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A Note on the Asymptotic Covariance in Fama-MacBeth Regression

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IN THEOREM 2 (page 44, Appendix B) of "The conditional CAPM and the cross-section of expected returns," which appeared in the March 1996 issue of *The Journal of Finance* (pages 3–53), we derive the sampling distribution of the parameters estimated using the cross-sectional regression approach. Unfortunately, there is an error in that derivation. We correct the error in this note.

In our derivations we assume that the residual from the regression of an asset return on one factor (say, factor y_j) is orthogonal to the factor y_j as well as to any other factor (say, factor y_k with $k \neq j$). In particular, using the notations in the paper, we assume that

$$E\left[e_{ikt} \,|\, \{y_{ns}\}_{n=1,\ldots,K_2; s=1,\ldots,T}\right] = 0 \tag{1}$$

$$E\left[e_{ikt} e_{jlt} \,|\, \{y_{ns}\}_{n=1,\ldots,K_2; s=1,\ldots,T}\right] = \sigma_{ijkl}, \tag{2}$$

where e_{ikt} is the residual in the time-series regression of return R_{it} on the factor y_{kt} as defined in Appendix B of the paper.

This is an erroneous assumption.[1] For example, such an assumption will not be satisfied when returns and factors have an i.i.d. joint Normal distribution. Since residuals are, by definition, orthogonal to regressors, we have:

$$E\left[e_{ikt} y_{kt}\right] = 0. \tag{3}$$

In the simple case with i.i.d. Normal observations, equation (3) implies that the residual e_{ikt} and the factor y_{kt} are independent. It follows that

$$E\left[e_{ikt} \,|\, y_{kt}\right] = 0 \tag{4}$$

$$E\left[e_{ikt} e_{jkt} \,|\, y_{kt}\right] = \tau_{ijk}. \tag{5}$$

However, when conditioned on those factors that are not regressors, the moments of residual will in general be a nonconstant function of those factors. Therefore, equations (1) and (2) do not hold even for this simple case.

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[1] We thank Raymond Kan for pointing out the inadmissibility of Assumption 4.

In view of this it is necessary to replace equations (1) and (2) in Assumption 4 by equations (4) and (5). We need the following additional notation. Define an NK_2-dimensional vector $\boldsymbol{S}$ such that its elements are of the form $Y_k'e_{ik}/T$ for $i = 1,\ldots,N$ and $k = 1,\ldots,K_2$, where $Y_k = (y_{k1} - \bar{y}_k,\ldots,y_{kT} - \bar{y}_k)'$, $\bar{y}_k$ is the time-series mean of y_{kt}, and $e_{ik} = (e_{ik1},\ldots,e_{ikT})'$.

ASSUMPTION 4′: *For each i,j,k and t, $E[e_{ikt}|Y_k] = 0$ and $E[e_{ikt}e_{jkt}|Y_k] = \tau_{ijk}$ where τ_{ijk} is independent of t. When $T \to \infty$, the vector $\boldsymbol{S}$ converges to zero in probability, and $\sqrt{T}\hat{\boldsymbol{S}}$ converges to a normal distribution with mean zero and some covariance matrix* Π.

It follows from equations (4) and (5) and the Lindeberg–Lèvy Central Limit Theorem that this assumption will be satisfied when returns and factors are independently distributed over time and drawn from the same joint Normal distribution every period.

Under Assumption 4′, the formula for W in Theorem 2 (equation B18 in Appendix B of the paper) becomes

$$W = (X'X)^{-1}X'\left(\sum_{k,l=1}^{K_2} c_{2k}c_{2l}\omega_{kk}^{-1}\Pi_{kl}\omega_{ll}^{-1}\right)X(X'X)^{-1}, \tag{6}$$

where Π_{kl} is an $N \times N$ matrix, with the element π_{ijkl} being the limiting value of the covariance between $Y_k'e_{ik}/\sqrt{T}$ and $Y_l'e_{jl}/\sqrt{T}$ as T becomes large.

Although these corrections lead to changes in the "corrected t-values" and the "corrected p-values" reported in Tables II through V in the paper, the magnitude of the changes is so small that none of the conclusions in the paper change.

Derivation of equation (6): It is sufficient to show that the asymptotic variance of $\sqrt{T}(\hat{c} - c)$ is $V + W$. We showed in the paper that

$$\bar{R}_i - E[R_i] = (\bar{y}_k - E[y_{kt}])\beta_{ik} + e_{ik}'1_T/T \tag{7}$$

and

$$\hat{\beta}_{ik} - \beta_{ik} = (Y_k'Y_k)^{-1}Y_k'e_{ik}. \tag{8}$$

Then, by Assumption 4′,

$$\begin{aligned} TE[(\hat{\beta}_{ik} - \beta_{ik})(\bar{R}_j - E[R_j])] &= E[(Y_k'Y_k)^{-1}Y_k'E[e_{ik}e_{jk}'|Y_k]1_T] \\ &= E[\tau_{ijk}(Y_k'Y_k)^{-1}Y_k'1_T] = 0. \end{aligned} \tag{9}$$

Hence, $\hat{B} - B$ is uncorrelated with $\bar{R} - E[R]$. Therefore, u and Hc_2 should be uncorrelated with each other, where $\sqrt{T}(\bar{R} - E[R])$ and $\sqrt{T}(\hat{B} - B)$ converge to the limiting random variables u and H as T becomes large. Thus, the asymptotic variance of $\sqrt{T}(\hat{c} - c)$ should be

$$(X'X)^{-1}X'[\mathrm{var}(u) + \mathrm{var}(Hc_2)]X(X'X)^{-1}. \tag{10}$$

It follows from equation (B15) in the paper that

$$(X'X)^{-1}X'\mathrm{var}(u)X(X'X)^{-1} = V. \tag{11}$$

Let S denote the limiting distribution of $\sqrt{T}\hat{S}$ and s_{ik} denote the limiting distribution of $Y_k' e_{ik}/\sqrt{T}$. Then, by Assumption 5 in the paper and the above Assumption 4′, $\sqrt{T}(\hat{\beta}_{ik} - \beta_{ik})$ converges to $\omega_{kk}^{-1}s_{ik}$ in distribution. Hence, the asymptotic covariance between $\sqrt{T}(\hat{\beta}_{ik} - \beta_{ik})$ and $\sqrt{T}(\hat{\beta}_{jl} - \beta_{jl})$ is

$$\omega_{kk}^{-1}E[s_{ik}s_{jl}]\omega_{ll}^{-1} = \omega_{kk}^{-1}\pi_{ijkl}\omega_{ll}^{-1}. \tag{12}$$

It follows that

$$\mathrm{var}(Hc_2) = \sum_{k,l=1}^{K_2} c_{2k}c_{2l}\omega_{kk}^{-1}\Pi_{kl}\omega_{ll}^{-1}. \tag{13}$$

Hence,

$$(X'X)^{-1}X'\mathrm{var}(Hc_2)X(X'X)^{-1} = W. \tag{14}$$