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Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates

KRISTIAN R. MILTERSEN, KLAUS SANDMANN, and DIETER SONDERMANN

ABSTRACT

We derive a unified model that gives closed form solutions for caps and floors written on interest rates as well as puts and calls written on zero-coupon bonds. The crucial assumption is that simple interest rates over a fixed finite period that matches the contract, which we want to price, are log-normally distributed. Moreover, this assumption is shown to be consistent with the Heath–Jarrow–Morton model for a specific choice of volatility.

CLOSED FORM SOLUTIONS FOR interest rate derivatives, in particular caps, floors, and bond options, have been obtained by a number of authors for Markovian term structure models with normally distributed interest rates or alternatively log-normally distributed bond prices (see, for example, Jamshidian (1989, 1991a); Heath, Jarrow, and Morton (1992); Brace and Musiela (1994); Geman, El Karoui, and Rochet (1995)). These models support Black–Scholes type formulas most frequently used by practitioners for pricing bond options and swaptions. Unfortunately, these models imply negative interest rates with positive probabilities, and hence they are not arbitrage free in an economy with opportunities for riskless and costless storage of money. Briys, Crouhy, and Schöbel (1991) apply the Gaussian framework to derive closed form solutions for caps, floors, and European zero-coupon bond options. To exclude the influence of negative forward rates on the pricing of zero-coupon bond options, they introduce an additional boundary condition. As shown by Rady and Sandmann (1994) these pricing formulas are only supported by a term structure model with an absorbing boundary for the forward rate at zero, where the absorbing probability is not negligible, which for a term structure model is a quite problematic assumption.

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Alternatively, modeling log-normally distributed interest rates avoids the problems of negative interest rates. However, as shown by Morton (1988) and Hogan and Weintraub (1993), these rates explode with positive probability, implying zero prices for bonds and hence also arbitrage opportunities. Furthermore, so far, no closed form solutions are known for these models.

As observed by Sandmann and Sondermann (1994), the problems of exploding interest rates result from an unfortunate choice of compounding period of the interest rates modeled, namely the continuously compounding rate. Assuming that the continuously compounded interest rate is log-normally distributed results in "double exponential" expressions, i.e., the exponential function is itself an argument of an exponential function, thus giving rise to infinite expectations of the accumulation factor and of inverse bond prices under the martingale measure. The problem disappears as shown in Sandmann and Sondermann (1994) if, instead of assuming that the continuously compounded interest rates are log-normally distributed, one assumes that simple interest rates over a fixed finite period are log-normally distributed. In practice, interest rates, both spot and forward, are quoted as simple rates per annum (yearly), even if the finite period is different from one year, for example, three months. Moreover, effective annual rates¹ are calculated and used as *the* benchmark for comparing simple rates over different finite periods. Hence, simple interest rates over finite periods are directly observable in the market and form a natural starting point for modeling the term structure. We are aware of two alternative approaches that are similar to our approach and also avoid the problem of exploding rates: (i) Musiela (1994) models instantaneous forward rates with noncontinuous compounding as log-normal and finds the corresponding dynamics of the continuously compounding rates. (ii) Ho, Stapleton, Subrahmanyam, and Thanassoulas (1994) model "bankers discount" rates as log-normal. However, this latter approach implies negative bond prices with positive probability.

The main result of this article is a *unified* model that provides closed form solutions for interest rate caps and floors as well as puts and calls written on zero-coupon bonds within the context of a log-normal interest rate model. These solutions coincide with modifications of the Black-Scholes formula. In particular, for caps and floors with payment periods of the same length as the fixed period of the underlying simple interest rates we obtain the Black formula often used by market practitioners *without* making the unrealistic assumption that forward rates are independent of the accumulation process.² Thus, in this case our model supports market practice. For call and put options on zero-coupon bonds, our derived closed form solution matches the formula

¹ By effective annual rates we mean the annually compounded rate which yields the same return as the original rate compounded appropriately.

² Hull (1993, p.375).

derived in Käsler (1991).³ Käsler (1991) derives the formula using no-arbitrage arguments on two bond prices only. In this article, we provide a supporting no-arbitrage term structure model. Moreover, the log-normal assumption is shown to be consistent with the Heath–Jarrow–Morton model for a specific choice of volatility structure.

The article is organized as follows. Section I presents the model. Solutions for interest rate derivatives are then derived in Section II. The relation to the Heath–Jarrow–Morton model is found in Section III, and a discussion of the limitations of the model is found in the conclusion, Section IV. Finally, some proofs are deferred to the Appendix.

I. A Model for Simple Forward Rates over a Fixed Period

Let $P(t, T)$ denote the price, at date t , of a (default-free) zero-coupon bond that pays one dollar at maturity date T . Let $f(t, T, \alpha)$ denote the simple *forward rate* at date t over a fixed period of length α prevailing at date t for the future time interval $[T, T + \alpha]$. That is,

$$P(t, T + \alpha) = P(t, T) \frac{1}{1 + \alpha f(t, T, \alpha)} . \quad (1)$$

The limit case $\alpha = 0$ corresponds to the continuously compounding forward rate. This rate has to be treated as a special case in the following way:

$$f(t, T, 0) \equiv \lim_{\alpha \rightarrow 0} f(t, T, \alpha),$$

deduced using l'Hospital's rule from the following definition of a continuously compounded forward rate:

$$f(t, T, 0) = - \frac{(\partial/\partial T)P(t, T)}{P(t, T)} .$$

Consider at date t an agreement between two parties to sell or buy the zero-coupon bond with maturity $T + \alpha$ at the future date T , which is known as a forward contract. The forward price $F(t, T, \alpha)$ of the contract is defined as the fixed price which the buyer agrees to pay at date T for the bond with maturity $T + \alpha$, such that the value of the forward contract at date t is zero. No-arbitrage implies

$$F(t, T, \alpha) = \frac{P(t, T + \alpha)}{P(t, T)} = \frac{1}{1 + \alpha f(t, T, \alpha)} . \quad (2)$$

³ This formula is published in Käsler's Ph.D. dissertation written in German. The formula appears in the English manuscript Rady and Sandmann (1994), which is a comparative study of different bond based no-arbitrage models.

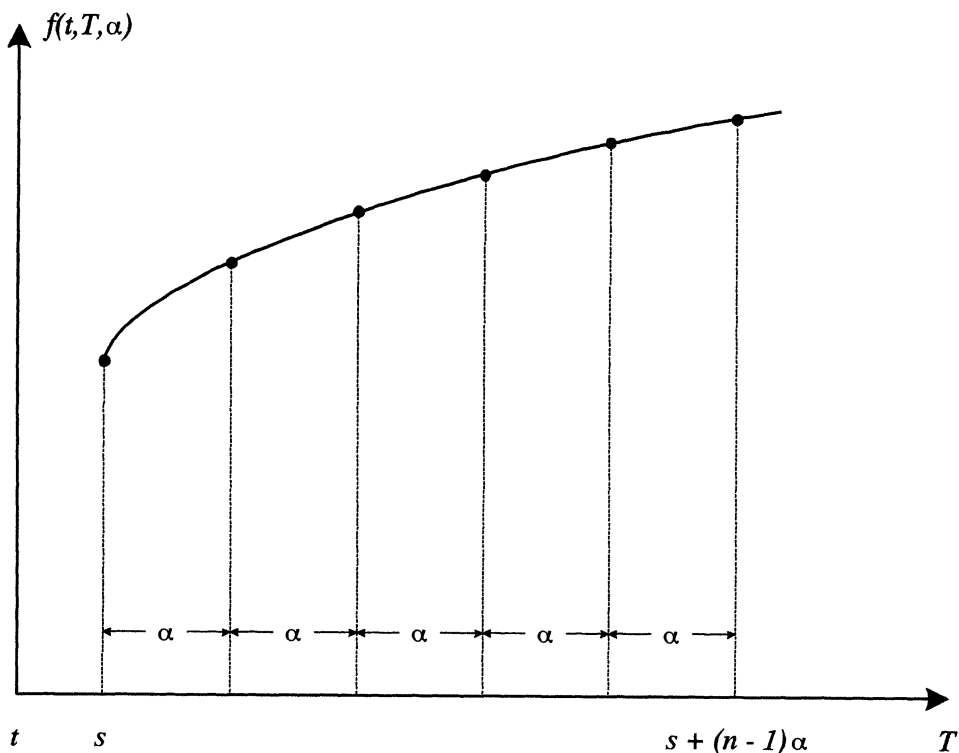


Figure 1. Curve of the simple forward rates with fixed period length α .

Note that, at each date t , bond prices, forward prices, and forward rates are related by

$$\begin{aligned}
 P(t, s + n\alpha) &= P(t, s) \prod_{i=0}^{n-1} \frac{P(t, s + (i+1)\alpha)}{P(t, s + i\alpha)} \\
 &= P(t, s) \prod_{i=0}^{n-1} \frac{1}{1 + \alpha f(t, s + i\alpha, \alpha)}, \quad (3)
 \end{aligned}$$

for $n = 1, \dots$, and $s \in [t, t + \alpha)$, where, for pure simplicity, we have chosen the same fixed period length α for each interval. Figure 1 shows the points on the forward rate curve used to price the bond with maturity $s + n\alpha$ in the above situation.

In our model, the stochastic behavior of the term structure of interest rates is determined by the simple forward rates. We assume that, at each date t , we observe several simple forward rates that are different with respect to the length and position of their compounding interval. The set of simple forward rates, at date t , can be expressed by the set of their compounding intervals

$\{[T_i, T_i + \alpha_i]_{i \in \mathbb{I}}\}$, where $T_i < T_j$, for $i, j \in \mathbb{I}$ such that $i < j$, $\alpha_i > 0$ for $i \in \mathbb{I}$, and $\mathbb{I}$ is an ordered index set that may be infinite.

In practice, this set of intervals is determined by the observed simple forward rates and therefore by maturities of existing bonds or interest rate futures. We model the processes of the simple forward rates in this set as log-normal diffusions,⁴ i.e.,

$$df(\cdot, T, \alpha)_t = \mu(t, T, \alpha)f(t, T, \alpha)dt + \gamma(t, T, \alpha)f(t, T, \alpha)dW_t, \quad (4)$$

where $\{f(\cdot, T, \alpha)_t\}$ is initiated using the term structure of interest rates observable at date zero

$$f(0, T, \alpha) = \frac{1}{\alpha} \left(\frac{P(0, T)}{P(0, T + \alpha)} - 1 \right).$$

No arbitrage implies that the set of forward rate processes, satisfying the above log-normal assumption, is restricted to those processes that cannot replicate each other. The mathematical reason is that the sum of log-normally distributed variables is itself not log-normally distributed. From the economic point of view, out of all traded simple forward rates we have to choose a nonredundant subset of processes that can be modeled as log-normal diffusions. We are free in the choice of this subset. For instance we can choose the forward rates given by three month Eurodollar futures contracts.⁵ In this case T_i would correspond to the settlement date of the i -th contract, and $\alpha_i = T_{i+1} - T_i$ to the length of the period covered by this contract. Therefore, purely in order to simplify the notation, we sometimes take the length of the compounding period α as fixed.

The existence of a unique nonnegative solution of the stochastic differential equation, SDE (4), is proven (under suitable regularity conditions)⁶ in Brace, Gatarek, and Musiela (1995).

To complete the model, consider the situation where SDE (4) is satisfied for all T and one fixed α . Then we have not only specified the stochastic model for the simple forward rates with compounding period α , but simultaneously we

⁴ Under the usual regularity conditions we can extend this to a multidimensional Wiener process. Similar closed form solutions can be derived in this situation. For simplicity of exposition we are concentrating on the one-dimensional case. The Heath, Jarrow, and Morton (1992) model use the continuously compounded forward rates as a starting point, whereas our modeling assumption is based on the simple forward rates. The relationship between both approaches is discussed in Section III.

⁵ A three month Eurodollar futures quote of 94.47 corresponds to a three month forward LIBOR rate of 5.53 (compare Hull (1993, p.99)). This is common market practice, i.e., the market neglects the stochastic effect of margin payments and thus the difference between a futures contract and a forward rate agreement (FRA) based on the futures quote. With reference to the discussion in Section II, it is an assumption of the model that the underlying interest rate is default free.

⁶ Taking up the ideas from our model, Brace, Gatarek, and Musiela (1995, Theorem 2.1) have shown existence for a bounded and (piecewise) continuous volatility function $\gamma : \mathbb{R}_+^2 \rightarrow \mathbb{R}$. The existence of a solution to the SDE (4) under the original probability measure is well-known. The question solved in their article is the existence under the equivalent martingale measure.

have determined the stochastic model for all rates with any compounding period (including the continuously compounding rates) through the bond prices. That is, bond prices are calculated using equation (3). Given the bond prices, forward rates can be calculated with any different compounding period than the chosen α using equation (1). However, the domain of this stochastic description is only the time interval $[t + \alpha, \infty)$; compare Figure 1. That is, for rates with shorter compounding periods, β , than α , we have not determined the stochastic model of simple forward rates with compounding period β (including the continuously compounding rates, i.e., $\beta = 0$) in the time interval $[t, t + \alpha - \beta]$.

Using Itô's lemma on the forward price process from equation (2) gives

$$\begin{aligned} \text{vol}(dF(\cdot, T, \alpha)_t) &= -F(t, T, \alpha)^2 \alpha \gamma(t, T, \alpha) f(t, T, \alpha) dW_t \\ &= -F(t, T, \alpha)(1 - F(t, T, \alpha)) \gamma(t, T, \alpha) dW_t, \end{aligned} \quad (5)$$

where we are only calculating the diffusion part of the Itô processes in this article, since we know from Harrison and Kreps (1979) and Harrison and Pliska (1981) that the drift part will not play any role for the pricing of contingent claims. For that purpose, we have introduced the obvious notation "vol." That is, for the Itô process

$$dX_t = \xi(X_t, t)dt + \delta(X_t, t)dW_t \quad \text{we define} \quad \text{vol}(dX_t) \equiv \delta(X_t, t)dW_t.$$

II. Closed Form Solutions for Interest Rate Derivatives

In this section, we focus on the arbitrage price of interest rate derivatives. More precisely, we consider two special interest rate derivatives: interest rate caps and floors, and European debt options where the underlying security is a zero-coupon bond. Since the construction of the underlying term structure model is very closely related to the Black-Scholes model, we should expect similar pricing formulas for these derivatives within our model.

Caps and floors are special types of options where a nominal interest rate is the underlying security. The underlying interest rate could be, for example, the three or six month London Inter Bank Offered Rate (LIBOR). A cap is an insurance against upward movements in the interest rate, and a floor is an insurance against downward movements in the interest rate. Let $\{r_t\}$ be a nominal interest rate process with compounding period α , for instance, for $\alpha = 1/4$ the process $\{r_t\}$ is the quoted three-month LIBOR. It is an assumption of the model that the underlying interest rate, $f(\cdot, \cdot, \alpha)$, is default-free since it is used for pricing default-free bonds. In practice, the LIBOR is based on a "replenished" AA rate and, hence, not default-free. However, (i) assuming that the short position of the cap or floor contract has the same credit quality as the one on which LIBOR is based, and (ii) modeling the default risk as in Duffie and Singleton (1994) and Duffie, Schroder, and Skiadas (1994), the same formulas apply with the volatility process adjusted to include the default spread on LIBOR. As it is shown in Duffie (1994), the volatility of the credit spread and

of the default-free rate simply adds together to give the volatility of the defaultable rate. This result also applies to our model, SDE (4), with appropriate dynamics of the default risk. Duffie and Singleton (1994) then show that options, etc., written on defaultable interest rates can be priced using standard option pricing techniques, such as valuing expectations under an equivalent martingale measure or solving partial differential equations (PDE)s, by (i) simply substituting the default-free volatility with the volatility of the defaultable rate and (ii) using the defaultable rate as the short rate in the option pricing model.⁷

Let $t_0 < t_1 < \dots < t_N$ be a set of dates and define $\alpha_i = t_{i+1} - t_i$. A cap contract with level L , face value V , underlying nominal interest rate process $\{r_t\}$, and payment dates $t_1, \dots, t_N$ is defined by the payoff at all dates t_{i+1}

$$V\alpha_i[r_{t_i} - L]^+ \equiv V\alpha_i \max\{r_{t_i} - L, 0\},$$

if payments are made in arrear. A cap with one payment date is called a *caplet*. Since all caps are portfolios of caplets, we concentrate on pricing a caplet. Clearly, $r_{t_i} = f(t_i, t_i, \alpha_i)$. Since this rate is known at date t_i , the payoff of a caplet at date t_{i+1} is also known at date t_i , hence the present value of this payoff, at date t_i , is equal to

$$\begin{aligned} P(t_i, t_{i+1})V\alpha_i[f(t_i, t_i, \alpha_i) - L]^+ \\ &= \frac{V}{1 + \alpha_i f(t_i, t_i, \alpha_i)}[1 + \alpha_i f(t_i, t_i, \alpha_i) - (1 + \alpha_i L)]^+ \\ &= V \left[1 - \frac{1 + \alpha_i L}{1 + \alpha_i f(t_i, t_i, \alpha_i)} \right]^+ \\ &= V(1 + \alpha_i L) \left[\frac{1}{1 + \alpha_i L} - F(t_i, t_i, \alpha_i) \right]^+. \end{aligned} \quad (6)$$

The floor is just the opposite contract, and the present value at date t_i is given by

$$V(1 + \alpha_i L) \left[F(t_i, t_i, \alpha_i) - \frac{1}{1 + \alpha_i L} \right]^+. \quad (7)$$

The payoff of a cap or a floor, at each date t_i , is equivalent to $V(1 + \alpha_i L)$ times the payoff of a European put option or a European call option, respectively, with exercise date t_i , exercise price $K = 1/(1 + \alpha_i L)$, and a zero-coupon bond with maturity $t_{i+1} = t_i + \alpha_i$ as the underlying security. Thus, the arbitrage price of a cap or a floor is equal to the arbitrage price of a portfolio of European put options or European call options, respectively.

⁷ We are indebted to Darrell Duffie for pointing this out to us.

PROPOSITION 1: Consider a European call option with exercise price K and exercise date T written on a zero-coupon bond with maturity date $T + \alpha$. If the simple forward rate process $\{f(t, T, \alpha)\}_{t \in [0, T]}$ is log-normally distributed, i.e. satisfies SDE (4), then the arbitrage price is

$$\begin{aligned} \text{Call} &= P(t, T + \alpha)N(e_1) - KP(t, T)N(e_2) - KP(t, T + \alpha)(N(e_1) - N(e_2)) \\ &= (1 - K)P(t, T + \alpha)N(e_1) - K(P(t, T) - P(t, T + \alpha))N(e_2), \end{aligned} \quad (8)$$

with

$$e_{1,2} = \frac{1}{\sigma(t, T, \alpha)} \left(\ln \frac{P(t, T + \alpha)(1 - K)}{(P(t, T) - P(t, T + \alpha))K} \pm \frac{\sigma^2(t, T, \alpha)}{2} \right),$$

$$\sigma^2(t, T, \alpha) = \int_t^T \gamma^2(s, T, \alpha) ds,$$

where $N(\cdot)$ denotes the standard normal distribution.

Proof: The proof consists of two steps. First, we consider a self-financing portfolio strategy on the bond market that duplicates the payoff of the European call option. The resulting partial differential equation will be solved in the second step.

Assume that there exists a self-financing portfolio strategy $(\phi^1, \phi^2) \equiv \{(\phi_t^1, \phi_t^2)\}_{t \in [0, T]}$ on the bond market with value process $V \equiv \{V_t\}_{t \in [0, T]}$. Then the dynamics of the value process, V , is according to Itô's lemma

$$dV_t = \phi_t^1 dP(t, T + \alpha) + \phi_t^2 dP(t, T) \quad \text{with} \quad V_T = [P(T, T + \alpha) - K]^+.$$

By no arbitrage, the value process of the call option is then equal to the value process of the portfolio strategy. Consider instead the forward value process $\hat{V}$ defined by

$$\hat{V}_t \equiv \frac{V_t}{P(t, T)} = \phi_t^1 F(t, T, \alpha) + \phi_t^2.$$

If the portfolio strategy (ϕ^1, ϕ^2) is self-financing on the bond market we derive by Itô's lemma that

$$d\hat{V}_t = \phi_t^1 dF(\cdot, T, \alpha)_t \quad \text{with} \quad \hat{V}_T = [P(T, T + \alpha) - K]^+ = [F(T, T, \alpha) - K]^+.$$

Thus, ϕ_t^1 can be interpreted as the number of T forward contracts to hold at date t committing us to buy, at time T , a zero-coupon bond with maturity $T + \alpha$. Define the forward price of the call option as

$$\hat{c}(t, F(t, T, \alpha)) \equiv \frac{1}{P(t, T)} \text{Call}.$$

By no arbitrage, the forward value of the portfolio strategy is equal to the forward price of the call option. This corresponds exactly to a change of measure from the martingale measure to the T forward risk adjusted measure with the zero-coupon bond $P(\cdot, T)$ as numeraire (compare Geman (1989), Jamshidian (1991b), or Geman, El Karoui, and Rochet (1995)). That is,

$$\hat{V}_t = \phi_t^1 F(t, T, \alpha) + \phi_t^2 = \hat{c}(t, F(t, T, \alpha))$$

implying that

$$\begin{aligned} d\hat{V}_t &= \phi_t^1 dF(\cdot, T, \alpha)_t = d\hat{c}(\cdot, F(\cdot, T, \alpha))_t \\ &= \hat{c}_t(t, F)dt + \frac{1}{2}\hat{c}_{FF}(t, F)d\langle F(\cdot) \rangle_t + \hat{c}_F(t, F)dF. \end{aligned}$$

In particular, the self-financing portfolio strategy is determined by $\phi_t^1 = \hat{c}_F(t, F(t, T, \alpha))$ and $\phi_t^2 = \hat{c}(t, F(t, T, \alpha)) - \phi_t^1 F(t, T, \alpha)$. Furthermore, the forward price process of the call option is a solution of the PDE

$$\hat{c}_t(t, F(t, T, \alpha)) + \frac{1}{2}\hat{c}_{FF}(t, F(t, T, \alpha))d\langle F(\cdot, T, \alpha) \rangle_t = 0, \quad (9)$$

with boundary condition $\hat{c}(T, F(T, T, \alpha)) = [F(T, T, \alpha) - K]^+$. From equation (5) the process of the quadratic variation of the forward price is known. Hence,

$$\begin{aligned} \hat{c}_t(t, F(t, T, \alpha)) + \frac{1}{2}\hat{c}_{FF}(t, F(t, T, \alpha))\gamma^2(t, T, \alpha)F^2(t, T, \alpha) \\ \cdot (1 - F(t, T, \alpha))^2 dt = 0. \end{aligned} \quad (10)$$

The solution of the PDE (10) follows the presentation in Rady and Sandmann (1994). For completeness, we give the outline of the proof in the Appendix. Q.E.D.

Note that the self-financing portfolio strategy is given by

$$\begin{pmatrix} \phi_t^1 \\ \phi_t^2 \end{pmatrix} = \begin{pmatrix} (1 - K)N(e_1) + KN(e_2) \\ -KN(e_2) \end{pmatrix},$$

where ϕ_t^1 (ϕ_t^2) is the number of bonds with maturity $T + \alpha$ (T) to hold at date t . The equivalent hedge on the forward market, at any date $0 \leq t \leq T$, consists of holding a long position of ϕ_t^1 forward contracts with forward price $F(t, T, \alpha) = P(t, T + \alpha) \cdot P(t, T)^{-1}$ and holding $\hat{V}_t = V_t \cdot P(t, T)^{-1}$ bonds with maturity T . As shown in the proof of Proposition 1 this strategy is self-financing, duplicates

the call value V_t , and requires no cash transactions between the start of the option and its settlement date. Using put-call parity, the value for the put option is

$$\text{Put} = K(P(t, T) - P(t, T + \alpha))N(-e_2) - (1 - K)P(t, T + \alpha)N(-e_1), \quad (11)$$

where $N(\cdot)$, e_1 , e_2 , and σ are as defined in Proposition 1. For the put option, the self-financing portfolio strategy on the bond market is given by $1 - \phi_t^1$ and $1 - \phi_t^2$, respectively.

We have written two versions of the closed form solutions in equation (8). The first version has three terms, whereas the first two terms look similar to the Black-Scholes formula (but note that e_1 and e_2 are not the usual arguments of $N(\cdot)$), and then there is a third correction term. The second version is in structure a Black-Scholes formula, where $(1 - K)P(t, T + \alpha)$ should be interpreted as the price of the underlying security and $K(P(t, T) - P(t, T + \alpha))$ as the present value of the exercise price.

A closed form solution of the type equation (8) was first derived by Käsler (1991) under specific assumptions for the two underlying zero-coupon bonds. A discussion of this model relative to other bond price based models⁸ can be found in Rady and Sandmann (1994). In a pure probabilistic framework for zero-coupon bonds, Rady (1995) has recently derived the same pricing formulas for zero-coupon bond options using the change of measure technique. This approach is based on the fact that under T forward risk adjusted measure, Q_T , the forward price of the call option is a martingale; i.e.,

$$\hat{c}(t, x) = E_{Q_T}[[F(T, T, \alpha) - K]^+ | F(t, T, \alpha) = x].$$

The main difficulty is to determine the transition density $q(v; t, x)$. Given the solution of the PDE (10) (see the Appendix) the transition density for $v \in]0, 1[$ is equal to

$q(v; t, x)$

$$\equiv \frac{1}{-\sqrt{2\pi\sigma^2(t)}} \cdot \frac{x}{v^2(1-v)} \cdot \exp\left(-\frac{[\ln((1-v)/v) - \ln((1-x)/x) + \frac{1}{2}\sigma^2(t)]^2}{2\sigma^2(t)}\right) \quad (12)$$

with

$$\sigma^2(t) = \int_t^T \gamma^2(u, T, \alpha) du.$$

⁸ By the term bond price based model we understand a model in which the bond prices are the exogenously given basic underlying variables. This is in contrast to the model of this article—and many other models—where the exogenously given basic underlying variable is the term structure of interest rates from which the bond prices are then given endogeneously.

Applying the substitution $u = (1 - v)/v$ with $du = -(1/v^2) dv$ and $(u + 1)/u = 1/(1 - v)$ yields that

$$\int_0^1 q(v; t, x) dv = \int_0^1 \frac{1}{-\sqrt{2\pi\sigma^2(t)}} \cdot x \cdot \frac{1+u}{u} \cdot \exp\left(-\frac{[\ln(u) - \ln((1-x)/x) + \frac{1}{2}\sigma^2(t)]^2}{2\sigma^2(t)}\right) du = 1,$$

which implies that $q(v; t, x)$ is indeed a density. Furthermore, the forward price $F(t, T, \alpha)$ is a martingale under Q_T ; i.e. $x = E_{Q_T}[F(T, T, \alpha) | F(t, T, \alpha) = x]$, and the PDE (10) is satisfied by the function $\hat{c}(t, x) = \int_0^1 [v - K]^+ q(v; t, x) dv$. Since x is equal to the forward price at time t ; i.e. $x = 1/[1 + \alpha f(t, T, \alpha)]$ the substitution $\rho = \ln[(1 - v)/\alpha v]$ yields

$$\hat{c}(t, x) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \int_{-\infty}^{+\infty} \left[\frac{1}{1 + \alpha e^\rho} - K \right]^+ \frac{(1 + \alpha e^\rho)}{(1 + \alpha f)} \cdot \exp\left\{-\frac{(\rho - \ln f + \frac{1}{2}\sigma^2(t))^2}{2\sigma^2(t)}\right\} d\rho.$$

The solution can be calculated in the same way as the one in the Appendix.⁹ We can now apply Proposition 1 and equation (11) to the pricing of interest rate caps and floors.

PROPOSITION 2: *Consider a cap with interest rate level L , face value V , and (arrear) payment dates $t_1, \dots, t_N$. Define $\alpha_i = t_{i+1} - t_i$, for $i = 0, \dots, N - 1$. If the simple forward rate processes $\{f(t, t_i, \alpha_i)\}_{t \in [0, t_i]}$ satisfy the SDE (4), then the arbitrage price of the cap at date $t \leq t_0 = t_1 - \alpha_0$ is*

$$\text{Cap} = V \sum_{i=0}^{N-1} \alpha_i P(t, t_{i+1}) (f(t, t_i, \alpha_i) N(d_1(t, t_i, \alpha_i)) - L N(d_2(t, t_i, \alpha_i))), \quad (13)$$

⁹ This is not only an alternative way to prove Proposition 1. The arbitrage price of any European type contingent claim with Borel measurable payoff function $h(\cdot)$ at time T on the zero coupon bond with maturity $T + \alpha$ is determined by

$$P(t, T + \alpha) \cdot E_{Q_t}[h(F(T, T, \alpha)) | F(t, T, \alpha) = x] = P(t, T + \alpha) \cdot \int_0^1 h(v) q(v; t, x) dv.$$

Hence, the pricing of such claims is reduced to the (numerical) calculation of a one-dimensional integral equation.

with

$$d_{1,2}(t, s, \alpha) = \frac{1}{\sigma(t, s, \alpha)} \left(\ln \frac{f(t, s, \alpha)}{L} \pm \frac{\sigma^2(t, s, \alpha)}{2} \right),$$

$$\sigma^2(t, s, \alpha) = \int_t^s \gamma^2(u, s, \alpha) du,$$

where $N(\cdot)$ denotes the standard normal distribution.

Proof: By arbitrage, we can reduce the problem to the pricing of a caplet with arrear payment date $t_{i+1} = t_i + \alpha_i$. From equation (6), the payoff of the caplet is equivalent to $V(1 + \alpha_i L)$ times the payoff of a European put option with exercise price $1/(1 + \alpha_i L)$, where the underlying security is a zero-coupon bond with maturity date t_{i+1} . Then, according to equation (11), the arbitrage price of the caplet is

$$\begin{aligned} \text{Caplet} &= V(1 + \alpha_i L) \left(\frac{1}{1 + \alpha_i L} (P(t, t_i) - P(t, t_{i+1})) N(-e_2) \right. \\ &\quad \left. - \left(1 - \frac{1}{1 + \alpha_i L} \right) P(t, t_{i+1}) N(-e_1) \right) \\ &= VP(t, t_{i+1}) \left(\left(\frac{P(t, t_i)}{P(t, t_{i+1})} - 1 \right) N(-e_2) - \alpha_i L N(-e_1) \right) \\ &= VP(t, t_{i+1}) (\alpha_i f(t, t_i, \alpha_i) N(-e_2) - \alpha_i L N(-e_1)), \end{aligned} \quad (14)$$

with

$$\begin{aligned} -e_2 &= \frac{-1}{\sigma(t, t_i, \alpha_i)} \left(\ln \frac{P(t, t_{i+1})(1 - 1/(1 + \alpha_i L))}{(P(t, t_i) - P(t, t_{i+1}))[1/(1 + \alpha_i L)]} - \frac{\sigma^2(t, t_i, \alpha_i)}{2} \right) \\ &= \frac{-1}{\sigma(t, t_i, \alpha_i)} \left(\ln \frac{\alpha_i L}{\alpha_i f(t, t_i, \alpha_i)} - \frac{\sigma^2(t, t_i, \alpha_i)}{2} \right) = d_1(t, t_i, \alpha_i) \end{aligned}$$

By summing the respective caplets, this yields the pricing formula for a cap. Q.E.D.

The hedge strategy for each caplet is determined by a long and a short position on the bond market. For the caplet with payment date t_{i+1} the hedge strategy for $t \leq t_i$ is equal to a short position ϕ_t^1 in the bond with maturity t_{i+1} and a long position ϕ_t^2 in the bond with maturity t_i , with

$$\begin{pmatrix} \phi_t^1 \\ \phi_t^2 \end{pmatrix} = \begin{pmatrix} -V[N(d_1(t, t_i, \alpha_i)) + \alpha_i L N(d_2(t, t_i, \alpha_i))] \\ VN(d_1(t, t_i, \alpha_i)) \end{pmatrix}.$$

An equivalent hedge on the forward market is to hold a short position of ϕ_t^1 forward contracts settled at t_i on the bond maturing at $t_i + \alpha_i$, and to invest the caplet premium into bonds with maturity t_i . Again, no cash transactions are needed between 0 and t_i .

A similar proof gives, under the same conditions as in Proposition 2, that the price of a floor is

$$\text{Floor} = V \sum_{i=0}^{N-1} \alpha_i P(t, t_{i+1}) (LN(-d_2(t, t_i, \alpha_i)) - f(t, t_i, \alpha_i) N(-d_1(t, t_i, \alpha_i))), \quad (15)$$

with $d_{1,2}$ and σ are as defined in Proposition 2.

The formulas (13) and (15) coincide with the formulas frequently used by market practitioners for the pricing of caps and floors. So far, there was no convincing theoretical argument justifying this practice. In particular, it was an open question whether there exists a consistent term structure model in which these formulas are valid, ruling out arbitrage across different maturities.

The proof of Proposition 2 uses the relationship between a caplet and a European zero coupon put option. Alternatively, one can calculate the arbitrage price of a caplet directly under the T forward risk adjusted measure Q_T . The transition probability $q(v; t, x)$, where x is the time t forward price is given by equation (12). With $x = 1/(1 + \alpha f)$ and the substitution $\rho = \ln[(1 - v)/\alpha v]$ this implies:

$$\begin{aligned} \text{Caplet} &= P(t, t_i) V (1 + \alpha_i L) E_{Q_T} \left[\left[\frac{1}{1 + \alpha L} - F(T, T, \alpha) \right]^+ \middle| F(t, T, \alpha) = x \right] \\ &= P(t, t_i) V \int_{-\infty}^{+\infty} \left[1 - \frac{1 + \alpha L}{1 + \alpha e^\rho} \right]^+ \frac{1 + \alpha e^\rho}{1 + \alpha f} \exp \left\{ -\frac{(\rho - \ln f + \frac{1}{2} \sigma^2)^2}{2\sigma^2} \right\} d\rho \\ &= \frac{P(t, t_i) V \alpha}{1 + \alpha f} \int_{\ln L}^{+\infty} (e^\rho - L) \exp \left\{ -\frac{(\rho - \ln f + \frac{1}{2} \sigma^2)^2}{2\sigma^2} \right\} d\rho \\ &= P(t, t_{i+1}) V \alpha [f N(d_1) - LN(d_2)] \end{aligned}$$

where, for simplicity, we have set

$$f \equiv f(t, T, \alpha), \quad \sigma^2 \equiv \int_t^T \gamma^2(u, T, \alpha) du \quad \text{and} \quad d_{1,2} \equiv \frac{1}{\sigma} \left(\ln \left(\frac{f}{L} \right) \pm \frac{\sigma^2}{2} \right).$$

However, under Q_T the simple forward rate $f(t, T, \alpha)$ is not a martingale since

$$E_{Q_T}[f(T, T, \alpha) | F(t, T, \alpha) = x] = f(t, T, \alpha) \cdot \frac{1 + \alpha f(t, T, \alpha) e^{\sigma^2(t, T, \alpha)}}{1 + \alpha f(t, T, \alpha)}. \quad (16)$$

In order to clarify the relation between the different forward rates and to strengthen the intuition behind these formulas we pick up a suggestion made by a referee of this journal. It was observed by the referee (and by Brace et al. (1995)) that a probabilistic proof of the cap pricing formula—without recurrence to the PDE (10) and the put option formula (11)—may be obtained as follows: “show that under an appropriate change of measure the caplet formula (14) is the expected value of its payoff at maturity under this measure. . . . This involves identifying the change of measure; explicitly identifying the distribution under the change of measure; and explicitly identifying how to evaluate the integral. . . . Given expression (4), show the form of equation (4) under the change of measure. Show how to compute the expression for the cap.”¹⁰ Let

$$df(t, T, 0) = \mu(t, T, f)dt + \eta(t, T, f)dW_t \quad (17)$$

be the corresponding Heath, Jarrow, and Morton (1992) model for the continuously compounding forward rates (compare Section III). Denote by P^* the risk-neutral probability measure of this model, and by $E^*[\cdot | \mathcal{F}_t]$ the conditional expectation under P^* with respect to the σ -algebra $\mathcal{F}_t$ at date t (compare Heath, Jarrow, and Morton (1992)). Then $f(t, T, 0)$ is the continuously compounding spot rate. It is well-known that under the risk-neutral measure P^* the value $c(t)$ of the caplet (with face value one dollar), at any date $t \leq T$, is given by

$$c(t) = E^* \left[\exp \left\{ - \int_t^{T+\alpha} f(u, u, 0) du \right\} \cdot \alpha \cdot [f(T, T, \alpha) - L]^+ | \mathcal{F}_t \right]. \quad (18)$$

This expression can be evaluated by a change of measure from P^* to the $(T + \alpha)$ forward risk adjusted measure $Q_{T+\alpha}$

$$Q_{T+\alpha} = (P(0, T + \alpha) \cdot \beta(T + \alpha))^{-1} P^*,$$

where $\beta(T + \alpha) = \exp \int_0^{T+\alpha} f(u, u, 0) du$ is the accumulation factor up to date $T + \alpha$ (see for example Geman (1989), Jamshidian (1991a) or Geman, El Karoui, and Rochet (1995)). Denoting by $E_{T+\alpha}[\cdot | \mathcal{F}_t]$ the conditional expectation under $Q_{T+\alpha}$, the expression (18) transforms to

$$c(t) = P(t, T + \alpha) E_{T+\alpha} [\alpha [f(T, T, \alpha) - L]^+ | \mathcal{F}_t].$$

According to the following Lemma, $f(t, T, \alpha)$ is a log-normal martingale under $Q_{T+\alpha}$. Hence Black's formula gives immediately

$$c(t) = \alpha \cdot P(t, T + \alpha) \{ f(t, T, \alpha) \cdot N(d_1) - L \cdot N(d_2) \}$$

with $d_{1,2}$ as defined in Proposition 2.

¹⁰ Quotations from a referee's report on an earlier version of this article.

LEMMA 3: *The forward rate process equation (4) satisfies the stochastic differential equation*

$$\frac{df(\cdot, T, \alpha)_t}{f(t, T, \alpha)} = \gamma(t, T, \alpha) dW_{T+\alpha}(t),$$

where $W_{T+\alpha}(t)$ is a Wiener process under the $(T + \alpha)$ forward measure $Q_{T+\alpha}$. (Proof: see the Appendix.)

III. The Corresponding Term Structure Model for the Continuously Compounding Interest Rates

We want to show how to specify the volatility of the Heath–Jarrow–Morton model such that this model will give the crucially needed log-normal simple forward rates. In the Heath–Jarrow–Morton model the continuously compounding forward rate, $\{f(t, T, 0)\}_{t \in [0, T]}$, for $T \in [0, \tau]$, is the basic modeling element. This process is modeled as an Itô process in the following way

$$df(\cdot, T, 0)_t = \mu(t, T, f(t, T, 0))dt + \eta(t, T, f(t, T, 0))dW_t.$$

The relation between the continuously compounding forward rates and the simple forward rates over a fixed period α is given by

$$\frac{1}{1 + \alpha f(t, T, \alpha)} = F(t, T, \alpha) = \exp\left(-\int_T^{T+\alpha} f(t, s, 0)ds\right) \quad t \leq T.$$

On the first hand, define $Y(t, T, \alpha) = -\ln F(t, T, \alpha)$ then

$$\frac{\partial}{\partial T} Y(t, T, \alpha) = f(t, T + \alpha, 0) - f(t, T, 0). \quad (19)$$

On the other hand, $Y(t, T, \alpha) = \ln(1 + \alpha f(t, T, \alpha))$, therefore,

$$\frac{\partial}{\partial T} Y(t, T, \alpha) = \frac{1}{1 + \alpha f(t, T, \alpha)} \alpha f_T(t, T, \alpha) = F(t, T, \alpha) \alpha f_T(t, T, \alpha), \quad (20)$$

where $f_T(t, T, \alpha)$ denotes $(\partial/\partial T)f(t, T, \alpha)$. Combining equations (19) and (20) yields

$$f(t, T + \alpha, 0) - f(t, T, 0) = \alpha F(t, T, \alpha) f_T(t, T, \alpha). \quad (21)$$

Solving the simple difference equation (21) gives

$$f(t, s + n\alpha, 0) = f(t, s, 0) + \sum_{i=0}^{n-1} \alpha F(t, s + i\alpha, \alpha) f_T(t, s + i\alpha, \alpha),$$

$$s \in [t, t + \alpha), \quad (22)$$

with initial condition

$$\int_t^{t+\alpha} f(t, s, 0) ds = \ln(1 + \alpha f(t, t, \alpha)). \quad (23)$$

This is compatible with our earlier findings in Section I; that is, when specifying the Itô process of the simple forward rates with fixed period length α , we do not specify the continuously compounding interest rates in the time interval $[t, t + \alpha]$. So any (nonnegative) value of the continuously compounding forward rate in that interval, fulfilling the initial condition (23), is valid, because the continuously compounded rates specified in the time interval $[t + \alpha, \tau]$ by equation (22) take care of integrating up to the right bond prices. Again, the reader is referred to Figure 1 to get the intuition.

To find the volatility of the corresponding continuously compounding interest rates we just have to find $\text{vol}(dX(\cdot, T, \alpha)_t)$, where

$$X(t, T, \alpha) = \alpha f_T(t, T, \alpha) F(t, T, \alpha)$$

and then use equation (22). We already know $\text{vol}(dF(\cdot, T, \alpha)_t)$ from equation (5). Moreover, using the Itô process description from equation (4) and a result from Fernique et al. (1983, Chapter 2), $\text{vol}(df_T(\cdot, T, \alpha)_t)$ can be calculated as

$$\begin{aligned} \text{vol}(df_T(\cdot, T, \alpha)_t) &= \frac{\partial}{\partial T}(f(t, T, \alpha)\gamma(t, T, \alpha))dW_t \\ &= (f_T(t, T, \alpha)\gamma(t, T, \alpha) + f(t, T, \alpha)\gamma_T(t, T, \alpha))dW_t. \end{aligned} \quad (24)$$

Finally, using Itô's lemma and the relation $\alpha \cdot f_T(t, T, \alpha) \cdot F(t, T, \alpha) = 1 - F(t, T, \alpha)$

$$\begin{aligned} \text{vol}(dX(\cdot, T, \alpha)_t) &= \alpha(-f_T(t, T, \alpha)F(t, T, \alpha)(1 - F(t, T, \alpha))\gamma(t, T, \alpha) \\ &\quad + F(t, T, \alpha)(f_T(t, T, \alpha)\gamma(t, T, \alpha) + f(t, T, \alpha)\gamma_T(t, T, \alpha)))dW_t \\ &= (\alpha f_T(t, T, \alpha)F^2(t, T, \alpha)\gamma(t, T, \alpha) \\ &\quad + (1 - F(t, T, \alpha))\gamma_T(t, T, \alpha))dW_t \\ &= (-F_T(t, T, \alpha)\gamma(t, T, \alpha) + (1 - F(t, T, \alpha))\gamma_T(t, T, \alpha))dW_t \\ &= \frac{\partial}{\partial T}((1 - F(t, T, \alpha))\gamma(t, T, \alpha))dW_t. \end{aligned} \quad (25)$$

Using Itô's lemma on equation (22) and the result of equation (25) yields the volatility of the corresponding continuously compounding forward rate model. However, it should be emphasized that this volatility process of the Heath–Jarrow–Morton model is state dependent.

IV. Conclusion

Under the assumption of log-normality of simple rates, we have derived intuitive closed form solutions for pricing and hedging caps and floors in a consistent arbitrage-free term structure model of the Heath–Jarrow–Morton type. These formulas support common market practice to price caplets by a naive application of Black's formula on interest rates. However, there is another common market practice that is inconsistent with the caplet formula: namely, to apply Black's formula also to the pricing of bond options and swaptions. We have shown how the formula for calls and puts on zero coupon bonds has to be modified in order to be consistent with the caplet formula.

However, our model does not strictly support the application of the Black type formulas to all interest rate derivatives in large portfolios. Recall the assumption of log-normality of the α_i simple forward rates, where we are free to choose the interval lengths α_i . In practice, one would choose these intervals according to the most liquid markets for forward rates; for instance first model the three-month forward rates using either the Eurodollar futures—or the FRA—market for the first two or three years, then space the α_i according to market information on longer forward rates. But, if for example, two consecutive three-month rates are log-normal, the six-month rate for the same period cannot be log-normal at the same time. Thus, if we have priced the two three-month caplets consistently, the caplet formula for the six-month caplet must be considered with caution, since Black's formula will not give the exact arbitrage-free price. But the Heath–Jarrow–Morton solution to our term structure model is valid for *all* forward rates and provides the exact no-arbitrage price for any composed or interpolated rate. However, this solution may require numerical techniques.

In general, if some of the needed forward rate processes are replicable, there are two ways to proceed: either (i) one can solve the corresponding PDE by numerical procedures, or (ii) one can inconsistently assume that the true underlying simple interest rate process is log-normally distributed. Surely, at first glance (ii) is questionable. However, the question that arises is: how big is the mispricing if, in spite of the inconsistency, one uses the closed form solutions to price different options? Further research is needed to measure the size of this problem. This mispricing should be counterbalanced with the extra calculations needed to do numerical procedures.

The second problem is analogous to the problem of using the Black–Scholes formula on individual assets simultaneously while using the Black–Scholes formula on an arithmetic index of the same assets. An inconsistency problem that practitioners do not care much about, because the magnitude of this

problem is smaller than many other theoretically inconsistent problems of using the Black–Scholes formula.

Appendix

Solution of the PDE (10):

Proof: Given the assumptions of Proposition 1, we have to solve the PDE (10) on $[0, T] \times (0, 1)$ where, for simplicity, we omit the period length α , i.e.,

$$\hat{c}_t(t, x) + (1/2)\gamma^2(t, T)x^2(1-x)^2\hat{c}_{xx}(t, x) = 0$$

with

$$\hat{c}(T, x) = [x - K], \quad x \in [0, 1],$$

where $\hat{c}(t, x)$ is the date T forward value of the option contract. This problem is transformed by introducing the new time variable

$$s = s(t, T) = \int_t^T \gamma^2(r, T) \, dr$$

and the new space variable

$$z = \ln \frac{x}{1-x} \Leftrightarrow x = \frac{1}{1 + \exp(-z)}$$

and, finally, setting $\hat{c}(t, x) = a(z)b(s)h(s, z)$. The idea is now to choose differentiable functions $a(\cdot)$ and $b(\cdot)$ in such a way that any solution $h(\cdot, \cdot)$ of the heat equation yields a solution $\hat{c}(\cdot, \cdot)$ of the original PDE. As shown in Rady and Sandmann (1994) this can be done by setting

$$a(z) \equiv \frac{1}{\exp(z/2) + \exp(-z/2)} \quad \text{and} \quad b(s) = e^{-s/8}.$$

That is,

$$\hat{c}(t, x) = \frac{1}{\exp(z/2) + \exp(-z/2)} e^{-s/8} h(s, z).$$

The transformed problem on $[0, T] \times \mathbb{R}$ is

$$\frac{1}{2} h_{zz} - h_s = 0, \quad \text{with} \quad h(0, z) = (e^{z/2} + e^{-z/2}) \left[\frac{1}{1 + \exp(-z)} - K \right]^+.$$

This equation is known as the Heat Equation (compare for example Merton (1973, p. 225) or McKean (1965)) with the solution

$$\begin{aligned}
 h(s, z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(0, z + \rho \sqrt{s}) e^{-\rho^2/2} d\rho \\
 &= \frac{1}{\sqrt{2\pi}} \int_{(1/\sqrt{s})(\ln[K/(1-K)]-z)}^{\infty} (e^{(z+\rho\sqrt{s})/2} + e^{-(z+\rho\sqrt{s})/2}) \\
 &\quad \cdot \left(\frac{1}{1 + \exp(-(z + \rho\sqrt{s}))} - K \right) e^{-\rho^2/2} d\rho \\
 &= (1-K)I_1 - KI_2,
 \end{aligned}$$

with

$$\begin{aligned}
 I_1 &= \frac{1}{\sqrt{2\pi}} \int_{(1/\sqrt{s})(\ln[K/(1-K)]-z)}^{\infty} e^{(z+\rho\sqrt{s})/2} e^{-\rho^2/2} d\rho = e^{z/2} e^{s/8} N\left(\frac{1}{\sqrt{s}}\left(z + \ln \frac{1-K}{K} + \frac{s}{2}\right)\right), \\
 I_2 &= \frac{1}{\sqrt{2\pi}} \int_{(1/\sqrt{s})(\ln[K/(1-K)]-z)}^{\infty} e^{-(z+\rho\sqrt{s})/2} e^{-\rho^2/2} d\rho = e^{-z/2} e^{s/8} N\left(\frac{1}{\sqrt{s}}\left(z + \ln \frac{1-K}{K} - \frac{s}{2}\right)\right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \hat{c}(t, x) &= \frac{\exp(-s/8)}{\exp(z/2) + \exp(-z/2)} h(s, z) \\
 &= (1-K) \underbrace{\frac{\exp(z/2)}{\exp(z/2) + \exp(-z/2)}}_{=x} N(e_1) - K \underbrace{\frac{\exp(-z/2)}{\exp(z/2) + \exp(-z/2)}}_{=1-x} N(e_2)
 \end{aligned}$$

and since

$$P(t, T + \alpha) = P(t, T)F(t, T, \alpha) = P(t, T) \frac{1}{1 + \alpha f(t, T, \alpha)},$$

the spot arbitrage price of the European call option is

$$\text{Cal} = P(t, T) \hat{c}(t, F(t, T, \alpha)). \quad \text{Q.E.D.}$$

Proof of Lemma 3

Proof: From the Heath–Jarrow–Morton model we know that

$$\frac{dP(\cdot, T)_t}{P(t, T)} = f(t, t, 0)dt - \delta(t, T)dW_t,$$

where W is a Wiener process under P^* and

$$\delta(t, T) = \int_t^T \eta(t, u) du.$$

Using $P(t, T)$ as numeraire (i.e., dollars delivered at date T), Girsanov's theorem implies that

$$W_T(t) = W(t) + \int_0^t \delta(u, T) du$$

is a Wiener process under Q_T (compare for example Duffie (1992) or Karatzas and Shreve (1988)). Itô's lemma applied to the T forward price process $F(t, T, \alpha) = P(t, T + \alpha)/P(t, T)$ gives

$$\begin{aligned} \frac{dF(\cdot, T, \alpha)_t}{F(t, T, \alpha)} &= (\delta(t, T) - \delta(t, T + \alpha))(dW(t) + \delta(t, T)dt) \\ &= - \int_T^{T+\alpha} \eta(t, u) du dW_T(t) \end{aligned} \quad (\text{A1})$$

and by comparison of $\text{vol}(dF)$ with equation (5)

$$\int_T^{T+\alpha} \eta(t, u) du = (1 - F(t, T, \alpha)) \cdot \gamma(t, T, \alpha).$$

Since

$$f(t, T, \alpha) = \frac{1}{\alpha} \left(\frac{1}{F(t, T, \alpha)} - 1 \right),$$

Itô's lemma in connection with equation (A1) implies

$$\begin{aligned} df &= \frac{1}{\alpha} \left(-\frac{1}{F^2} dF + \frac{1}{F^3} d\langle F \rangle \right) \\ &= \frac{1}{\alpha F} \left(\int_T^{T+\alpha} \eta(t, u) du dW_T + \left(\int_T^{T+\alpha} \eta(t, u) du \right)^2 dt \right) \\ &= \frac{1}{\alpha F} \int_T^{T+\alpha} \eta(t, u) du dW_{T+\alpha} = \frac{1-F}{\alpha f} \gamma(t, T, \alpha) dW_{T+\alpha} \\ &= f \cdot \gamma(t, T, \alpha) dW_{T+\alpha}. \end{aligned} \quad \text{Q.E.D.}$$

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