigma_1 < 0$) of an increasing maturity T. As they estimate volatilities for the upward-sloping part of the humped volatility structure, they expect a positive σ_1.[18] Our result is that the volatility structure increases with T ($\sigma_1 > 0$) for the reasons we have discussed above.

IV. Valuation Results

Here we examine the empirical quality of the models by directly comparing model prices to market prices. Table VI presents a first impression on the performance of the seven models. This table gives some summary statistics for the deviations between theoretical values and market prices of the warrants.

The mean option price of the sample is DM 3.13. The average absolute pricing errors range from DM 0.30 for the best models to DM 0.37 for the worst model. The third column indicates that with the exception of the one-

[17] The endogenous short-rate volatility depends strongly on the current short-rate r, but the historical volatility does not.

[18] The short-term Eurodollar data used by Amin and Morton only show the upward sloping part of the hump. See Amin and Morton (1994), p. 160.

Table VI

Deviation between Model and Market Values

This table presents summary statistics for the deviations between theoretical values and market prices of the warrants. Columns 1 to 5 report the average absolute pricing errors, the average pricing errors (defined as model value minus market value), and the average percentage absolute pricing errors. Column 6 shows the standard deviation of the absolute pricing error. The sample period is January 1990, to November 1993. There are 1,037 call prices and 744 put prices in the sample. To calculate the relative percentage errors, we remove all observations in which the market price of the option is less than DM 0.10. The total number of observations eliminated is 228, of which 81 are calls and 147 are puts. Absolute I is the one-factor forward-rate model with constant volatility. Linear proportional denotes the one-factor forward-rate model with linear proportional volatility. Absolute II is the two-factor forward-rate model in which both volatility functions are independent of the level of the forward rate. In the two-factor forward-rate model Proportional II, both volatility functions are proportional to the level of the forward rate. Short rate is the one-factor spot-rate model. Long rate and spread denotes the two-factor spot-rate model with the long rate and the spread as stochastic factors. The stochastic volatility model is denoted as Short rate and volatility.

Model	Mean Absolute Deviation (DM)	Mean Deviation (DM)	Mean Percentage Absolute Deviation			Standard Deviation (DM)
			All	Calls	Puts	
Absolute I	0.31	−0.15	0.24	0.21	0.28	0.49
Linear proportional	0.30	−0.17	0.24	0.22	0.27	0.46
Absolute II	0.36	−0.27	0.31	0.26	0.40	0.54
Proportional II	0.37	−0.28	0.32	0.27	0.41	0.54
Short rate	0.35	0.13	0.37	0.31	0.48	0.47
Long rate and spread	0.30	−0.17	0.21	0.17	0.26	0.44
Short rate and volatility	0.30	−0.09	0.23	0.17	0.31	0.41

factor spot-rate model, on average, all models underprice the options. The one-factor spot-rate model results in an average overpricing of DM 0.13. The underpricing of the other models ranges from the best value of DM −0.09 to the worst value of DM −0.28.

The average absolute percentage pricing errors vary between 21 percent and 37 percent. A comparison of calls and puts shows that the absolute percentage pricing error is uniformly lower for calls than for puts. A reason for this might be that the calls' average price of DM 3.51 is higher than our sample's puts' average price of DM 2.56. Amin and Morton (1994) report average fractional absolute deviations of 15.2 percent for the linear proportional model and 21.1 percent for the absolute model, and therefore lower errors.

To put these figures into perspective, we note two important differences between Amin and Morton's (1994) study and ours. First, because they use implied volatilities, Amin and Morton carry out a "local" test of the HJM model. Second, they are able to value the option on Eurodollar futures without any error in the prices of the underlyings. In our study, we must accept an absolute valuation error in the underlying bond averaging 0.15 percent. Since interest-rate warrants have a high elasticity compared with stock options, a correction for this error in the underlying reduces the mispricing.

The two one-factor forward-rate models show similar patterns of mispricing. The observation also holds true for the two-factor forward-rate models but, surprisingly, the two-factor models perform uniformly worse than the one-factor models. This unexpected bad performance can be attributed to two reasons. The first is the low volatility estimates compared with the one-factor forward-rate models (see Table IV). These result in systematically lower option values compared with the other models (column three of Table VI) and in higher absolute deviations (column two). This interpretation of the aggregated deviations proves very accurate if we analyze the deviations for each interest-rate warrant individually. Second, the higher variability of the volatility estimates in the two-factor models presented in Table IV could indicate an overfitting problem related to our use of the principal component analyses to estimate the input data for the HJM two-factor models.

For the spot-rate models, the relation between the one- and two-factor models is as expected. If measured by the mean absolute deviation, the two-factor models uniformly outperform the one-factor model. Furthermore, the one-factor spot-rate model is the only model that on average overvalues the warrants.

The fundamental reason for this comparatively bad performance comes from fitting the volatility curve by using the mean-reversion parameter κ. First, Panel A of Table AIV in the Appendix shows 0.06 as average value for κ. This value is very low if compared with an unbiased estimate for κ of 0.25 from a time-series of one-month rates for the period from 1970 to 1993. Since the endogenous volatilities of medium- and long-term yields increase if κ decreases, the low κ-values explain the relatively high option values for the one-factor spot-rate model. Second, for some out-of-the-money options, the model prices are close to zero even though the market prices are greater than one DM. This breakdown of the model occurs in periods with a sharp decline in the ratio of the long- and short-term volatility. To capture a sharp decline of the relative volatility, κ must take on relatively high values that result in low time values for the options.

Table VII reports correlations between pricing errors across models. We find the highest correlation, 0.99, between the two-factor HJM models, the second highest, 0.97, between the one-factor HJM models. In contrast, the one-factor spot-rate model has a very low correlation with all other models. Also, the spot-rate model with stochastic volatility shows a comparably low correlation to other models. Surprisingly, the HJM one-factor absolute model has a relatively high correlation with all other models, apart from the one-factor spot-rate model. Compared with the correlations reported by Amin and Morton (1994) for one-factor HJM models, we find considerably higher correlations because we estimate volatilities historically.

Table VII

Correlation between Pricing Errors across Models

This table reports the Bravais–Pearson correlation coefficients between pricing errors across models for the period from January 1990 to November 1993. Pricing errors are defined as model value minus market value. There are 1,751 observations for each model. Absolute I is the one-factor forward-rate model with constant volatility. Linear proportional denotes the one-factor forward-rate model with linear proportional volatility. Absolute II is the two-factor forward-rate model in which both volatility functions are independent of the level of the forward rate. In the two-factor forward-rate model Proportional II, both volatility functions are proportional to the level of the forward rate. Short rate is the one-factor spot-rate model. Long rate and spread denotes the two-factor spot-rate model with the long rate and the spread as stochastic factors. The stochastic volatility model is denoted as Short rate and volatility.

Model	Abs. I	Linear Prop.	Abs. II	Prop. II	Short Rate	Long Rate and Spread	Short Rate and Volatility
Absolute I	1						
Linear proportional	0.97	1					
Absolute II	0.86	0.93	1				
Proportional II	0.84	0.92	0.99	1			
Short rate	0.24	0.32	0.28	0.30	1		
Long rate and spread	0.87	0.90	0.91	0.90	0.34	1	
Short rate and volatility	0.86	0.79	0.66	0.63	0.27	0.70	1

To study the model performance in more detail, we analyze pricing errors for the different models. We first regress absolute pricing errors on moneyness (measured in DM) and on the maturity of an option. In addition to these two fundamental option characteristics, we include dummy variables for the years 1990, 1991, and 1992, and a dummy variable for a call. We introduce the calendar dummy variables to test whether the market for German interest-rate warrants, which started in 1989, shows maturity effects similar to those that have been reported for other markets. The dummy variable for calls allows us to test whether pricing errors are systematically different for calls and puts.

The results of the regressions are summarized in Table VIII. These show that the calendar dummy variables α_1, α_2, and α_3 are significant at a 1 percent level for almost all models. In addition, the dummy variable for 1990, α_1, is always higher than the dummy variables for the years 1991 and 1992, α_2 and α_3; often, it is twice as high as the dummy variable for the year 1992, α_2. This illustrates that absolute pricing errors are obviously very high in 1990 and that the errors reduce significantly over time for all the models. Since these results are very similar across models, there does not seem to be a model-specific effect, but rather a common component in pricing errors due to the presence of market imperfections.

The sign of the moneyness variable, α_4, differs across models. In addition, it is insignificant at the five percent level for all but one of the models. From this result we conclude that absolute pricing errors are not significantly

Table VIII

Regression Results for Absolute Pricing Errors

This table summarizes the regression results for

$$Abs.\ Deviation = \alpha_0 + \alpha_1 D90 + \alpha_2 D91 + \alpha_3 D92 + \alpha_4 Moneyness + \alpha_5 Maturity + \alpha_6 Dcall + \epsilon,$$

where $D90$, $D91$, and $D92$ are dummy variables for the calendar years 1990, 1991, and 1992, respectively. *Dcall* is a dummy variable for calls. *Moneyness* is defined as *bond price − exercise price* for calls and as *exercise price − bond price* for puts. Bond and exercise prices are expressed in percentage terms. The *Maturity* of the option is in years. ϵ is an error term. The sample period is from January 1990 to November 1993. There are 1,751 observations for each model. The t-statistics on the regression coefficients, adjusted for heteroskedasticity and 5th degree autocorrelation in residuals based on Newey and West (1987), are given below the coefficient values in parentheses. Absolute I is the one-factor forward-rate model with constant volatility. Linear proportional denotes the one-factor forward-rate model with linear proportional volatility. Absolute II is the two-factor forward-rate model in which both volatility functions are independent of the level of the forward rate. In the two-factor forward-rate model Proportional II, both volatility functions are proportional to the level of the forward rate. Short rate is the one-factor spot-rate model. Long rate and spread denotes the two-factor spot-rate model with the long rate and the spread as stochastic factors. The stochastic volatility model is denoted as Short rate and volatility.

Model	α_0	α_1	α_2	α_3	α_4	α_5	α_6	R^2
Absolute I	0.062	0.456	0.208	0.107	0.003	0.152	−0.070	0.25
	(2.42)	(7.46)	(5.56)	(3.55)	(0.89)	(5.35)	(−1.85)	
Linear proportional	0.047	0.405	0.216	0.126	0.002	0.123	−0.015	0.22
	(1.97)	(7.47)	(5.62)	(3.97)	(0.67)	(4.74)	(−0.43)	
Absolute II	0.050	0.398	0.182	0.230	−0.002	0.207	−0.041	0.23
	(1.69)	(6.66)	(4.58)	(5.08)	(−0.56)	(6.61)	(−1.03)	
Proportional II	0.053	0.369	0.183	0.235	−0.003	0.219	−0.039	0.23
	(1.75)	(6.40)	(4.39)	(4.98)	(−0.80)	(6.95)	(−0.97)	
Short rate	0.116	0.253	0.297	0.074	0.001	0.151	−0.027	0.17
	(3.88)	(5.22)	(6.45)	(2.27)	(0.50)	(6.69)	(−0.80)	
Long rate and spread	0.063	0.336	0.180	0.164	0.008	0.140	−0.044	0.19
	(2.40)	(6.16)	(4.64)	(4.02)	(2.60)	(4.81)	(−1.19)	
Short rate and volatility	0.016	0.485	0.174	0.089	0.004	0.239	−0.128	0.30
	(0.69)	(6.60)	(4.80)	(3.23)	(1.24)	(7.61)	(−2.94)	

influenced by the moneyness of the options. Contrary to the moneyness, the influence of the time-to-maturity of an option on the absolute pricing error is significant and positive for all models.

The estimate of the dummy variable for calls, α_6, is negative for all models. This means that on average, calls result in lower absolute errors than puts. However, the results are only significant for the two-factor model with stochastic volatility.

To study possible systematic biases of each model, we regress the pricing error, defined as the difference between market and model prices, on the same exogenous variables as in Table VIII. The results appear in Table IX. The estimate of the calendar dummy variable, β_1, indicates that in 1990, all but the one-factor spot-rate model significantly underprice the options. However, the dummy variables for the years 1991 and 1992, β_2 and β_3, show that

Table IX

Regression Results for Pricing Errors

This table summarizes the regression results for

$$Deviation = \beta_0 + \beta_1 D90 + \beta_2 D91 + \beta_3 D92 + \beta_4 Moneyness + \beta_5 Maturity + \beta_6 Dcall + \epsilon,$$

where *Deviation* is defined as *market price of the option* − *model price*. $D90$, $D91$, and $D92$ are dummy variables for the calendar years 1990, 1991, and 1992, respectively. *Dcall* is a dummy variable for calls. *Moneyness* is defined as *bond price* − *exercise price* for calls and as *exercise price* − *bond price* for puts. Bond and exercise prices are expressed in percentage terms. The *Maturity* of the option is in years. ϵ is an error term. The sample period is from January 1990 to November 1993. There are 1,751 observations for each model. The t-statistics on the regression coefficients, adjusted for heteroskedasticity and 5th degree autocorrelation in residuals based on Newey and West (1987), are given below the coefficient values in parentheses. Absolute I is the one-factor forward-rate model with constant volatility. Linear proportional denotes the one-factor forward-rate model with linear proportional volatility. Absolute II is the two-factor forward-rate model in which both volatility functions are independent of the level of the forward rate. In the two-factor forward-rate model Proportional II, both volatility functions are proportional to the level of the forward rate. Short rate is the one-factor spot-rate model. Long rate and spread denotes the two-factor spot-rate model with the long rate and the spread as stochastic factors. The stochastic volatility model is denoted as Short rate and volatility.

Model	β_0	β_1	β_2	β_3	β_4	β_5	β_6	R^2
Absolute I	−0.074	0.463	0.156	−0.136	−0.014	0.195	−0.060	0.26
	(−2.11)	(6.91)	(2.79)	(−2.95)	(−3.35)	(5.51)	(−1.27)	
Linear proportional	−0.040	0.364	0.118	−0.072	−0.016	0.197	−0.059	0.22
	(−1.24)	(6.18)	(2.09)	(−1.58)	(−4.35)	(6.04)	(−1.37)	
Absolute II	−0.038	0.335	0.109	0.061	−0.019	0.319	−0.092	0.25
	(−1.07)	(5.29)	(2.01)	(1.05)	(−5.09)	(8.72)	(−1.96)	
Proportional II	−0.033	0.304	0.110	0.070	−0.020	0.330	−0.089	0.24
	(−0.91)	(4.92)	(1.95)	(1.17)	(−5.34)	(8.93)	(−1.91)	
Short rate	0.210	−0.299	−0.443	−0.168	−0.007	−0.031	−0.219	0.18
	(5.31)	(−4.13)	(−7.40)	(−3.34)	(−1.88)	(1.00)	(−4.41)	
Long rate and spread	−0.083	0.312	0.110	0.073	−0.007	0.230	−0.071	0.16
	(−2.48)	(4.95)	(1.94)	(1.39)	(−1.80)	(6.41)	(−1.55)	
Short rate and volatility	0.072	0.442	0.021	−0.262	−0.008	0.096	−0.169	0.25
	(1.73)	(4.94)	(0.42)	(−6.07)	(−1.64)	(2.05)	(−3.13)	

underpricing fell over time. In addition, we find that in 1993, on average the pricing error was close to zero for all models. Thus, we can conjecture that at the beginning of the new market segment, which is covered by our sample period, German interest-rate warrants were considerably overpriced and some (presumably inexperienced) market participants were willing to accept these prices. This conjecture is reinforced by discussions with traders in the Frankfurt market who were able to hedge their short positions in London at better prices.

The estimate of the moneyness variable, β_4, is significantly negative for all forward-rate models. For the spot-rate models, the β_4 estimate is also negative, but much lower in absolute terms and insignificant. Consequently, all models seem to underprice out-of-the-money options and to overprice in-the-money options. A more detailed analysis of the moneyness effect reveals that it is stronger for puts than for calls. These results are similar to those of Amin and Morton (1994).

The reason for the differences between the β_4-estimates in Table IX are related to the different volatilities implicitly used in the models. Lower volatilities result in lower values for out-of-the-money options. For in-the-money options, which are close to the exercise boundary, a reduction of volatility has only a small effect on the option price. Therefore, those models that implicitly use the lowest volatilities undervalue out-of-the-money options most and should show the (absolute) maximum moneyness effect. Table IX demonstrates exactly this effect.

The regression coefficient β_5, which measures the influence of maturity on mispricing, has a positive sign whenever it is significant. All models except the one-factor spot-rate model underprice options more if maturity increases. This effect could be attributed to the lower liquidity of warrants with long maturities. We find particularly high estimates for the HJM two-factor models, again a consequence of the low volatility estimates described in Table IV.

Finally, the coefficient β_6 for the call dummy variable indicates that undervaluation is stronger for puts in general. However, this effect is significant for only two of the seven models.

The regression results show that the mispricing of the different models can be explained to some extent by common factors, but it is also evident that some models are more susceptible than others to certain influencing factors. Because our main objective is to compare the ex ante predictabilities of different models, we try to separate model effects from common factors in pricing errors. Therefore, our focus now is on a paired comparison of absolute pricing errors. For this purpose, we compute the differences of absolute mispricings for each paired model and test whether the mean of these paired differences is equal to zero. Panel A, Table X, summarizes the results of this comparison.

The model with the lowest absolute deviation is the linear proportional one-factor HJM model. For this model, we can reject the hypothesis with a probability of at least 99 percent that the paired differences between each of the two-factor HJM models and with the one-factor spot-rate model are zero. In addition, the difference with the absolute one-factor HJM model is different from zero at the 5 percent level. However, no significant difference is found if we compare it with the two-factor spot-rate models.

For the two-factor spot-rate models, which are the models with the second and third lowest absolute deviations, we can also reject the hypothesis, with a probability of at least 99 percent, that the paired differences between each of the two-factor forward-rate models and with the one-factor spot-rate model are zero.

The model with the largest absolute deviation is the two-factor HJM model with proportional volatility. As we have noted, this model performs significantly worse than the linear proportional one-factor HJM model and the two two-factor spot-rate models. We also find a significant difference compared with the one- and two-factor HJM models with absolute volatility.

Table X

Comparison of Absolute Pricing Errors in Pairs

This table summarizes the results of the comparison of absolute pricing errors in pairs. The ijth entry shows the estimate of $Abs.\ Deviation_{model\,i} - Abs.\ Deviation_{model\,j}$, $i > j$. $Abs.\ Deviation_{model\,i}$ denotes the absolute deviation between the value of model i and the market value. The t-statistics, adjusted for heteroskedasticity and 5th degree autocorrelation in residuals based on Newey and West (1987), are given below the coefficient values in parentheses. Panel A provides the results for the whole period from January 1990 to November 1993. The results of Panel B refer to only the last year. The two bold figures in Panel B indicate the two main differences to the four-year period results of Panel A. Absolute I is the one-factor forward-rate model with constant volatility. Linear proportional denotes the one-factor forward-rate model with linear proportional volatility. Absolute II is the two-factor forward-rate model in which both volatility functions are independent of the level of the forward rate. In the two-factor forward-rate model Proportional II, both volatility functions are proportional to the level of the forward rate. Short rate is the one-factor spot-rate model. Long rate and spread denotes the two-factor spot-rate model with the long rate and the spread as stochastic factors. The stochastic volatility model is denoted as Short rate and volatility.

Model i / j	Abs. I	Linear Prop.	Abs. II	Prop. I	Short Rate	Long Rate and Spread
Panel A: Sample Period of January 1990 to November 1993 (1751 observations)						
Linear proportional	−0.013					
	(−2.22)					
Absolute II	0.056	0.069				
	(5.34)	(8.46)				
Proportional II	0.065	0.078	0.009			
	(5.64)	(8.71)	(4.38)			
Short rate	0.044	0.057	−0.012	−0.021		
	(2.33)	(2.94)	(−0.56)	(−0.99)		
Long rate and spread	−0.011	0.001	−0.067	−0.077	−0.055	
	(−1.14)	(0.16)	(−7.41)	(−8.34)	(−2.66)	
Short rate and volatility	−0.009	0.003	−0.065	−0.075	−0.054	0.002
	(−0.89)	(0.24)	(−4.40)	(−4.80)	(−2.59)	(0.13)
Panel B: Sample Period of January 1993 to November 1993 (685 observations)						
Linear proportional	−0.009					
	(–1.49)					
Absolute II	0.046	0.055				
	(3.90)	(6.15)				
Proportional II	0.060	0.069	0.014			
	(4.64)	(6.69)	(5.10)			
Short rate	0.076	0.085	0.013	0.016		
	(4.37)	(5.34)	(1.15)	(1.25)		
Long rate and spread	−0.013	0.022	−0.032	−0.046	−0.063	
	(−1.70)	**(3.74)**	(−3.11)	(−4.06)	(−3.84)	
Short rate and volatility	−0.000	0.009	−0.046	−0.060	−0.076	−0.013
	(−0.00)	(1.33)	(−4.12)	(−4.85)	(−5.23)	(−1.45)

To analyze the stability of these results, we perform the same comparisons for varying time periods. We successively leave out the beginning of the original four-year time span and consider the periods 1991–1993, 1992–1993, and 1993. A comparison of the results for these three periods shows a remarkable stability of the relative performance of the seven models. In each period, the two-factor forward-rate models, the one-factor spot-rate model, and the two-factor spot-rate model with long rate and spread are significantly dominated by at least one other model. The two one-factor forward-rate models and the second two-factor spot-rate model are never dominated. Since the results are qualitatively identical for the time periods beginning after 1990, Panel B of Table X only shows the model's relative performance for the last year.

Compared with the results of the whole period, which we present in Panel A of Table X, the time period for the more mature market exhibits two main differences: First, the one-factor forward-rate model with absolute volatility is still dominated by its linear proportional counterpart, but it is no longer significantly dominated. Second, the two-factor spot-rate model with long rate and spread is now significantly outperformed.

In summary, based on the total four-year period, the seven models tested can be grouped into two sets:

Set One: One-factor forward-rate model with linear proportional volatility, two-factor spot-rate model with long rate and spread, two-factor spot-rate model with stochastic volatility.

Set Two: One-factor forward-rate model with absolute volatility, one-factor spot-rate model, two-factor forward-rate model with absolute volatility, two-factor forward-rate model with proportional volatility.

The first set contains those models that are never significantly outperformed by any of the other models. In contrast, the second set consists of those models that are significantly outperformed by at least one other model.

Using the results of the stability analysis, we can further assess our models. From the original three models of the first set, two models remain that are not significantly outperformed by any of the other models in any of the four different periods considered. These are the one-factor forward-rate model with linear proportional volatility and the two-factor spot-rate model with stochastic volatility. The other five models have been significantly outperformed at least once. Taking into consideration the fact that more mature market periods result in smaller pricing errors, these latter results are particularly important.

V. Summary and Conclusions

This study presents an extensive empirical test of those valuation models for interest-rate options that dominate the current theoretical discussion.

We perform empirical tests on four forward-rate models and three spot-rate models. These models are potential candidates for measuring, controlling, and supervising interest-rate risk within a risk management system, and should be able to value very different interest-rate derivatives consistently across different markets. This intended application of a valuation model should be strictly separated from its usage as a fine-tuned trading oriented model.

With this application in mind, we make two important decisions concerning the test methodology. First, we estimate input data from time-series, not implicitly. Second, we select as the dominant assessment criterion the ex ante predictability of a model and not its ability to identify mispriced options.

Using this evaluation criterion, the one-factor forward rate model with linear proportional volatility and the two spot-rate models with two factors significantly outperform the other four models for the four-year period from 1990 to 1993.

If we take into account the results for the later time periods, we can make an even stronger assessment of the models. Applying the criterion, "A model is not significantly dominated by any of the other models in any of the four time periods," two of the three models remain, the one-factor forward-rate model with linear proportional volatility and the two-factor spot-rate model with stochastic volatility.

Valuation models for interest-rate derivatives, which could be used in a risk management system, must satisfy additional criteria. Besides the ex ante predictability, differences in estimating the input data, in fitting the model to the current market information, and in numerically valuing the warrants, should be reflected in the overall assessment. When we consider the robustness of the estimation procedure for the input data, the one-factor forward-rate model outperforms the two-factor spot-rate model. In fitting a model to the current term structure of interest rates and volatilities, the forward-rate model is also superior to the other model. In terms of computing the option values numerically, none of the models are especially cumbersome. If we take into account these additional assessment criteria, we can conclude that the one-factor forward-rate model with linear proportional volatility outperforms all other models.

Appendix

Table AI

Bond and Interest-Rate Warrant Data

This table gives a summary of interest-rate warrant terms, including the average number of daily trades and the average daily turnover in thousands of DM from 1990 to 1993. Am. denotes American-style options; Eu. denotes European-style options. The securities underlying these warrants are either long- (BUND) or medium-term (BOBL) German government bonds. The bond's coupon size and maturity are also provided.

Type	Strike Price (%)	Maturity	Underlying Bond		Average Daily Trades	Average Daily Turnover
Am. Call	100	25/10/89–25/10/90	BUND 7%	89/99	30.1	154.82
Am. Put	100	25/10/89–25/10/90	BUND 7%	89/99	3.3	54.22
Am. Call	100	23/01/90–14/01/91	BUND 7,25%	90/00	5.3	16.65
Am. Put	100	23/01/90–14/01/91	BUND 7,25%	90/00	4.2	40.39
Am. Call	100	21/05/90–05/12/91	BUND 8,75%	90/00	9.9	64.51
Am. Put	100	21/05/90–05/12/91	BUND 8,75%	90/00	11.6	60.90
Am. Call	101	01/08/90–30/07/91	BOBL 8,75%	90/95	3.4	10.09
Am. Put	99	01/08/90–30/07/91	BOBL 8,75%	90/95	7.1	19.61
Am. Call	99	29/01/90–05/03/91	BUND 7,25%	90/00	10.6	66.59
Am. Call	99	13/07/92–01/10/93	BOBL 8%	92/97	5.1	62.38
Am. Put	98	13/07/92–01/10/93	BOBL 8%	92/97	0.4	0.42
Am. Call	100	07/11/91–06/11/92	BUND 8,25%	91/01	3.7	50.04
Am. Put	100	07/11/91–06/11/92	BUND 8,25%	91/01	2.7	7.96
Am. Call	101	24/04/92–24/06/93	BUND 8%	92/02	3.7	26.01
Am. Put	101	24/04/92–24/06/93	BUND 8%	92/02	3.0	6.57
Am. Call	102	24/04/92–07/04/94	BUND 8%	92/02	16.1	177.32
Am. Put	102	24/04/92–07/04/94	BUND 8%	92/02	5.0	13.55
Am. Call	100	27/08/92–17/08/95	BUND 8%	92/02	4.1	48.47
Am. Call	101	27/08/92–17/08/95	BUND 8%	92/02	8.4	97.33
Am. Call	102	27/08/92–17/08/95	BUND 8%	92/02	14.9	186.90
Am. Call	102	23/12/92–27/01/94	BOBL 7,5%	92/97	4.2	40.46
Am. Put	102	23/12/92–27/01/94	BOBL 7,5%	92/97	1.0	1.94
Am. Call	103	23/12/92–27/01/94	BOBL 7,5%	92/97	3.8	44.39
Am. Put	103	23/12/92–27/01/94	BOBL 7,5%	92/07	1.7	5.44
Am. Call	103	05/04/93–02/03/94	BUND 7,125%	92/02	2.4	43.42
Am. Put	103	05/04/93–02/03/94	BUND 7,125%	92/02	1.9	8.13
Am. Call	104	05/04/93–02/03/94	BUND 7,125%	92/02	2.7	39.34
Am. Put	104	05/04/93–02/03/94	BUND 7,125%	92/02	3.1	17.65
Am. Call	103	30/09/93–19/09/94	BUND 6,5%	93/03	2.5	38.44
Am. Put	103	30/09/93–19/09/94	BUND 6,5%	93/03	2.2	9.06
Eu. Call	100	29/06/90–10/02/92	BUND 7,75%	90/00	5.2	31.52
Eu. Put	100	29/06/90–10/02/92	BUND 7,75%	90/00	3.9	20.76
Eu. Call	100	03/01/91–15/07/92	BUND 9%	90/00	13.5	1295.08

Table AII

Parameter Estimates of the GARCH Model

Maximum likelihood estimates of the generalized autoregressive conditional heteroskedastic (GARCH) model

$$r_t - r_{t-1} = b_0 + b_1 r_{t-1} + b_2 V_t + \epsilon_t$$

$$\epsilon_t \sim N(0, V_t)$$

$$V_t = a_0 + a_1 r_{t-1} + a_2 V_{t-1} + a_3 \epsilon_{t-1}^2$$

for weekly observations of one-month money market rates from January 1970 to December 1989. b_i, $i = 0, 1, 2$ and a_j, $j = 0, 1, 2, 3$ denote the parameters of the GARCH model. r_t is the short rate at time t. The error term ϵ_t is (conditional) normally distributed with zero mean and variance V_t. The observations of the first three months are used to estimate an initial volatility V_0. The remaining 1,031 rates are used as input for the GARCH estimation.

Variable	Estimate	Standard Error
b_0	0.00037395	0.00016744
b_1	−0.00816464	0.00324843
b_2	10.84576386	2.83996069
a_0	−0.00000031	0.00000060
a_1	0.00001391	0.00001133
a_2	0.72163780	0.18557610
a_3	0.27047252	0.03351277

Table AIII

Parameter Estimates for the Volatility Parameters within the Forward-Rate Models

This table summarizes the parameter estimates for the volatility parameters of the forward-rate models. The positive parameter σ is the volatility parameter for the one-factor model with constant volatility (Absolute I). σ_0 and σ_1 are two positive parameters for the one-factor forward-rate model with linear proportional volatility (Linear proportional). $\sigma_{factor1}$ and $\sigma_{factor2}$ are the volatilities of the two driving factors for each of the two two-factor models. These are the Absolute II model, in which both volatility functions are independent of the level of the forward rate, and the Proportional II model, in which both volatility functions are proportional to the level of the forward rate. The volatility functions are obtained by multiplying $\sigma_{factor1}$ and $\sigma_{factor2}$ with the factor loadings obtained from the principal component analyses. The parameters are estimated for 204 Fridays within the valuation period from January 5, 1990, through November 16, 1993.

Model	Parameter	Mean	Minimum	Maximum	Standard Deviation
Absolute I	σ	0.0128	0.0097	0.0159	0.0017
Linear proportional	σ_0	0.1081	0.0476	0.1994	0.0363
	σ_1	0.0168	0.0055	0.0270	0.0056
Absolute II	$\sigma_{factor1}$	1.029	0.9810	1.0408	0.0129
	$\sigma_{factor2}$	1.017	0.9258	1.0408	0.0273
Proportional II	$\sigma_{factor1}$	1.0271	0.9686	1.0408	0.0154
	$\sigma_{factor2}$	1.0183	0.9308	1.0408	0.0259

Table AIV

Parameter Estimates for the Volatility Parameters of the Spot-Rate Models

This table summarizes the parameter estimates for the volatility parameters of the spot-rate models. Panel A represents the estimation results for the mean-reversion parameter κ and the volatility parameter σ of the one-factor spot-rate model $dr = \kappa(\gamma - r)\,dt + \sigma r^{\epsilon}\,dz$. From this equation, the long-term mean γ and the elasticity parameter ϵ have already been estimated as structural parameters in a previous step. σ is estimated by means of the maximum likelihood method. κ is determined (implicitly) such that the endogenous volatility of the long rate equals the historically estimated volatility. Panel B summarizes the results for the two-factor spot-rate model, using the long rate l and the spread s as stochastic factors. The volatility parameters σ_l and σ_s are estimated for the long-rate process $dl = \sigma_l\sqrt{l}\,dz_1$ and the spread process $ds = (\gamma_s - s)\,dt + \sigma_s\,dz_2$ by means of the maximum likelihood method. Panel C summarizes the parameter estimates for the volatility parameters of the two-factor spot-rate model with stochastic volatility. $E(V_\infty)$, $Var(V_\infty)$ are the sample mean and variance of the volatility of the short rate. $\min_{ratio}$ and $\max_{ratio}$ are the minimum and maximum of the volatility–spot rate ratio $\sqrt{V(t)/r(t)}$ within the historical time series. We also provide the parameter estimates for the stochastic processes $dx = \kappa_x(\gamma_x - x)\,dt + \sigma_x\sqrt{x}\,dz_1$ and $dy = \kappa_y(\gamma_y - y)\,dt + \sigma_y\sqrt{y}\,dz_2$ of the abstract factors x and y. These six parameters κ_x, γ_x, σ_x, κ_y, γ_y, and σ_y are obtained as a solution to the following system of equations, initially derived by Longstaff and Schwartz (1993):

$$E(r_\infty) = \gamma_x + \gamma_y, \qquad \mathrm{Var}(r_\infty) = \frac{\sigma_x^2\,\gamma_x}{2\kappa_x} + \frac{\sigma_y^2\,\gamma_y}{2\kappa_y},$$

$$E(V_\infty) = \sigma_x^2\,\gamma_x + \sigma_y^2\,\gamma_y, \qquad \mathrm{Var}(V_\infty) = \sigma_x^6\,\frac{\gamma_x}{2\kappa_x} + \sigma_y^6\,\frac{\gamma_y}{2\kappa_y},$$

$$\min_{ratio} = \sigma_y, \qquad \max_{ratio} = \sigma_x.$$

All parameters are estimated for 204 Fridays within the valuation period from January 5, 1990, through November 16, 1993.

Parameter	Mean	Minimum	Maximum	Standard Deviation
		Panel A: One-Factor Spot-Rate Model		
σ	0.1167	0.0592	0.1954	0.0382
κ	0.0586	0.0002	0.2107	0.0399
		Panel B: Two-Factor Spot-Rate Model with Long Rate and Spread		
σ_l	0.0270	0.0450	0.0169	0.0081
σ_s	0.0124	0.0175	0.0081	0.0024
		Panel C: Two-Factor Spot-Rate Model with Stochastic Volatility		
$E(V_\infty)$	0.00035	0.00028	0.00048	0.00014
$\mathrm{Var}(V_\infty)$	2×10^{-8}	5×10^{-9}	5×10^{-8}	1×10^{-9}
$\min_{ratio}$	0.048	0.045	0.052	0.002
$\max_{ratio}$	0.106	0.082	0.125	0.015
κ_x	1.032	0.539	1.540	0.272
γ_x	0.026	0.017	0.034	0.006
σ_x	0.106	0.082	0.125	0.015
κ_y	0.082	0.054	0.113	0.015
γ_y	0.040	0.027	0.043	0.006
σ_y	0.048	0.045	0.052	0.002

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