THE JOURNAL OF FINANCE • VOL. LII, NO. 2 • JUNE 1997

Equilibrium Valuation of Foreign Exchange Claims

GURDIP S. BAKSHI and ZHIWU CHEN*

ABSTRACT

This article studies the equilibrium valuation of foreign exchange contingent claims. Within a continuous-time Lucas (1982) two-country model, exchange rates, interest rates and, in particular, factor risk prices are all endogenously and jointly determined. This guarantees the internal consistency of these price processes with a general equilibrium. In the same model, closed-form valuation formulas are presented for currency options and currency futures options. Common to these formulas is that stochastic volatility and stochastic interest rates are admitted. Hedge ratios and other comparative statics are also provided analytically. It is shown that most existing currency option models are included as special cases.

THIS ARTICLE STUDIES THE equilibrium valuation and hedging of foreign exchange contingent claims. The framework for analysis is the continuous-time counterpart of the well known two-country monetary model of Lucas (1982).[1] That is, we use the modeling structure of Lucas (1982) and adopt his representative-agent perfect-pooling equilibrium notion, rather than extending the Lucas setup to a Cox, Ingersoll, and Ross (1985a) (henceforth, CIR) production economy as is done in Nielsen and Saá-Requejo (1993). For our purpose, the Lucas framework offers the desired simplicity and tractability. We assume a stochastic structure for the international economy such that the resulting nominal interest rates and exchange rate dynamics exhibit similar, empirically plausible features as the corresponding ones in Nielsen and Saá-Requejo (1993). For example, the endogenous interest rates are positively correlated across national borders (Campbell and Clarida (1987) and Ilmanen (1995)). As a predictor of future spot exchange rates, the equilibrium forward exchange rate is biased in a direction consistent with the "forward premium puzzle"

* Bakshi is from the University of Maryland, and Chen is from the Ohio State University. We thank David Backus, Suleyman Basak, Glenn Boyle, Charles Cao, Jia He, Jimmy Hilliard, Louis Scott, and especially René Stulz (the editor) and the referee for helpful comments and suggestions. We also gratefully acknowledge comments by seminar participants at the University of Georgia, Louisiana State University, University of New Orleans, and Tulane University. Any remaining errors are the authors' alone.

[1] Other two-country monetary models include Stulz (1987, 1992) and Svensson (1985). The focus in these related articles is on (i) characterizing equilibrium interest rates and exchange rates and/or (ii) studying the nature of the forward exchange rate biases. Exchange rate derivatives such as foreign exchange options contracts are, however, not studied in the existing literature.

(Fama (1984)). The term structure and exchange rate models each have time-varying drift and stochastic volatility, a feature in line with the empirical evidence in, for example, Hodrick (1987) and Saá-Requejo (1994a,b). Yet, going beyond the analysis of Nielsen and Saá-Requejo (1993), we derive closed-form pricing formulas for currency options and currency futures options within the same general equilibrium framework. Thus, in some sense, our study and theirs complement each other. This is true even though the underlying economy in our model is distinctly different from theirs.

Our currency option pricing model differs from existing ones in several important ways. First, as the processes for exchange rates, domestic and foreign interest rates, and factor risk prices are all endogenously and jointly determined in equilibrium, the internal consistency is guaranteed. In contrast, existing models often take a partial equilibrium approach.[2] Second, interest rates and currency volatility underlying our option model are both stochastic over time,[3] whereas in existing currency option models they are not. The general model in Heston (1993) does allow both interest rates and volatility to be stochastic, but it does not offer a closed-form solution for European currency option prices. Third, any degree of correlation between domestic and foreign interest rates is permitted in our model, which contrasts with Heston's (1993) assumption of perfectly correlated cross-border interest rates. Next, the endogenous exchange rate process has the potential to reconcile the Black–Scholes (1973) formula-generated "volatility smiles," as the covariance between movements in exchange rate and exchange-rate volatility can take either sign, depending on the monetary state of the economies as well as on the structural parameter values. Finally, according to existing empirical research, currency option models with stochastic volatility but constant interest rates can price short-term options well but misprice long-term options (e.g., Bates (1995, 1996), Bodurtha and Courtadon (1987), and Knoch (1992)). This finding is not surprising given the sensitivity of long-term options to interest rates. Therefore, incorporating both stochastic interest rates and stochastic currency volatility should help improve the model's ability to price and hedge long-term currency options.

[2] For such models, see, for a partial list, Amin and Jarrow (1991), Bates (1996), Biger and Hull (1983), Chesney and Scott (1989), Garman and Kohlhagen (1983), Grabbe (1983), Heston (1993), Hilliard, Madura, and Tucker (1991), Ingersoll (1990), and Melino and Turnbull (1990, 1995). CIR (1985b, Section 5) provide an example illustrating the relative advantages of the general equilibrium approach over its partial equilibrium counterpart. Related discussion in Longstaff and Schwartz (1992) also applies here. In addition, a general equilibrium framework may afford insights that are not obtainable in a partial equilibrium setup. For example, Bailey and Stulz (1989) show that in a general equilibrium, an increase in stock volatility is not necessarily accompanied by an increase in the price of an option written on the stock, a conclusion contrary to what one would draw from the arbitrage-based Black-Scholes model.

[3] See Bakshi and Chen (1995) and Scott (1996) for equity option pricing models that also allow for stochastic volatility and stochastic interest rates.

The closed-form nature of our currency option formulas makes it possible to derive hedge ratios analytically. Note that in addition to the spot exchangerate, there are two state variables driving currency option prices in our model. For this reason, the model should possess better *dynamic properties* than existing single-factor option models. It is known that when one uses the implied volatility from observed option prices to value other options written on the same underlying asset and traded at the same time, even the constant-volatility and constant-interest-rate Black–Scholes model may exhibit practically acceptable performance in pricing options *cross-sectionally* (e.g., Melino and Turnbull (1995)). That, however, only reflects the *static performance*, not the *dynamic performance*, of a model. If an option pricing model is to perform well, it should be able not only to value traded options with "small" pricing errors cross-sectionally, but also to lead to hedging strategies that generate "small" hedging errors over time (see Melino and Turnbull (1995) for a related discussion). In principle, one can expect our model to offer better-performing hedging strategies.

The article is organized as follows. Section I outlines a two-country economy and presents the valuation framework for nominal contingent claims. It derives and characterizes the exchange rate and the domestic and foreign term structures of nominal interest rates. In Section II, we present the currency option pricing formula and its comparative statics and discuss the implementational aspects of the model. Section III develops an equilibrium pricing formula for foreign exchange futures options. Section IV offers concluding remarks. The Appendix contains a proof of each result.

I. Valuation Framework for Foreign Exchange Claims

Consider a continuous-time Lucas (1982) two-country economy in which each country produces a unique perishable good and adopts its own monetary policy. Information flow in this infinite-horizon economy is generated by a four-dimensional standard Brownian motion $\omega = (\omega_x, \omega_{x^*}, \omega_1, \omega_2)$, where the component processes in ω are mutually independent. Agents in both countries share the same information as generated by ω.

The output flow of the home country is $2x(t)$ at time t, and $x(t)$ follows a diffusion:

$$dx(t) = \mu_x(t, x)dt + \sigma_x(t, x)d\omega_x(t), \tag{1}$$

where $\mu_x(t, x)$ and $\sigma_x(t, x)$ are respectively the drift and diffusion terms of $x(t)$. *They can be functions of any other state variables*, but, as shown later, $\mu_x(t, x)$ and $\sigma_x(t, x)$ will not affect any nominal rates or nominal prices in this economy. Similarly, the foreign country produces $2x^*(t)$ units of output of the foreign good, with $x^*(t)$ following a similar process.

Each country has its own currency and conducts its own monetary policy. Money supply in each country is affected by two state variables, $Y(t)$ and $Z(t)$, where

$$dY(t) = (\theta_y - \kappa_y Y(t))dt + \sigma_y \sqrt{Y(t)} d\omega_1(t), \tag{2}$$

$$dZ(t) = (\theta_z - \kappa_z Z(t))dt + \sigma_z \sqrt{Z(t)} d\omega_2(t). \tag{3}$$

The domestic monetary policy results in money supply, $M(t)$, satisfying

$$\frac{dM(t)}{M(t)} = [\mu_m + \eta_1 Y(t) + \eta_2 Z(t)]dt + \sigma_1 \sqrt{Y(t)} d\omega_1(t) + \sigma_2 \sqrt{Z(t)} d\omega_2(t). \tag{4}$$

This money supply process is empirically plausible, as seen from the discussion in Bakshi and Chen (1996). It also has the same stochastic structure as the real output process assumed in Nielsen and Saá-Requejo (1993), in that the conditional mean and variance of the growth rate are both linear in two independent square-root state variables. The foreign money supply, $M^*(t)$, is assumed to have the same structure as in equation (4), with the corresponding coefficients each having a "$*$" superscript (which is a convention followed throughout). Differences between domestic and foreign monetary policies are reflected by distinct structural parameter values.

As in Lucas (1982), each country has a representative agent and a single claim traded against the whole output ($2x(t)$ for the home and $2x^*(t)$ for the foreign country) of its sole production technology. This claim is initially held by the country's representative agent. In addition, there are zero-net-supply securities traded. The two goods are traded in the respective countries and only in the producing country's currency. For example, the home country's output $2x(t)$ can only be purchased and consumed by paying a total of $2x(t)P(t)$ dollars, where $P(t)$ is the domestic currency price per unit of domestic good. Let $P^*(t)$ be the foreign currency price for the foreign good. Capital and goods are perfectly mobile across national borders.

To define the domestic agent's problem, let $\mathrm{w}(t)$ be the agent's time-t real wealth in terms of units of the domestic good, $\alpha_x(t)[\alpha_{x^*}(t)]$ the number of shares of the domestic [foreign] equity claim held, and $N(t)$ $[N^*(t)]$ the amount of the home [foreign] currency demanded for domestic [foreign] transactions. Let $S(t)$ denote the domestic currency (say, dollars) price of a unit of the foreign currency (say, Deutschemarks), and $B(t, \tau)$ $[B^*(t, \tau)]$ the time t dollar (Deutschemark) price of a nominal discount bond that pays one dollar (Deutschemark) in τ periods. To simplify the budget constraint, assume for now that the agent invests only in the two equity markets as well as in the domestic bond market, with the understanding that the investment universe actually includes foreign bonds and other financial assets. Assuming that both countries' agents have the same logarithmic preferences over the two goods, define the domestic agent's life time problem as

$$\max_{c, c^*, \alpha_x, \alpha_{x^*}} E_0 \int_0^\infty e^{-\rho t}\{\gamma \ln[c(t)] + (1 - \gamma)\ln[c^*(t)]\}\, dt, \tag{5}$$

subject to

$$d\mathrm{w}(t) = \{\mathrm{w}(t) - \alpha_x(t)p_x(t) - \alpha_{x^*}(t)p_{x^*}(t)\}\left[\frac{B(t,\tau) + dB(t,\tau)}{B(t,\tau)}\frac{P(t)}{P(t) + dP(t)} - 1\right] + \alpha_x(t)dp_x(t) + \alpha_{x^*}(t)dp_{x^*}(t) - \frac{N(t)}{P(t)}dt - \frac{N^*(t)S(t)}{P(t)}dt, \tag{6}$$

$$N(t) = P(t)c(t), \tag{7}$$

$$N^*(t) = P^*(t)c^*(t), \tag{8}$$

with $\mathrm{w}(0) = p_x(0)$, where $p_x(t)$ and $p_{x^*}(t)$ are respectively the prices of the home and the foreign equity share in terms of units of the domestic good, ρ is the time preference parameter, and γ the expenditure share on the domestic good. A log utility function in domestic and foreign goods is also used in Nielsen and Saá-Requejo (1993) and Stulz (1987). Conditions (7) and (8) respectively reflect the cash-in-advance constraint for the domestic and the foreign goods markets.[4] The foreign representative agent's problem is similarly defined.

Given the above setup, a Lucas-type perfect-pooling equilibrium exists where the domestic and foreign representative agents each consume half of each of the two goods, $c(t) = x(t)$ and $c^*(t) = x^*(t)$, and where each agent holds half of the domestic and foreign equity share, $\alpha_x(t) = 1/2$ and $\alpha_{x^*}(t) = ½$, for every time t. Then, the equilibrium demand for the home currency is $N(t) = x(t)P(t)$, each by the domestic and the foreign agent, implying a total demand of $2x(t)P(t)$ dollars. Similarly, the total demand for the foreign currency is $2x^*(t)P^*(t)$. In equilibrium, the monetary policy for each country should be such that its money supply equal demand: $M(t) = 2x(t)P(t)$ and $M^*(t) = 2x^*(t)P^*(t)$. For variants of the Lucas (1982) model, see Bekaert, Hodrick, and Marshall (1995), Boyle (1990), and Stulz (1987, 1992).

To value foreign exchange contingent claims in this perfect-pooling international equilibrium, it is important to first understand how risk factors are rewarded. Let $R(t)$ be the domestic instantaneous nominal interest rate, and $Q(t)$ the nominal price of any contingent claim, at time t. In the Appendix, we follow the variational approach of Grossman and Shiller (1982) to show that the time t equilibrium risk premium for the claim must satisfy

$$\mu_q(t) - R(t) = \mathrm{Cov}_t\left(\frac{dM(t)}{M(t)}, \frac{dQ(t)}{Q(t)}\right), \tag{9}$$

[4] Conditions (7) and (8) reflect only the "cash requirement" aspect of the "cash-in-advance" constraint, and they do not incorporate the "in-advance" aspect. This treatment of the cash-in-advance constraint was first used by Lucas (1982). The later treatment by Lucas (1990) is such that one first has to have cash before making any consumption purchase. While that timing convention can be implemented in a discrete-time framework, it is technically difficult, if even possible, in a continuous-time framework. For this reason, we adopt the treatment in Lucas (1982).

where $\mu_q(t)$ is the expected instantaneous rate of return from holding the claim, and $\mathrm{Cov}_t(\cdot, \cdot)$ is the usual conditional covariance between the two arguments divided by dt. The nominal risk premium is thus merely a function of, and increasing in, the covariance between the return on a contingent claim and domestic money growth. This result is obtained because the nominal intertemporal marginal rate of substitution in this logarithmic economy depends solely on the money growth process. In existing partial equilibrium models of currency option valuation with stochastic volatility, for instance, the market price of volatility risk is often taken to be zero (or a constant) (e.g., Melino and Turnbull (1990), and Stein and Stein (1991)). In light of the above equation, the zero volatility risk premium assumption holds only when the option's price movements are uncorrelated with money growth.

A. The Nominal Exchange Rate

Exploiting the agent's first-order conditions, we obtain the equilibrium nominal exchange rate below:

$$S(t) = \frac{(1-\gamma)x(t)}{\gamma x^*(t)} \frac{P(t)}{P^*(t)} = \frac{1-\gamma}{\gamma} \frac{M(t)}{M^*(t)}, \quad \forall t. \tag{10}$$

The nominal exchange rate is thus decreasing in the expenditure share of the domestic good and proportional to the ratio of domestic to foreign money supply. Surprisingly, it is independent of the production output in either country and completely driven by monetary policies. This detachment of exchange rates from the real side is obtained due to the perfect-pooling assumption and the "cash-only" requirement for consumption transactions. In a perfect-pooling equilibrium, cross-border trade in goods occurs irrespective of exchange rates. Consequently, the cash-in-advance constraint and monetary policies become the sole determinants of exchange rates. Applying Ito's lemma to equation (10) yields

$$\frac{dS(t)}{S(t)} = \mu_s(t)dt + \sigma_s\sqrt{Y(t)}d\omega_1(t) - \sigma_s^*\sqrt{Z(t)}d\omega_2(t), \tag{11}$$

where $\mu_s(t) \equiv \mu_m - \mu_m^* + (\eta_1 - \eta_1^* + \sigma_1^{*2} - \sigma_1\sigma_1^*)Y(t) + (\eta_2 - \eta_2^* + \sigma_2^{*2} - \sigma_2\sigma_2^*)Z(t)$, $\sigma_s \equiv \sigma_1 - \sigma_1^*$, and $\sigma_s^* \equiv \sigma_2^* - \sigma_2$. The expected depreciation rate of the domestic currency is therefore time-varying and monetary state-dependent. The exchange rate volatility, $V(t)$, taken here as the variance of the spot depreciation rate, is $V(t) = \sigma_s^2 Y(t) + \sigma_s^{*2}Z(t)$, while the conditional covariance between changes in the spot rate and those in volatility is $\mathrm{Cov}_t(dS/S, dV(t)) = \sigma_y\sigma_s^3 Y(t) - \sigma_z\sigma_s^{*3}Z(t)$. This endogenously determined covariance can be negative or positive, depending on the structural parameter values and on the monetary states of the countries. This fact is empirically plausible, because, according to Bates (1995, 1996), nonzero correlations between volatility and exchange rate changes may be necessary to reconcile the "volatility smile" puzzle associated with the Black-Scholes formula for currency options.

B. The Nominal Interest Rates

Consider a pure discount bond that pays one dollar in τ periods and determine its price $B(t, \tau)$. Via a variational argument, $B(t, \tau)$ must satisfy

$$B(t, \tau) = E_t\left(e^{-\rho\tau}\frac{x(t)}{x(t+\tau)}\frac{P(t)}{P(t+\tau)}\right) = E_t\left(e^{-\rho\tau}\frac{M(t)}{M(t+\tau)}\right). \qquad (12)$$

Evaluating this conditional expectation results in the closed-form bond price:

$$B(t, \tau) = \exp[-(\rho + \mu_m)\tau - \alpha_y(\tau) - \alpha_z(\tau) - \varrho_y(\tau)Y(t) - \varrho_z(\tau)Z(t)], \qquad (13)$$

where $\lambda_y \equiv \sigma_1\sigma_y$, $\lambda_z \equiv \sigma_2\sigma_z$, $\vartheta_y \equiv [(\kappa_y + \lambda_y)^2 + 2\sigma_y^2(\eta_1 - \sigma_1^2)]^{1/2}$, $\vartheta_z \equiv [(\kappa_z + \lambda_z)^2 + 2\sigma_z^2(\eta_2 - \sigma_2^2)]^{1/2}$,

$$\varrho_y(\tau) = \frac{2(\eta_1 - \sigma_1^2)(1 - e^{-\vartheta_y\tau})}{2\vartheta_y + (\kappa_y + \lambda_y - \vartheta_y)(1 - e^{-\vartheta_y\tau})},$$

$$\varrho_z(\tau) = \frac{2(\eta_2 - \sigma_2^2)(1 - e^{-\vartheta_z\tau})}{2\vartheta_z + (\kappa_z + \lambda_z - \vartheta_z)(1 - e^{-\vartheta_z\tau})},$$

$$\alpha_g(\tau) = \frac{2\theta_g}{\sigma_g^2}\left\{\ln\left[1 + \frac{(1 - e^{-\vartheta_g\tau})(\kappa_g + \lambda_g - \vartheta_g)}{2\vartheta_g}\right] + \frac{1}{2}[\vartheta_g - \kappa_g - \lambda_g]\tau\right\},$$

for $g = y, z$. The foreign bond price, denoted by $B^*(t, \tau)$, has a similar structure, except that λ_y and λ_z should be respectively replaced by $\lambda_y^* \equiv \sigma_1^*\sigma_y$, $\lambda_z^* \equiv \sigma_2^*\sigma_z$; $\eta_1 - \sigma_1^2$ replaced by $\eta_1^* - \sigma_1^{*2}$; and $\eta_2 - \sigma_2^2$ replaced by $\eta_2^* - \sigma_2^{*2}$. The implied τ-period nominal yields to maturity in the two countries are, respectively,

$$R(t, \tau; Y, Z) = \rho + \mu_m + \frac{\alpha_y(\tau) + \alpha_z(\tau)}{\tau} + \frac{\varrho_y(\tau)}{\tau}Y(t) + \frac{\varrho_z(\tau)}{\tau}Z(t), \qquad (14)$$

$$R^*(t, \tau; Y, Z) = \rho + \mu_m^* + \frac{\alpha_y^*(\tau) + \alpha_z^*(\tau)}{\tau} + \frac{\varrho_y^*(\tau)}{\tau}Y(t) + \frac{\varrho_z^*(\tau)}{\tau}Z(t), \qquad (15)$$

with the corresponding instantaneous nominal interest rates given by

$$R(t) = \rho + \mu_m + (\eta_1 - \sigma_1^2)Y(t) + (\eta_2 - \sigma_2^2)Z(t), \qquad (16)$$

$$R^*(t) = \rho + \mu_m^* + (\eta_1^* - \sigma_1^{*2})Y(t) + (\eta_2^* - \sigma_2^{*2})Z(t). \qquad (17)$$

Therefore, both countries' nominal term structures are driven by two common factors. Several features can be noted. First, nominal interest rates in both countries are correlated, consistent with the findings in Campbell and Clarida (1987), Ilmanen (1995), and Nielsen and Saá-Requejo (1993). The two countries' term structures will be uncorrelated when, for instance, $\eta_2 - \sigma_2^2 = \eta_1^* - \sigma_1^{*2} = 0$, or when $\eta_1 - \sigma_1^2 = \eta_2^* - \sigma_2^{*2} = 0$. Changes in both domestic and foreign interest rates are also correlated with the currency depreciation rate $dS(t)/S(t)$, as they are all driven by $Y(t)$ and $Z(t)$. The signs of these correlations can be positive or negative.

Next, the nominal interest rates, both domestic and foreign, are completely driven by monetary shocks and hence detached from the real sector of either country. This detachment seems to be a common property of monetary economies with log utility functions. In the context of a one-country monetary economy, Bakshi and Chen (1996) show that when investors care about both consumption and real cash holdings and when their preferences are representable by a log utility, the real and the nominal term structures of interest rates will have totally different risk factors, where the nominal rates are driven by monetary shocks and the real rates by real shocks. In their case, if monetary and output shocks are statistically independent of one another, the nominal and real term structure will also be.

Finally, our nominal term structure model, domestic or foreign, is structurally similar to the real term structure model of Nielsen and Saá-Requejo (1993), in that both domestic and foreign interest rates are linear functions of two square-root state variables, $Y(t)$ and $Z(t)$. For this reason, their characterizations and results concerning the term structures of the two countries as well as the spot, forward, and futures exchange rates carry over to our nominal economy as well. Likewise, our closed-form valuation formulas to be derived shortly for various currency options apply to their setup. In this sense, our work and theirs complement each other. It should, however, be noted that the underlying real economy in our model is distinctly different from their real economy, and only our nominal economy shares many features with their real economy. In addition, the international equilibrium in Nielsen and Saá-Requejo (1993) is more appropriately Walrasian, whereas the counterpart in this article is an exogenously imposed Lucas-type perfect-pooling equilibrium. This difference makes our valuation framework relatively more tractable.

II. Valuing Currency Options

Consider a European call option on the exchange rate with strike price K and τ periods to expiration. At expiration the dollar payoff to the call is $\max(0, S(t+\tau) - K)$. Then, under the log utility, its time t price is given by

$$C(t, \tau) = E_t\left[e^{-\rho\tau}\frac{M(t)}{M(t+\tau)}\max(0, S(t+\tau) - K)\right]. \tag{18}$$

By the money supply and exchange rate dynamics respectively specified in equations (4) and (11), the above conditional expectation must be independent of $\{x(t), x^*(t)\}$ and can only be a function of $\{S(t), Y(t), Z(t)\}$. This allows us to write the nominal price of the call as $C(t, S, Y, Z, \tau)$. Applying Ito's lemma to

$C(t, S, Y, Z, \tau)$ and substituting the resulting expression into equation (9) yields the partial differential equation (PDE) below:

$$\begin{aligned}(\tfrac{1}{2})(\sigma_s^2 Y + \sigma_s^{*2} Z)S^2 C_{SS} + S(R - R^*)C_S + \tfrac{1}{2}\sigma_y^2 Y C_{YY} \\ + [\theta_y - (\kappa_y + \lambda_y)Y]C_Y + \tfrac{1}{2}\sigma_z^2 Z C_{ZZ} + [\theta_z - (\kappa_z + \lambda_z)Z]C_Z \\ + \sigma_y\sigma_s SYC_{SY} - \sigma_z\sigma_s^* SZC_{SZ} - C_\tau - RC = 0, \quad (19)\end{aligned}$$

which is subject to the boundary condition: $C(t + \tau, 0) = \max(0, S(t + \tau) - K)$, where subscripts on upper-case letters denote partial derivatives with respect to the corresponding variables. As shown in the Appendix, the solution to this PDE is

$$C(t, \tau) = S(t)B^*(t, \tau)\Pi_1(t, S, Y, Z, \tau) - KB(t, \tau)\Pi_2(t, S, Y, Z, \tau), \quad (20)$$

where Π_1 and Π_2 may be interpreted as "risk-neutral" probabilities that the option expires in the money, and they are respectively given by

$$\Pi_j(t, S, Y, Z, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln[K]}f_j(t, S, Y, Z, \tau; \phi)}{i\phi}\right] d\phi, \quad (21)$$

for $j = 1, 2$, where Re[·] stands for the real value part of the expression and i for imaginary numbers, and the characteristic functions, f_1 and f_2, are given below:

$$\begin{aligned} f_1 = \exp\Bigg\{ & -\frac{\theta_y}{\sigma_y^2}\left[2\ln\left(1 + \frac{[\kappa_y + \lambda_y - (1 + i\phi)\sigma_y\sigma_s - \xi_y](1 - e^{-\xi_y\tau})}{2\xi_y}\right)\right. \\ & \left. \qquad - (\kappa_y + \lambda_y - (1 + i\phi)\sigma_y\sigma_s - \xi_y)\tau\right] \\ & - \frac{\theta_z}{\sigma_z^2}\left[2\ln\left(1 + \frac{[\kappa_z + \lambda_z + (1 + i\phi)\sigma_z\sigma_s^* - \xi_z](1 - e^{-\xi_z\tau})}{2\xi_z}\right)\right. \\ & \left. \qquad - (\kappa_z + \lambda_z + (1 + i\phi)\sigma_z\sigma_s^* - \xi_z)\tau\right] \\ & + i\phi\ln[S(t)] - \ln[B^*(t, \tau)] - (\rho + \mu_m)\tau + (1 + i\phi)(\mu_m - \mu_m^*)\tau \\ & + \frac{[(1 + i\phi)\{\eta_1 - \sigma_1^2 - \eta_1^* + \sigma_1^{*2} + \tfrac{1}{2}\sigma_s^2 i\phi\} - \eta_1 + \sigma_1^2](1 - e^{-\xi_y\tau})}{\xi_y + \tfrac{1}{2}[\kappa_y + \lambda_y - (1 + i\phi)\sigma_y\sigma_s - \xi_y](1 - e^{-\xi_y\tau})} Y(t) \\ & - \frac{[(1 + i\phi)\{\eta_2^* - \sigma_2^{*2} - \eta_2 + \sigma_2^2 - \tfrac{1}{2}\sigma_s^{*2} i\phi\} + \eta_2 - \sigma_2^2](1 - e^{-\xi_z\tau})}{\xi_z + \tfrac{1}{2}[\kappa_z + \lambda_z + (1 + i\phi)\sigma_z\sigma_s^* - \xi_z](1 - e^{-\xi_z\tau})} Z(t)\Bigg\} \end{aligned} \quad (22)$$

and

$$f_2 = \exp\Bigg\{ -\frac{\theta_y}{\sigma_y^2}\left[2\ln\left(1 + \frac{[\kappa_y + \lambda_y - i\phi\sigma_y\sigma_s - \xi_y^*](1 - e^{-\xi_y^*\tau})}{2\xi_y^*}\right) - (\kappa_y + \lambda_y - i\phi\sigma_y\sigma_s - \xi_y^*)\tau\right]$$

$$- \frac{\theta_z}{\sigma_z^2}\left[2\ln\left(1 + \frac{[\kappa_z + \lambda_z + i\phi\sigma_z\sigma_s^* - \xi_z^*](1 - e^{-\xi_z^*\tau})}{2\xi_z^*}\right) - (\kappa_z + \lambda_z + i\phi\sigma_z\sigma_s^* - \xi_z^*)\tau\right]$$

$$+ i\phi\ln[S(t)] - \ln[B(t,\tau)] - (\rho + \mu_m)\tau + i\phi(\mu_m - \mu_m^*)\tau$$

$$+ \frac{[i\phi\{\eta_1 - \sigma_1^2 - \eta_1^* + \sigma_1^{*2} - \tfrac{1}{2}\sigma_s^2\} - \eta_1 + \sigma_1^2 + \tfrac{1}{2}(i\phi)^2\sigma_s^2](1 - e^{-\xi_y^*\tau})}{\xi_y^* + \tfrac{1}{2}[\kappa_y + \lambda_y - i\phi\sigma_y\sigma_s - \xi_y^*](1 - e^{-\xi_y^*\tau})} Y(t)$$

$$- \frac{[i\phi\{\eta_2^* - \sigma_2^{*2} - \eta_2 + \sigma_2^2 + \tfrac{1}{2}\sigma_s^{*2}\} + \eta_2 - \sigma_2^2 - \tfrac{1}{2}(i\phi)^2\sigma_s^{*2}(1 - e^{-\xi_z^*\tau})}{\xi_z^* + \tfrac{1}{2}[\kappa_z + \lambda_z + i\phi\sigma_z\sigma_s^* - \xi_z^*](1 - e^{-\xi_z^*\tau})} Z(t)\Bigg\}, \tag{23}$$

with

$$\xi_y = [(\kappa_y + \lambda_y - (1 + i\phi)\sigma_y\sigma_s)^2 - 2\sigma_y^2((1 + i\phi)\{\eta_1 - \sigma_1^2 - \eta_1^* + \sigma_1^{*2} + \tfrac{1}{2}\sigma_s^2 i\phi\} - \eta_1 + \sigma_1^2)]^{1/2},$$

$$\xi_z = [(\kappa_z + \lambda_z + (1 + i\phi)\sigma_z\sigma_s^*)^2 + 2\sigma_z^2((1 + i\phi)\{\eta_2^* - \sigma_2^{*2} - \eta_2 + \sigma_2^2 - \tfrac{1}{2}\sigma_s^{*2} i\phi\} + \eta_2 - \sigma_2^2)]^{1/2},$$

$$\xi_y^* = [(\kappa_y + \lambda_y - i\phi\sigma_y\sigma_s)^2 - 2\sigma_y^2(i\phi\{\eta_1 - \sigma_1^2 - \eta_1^* + \sigma_1^{*2} - \tfrac{1}{2}\sigma_s^2\} - \eta_1\sigma_1^2 + \tfrac{1}{2}(i\phi)^2\sigma_s^2)]^{1/2},$$

$$\xi_z^* = [(\kappa_z + \lambda_z + i\phi\sigma_z\sigma_s^*)^2 + 2\sigma_z^2(i\phi\{\eta_2^* - \sigma_2^{*2} - \eta_2 + \sigma_2^2 + \tfrac{1}{2}\sigma_s^{*2}\} + \eta_2 - \sigma_2^2 - \tfrac{1}{2}(i\phi)^2\sigma_s^{*2})]^{1/2}.$$

Taking the integration in equation (21) to find the two probabilities represents a convenient alternative to solving the original PDEs for Π_1 and Π_2 (Heston (1993) and Scott (1996)). The currency option pricing formula in equation (20) includes most existing currency option models as special cases. To see this, note the following:

1. The Garman and Kohlhagen (1983) currency option model is obtained if we set, in equation (4) and its foreign counterpart, $Y(t) = Z(t) = 1$ for all t, making interest rates and volatility both constant;
2. The currency option models with constant interest rates but stochastic volatility of Chesney and Scott (1989), Heston (1993), Melino and Turnbull (1990), and Scott (1987) are obtained from the formula in equation (20) if we set, in equation (4) and its foreign counterpart, $\eta_1 = \sigma_1^2$, $\eta_2 = \sigma_2^2$, $\eta_1^* = \sigma_1^{*2}$, $\eta_2^* = \sigma_2^{*2}$;
3. The Amin and Jarrow (1991) currency option model with stochastic interest rates but constant volatility becomes a special case if we set $\sigma_1 = \sigma_1^*$ and $\sigma_2 = \sigma_2^*$ (i.e., if the home and foreign countries have perfectly correlated money growth rates);
4. The currency option model of Heston (1993) in which the stochastic interest rates (both domestic and foreign) and the stochastic currency volatility are perfectly correlated is a special case of the formula in equation (20), if we set $\sigma_2 = \sigma_2^*$, $\eta_2 = \sigma_2^2$ and $\eta_2^* = \sigma_2^{*2}$. It should also be noted that in Heston's Section 2, he does not have a closed-form expression for the characteristic functions in his equation (25), whereas we do here for the probabilities in equation (20). In this sense, we have provided a closed-form solution for the option valuation problem stated in Heston's Section 2.

The relative strength and weakness of each class of models can also be noted. First, the Garman-Kohlhagen model is probably the most widely applied currency option formula. Like the original Black-Scholes formula, its strength is derived mainly from its simplicity and ease to implement. Its relatively weak empirical performance then comes as no surprise. Among other things, its constant interest rates assumption implies that forward and futures exchange rates are not different from each other, which trivializes the practical differences between forward and futures contracts.

The next class of models assumes constant interest rates but stochastic volatility. As expected, these models have performed better than the Garman–Kohlhagen model in explaining currency option prices, in particular those with short terms to expiration (e.g., Bates (1995, 1996), Knoch (1992), and Melino and Turnbull (1990)). Using numerical procedures, for instance, Knoch (1992) demonstrates that for relatively short-term currency options with the appropriate choice of the structural parameter values, a currency option model with stochastic volatility can explain most of the known strike price-related biases associated with the Black–Scholes model and its currency option variant. It has difficulty explaining longer-term currency options, however. From the standpoint of modeling, it also requires a special set of restrictions on the underlying international economy. In particular, the constant interest rates assumption still trivializes the differences between forward and futures contracts.

In contrast, our general model allows domestic and foreign interest rates to be not only stochastic but also correlated across national borders (with any degree of correlation). Furthermore, volatility risk is priced in an internally consistent manner. The covariance between changes in exchange rate and in currency volatility can be negative and is time-varying, which offers more flexibility in resolving empirical biases of existing models. Since our model includes all the above mentioned ones as special cases, we should expect its empirical performance to be better. In addition, as the currency option price in our model is driven by two state variables, our model should be able to explain more time-series movements in option prices than, for instance, Heston's (1993) single-factor model. This is in the same spirit as in the term structure literature, where a two-factor term structure model is known generally to perform better than its single-factor counterparts (e.g., Longstaff and Schwartz (1992)).

Finally, the formula in equation (20) applies to European currency options. In practice, most traded options are American. For American options that are unlikely to be exercised early, the European option pricing model will still work. For example, as Grabbe (1983) and Ingersoll (1990) demonstrate, it will never be optimal to exercise an American currency call early when the domestic interest rates exceed their foreign counterpart; the same can be said about American foreign exchange puts when the domestic interest rates are lower than their foreign counterpart. For those with early exercise potential, one may use the early-exercise-premium approximation proposed in Barone-Adesi and Whaley (1987).

A. Implementation Considerations

The European currency option price in equation (20) is driven by, in addition to $S(t)$, two state variables, $Y(t)$ and $Z(t)$. These two are not observable in practice, which may hamper the implementation of the model. To resolve this problem, one can follow Nielsen and Saá-Requejo (1993), CIR (1985a,b), and Longstaff and Schwartz (1992) to replace them by two observable endogenous variables. In doing so, one faces a few alternative choices. For example, since both the domestic interest rate $R(t)$ and the exchange rate volatility $V(t)$ are linear in $Y(t)$ and $Z(t)$, one can use $R(t)$ and $V(t)$ to substitute out $Y(t)$ and $Z(t)$ in equation (20) so that the option price is expressed in terms of $S(t)$, $R(t)$, and $V(t)$. This substitution scheme would be consistent with the ones used in Bakshi and Chen (1995) (for equity options) and Longstaff and Schwartz (1992) (for bond options).

Alternatively, we can follow Nielsen and Saá-Requejo (1993) to substitute out $Y(t)$ and $Z(t)$ by $R(t)$ and $R^*(t)$. A major advantage of this scheme is that both $R(t)$ and $R^*(t)$ are directly observable quantities during trading hours. Solving the system of linear equations in (16) and (17) for $Y(t)$ and $Z(t)$ gives

$$Y(t) = \psi_0 + \psi_1 R(t) + \psi_2 R^*(t), \tag{24}$$

$$Z(t) = \psi_0^* + \psi_1^* R(t) + \psi_2^* R^*(t), \tag{25}$$

where

$$\psi_0 \equiv \frac{(\mu_m + \rho)(\sigma_2^{*2} - \eta_2^*) - (\mu_{m^*} + \rho)(\sigma_1^{*2} - \eta_1^*)}{\Phi},$$

$$\psi_0^* \equiv \frac{(\mu_m + \rho)(\eta_2 - \sigma_2^2) - (\mu_{m^*} + \rho)(\eta_1 - \sigma_1^2)}{\Phi},$$

$$\psi_1 \equiv \frac{\eta_2^* - \sigma_2^{*2}}{\Phi}, \quad \psi_2 \equiv \frac{\sigma_1^{*2} - \eta_1^*}{\Phi}, \quad \psi_1^* \equiv \frac{\sigma_2^2 - \eta_2}{\Phi}, \quad \psi_2^* \equiv \frac{\eta_1 - \sigma_1^2}{\Phi},$$

and

$$\Phi \equiv (\eta_1 - \sigma_1^2)(\eta_2^* - \sigma_2^{*2}) - (\eta_2 - \sigma_2^2)(\eta_1^* - \sigma_1^{*2}).$$

Then, substituting this solution into equation (20) and making appropriate changes in the characteristic functions, one has the currency option price expressed as a function of $S(t)$, $R(t)$, and $R^*(t)$.

B. Hedging and Hedge Ratios

One advantage offered by a closed-form option pricing formula is the ability to derive hedge ratios analytically. In the present context, currency option comparative statics are all different from their counterparts based on the Garman and Kohlhagen (1983) model. In particular, hedging strategies based on the latter will be severely biased. To demonstrate this, first take as an example the sensitivity to strike price: $\partial C(t, \tau)/\partial K = -B(t, \tau)\Pi_2 < 0$, where both $B(t, \tau)$ and Π_2 now depend on the nominal state of the economy and are hence time-varying. Next, there are three option delta measures, respectively with respect to $S(t)$, $Y(t)$, and $Z(t)$:

$$\Delta_S \equiv \frac{\partial C(t, \tau)}{\partial S} = B^*(t, \tau)\Pi_1 > 0, \tag{26}$$

$$\Delta_Y \equiv \frac{\partial C(t, \tau)}{\partial Y} = S(t)B^*(t, \tau)\left\{\frac{\partial \Pi_1}{\partial Y} - \varrho_y^*(\tau)\Pi_1\right\} - KB(t, \tau)\left\{\frac{\partial \Pi_2}{\partial Y} - \varrho_y(\tau)\Pi_2\right\}, \tag{27}$$

$$\Delta_Z \equiv \frac{\partial C(t, \tau)}{\partial Z} = S(t)B^*(t, \tau)\left\{\frac{\partial \Pi_1}{\partial Z} - \varrho_z^*(\tau)\Pi_1\right\} - KB(t, \tau)\left\{\frac{\partial \Pi_2}{\partial Z} - \varrho_z(\tau)\Pi_2\right\}, \tag{28}$$

where

$$\frac{\partial \Pi_j}{\partial L} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[e^{-i\phi \ln[K]}(i\phi)^{-1}\frac{\partial f_j}{\partial L}\right] d\phi, \qquad \text{for} \quad j = 1, 2, \quad \text{and} \quad L = Y, Z, \tag{29}$$

and the two partial derivatives, $\partial f_j/\partial Y$ and $\partial f_j/\partial Z$, can be easily obtained from equations (22) and (23). The gamma measures with respect to the three variables can also be derived in closed-form but are not given here to save space.

The analytical expressions for the deltas have important implications for the construction and execution of hedging strategies involving currency options and other contingent claims. This can be best demonstrated by considering a standard delta-neutral hedge with the target being a currency call option itself. To do so, one has three sources of uncertainty to control for: $dS(t)$, $dY(t)$, and $dZ(t)$, or equivalently $dS(t)$, $dR(t)$, and $dR^*(t)$. Then, consider a replicating portfolio consisting of (i) $\bar{\Delta}_S(t)$ units of the foreign currency (to control for $dS(t)$), (ii) $\bar{\Delta}_B(t)$ units of the domestic bond (to control for $dR(t)$), (iii) $\bar{\Delta}_{B^*}(t)$ units of the foreign bond (to control for $dR^*(t)$), and (iv) a residual cash position $\bar{\Delta}_0(t)$ earning the instantaneous interest rate $R(t)$. Using a standard argument, we find the following solution for a delta-neutral hedge:

$$\bar{\Delta}_S(t) = \Delta_S - \frac{\varrho_z(\tau)\Delta_Y - \varrho_y(\tau)\Delta_Z}{S(t)\{\varrho_y(\tau)\varrho_z^*(\tau) - \varrho_z(\tau)\varrho_y^*(\tau)\}}, \tag{30}$$

$$\bar{\Delta}_B(t) = \frac{\varrho_y^*(\tau)\Delta_Z - \varrho_z^*(\tau)\Delta_Y}{B(t,\ \tau)\{\varrho_y(\tau)\varrho_z^*(\tau) - \varrho_z(\tau)\varrho_y^*(\tau)\}}, \tag{31}$$

$$\bar{\Delta}_{B^*}(t) = \frac{\varrho_z(\tau)\Delta_Y - \varrho_y(\tau)\Delta_Z}{S(t)B^*(t,\ \tau)\{\varrho_y(\tau)\varrho_z^*(\tau) - \varrho_z(\tau)\varrho_y^*(\tau)\}}, \tag{32}$$

$$\bar{\Delta}_0(t) = C(t,\ \tau) - \{\bar{\Delta}_S(t)S(t) + \bar{\Delta}_B(t)B(t,\ \tau) + \bar{\Delta}_{B^*}(t)B^*(t,\ \tau)\}. \tag{33}$$

Taking these positions results in a locally risk-free replicating portfolio of the target. The same logic of reasoning can be used to devise analytical delta-neutral hedges for other types of foreign exchange contingent securities and portfolios.

C. A "Watered-Down" Version of the General Model

As noted before, the general currency option model in equation (20) has many appealing properties and includes existing models as special cases. However, when we substitute the solution in equations (24) and (25) into equations (16) and (17) and apply Ito's lemma to derive the dynamics for the resulting interest rates $R(t)$ and $R^*(t)$, we find that the processes followed by $R(t)$ and $R^*(t)$ are interrelated in a sufficiently complex manner so that estimating the structural parameters in practice becomes a formidable task.

To enhance applicability, we offer a specialized, parsimonious version of our general model such that (i) it still allows interest rates and currency volatility to be stochastic and yet (ii) the number of parameters to be estimated is as small as possible. To achieve these goals, set $\mu_m = \mu_{m^*} = -\rho$ and $\eta_2 = \sigma_2 = \eta_1^* = \sigma_1^* = 0$ in equation (4) and its foreign counterpart, so that domestic money supply is totally driven by $Y(t)$ and its foreign counterpart by $Z(t)$. Under these restrictions, we obtain $R(t) = (\eta_1 - \sigma_1^2)Y(t)$ and $R^*(t) = (\eta_2^* - \sigma_2^{*2})Z(t)$. To ensure positive interest rates, we require that $\eta_1 > \sigma_1^2$ and $\eta_2^* > \sigma_2^{*2}$. The resulting interest rate dynamics are

$$dR(t) = [\theta_R - \kappa_R R(t)]dt + \sigma_R \sqrt{R(t)}d\omega_1(t), \tag{34}$$

$$dR^*(t) = [\theta_R^* - \kappa_R^* R^*(t)]dt + \sigma_R^* \sqrt{R^*(t)}d\omega_2(t), \tag{35}$$

where

$$\kappa_R \equiv \kappa_y, \quad \theta_R \equiv \theta_y(\eta_1 - \sigma_1^2), \quad \sigma_R \equiv \sigma_y \sqrt{\eta_1 - \sigma_1^2}, \quad \kappa_R^* \equiv \kappa_z,$$

$$\theta_R^* \equiv \theta_z(\eta_2^* - \sigma_2^{*2}), \quad \text{and} \quad \sigma_R^* \equiv \sigma_z \sqrt{\eta_2^* - \sigma_2^{*2}}.$$

In practice, the six coefficients, κ_R, θ_R, σ_R, κ_R^*, θ_R^* and σ_R^*, are to be directly estimated from the observed interest rate processes. The domestic τ-period bond price is

$$B(t, \tau; R) = \exp[-\alpha_R(\tau) - \beta_R(\tau)R(t)], \tag{36}$$

where, with $\lambda_R \equiv \sigma_R\sigma_s$ and $\varsigma_R \equiv \sqrt{(\kappa_R+\lambda_R)^2+2\sigma_R^2}$, we set

$$\alpha_R(\tau) = \frac{\theta_R}{\sigma_R^2}\left\{(\varsigma_R - \kappa_R - \lambda_R)\tau + 2\ln\left[1 - \frac{(1 - e^{-\varsigma_R\tau})(\varsigma_R - \kappa_R - \lambda_R)}{2\varsigma_R}\right]\right\},$$

$$\beta_R(\tau) = \frac{2(1 - e^{-\varsigma_R\tau})}{2\varsigma_R - [\varsigma_R - \kappa_R - \lambda_R](1 - e^{-\varsigma_R\tau})}.$$

Adding a "*" superscript to each parameter and variable gives the foreign bond counterpart. The simplified exchange rate dynamics becomes

$$\frac{dS(t)}{S(t)} = [R(t) - R^*(t) + \sigma_s^2 R(t)]dt + \sigma_s \sqrt{R(t)}d\omega_1(t) - \sigma_s^* \sqrt{R^*(t)}d\omega_2(t), \tag{37}$$

where $\sigma_s \equiv \sigma_1/\sqrt{\eta_1 - \sigma_1^2}$, $\sigma_s^* \equiv \sigma_2^*/\sqrt{\eta_2^* - \sigma_2^{*2}}$, *and both can be directly estimated from the actual exchange rate process. The stochastic currency volatility is now* $\sigma_s^2 R(t) + \sigma_s^{*2} R^*(t)$.

Incorporating these restrictions into equation (20) and replacing $Y(t)$ and $Z(t)$ respectively by $R(t)/(\eta_1 - \sigma_1^2)$ and $R^*(t)/(\eta_2^* - \sigma_2^{*2})$, we arrive at the currency call option pricing formula below:

$$C(t, \tau) = S(t)B^*(t, \tau)\Pi_1(t, S, R, R^*, \tau) - KB(t, \tau)\Pi_2(t, S, R, R^*, \tau), \tag{38}$$

where the probabilities, Π_1 and Π_2, are inverted, as in equation (21), from their respective characteristic functions given in equations (A8) and (A9) of the Appendix. This currency option formula is a relatively simple function of $S(t)$, $R(t)$, and $R^*(t)$, and it depends only on eight structural parameters. The hedge ratios based on this simplified version can be similarly derived as before. To

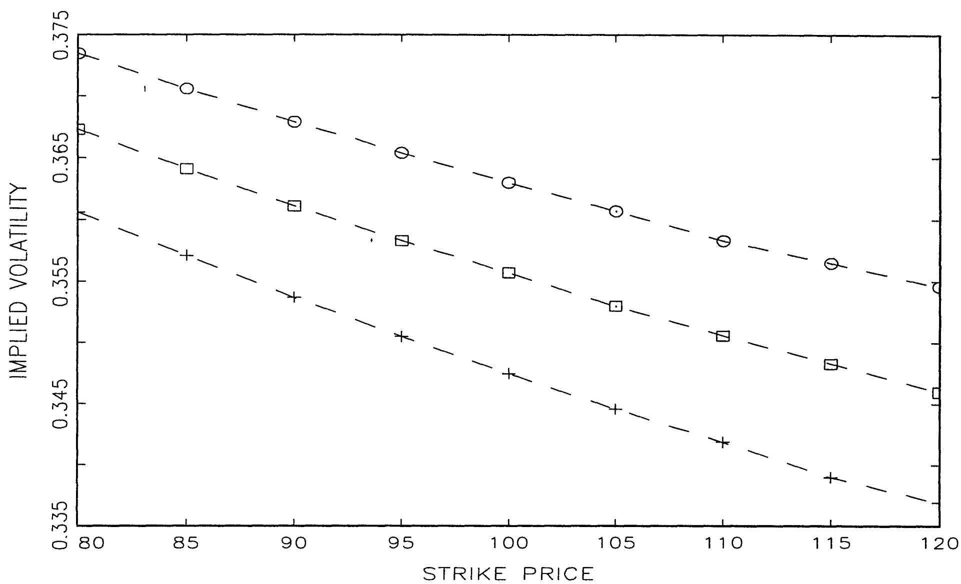

Figure 1. Implied volatility. Implied volatility is obtained by inverting the Garman–Kohlhagen currency option formula, with the call price input determined by formula (20). The + *curve*, the □ *curve*, and the ○ *curve* plot the relationship between strike price and implied volatility, respectively for options with terms-to-expiration of 20, 180, and 360 days. The spot exchange rate assumed is 100, $Y(t) = 0.75$, and $Z(t) = 1.00$. Other structural parameter values used in the calculations are: $\rho = 0.050$, $\mu_m = -0.050$, $\eta_1 = 0.135$, $\eta_2 = 0.048$, $\sigma_1 = 0.300$, $\sigma_2 = 0.00$, $\mu_m^* = -0.050$, $\eta_1^* = 0.052$, $\eta_2^* = 0.110$, $\sigma_1^* = 0.100$, $\sigma_2^* = 0.300$, $\kappa_y = 0.100$, $\theta_y = 0.120$, $\sigma_y = 0.150$, $\kappa_z = 0.500$, $\theta_z = 0.950$, and $\sigma_z = 0.40$.

implement this currency option formula for pricing and hedging purposes, one can consult with Bakshi, Cao, and Chen (1996) and Bates (1996).

D. Properties of Currency Options in an Example Economy

Without conducting a complete empirical test (which is beyond the scope of the present article), this subsection provides an example economy to illustrate the point that our currency option pricing model in equation (20) possesses empirically plausible features. Structural parameter values used in the example are given in the note to Figure 1. Given the parameter values, (i) the expected currency depreciation rate is 7.52 percent with a standard deviation of 34.60 percent; (ii) the instantaneous domestic and foreign interest rates are respectively 8.17 percent and 5.15 percent; (iii) Interest rates on longer maturity domestic bonds are R(t, 0.25) = 8.43 percent, R(t, 0.5) = 8.68 percent, R(t, 1) = 9.11 percent, R(t, 5) = 10.95 percent, and R(t, 30) = 12.30 percent, which represents an upward sloping yield curve; The foreign yield curve is also upward sloping, with $R^*(t, 0.25) = 5.25$ percent, $R^*(t, 0.5) = 5.34$ percent, $R^*(t, 1) = 5.48$ percent, $R^*(t, 5) = 6.16$ percent, and $R^*(t, 30) = 6.95$ percent; (iv) Due to positive yield differentials, the time t term structure of forward rates is also

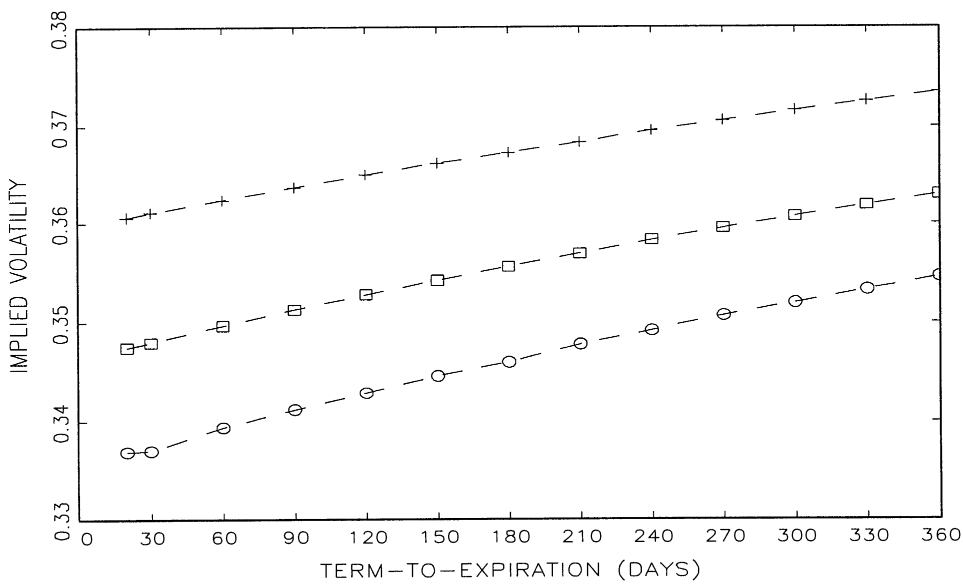

Figure 2. Term structure of implied volatility. Implied volatility is obtained by inverting the Garman–Kohlhagen currency option formula, with the call price input determined by formula (20). The + *curve*, the □ *curve*, and the ○ *curve* correspond to the relationship between maturity and implied volatility, respectively for currency calls with strike prices K = 80, K = 100, and K = 120. The spot exchange rate assumed is 100. All other structural parameter values used in the calculations are the same as assumed for Figure 1.

upward sloping. The currency call option price is increasing in θ_y and θ_z, and decreasing in the mean-reversion coefficients, κ_y and κ_z. Among other features, domestic and foreign interest rates are positively correlated with currency depreciation rate, and changes in currency volatility are negatively correlated with exchange rate fluctuations.

An often cited empirical regularity in the currency option literature is that when actual option prices are substituted into the Garman–Kohlhagen (1983) formula to back out the implied volatility, the resulting estimate varies across strike prices and across terms to expiration (i.e., *volatility smiles*). As pointed out by Cao (1993), the volatility smiles are time-dependent and can exhibit upward-sloping, downward-sloping, or U-shaped patterns.

We conduct the following exercise to demonstrate that currency option prices generated by our model can be consistent with their real-life counterparts. First, compute the theoretical option price based on formula (20) with the structural parameters as given. Next, substitute the computed option price into the Garman–Kohlhagen formula to back out the implied currency volatility. Finally, repeat these two steps for many options to arrive at implied-volatility patterns in relation to either strike prices or maturities. Figure 1 shows three strike price-implied volatility patterns: one each for short-term (20-day), medium term (180-day), and long-term (360-day) calls. Figure 2 presents three maturity-implied volatility patterns. In Figure 2, for instance, the maturity-implied volatility shape is upward sloping, with longer-term calls having higher implied volatility. Holding term-to-expiration fixed, out-of-the-money calls in Figure 1 have lower implied volatilities than both at-the-money and in-the-money calls. We also tried many other parameter value combinations (not reported here) and obtained a rich variety of possible shapes for the two relationships. Overall, the conclusion that can be drawn is that, depending on the state of the international economy and the structural parameter values, option prices obtained from formula (20) can produce any desired implied-volatility patterns in relation to moneyness and maturity, including the commonly observed upward- and downward-sloping smiles for currency options. For this reason, one can expect our currency option model to help mitigate moneyness- and maturity-related biases of existing models.

III. Currency Futures and Currency Futures Options

A τ-period currency futures contract represents an obligation to purchase in τ periods the said amount of the foreign currency at a contracted price (i.e., futures exchange rate). This type of contract is traded through the daily "marking-to-market" process. Let $G(t, \tau)$ be the futures exchange rate for delivery of one unit foreign currency τ periods from time t. Then, by Ito's lemma and the risk premium equation in (9), $G(t, \tau)$ must solve

$$\begin{aligned} &\tfrac{1}{2}(\sigma_s^2 Y + \sigma_s^{*2} Z) S^2 G_{SS} + S(R - R^*) G_S + \tfrac{1}{2}\sigma_y^2 Y G_{YY} \\ &\quad + [\theta_y - (\kappa_y + \lambda_y) Y] G_Y + \tfrac{1}{2}\sigma_z^2 Z G_{ZZ} + [\theta_z - (\kappa_z + \lambda_z) Z] G_Z + \sigma_y \sigma_s S Y G_{SY} \\ &\quad - \sigma_z \sigma_s^* S Z G_{SZ} - G_\tau = 0, \end{aligned} \tag{39}$$

subject to the boundary condition: $G(t + \tau, 0) = S(t + \tau)$. The solution to this PDE is

$$G(t, \tau) = S(t)\exp\{(\mu_m - \mu_m^*)\tau + \nu_y(\tau) + \nu_z(\tau) + \beta_y(\tau)Y(t) + \beta_z(\tau)Z(t)\}, \tag{40}$$

where

$$\nu_y(\tau) = -\frac{2\theta_y}{\sigma_y^2}\left\{\ln\left[1 + \frac{(1 - e^{-\zeta_y \tau})(\kappa_y + \lambda_y - \sigma_y\sigma_s - \zeta_y)}{2\zeta_y}\right] + \tfrac{1}{2}[\zeta_y - \kappa_y - \lambda_y + \sigma_y\sigma_s]\tau\right\},$$

$$\nu_z(\tau) = -\frac{2\theta_z}{\sigma_z^2}\left\{\ln\left[1 + \frac{(1 - e^{-\zeta_z \tau})(\kappa_z + \lambda_z + \sigma_z\sigma_s^* - \zeta_z)}{2\zeta_z}\right] + \tfrac{1}{2}[\zeta_z - \kappa_z - \lambda_z - \sigma_z\sigma_s^*]\tau\right\},$$

$$\beta_y(\tau) = \frac{2(\eta_1 - \sigma_1^2 - \eta_1^* + \sigma_1^{*2})(1 - e^{-\zeta_y \tau})}{2\zeta_y + (\kappa_y + \lambda_y - \sigma_y \sigma_s - \zeta_y)(1 - e^{-\zeta_y \tau})},$$

$$\beta_z(\tau) = -\frac{2(\eta_2^* - \sigma_2^{*2} - \eta_2 + \sigma_2^2)(1 - e^{-\zeta_z \tau})}{2\zeta_z + (\kappa_z + \lambda_z + \sigma_z \sigma_s^* - \zeta_z)(1 - e^{-\zeta_z \tau})},$$

$$\zeta_y \equiv \sqrt{(\kappa_y + \lambda_y - \sigma_y \sigma_s)^2 - 2\sigma_y^2\{\eta_1 - \sigma_1^2 - \eta_1^* + \sigma_1^{*2}\}},$$

$$\zeta_z \equiv \sqrt{(\kappa_z + \lambda_z + \sigma_z \sigma_s^*)^2 + 2\sigma_z^2\{\eta_2^* - \sigma_2^{*2} - \eta_2 + \sigma_2^2\}}.$$

A European call option on a currency futures gives the holder the right to take a long position in the futures contract at a prefixed price. Let τ be the time to expiration for such a futures option contract, and $\tilde{\tau}$ the time to maturity for the underlying futures, all from time t. Assume $\tau \leq \tilde{\tau}$. Let $H(t, \tau)$ be the time t price of the futures option. As before, $H(t, \tau)$ can only be a function of $S(t)$, $Y(t)$, and $Z(t)$, and it must solve the valuation PDE given in equation (19), subject to the terminal payoff condition: $H(t + \tau, 0) = \max(0, G(t + \tau, \tilde{\tau} - \tau) - K)$. As shown in the Appendix, the solution is

$$H(t, \tau) = G(t, \tilde{\tau})D(t, Y, Z)\Pi_1(t, S, Y, Z, \tau) - KB(t, \tau)\Pi_2(t, S, Y, Z, \tau), \tag{41}$$

where the "risk-neutral" probabilities are given by

$$\Pi_j(t, S, Y, Z, \tau) = 1/2 + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\tilde{K}} f_j(t, S, Y, Z, \tau; \phi)}{i\phi}\right] d\phi, \tag{42}$$

for $j = 1, 2$, where $\tilde{K} \equiv \ln[K] - (\mu_m - \mu_m^*)(\tilde{\tau} - \tau) - \nu_y(\tilde{\tau} - \tau) - \nu_z(\tilde{\tau} - \tau)$ and the characteristic functions, f_j for $j = 1, 2$, are given in equations (A11) and (A12) of the Appendix. The exercise region for this call is determined by the condition: $\beta_y(\tilde{\tau} - \tau)Y(t + \tau) + \beta_z(\tilde{\tau} - \tau)Z(t + \tau) + \ln[S(t + \tau)] \geq \tilde{K}$. In (41), $D(t, Y, Z)$ adjusts for the fact that the foreign currency pays a stochastic interest rate (which is like a stochastic dividend flow) and this adjustment function is given in equation (A10) of the Appendix. As $\tilde{\tau} \to \tau$, the futures option becomes a spot currency option and its price in equation (41) converges to its spot option counterpart in equation (20). For applications, one can, for instance, use the substitution scheme given in equations (24) and (25) to replace the unobservable $Y(t)$ and $Z(t)$ in equation (41), by $R(t)$ and $R^*(t)$.

One distinctive feature of the pricing formula in equation (41) is that the underlying futures exchange rate, domestic and foreign bond prices, and the futures option price are endogenously determined in the same equilibrium. In contrast, the well-known futures option model of Black (1976) is partial equilibrium in nature, and it only allows constant futures price volatility and constant interest rates. Amin and Jarrow's (1991) partial-equilibrium model for currency futures options does allow stochastic interest rates but only deterministic exchange rate volatility. Both the Black and the Amin-Jarrow models are nested within our currency futures option model.

We close this section by mentioning two more points. First, as in the previous section for spot currency options, hedge ratios and hence hedging strategies can be conveniently derived in closed-form for currency futures options. Next, by imposing the same set of restrictions required for the "watered-down" version of the spot currency option model, one can develop an easy-to-implement version of equation (41). Since the derivation of these formulas is essentially the same as for the previous case, the details are not repeated here.

IV. Conclusion

Extending the Lucas (1982) two-country economy to a continuous-time setup, we have developed an equilibrium valuation framework for foreign exchange contingent claims, especially for option-like contracts. The simple framework makes it convenient to derive analytical solutions for nominal valuation problems. For example, we derive and study closed-form pricing formulas respectively for spot exchange rate options, futures exchange rate, and currency futures options. The closed-form nature of these formulas not only renders the valuation straightforward and precise, but also makes it possible to analytically derive hedge ratios and hedging strategies.

The discussion in this article can be extended in several directions. First, cross-currency equity options and cross-currency equity futures options (e.g., the Chicago-traded Nikkei futures contract and options written on the contract) can be valued in the same context. Second, jumps in money supplies and hence in exchange rates can be incorporated along the lines of Bates (1996), Jorion (1988), and Scott (1996). Finally, the spot and the futures currency option formulas can be subjected to empirical tests. This exercise is pursued in a follow-up project.

Appendix: Proof of Results

Proof of Equation (9)

The proof involves applying the variational argument in Grossman and Shiller (1982), in which the time horizon is divided into many subperiods of length Δt. Let $U(x(t), x^*(t))$ be a general twice continuously differentiable period utility of consumption in the domestic and the foreign good. Observe that

$$\frac{P(t)}{P(t+\Delta t)} = 1 - \frac{\Delta P(t)}{P(t)} + \left(\frac{\Delta P(t)}{P(t)}\right)^2$$

and

$$\frac{\Delta P(t)}{P(t)} = \frac{\Delta M(t)}{M(t)} - \frac{\Delta x(t)}{x(t)} + \left(\frac{\Delta x(t)}{x(t)}\right)^2 + O(\Delta t)^{3/2}. \tag{A1}$$

Taking the Taylor series of the marginal utility $U_x(x(t + \Delta t), x^*(t + \Delta t))$ at $(x(t), x^*(t))$ yields

$$\begin{aligned}\frac{U_x(x(t + \Delta t),\, x^*(t + \Delta t))}{U_x(x(t),\, x^*(t))} &= 1 + \frac{U_{xx}}{U_x}\Delta x + \tfrac{1}{2}\frac{U_{xxx}}{U_x}(\Delta x)^2 + \frac{U_{xx^*}}{U_x}\Delta x^* \\ &\quad + \tfrac{1}{2}\frac{U_{xx^*x^*}}{U_x}(\Delta x^*)^2 + \frac{U_{xxx^*}}{U_x}(\Delta x \Delta x^*) + O(\Delta t)^{3/2}.\end{aligned} \tag{A2}$$

Then, for the discretized economy, the representative agent's first-order condition yields the risk premium on any contingent claim as determined by the Euler equation:

$$E_t\left[e^{-\rho\Delta t}\frac{U_x(x(t + \Delta t),\, x^*(t + \Delta t))}{U_x(x(t),\, x^*(t))}\frac{P(t)}{P(t + \Delta t)}\left(\frac{\Delta Q(t)}{Q(t)} - R(t)\Delta t\right)\right] = 0. \tag{A3}$$

Then, equation (9) is obtained by taking $\Delta t \to 0$ and repeatedly applying Ito's lemma to (A3), and by using equations (A1), (A2), and the log utility assumed in the model. Q.E.D.

Derivation of the Currency Option Formula in Equation (20)

The PDE in equation (19) can be expressed as

$$\begin{aligned}0 &= \tfrac{1}{2}(\sigma_s^2 Y + \sigma_s^{*2} Z)\frac{\partial^2 C}{\partial h^2} + [R - R^* - \tfrac{1}{2}(\sigma_s^2 Y + \sigma_s^{*2} Z)]\frac{\partial C}{\partial h} \\ &\quad + \tfrac{1}{2}\sigma_y^2 Y\frac{\partial^2 C}{\partial Y^2} + [\theta_y - (\kappa_y + \lambda_y)Y]\frac{\partial C}{\partial Y} + \tfrac{1}{2}\sigma_z^2 Z\frac{\partial^2 C}{\partial Z^2} \\ &\quad + [\theta_z - (\kappa_z + \lambda_z)Z]\frac{\partial C}{\partial Z} \\ &\quad + \sigma_y\sigma_s Y\frac{\partial^2 C}{\partial h\partial Y} - \sigma_s\sigma_s^* Z\frac{\partial^2 C}{\partial h\partial Z} - \frac{\partial C}{\partial \tau} - RC,\end{aligned} \tag{A4}$$

where for convenience, $h(t) \equiv \ln[S(t)]$. Then, substituting the conjecture (20) into equation (A4) yields the following PDE respectively for the probabilities,

Π_1 and Π_2:

$$\tfrac{1}{2}(\sigma_s^2 Y + \sigma_s^{*2} Z)\frac{\partial^2 \Pi_1}{\partial h^2} + \left[R - R^* + \tfrac{1}{2}\,\sigma_s^2 Y + \sigma_s^{*2} Z + \sigma_y \sigma_s Y \,\frac{1}{B^*}\,\frac{\partial B^*}{\partial Y} - \sigma_z \sigma_s^* Z\,\frac{1}{B^*}\,\frac{\partial B^*}{\partial Z}\right]\frac{\partial \Pi_1}{\partial h}$$
$$+ \tfrac{1}{2}\,\sigma_y^2 Y\,\frac{\partial^2 \Pi_1}{\partial Y^2} + \left[\theta_y - (\kappa_y + \lambda_y)Y + \sigma_y\sigma_s Y + \sigma_y^2 Y\,\frac{1}{B^*}\,\frac{\partial B^*}{\partial Y}\right]\frac{\partial \Pi_1}{\partial Y}$$
$$+ \sigma_y \sigma_s Y\,\frac{\partial^2 \Pi_1}{\partial h \partial Y} - \frac{\partial \Pi_1}{\partial \tau}$$
$$+ \tfrac{1}{2}\,\sigma_z^2 Z\,\frac{\partial^2 \Pi_1}{\partial Z^2} + \left[\theta_z - (\kappa_z + \lambda_z)Z - \sigma_z \sigma_s^* Z + \sigma_z^2 Z\,\frac{1}{B^*}\,\frac{\partial B^*}{\partial Z}\right]\frac{\partial \Pi_1}{\partial Z}$$
$$- \sigma_z \sigma_s^* Z\,\frac{\partial^2 \Pi_1}{\partial h \partial Z} = 0 \qquad \text{(A5)}$$

and

$$\tfrac{1}{2}\,(\sigma_s^2 Y + \sigma_s^{*2} Z)\frac{\partial^2 \Pi_2}{\partial h^2} + \left[R - R^* - \tfrac{1}{2}\,\sigma_s^2 Y - \sigma_s^{*2} Z + \sigma_y \sigma_s Y\,\frac{1}{B}\,\frac{\partial B}{\partial Y} - \sigma_z \sigma_s^* Z\,\frac{1}{B}\,\frac{\partial B}{\partial Z}\right]\frac{\partial \Pi_2}{\partial h} + \tfrac{1}{2}\,\sigma_y^2 Y\,\frac{\partial^2 \Pi_2}{\partial Y^2}$$
$$+ \left[\theta_y - (\kappa_y + \lambda_y)Y + \sigma_y^2 Y\,\frac{1}{B}\,\frac{\partial B}{\partial Y}\right]\frac{\partial \Pi_2}{\partial Y} + \sigma_y\sigma_s Y\,\frac{\partial^2 \Pi_2}{\partial h \partial Y} + \tfrac{1}{2}\,\sigma_z^2 Z\,\frac{\partial^2 \Pi_2}{\partial Z^2}$$
$$+ \left[\theta_z - (\kappa_z + \lambda_z)Z + \sigma_z^2 Z\,\frac{1}{B}\,\frac{\partial B}{\partial Z}\right]\frac{\partial \Pi_2}{\partial Z} - \sigma_z\sigma_s^* Z\,\frac{\partial^2 \Pi_2}{\partial h \partial Z} - \frac{\partial \Pi_2}{\partial \tau} = 0. \qquad \text{(A6)}$$

The PDEs in equations (A5) and (A6) must be separately solved subject to the terminal condition

$$\Pi_j(t + \tau,\, h,\, Y,\, Z,\, 0;\, \ln[K]) = 1_{h(t+\tau)\geq \ln[K]},$$

for $j = 1, 2$. As in Heston (1993), the characteristic functions corresponding to the two probabilities, $f_j(t, h, Y, Z, \tau;\, \phi)$ for $j = 1, 2$, must also satisfy the respective PDEs in equations (A5) and (A6), subject to the terminal condition

$$f_j(t + \tau,\, h,\, Y,\, Z,\, 0;\, \phi) = e^{i\phi h(t+\tau)}. \qquad \text{(A7)}$$

To solve these PDEs, make the substitution: $\hat{f}_1 = {}^*(t, \tau) f_1$ and $\hat{f}_2 = B(t, \tau) f_2$. Solving the PDEs separately results in the characteristic functions given respectively in equations (22) and (23). Q.E.D.

Characteristic Functions for the Watered-Down Option Formula in Equation (38)

The two characteristic functions are derived as in the general case and given below:

$$f_1 = \exp\Bigg\{ -\frac{\theta_R}{\sigma_R^2}\left[2\ln\left(1 + \frac{[\kappa_R - i\phi\lambda_R - \xi_1](1 - e^{-\xi_1\tau})}{2\xi_1}\right) - (\kappa_R - i\phi\lambda_R - \xi_1)\tau\right]$$
$$- \frac{\theta_{R^*}}{\sigma_{R^*}^2}\left[2\ln\left(1 + \frac{[\kappa_{R^*} + (2 + i\phi)\lambda_{R^*} - \xi_2](1 - e^{-\xi_2\tau})}{2\xi_2}\right)\right.$$
$$\left. - (\kappa_{R^*} + (2 + i\phi)\lambda_{R^*} - \xi_2)\tau\right] + \frac{[(1 + i\phi)(1 + \tfrac{1}{2}i\phi\sigma_s^2) - 1](1 - e^{-\xi_1\tau})}{\xi_1 + \tfrac{1}{2}[\kappa_R - i\phi\lambda_R - \xi_1](1 - e^{-\xi_1\tau})}R(t)$$
$$- \frac{[(1 + i\phi)(1 - \tfrac{1}{2}i\phi\sigma_s^{*2})](1 - e^{-\xi_2\tau})}{\xi_2 + \tfrac{1}{2}[\kappa_{R^*} + (2 + i\phi)\lambda_{R^*} - \xi_2](1 - e^{-\xi_2\tau})}R^*(t) + i\phi\ln[S(t)]$$
$$- \ln[B^*(t, \tau)]\Bigg\}, \quad \text{(A8)}$$

$$f_2 = \exp\Bigg\{ -\frac{\theta_R}{\sigma_R^2}\left[2\ln\left(1 + \frac{[\kappa_R + (1 - i\phi)\lambda_R - \xi_1^*](1 - e^{-\xi_1^*\tau})}{2\xi_1^*}\right)\right.$$
$$\left. - (\kappa_R + (1 - i\phi)\lambda_R - \xi_1^*)\tau\right]$$
$$- \frac{\theta_{R^*}}{\sigma_{R^*}^2}\left[2\ln\left(1 + \frac{[\kappa_{R^*} + (1 + i\phi)\lambda_{R^*} - \xi_2^*](1 - e^{-\xi_2^*\tau})}{2\xi_2^*}\right)\right.$$
$$\left. - (\kappa_{R^*} + (1 + i\phi)\lambda_{R^*} - \xi_2^*)\tau\right]$$
$$+ \frac{[(i\phi - 1)(1 + \tfrac{1}{2}i\phi\sigma_s^2)](1 - e^{-\xi_1^*\tau})}{\xi_1^* + \tfrac{1}{2}[\kappa_R + (1 - i\phi)\lambda_R - \xi_1^*](1 - e^{-\xi_1^*\tau})}R(t)$$
$$- \frac{[i\phi(1 + \tfrac{1}{2}(1 - i\phi)\sigma_s^{*2})](1 - e^{-\xi_2^*\tau})}{\xi_2^* + \tfrac{1}{2}[\kappa_{R^*} + (1 + i\phi)\lambda_{R^*} - \xi_2^*](1 - e^{-\xi_2^*\tau})}R^*(t) + i\phi\ln[S(t)]$$
$$- \ln[B(t, \tau)]\Bigg\}, \quad \text{(A9)}$$

where

$$\xi_1 = \{(\kappa_R - i\phi\lambda_R)^2 - 2\sigma_R^2[(1 + i\phi)(1 + \tfrac{1}{2}i\phi\sigma_s^2) - 1]\}^{1/2},$$
$$\xi_2 = \{(\kappa_{R^*} + (2 + i\phi)\lambda_{R^*})^2 + 2\sigma_{R^*}^2(1 + i\phi)(1 - \tfrac{1}{2}i\phi\sigma_s^{*2})\}^{1/2},$$
$$\xi_1^* = \{(\kappa_R + (1 - i\phi)\lambda_R)^2 - 2\sigma_R^2(i\phi - 1)(1 + \tfrac{1}{2}i\phi\sigma_s^2)\}^{1/2},$$
$$\xi_2^* = \{(\kappa_{R^*} + (1 + i\phi)\lambda_{R^*})^2 + 2\sigma_{R^*}^2 i\phi(1 + \tfrac{1}{2}(1 - i\phi)\sigma_s^{*2})\}^{1/2}.$$

Q.E.D.

Derivation of the Currency Futures Option Formula in Equation (41)

Conjecture the solution to the PDE in equation (19) is of the form as given in equation (41). Then, using the same steps as in the derivation of the spot currency option formula, we arrive at the adjustment function $D(t, Y, Z)$ and the two corresponding characteristic functions below:

$$
\begin{aligned}
D(t, Y, Z) = \exp\Bigg\{ & -\frac{2\theta_y}{\sigma_y^2}\ln\left(1 - \frac{(\upsilon_y - \kappa_y - \lambda_y + \sigma_y[\sigma_y\beta_y(\tilde{\tau}-\tau) + \sigma_s])(1 - e^{-\upsilon_y\tau})}{2\upsilon_y}\right) \\
& - (\rho + \mu_m)\tau + \nu_y(\tilde{\tau}-\tau) + \nu_z(\tilde{\tau}-\tau) - \nu_y(\tilde{\tau}) - \nu_z(\tilde{\tau}) + (\beta_y(\tilde{\tau}-\tau)\theta_y \\
& + \beta_z(\tilde{\tau}-\tau)\theta_z)\tau + [\beta_y(\tilde{\tau}-\tau) - \beta_y(\tilde{\tau})]Y(t) + [\beta_z(\tilde{\tau}-\tau) - \beta_z(\tilde{\tau})]Z(t) \\
& - \frac{\theta_y}{\sigma_y^2}(\upsilon_y - \kappa_y - \lambda_y + \sigma_y[\sigma_y\beta_y(\tilde{\tau}-\tau) + \sigma_s])\tau \\
& - \frac{2\theta_z}{\sigma_z^2}\ln\left(1 - \frac{(\upsilon_z - \kappa_z - \lambda_z + \sigma_z[\sigma_z\beta_z(\tilde{\tau}-\tau) - \sigma_S^*])(1 - e^{-\upsilon_z\tau})}{2\upsilon_z}\right) \\
& - \frac{\theta_z}{\sigma_z^2}(\upsilon_z - \kappa_z - \lambda_z + \sigma_z[\sigma_z\beta_z(\tilde{\tau}-\tau) - \sigma_S^*])\tau \\
& - \frac{(\eta_1^* - \sigma_1^{*2} + \frac{1}{2}\sigma_S^2 + (\kappa_y + \lambda_y)\beta_y(\tilde{\tau}-\tau) - \frac{1}{2}[\sigma_y\beta_y(\tilde{\tau}-\tau) + \sigma_s]^2}{\upsilon_y(1 - e^{-\upsilon_y\tau})^{-1} + \frac{1}{2}(\upsilon_y - \kappa_y - \lambda_y + \sigma_y[\sigma_y\beta_y(\tilde{\tau}-\tau) + \sigma_s])}Y(t) \\
& - \frac{(\eta_2^* - \sigma_2^{*2} + \frac{1}{2}\sigma_s^{*2} + (\kappa_z + \lambda_z)\beta_z(\tilde{\tau}-\tau) - \frac{1}{2}[\sigma_z\beta_z(\tilde{\tau}-\tau) - \sigma_s^*]^2)}{\upsilon_z(1 - e^{-\upsilon_z\tau})^{-1} + \frac{1}{2}(\upsilon_z - \kappa_z - \lambda_z + \sigma_z[\sigma_z\beta_z(\tilde{\tau}-\tau) - \sigma_s^*])}Z(t)\Bigg\},
\end{aligned}
\tag{A10}
$$

$$
\begin{aligned}
f_1 = \exp\Bigg\{ & -\frac{2\theta_y}{\sigma_y^2}\ln\left(1 + \frac{[\kappa_y + \lambda_y - (1 + i\phi)\sigma_y[\sigma_s + \sigma_y\beta_y(\tilde{\tau}-\tau)] - \varsigma_y](1 - e^{-\varsigma_y\tau})}{2\varsigma_y}\right) \\
& + i\phi\ln[S(t)] - \ln[D(t, Y, Z)] + \nu_y(\tilde{\tau}-\tau) + \nu_z(\tilde{\tau}-\tau) \\
& + (1 + i\phi)\beta_y(\tilde{\tau}-\tau)\{Y(t) + \theta_y\tau\} \\
& + (1 + i\phi)\beta_z(\tilde{\tau}-\tau)\{Z(t) + \theta_z\tau\} - (\mu_m - \mu_m^*)\tau - \nu_y(\tilde{\tau}) - \nu_z(\tilde{\tau}) \\
& - \beta_y(\tilde{\tau})Y(t) - \beta_z(\tilde{\tau})Z(t) + \frac{\theta_y}{\sigma_y^2}(\kappa_y + \lambda_y - (1 + i\phi)\sigma_y[\sigma_s + \sigma_y\beta_y(\tilde{\tau}-\tau)] - \varsigma_y)\tau \\
& + (\mu_m - \mu_m^*)(1 + i\phi)\tau - (\rho + \mu_m)\tau \\
& - \frac{2\theta_z}{\sigma_z^2}\ln\left(1 + \frac{[\kappa_z + \lambda_z + (1 + i\phi)\sigma_z[\sigma_s^* - \sigma_z\beta_z(\tilde{\tau}-\tau)] - \varsigma_z](1 - e^{-\varsigma_z\tau})}{2\varsigma_z}\right)
\end{aligned}
$$

$$+\frac{\sigma_1^2-\eta_1+(1+i\phi)[\tilde{\eta}-(\kappa_y+\lambda_y)\beta_y(\tilde{\tau}-\tau)] + \tfrac{1}{2}(1+i\phi)^2[\sigma_s+\sigma_y\beta_y(\tilde{\tau}-\tau)]^2}{(1-e^{-\varsigma_y\tau})^{-1}\varsigma_y+\tfrac{1}{2}\{\kappa_y+\lambda_y-(1+i\phi)\sigma_y[\sigma_s+\sigma_y\beta_y(\tilde{\tau}-\tau)]-\varsigma_y\}}Y(t)$$

$$-\frac{\eta_2-\sigma_2^2+(1+i\phi)[\tilde{\eta}^*+(\kappa_z+\lambda_z)\beta_z(\tilde{\tau}-\tau)] - \tfrac{1}{2}(1+i\phi)^2[\sigma_s^*-\sigma_z\beta_z(\tilde{\tau}-\tau)]^2}{(1-e^{-\varsigma_z\tau})^{-1}\varsigma_z+\tfrac{1}{2}\{\kappa_z+\lambda_z+(1+i\phi)\sigma_z[\sigma_s^*-\sigma_z\beta_z(\tilde{\tau}-\tau)]-\varsigma_z\}}Z(t)$$

$$\left.+\frac{\theta_z}{\sigma_z^2}(\kappa_z+\lambda_z+(1+i\phi)\sigma_z[\sigma_s^*-\sigma_z\beta_z(\tilde{\tau}-\tau)]-\varsigma_z)\tau\right\}, \tag{A11}$$

$$f_2=\exp\left\{-\frac{2\theta_y}{\sigma_y^2}\ln\left(1+\frac{[\kappa_y+\lambda_y-i\phi\sigma_y[\sigma_s+\sigma_y\beta_y(\tilde{\tau}-\tau)]-\varsigma_y^*](1-e^{-\varsigma_y^*\tau})}{2\varsigma_y^*}\right)\right.$$

$$+i\phi\ln[S(t)]-\ln[B(t,\tau)]+i\phi\beta_y(\tilde{\tau}-\tau)\{Y(t)+\theta_y\tau\}$$

$$+i\phi\beta_z(\tilde{\tau}-\tau)\{Z(t)+\theta_z\tau\}$$

$$+\frac{\theta_y}{\sigma_y^2}(\kappa_y+\lambda_y-i\phi\sigma_y[\sigma_s+\sigma_y\beta_y(\tilde{\tau}-\tau)]-\varsigma_y^*)\tau-(\rho+\mu_m)\tau$$

$$+(\mu_m-\mu_m^* i\phi\tau)$$

$$-\frac{2\theta_z}{\sigma_z^2}\ln\left(1+\frac{[\kappa_z+\lambda_z+i\phi\sigma_z[\sigma_s^*-\sigma_z\beta_z(\tilde{\tau}-\tau)]-\varsigma_z^*](1-e^{-\varsigma_z^*\tau})}{2\varsigma_z^*}\right)$$

$$+\frac{i\phi\tilde{\eta}-\eta_1+\sigma_1^2-i\phi(\kappa_y+\lambda_y)\beta_y(\tilde{\tau}-\tau) + \tfrac{1}{2}(i\phi)^2[\sigma_s+\sigma_y\beta_y(\tilde{\tau}-\tau)]^2}{(1-e^{-\varsigma_y^*\tau})^{-1}\varsigma_y^*+\tfrac{1}{2}\{\kappa_y+\lambda_y-i\phi\sigma_y[\sigma_s+\sigma_y\beta_y(\tilde{\tau}-\tau)]-\varsigma_y^*\}}Y(t)$$

$$-\frac{i\phi\tilde{\eta}^*+\eta_2-\sigma_2^2+i\phi(\kappa_z+\lambda_z)\beta_z(\tilde{\tau}-\tau)- \tfrac{1}{2}(i\phi)^2[\sigma_S^*-\sigma_z\beta_z(\tilde{\tau}-\tau)]^2}{(1-e^{-\varsigma_z^*\tau})^{-1}\varsigma_z^*+\tfrac{1}{2}\{\kappa_z+\lambda_z+i\phi\sigma_z[\sigma_s^*-\sigma_z\beta_z(\tilde{\tau}-\tau)]-\varsigma_z^*\}}Z(t)$$

$$\left.+\frac{\theta_z}{\sigma_z^2}(\kappa_z+\lambda_z+i\phi\sigma_z[\sigma_s^*-\sigma_z\beta_z(\tilde{\tau}-\tau)]-\varsigma_z^*)\tau\right\}, \tag{A12}$$

where $\tilde{\eta}\equiv\eta_1-\sigma_1^2-\eta_1^*+\sigma_1^{*2}-\tfrac{1}{2}\sigma_s^2$, $\tilde{\eta}^*\equiv\eta_2^*-\sigma_2^{*2}-\eta_2+\sigma_2^2+\tfrac{1}{2}\sigma_s^{*2}$,

$$v_y=\{(\kappa_y+\lambda_y-\sigma_y[\sigma_y\beta_y(\tilde{\tau}-\tau)+\sigma_s])^2+2\sigma_y^2(\eta_1^*-\sigma_1^{*2}+\tfrac{1}{2}\sigma_s^2 + (\kappa_y+\lambda_y)\beta_y(\tilde{\tau}-\tau)-\tfrac{1}{2}(\sigma_y\beta_y(\tilde{\tau}-\tau)+\sigma_s)^2)\}^{1/2},$$

$$v_z=\{(\kappa_z+\lambda_z-\sigma_z[\sigma_z\beta_z(\tilde{\tau}-\tau)-\sigma_s^*])^2+2\sigma_z^2(\eta_2^*-\sigma_2^{*2}+\tfrac{1}{2}\sigma_s^{*2} + (\kappa_z+\lambda_z)\beta_z(\tilde{\tau}-\tau)-\tfrac{1}{2}(\sigma_z\beta_z(\tilde{\tau}-\tau)-\sigma_s^*)^2)\}^{1/2},$$

$$\varsigma_y=\{(\kappa_y+\lambda_y-(1+i\phi)\sigma_y[\sigma_s+\sigma_y\beta_y(\tilde{\tau}-\tau)])^2 - 2\sigma_y^2((1+i\phi)\{\eta_1-\sigma_1^2-\eta_1^*+\sigma_1^{*2}-\tfrac{1}{2}\sigma_s^2\}$$

$$- \eta_1 + \sigma_1^2 - (1 + i\phi)(\kappa_y + \lambda_y)\beta_y(\tilde{\tau} - \tau) + \tfrac{1}{2}(1 + i\phi)^2[\sigma_s + \sigma_y\beta_y(\tilde{\tau} - \tau)]^2)\}^{1/2},$$

$$\varsigma_z = \{(\kappa_z + \lambda_z + (1 + i\phi)\sigma_z[\sigma_s^* - \sigma_z\beta_z(\tilde{\tau} - \tau)])^2 + 2\sigma_z^2((1 + i\phi) \cdot \{\eta_2^* - \sigma_2^{*2} - \eta_2 + \sigma_2^2 + \tfrac{1}{2}\sigma_s^{*2}\} + \eta_2 - \sigma_2^2 + (1 + i\phi)(\kappa_z + \lambda_z)\beta_z(\tilde{\tau} - \tau) - \tfrac{1}{2}(1 + i\phi)^2[\sigma_s^* - \sigma_z\beta_z(\tilde{\tau} - \tau)]^2)\}^{1/2},$$

$$\varsigma_y^* = \{(\kappa_y + \lambda_y - i\phi\sigma_y[\sigma_s + \sigma_y\beta_y(\tilde{\tau} - \tau)])^2 - 2\sigma_y^2(i\phi\{\eta_1 - \sigma_1^2 - \eta_1^* + \sigma_1^{*2} - \tfrac{1}{2}\sigma_s^2\} - \eta_1 + \sigma_1^2 - i\phi(\kappa_y + \lambda_y)\beta_y(\tilde{\tau} - \tau) + \tfrac{1}{2}(i\phi)^2[\sigma_s + \sigma_y\beta_y(\tilde{\tau} - \tau)]^2)\}^{1/2},$$

$$\varsigma_z^* = \{(\kappa_z + \lambda_z + i\phi\sigma_z[\sigma_s^* - \sigma_z\beta_z(\tilde{\tau} - \tau)])^2 + 2\sigma_z^2(i\phi\{\eta_2^* - \sigma_2^{*2} - \eta_2 + \sigma_2^2 + \tfrac{1}{2}\sigma_s^{*2}\} + \eta_2 - \sigma_2^2 + i\phi(\kappa_z + \lambda_z)\beta_z(\tilde{\tau} - \tau) - \tfrac{1}{2}(i\phi)^2[\sigma_s^* - \sigma_z\beta_z(\tilde{\tau} - \tau)]^2)\}^{1/2}.$$

Q.E.D.

REFERENCES

Amin, Kaushik, and Robert Jarrow, 1991, Pricing foreign currency options under stochastic interest rates, *Journal of International Money and Finance* 10, 310–329.

Bailey, Warren, and René Stulz, 1989, The pricing of stock index options in a general equilibrium model, *Journal of Financial and Quantitative Analysis* 24, 1–12.

Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 1996, Empirical performance of alternative option pricing models. *Journal of Finance*, forthcoming.

Bakshi, Gurdip, and Zhiwu Chen, 1996, Inflation, asset prices and the term structure of interest rates in monetary economies, *Review of Financial Studies* 9, 237–271.

Bakshi, Gurdip, and Zhiwu Chen, 1995, An alternative valuation model for contingent claims, *Journal of Financial Economics*, forthcoming.

Barone-Adesi, Giovanni, and Robert Whaley, 1987, Efficient analytic approximation of American option values, *Journal of Finance* 42, 301–320.

Bates, David, 1995, Testing option pricing models, in G. S. Maddla and C. R. Rao, Eds.: *Handbook of Statistics*, Vol. 15: Statistical Methods in Finance, 567–611.

Bates, David, 1996, Jumps and stochastic volatility: Exchange rate processes implicit in Deutschmark options, *Review of Financial Studies* 9, 69–108.

Bekaert, Geert, Robert Hodrick, and David Marshall, 1995, The implications of first-order risk aversion for asset market risk premiums, Working paper, Northwestern University.

Biger, Nahum, and John Hull, 1983, The valuation of currency options, *Financial Management* 12, 24–28.

Black, Fischer, 1976, The pricing of commodity contracts, *Journal of Financial Economics* 3, 167–179.

Black, Fischer, and Myron Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–659.

Bodurtha, James, and Georges Courtadon, 1987, Tests of an American option pricing model on the foreign exchange market, *Journal of Financial and Quantitative Analysis* 22, 153–167.

Boyle, Glenn, 1990, International interest rates, exchange rates, and the stochastic structure of supply, *Journal of Finance* 45, 655–671.

Campbell, John, and Richard Clarida, 1987, The term structure of EuroMarket interest rates: An empirical investigation, *Journal of Monetary Economics* 19, 25–44.

Cao, Charles, 1993, Pricing foreign currency options with stochastic volatility, Ph.D. Dissertation, University of Chicago.

Chesney, Marc, and Louis Scott, 1989, Pricing European currency options: A comparison of the modified Black-Scholes model and a random variance model, *Journal of Financial and Quantitative Analysis* 24, 267–284.

Cox, John, Jonathan Ingersoll, and Stephen Ross, 1985a, An intertemporal general equilibrium model of asset prices, *Econometrica* 53, 363–384.

Cox, John, Jonathan Ingersoll, and Stephen Ross, 1985b, A theory of the term structure of interest rates, *Econometrica* 53, 385–408.

Fama, Eugene, 1984, Forward and spot exchange rates, *Journal of Monetary Economics* 14, 319–338.

Garman, Mark, and Steven Kohlhagen, 1983, Foreign currency option values, *Journal of International Money and Finance* 2, 231–237.

Grabbe, Orlin, 1983, The pricing of call and put options on foreign exchange, *Journal of International Money and Finance* 2, 239–253.

Grossman, Sanford, and Robert Shiller, 1982, Consumption correlatedness and risk measurement in economies with non-traded assets and heterogeneous information, *Journal of Financial Economics* 10, 195–210.

Heston, Steven, 1993, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* 6, 327–343.

Hilliard, Jimmy, Jeff Madura, and Alan Tucker, 1991, Currency option pricing with stochastic domestic and foreign interest rates, *Journal of Financial and Quantitative Analysis* 26, 139–152.

Hodrick, Robert, 1987, *The Empirical Evidence on the Efficiency of Forward and Futures Foreign Exchange* (Harwood Academic Publishers, New York).

Ilmanen, Antti, 1995, Time-varying expected returns in international bond markets, *Journal of Finance* 50, 481–506.

Ingersoll, Jonathan, 1990, Contingent foreign exchange contracts with stochastic interest rates, Working paper, Yale University.

Jorion, Philippe, 1988, On jump processes in the foreign exchange and stock markets, *Review of Financial Studies* 1, 427–445.

Knoch, Hans, 1992, The pricing of foreign currency options, Ph.D. Dissertation, Yale University.

Longstaff, Francis, and Eduardo Schwartz, 1992, Interest rate volatility and the term structure: A two-factor general equilibrium model, *Journal of Finance* 47, 1259–1282.

Lucas, Robert, 1982, Interest rates and currency prices in a two-country world, *Journal of Monetary Economics* 10, 335–360.

Lucas, Robert, 1990, Liquidity and interest rates, *Journal of Economic Theory* 50, 237–264.

Melino, Angelo, and Stuart Turnbull, 1990, Pricing foreign currency options with stochastic volatility, *Journal of Econometrics* 45, 239–265.

Melino, Angelo, and Stuart Turnbull, 1995, Misspecification and the pricing and hedging of long-term foreign currency options, *Journal of International Money and Finance* 14, 373–393.

Nielsen, Lars, and Jesús Saá-Requejo, 1993, Exchange rate and term structure dynamics and the pricing of derivative securities, Working paper, University of Chicago.

Saá-Requejo, Jesús, 1994a, The dynamics and the term structure of risk premia in foreign exchange markets, Working paper, University of Chicago.

Saá-Requejo, Jesús, 1994b, On the characterization of exchange rate dynamics, Working paper, University of Chicago.

Scott, Louis, 1987, Option pricing when the variance changes randomly: Theory, estimation, and an application, *Journal of Financial and Quantitative Analysis* 22, 419–438.

Scott, Louis, 1996, Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: Application of Fourier inversion methods. *Mathematical Finance*, Forthcoming.

Stein, Elias, and Jeremy Stein, 1991, Stock price distributions with stochastic volatility, *Review of Financial Studies* 4, 727–752.

Stulz, René, 1987, An equilibrium model of exchange rate determination and asset pricing with nontraded goods and imperfect information, *Journal of Political Economy* 95, 1024–1040.

Stulz, René, 1992, International portfolio choice and asset pricing: An integrative survey, Working paper, Ohio State University.

Svensson, Lars, 1985, Currency prices, terms of trade, and interest rates: A general equilibrium cash-in-advance approach, *Journal of International Economics* 18, 29–43.