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Equilibrium Analysis of Portfolio Insurance

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ABSTRACT

A martingale approach is used to characterize general equilibrium in the presence of portfolio insurance. Insurers sell to noninsurers in bad states, and general equilibrium requires that the risk premium rises to induce noninsurers to increase their holdings. We show that portfolio insurance increases price volatility, causes mean reversion in asset returns, raises the Sharpe ratio and volatility in bad states, and causes volatility to be correlated with volume. We also explain why out-of-the-money S&P 500 put options trade at a higher volatility than do in-the-money puts.

WE CONSIDER A FINANCIAL MARKET with two types of risk managers: insurers and noninsurers. A noninsurer's objective is to maximize the expected utility of his terminal wealth, and an insurer's objective is to maximize the expected utility of his terminal wealth, subject to the constraint that in no circumstance, his portfolio wealth falls below a certain percentage of his initial capital ("floor"). Managers can invest in a risky asset and a risk-free asset. We develop a continuous time general equilibrium model of the price of the risky asset.

The purpose of the analysis is to increase our understanding of the impact on market equilibrium of dynamic trading strategies and the demand for options, as well as to show that the relatively high implied volatility of out-of-the-money put options, documented in Rubinstein (1994), can be explained by the use of portfolio insurance strategies by investors. Cox and Leland (1982) and Grossman and Vila (1989) show that if markets are dynamically complete, then portfolio insurance is efficiently achieved by purchasing put options; that is, the demand for an absolute level of price protection is equivalent to a demand for put options.[1] We show that a similar argument can be used in a general equilibrium context where the dynamic completeness of the market is determined endogenously.[2]

Grossman (1988) argues that the current demand for put options determines the realized subsequent stock price volatility. In a two-period example, he shows that an increase in the demand for put protection will increase realized stock price volatility. Here, we present a continuous time general equilibrium

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[1] See Brennan and Schwartz (1988) for a characterization of time invariant portfolio insurance strategies.

[2] Brennan and Schwartz (1989) have also analyzed the equilibrium implications of portfolio insurance. Their work, in contrast to ours, did not explicitly model the demand functions of the portfolio insurers as arising from an explicit floor-constrained expected utility optimization problem.

analysis with assumptions similar to that of the Black-Scholes model, except that volatility is an endogenous quantity determined both by information regarding final payoffs, and by the demand for put protection. The price of the stock is determined by the information about its payoff, and by the markets' required risk premium for holding this type of risk and return. The required risk premium is determined by the demand for portfolio insurance. Thus, the demand for the option is an important determinant of the price of the stock. It is thus misleading to think of the option as the "derivative" security whose price is determined by the exogenous stock price process. In some situations, such as around market crashes, the demand for option-like payoffs can be a more important determinant of the asset price than is information about payoffs.

Campbell, Grossman, and Wang (1992) present empirical evidence that the risk premium is related to trading volume. We show that risk is reallocated between insurers and noninsurers as a function of the size of the insurers' losses. This reallocation appears as trading volume between the two groups. The noninsurers require a higher risk premium, as they are induced to acquire the risky positions of the insurers subsequent to insurers' losses.

We find that:

1) The equilibrium price in the presence of insurers is lower than that in the absence of insurers in states where the insurers suffer losses.
2) The Sharpe ratio in the presence of insurers is higher than that in the absence of insurers in states where insurers suffer losses. Good news will lower the Sharpe ratio, while bad news will raise it.
3) Price volatility in the presence of insurers is higher than that in the absence of insurers, and when good news comes, volatility falls, while bad news serves to raise it.
4) There is mean reversion of asset returns.
5) When the price goes up, insurers buy the risky asset; while a fall in price will prompt the insurers to sell it.
6) The Black-Scholes implied volatility of out-of-the-money put options is higher than the implied volatility of in-the-money put options.
7) There is a positive correlation between the volume of trade and price volatility.

The change in the price level, Sharpe ratio, trading volume, and volatility is most dramatic around the point at which the wealth of the insurers is expected to equal the floor.

The above results are all a consequence of the fact that the use of an insurance strategy increases the sensitivity of an individual's risk aversion to changes in his wealth. His risk aversion is higher in bad states (i.e., when the price falls) and lower in good states (i.e., when price rises). Moreover, in bad states, the effective risk aversion in an economy with portfolio insurance is higher than in the economy without portfolio insurance. Thus, in bad states the economy's risk aversion and Sharpe ratio are higher, and the asset price is lower than would be the case in the absence of insurance strategies. The rise in volatility is caused by the fact that prices fall more in bad states and rise by more in good states, because the insurers' risk aversion rises as their wealth falls. The mean reversion of returns is caused by the fact that, in bad states, expected returns rise when insurers sell, as the price falls by enough to induce noninsurers to buy from the insurers. The high Black-Scholes implied volatility of out-of-the-money put options is due to the fact that the realized price volatility will be higher as the wealth of insurers approaches the floor. The positive correlation between volume and volatility is due to the fact that high volatility states are states in which a large price move is necessary in order to effect a reallocation of the risky asset between insurers and non-insurers.[3]

The structure of the article is as follows. Section I discusses the optimization problem of portfolio insurers. Section II sets out the basic equations for general equilibrium, and provides and uses the "equivalent martingale measure" approach to derive a simple formula for the equilibrium price process. Section III provides a detailed analysis of two examples. Example 1 assumes that the insurers and noninsurers have the same utility functions, while Example 2 postulates that the insurers are less risk averse in their utility function than noninsurers. Section IV contains conclusions.

I. Optimization

We consider a financial market where trading takes place continuously between time zero and time T. The traded securities are: (i) claims to a risky asset that is in positive supply, and (ii) a riskfree security, in zero supply, whose terminal payoff at T is always one unit of the consumption good. There is no consumption until time T, and there are no consumption goods until time T. At time T, a lump sum, liquidating dividend D_T is paid on the risky asset.

[3] It may be interesting to contrast our work with the simultaneously produced work of Başak (1995). Başak uses the martingale technique to study the consequences of insurance strategies, but in contrast to our work he assumes that the termination date of the portfolio insurance strategy occurs at a date T prior to the last date of trading T'. Başak (p. 2) finds that "... volatility and the risk premium are decreased by the presence of portfolio insurers." Başak (p. 2) gives the intuition for his result:

> *... portfolio insurers behave as more "risk averse" than the normal agents.... Hence, ... to clear the markets the risky securities must become more favorable which, within our set-up, is achieved by a decrease in the volatility (riskiness) of the equity market. As the market becomes less risky, we find that its risk premium must simultaneously decrease.*

We are primarily concerned with understanding how insurance changes the *sensitivity* of the risk premium and expected returns to good states and bad states. Başak is primarily concerned with understanding how the presence or absence of insurers changes the *level* of the risk premium and *volatility*.

In addition to having different objectives, Başak has a different notion of volatility than we have. In his model, the bond and stock price processes are discontinuous at date T as the portfolio insurance expires. It is difficult for us to describe an economy in which prices jump and interest rates become infinite to have a lower volatility than an economy where the processes follow a diffusion. In any case, it is difficult to map our experience of the world into his result that there is a *predictable* date at which prices jump and interest rates become infinite.

The lump sum dividend provides the consumption good at time T. The risky asset will not pay out any dividend before time T, and at time T, each unit of risky asset will pay out a lump sum dividend in the amount of D_T (in the unit of consumption good, denoted by "dollars"). The information flow concerning D_T is described by a filtration generated by a standard Brownian motion $\{Z_t\}$.

The price of the risky asset is expressed in units of the price of the safe asset. That is, the price of the safe asset serves as a numeraire for the price of the risky asset. By definition, the price of safe asset is always equal to one.

There are two groups of risk managers: *insurers* and *noninsurers*. The proportion of insurers in the market is denoted by λ, and the proportion of noninsurers is thus $(1 - \lambda)$, where λ is a constant between zero and one. At time zero, each manager is endowed with one unit of the risky asset.

A noninsurer chooses a nonanticipative trading strategy in the risky and risk free securities to maximize his expected utility of final wealth:

$$\max \mathrm{E}_0[U_1(W_T)],$$

subject to:

$$W_T = X_T D_T + B_T; \tag{1}$$

$$dW = X_t dP \tag{2}$$

where E_0 is an expectation operator conditional on time-zero information, W_T denotes the final wealth, W_0 is given, X_t is the amount owned of the risky security at time t, and B_t represents the investment in the risk-free security, and where there is no return to the risk-free security because it is the numeraire.

An insurer's optimization problem is the same as the noninsurer's, except that there is an additional constraint:

$$W_T \geq K \equiv \alpha W_0, \tag{3}$$

which requires that the manager is not allowed to lose more than $(1 - \alpha) \cdot 100$ percent of his initial wealth in any circumstance.[4] We denote the insurer's utility function by $U_2(W)$, and require that he choose a nonanticipative trading strategy to maximize $\mathrm{E}_0 U_2(W_T)$ subject to equations (1)–(3), where $\alpha \in [0, 1]$ and W_0 is given.

We assume that both types of managers have strictly increasing concave and twice differentiable utility functions.

[4] Note that the lower bound, αW_0, depends on the initial price of the risky asset. We choose a lower bound that is a percentage of initial wealth, rather than a fixed constant, because this is consistent with the type of risk control preferences we observe in the marketplace.

II. Definition of Equilibrium and Basic Equations

Let X_{1t} and X_{2t} denote the optimal holdings of the risky asset by the noninsurer and the insurer, respectively, at time t, when faced with a price process $\{P_t\}$. We define $\{P_t\}$ to be an equilibrium price process if

$$(1-\lambda)X_{1t}+\lambda X_{2t}=1, \qquad \forall t\in[0, T]. \tag{4}$$

Note that the market in the risk free asset clears if the risky asset market clears. Our approach to computing the equilibrium price process is to conjecture that the price process is a diffusion.

$$\frac{dP_t}{P_t}=\mu_t dt+\sigma_t dZ_t, \tag{5}$$

where μ_t and σ_t are dependent on the history $(Z_s)_{s=0}^t$, and then to verify that μ_t and σ_t can indeed be chosen so that equation (4) holds.

Let $\{P_t\}$ be a process. A positive-valued adapted process $\{\eta_t\}$ is called a pricing kernel of $\{P_t\}$ provided (i) $\eta_0 = 1$, (ii) η_t is a martingale; and (iii) $P_t\eta_t$ is a martingale.

If a price process is a diffusion, $dP_t/P_t = \mu_t dt + \sigma_t dZ_t$, then by standard arguments the pricing kernel of $\{P_t\}$ is unique and is given by: $\eta_t = E_t[\eta_T]$, where

$$\eta_T=\exp\left[-\frac{1}{2}\int_0^T \theta_t^2\, \mathrm{d}t-\int_0^T \theta_t\, \mathrm{d}Z_t\right], \tag{6}$$

where $\theta_t = \mu_t/\sigma_t$. The following lemma states that characterizing a price process is the same as characterizing the pricing kernel.

LEMMA 1: (a) *If $\{\eta_t\}$ is a pricing kernel of P_t where $P_T = D_T$, then P_t satisfies*

$$P_t=\frac{\mathrm{E}_t[D_T\eta_T]}{\mathrm{E}_t[\eta_T]}.$$

(b) *If there exists a process $\{\xi_t\}$ and a process $\{P_t\}$ such that:*

$$P_t=\frac{\mathrm{E}_t[D_T\xi_T]}{\mathrm{E}_t[\xi_T]},$$

then $\eta_t \equiv E_t[\xi_T]/E_0[\xi_T]$ is the pricing kernel for $\{P_t\}$.

Proof: To prove (a), note that $P_t\eta_t$ is a martingale, which implies that $P_t\eta_t = \mathrm{E}_t[P_T\eta_T]$. Using $\eta_t = \mathrm{E}_t[\eta_T]$ and $P_T = D_T$, we obtain: $P_t \cdot \mathrm{E}_t[\eta_T] = \mathrm{E}_t[D_T\eta_T]$, which gives

$$P_t=\frac{\mathrm{E}_t[D_T\eta_T]}{\mathrm{E}_t[\eta_T]}.$$

To prove (b), we note that the definition of η_t implies that $\eta_0 = 1$, and that η_t is a martingale. The fact that $P_t\eta_t$ is a martingale follows immediately from the observation that $P_t\eta_t = \mathrm{E}_t[D_T\eta_T]$. Q.E.D.

A trading strategy is a nonanticipating process $\{X_t\}$ such that

$$\mathrm{E}_0\left[\int_0^T X_t^2 P_t^2 \sigma_t^2 \mathrm{d}t\right] < \infty.$$

If X_t is a trading strategy, then, as is standard, we define the gains from trade by the stochastic integral $\int X \cdot dP$. We will say a final wealth $\tilde{W}_T$ is attainable if there exists a trading strategy X_t such that $dW_t = X_t \cdot dP_t$, and $W_T = \tilde{W}_T$. Clearly, if W_T is attainable with an initial wealth W_0, and η_T is the pricing kernel, then $\mathrm{E}_0[W_T\eta_T] = W_0$.

A. Characterizing the Optimal Trading Strategy

The characterization of equilibrium can be facilitated by using the equivalent martingale measure approach, which uses the fact that markets are dynamically complete to make the object of choice a final wealth random variable, instead of a trading strategy.

The following lemma states that if the solution to the insurer's optimization problem is attainable, then it can be achieved by synthesizing a put with strike K on the optimally controlled wealth obtained by ignoring constraint (3).

LEMMA 2: *The solution to an insurer's optimization problem,* $\max \mathrm{E}_0[U_2(W_T)]$ *subject to* $W_T \geq K$, *is given by the following expression for the optimal final wealth:* $W_T^* = \max\{K, I_2(x \cdot \eta_T)\}$, *provided that* W_T^* *is attainable, where* I_2 *is the inverse function of* U_2', *and* x *is a positive number chosen to satisfy the budget constraint* $\mathrm{E}_0[\eta_T \cdot W_T^*] = W_0$.

In Appendix 1, we give a proof which is a variant of the proof in Grossman and Vila (1989) and Cox and Leland (1982). Taking $K = 0$ in the above Lemma, we obtain the solution to the noninsurer's optimization problem.

COROLLARY 1: *The solution to the unconstrained problem,* $\max \mathrm{E}_0[U(W_T)]$ *with an initial wealth* W_0, *is given by the following expression for the optimal final wealth:* $W_T^* = I(y \cdot \eta_T)$, *where* $I(\cdot)$ *is the inverse function of* $U'(\cdot)$, *and* y *is a nonnegative number chosen to satisfy the budget constraint* $\mathrm{E}_0[\eta_T \cdot I(y \cdot \eta_T)] = W_0$. *Q.E.D.*

To understand the Corollary, y should be thought of as the Lagrange multiplier in the Lagrangian expression $L = \mathrm{E}_0 U(W_T) + y[W_0 - \mathrm{E}_0 W_T \eta_T]$. The maximization of L is achieved by choosing W_T in each state to maximize $U(W_T) - yW_T\eta_T$, leading to the first order condition $U'(W_T) = y\eta_T$, which then leads to the expression: $W_T^* = I(y \cdot \eta_T)$.

Using Corollary 1, we obtain that the optimal final wealth of a noninsurer can be written as:

$$W_T^U = I_1(y \cdot \eta_T), \tag{7}$$

where y is a constant to be chosen to satisfy the noninsurer's budget constraint,

$$\mathrm{E}_0[\eta_T I_1(y\eta_T)] = W_0$$

and I_1 is the inverse function of U_1'.

To understand Lemma 2, we find it useful to decompose the optimal final wealth of an insurer as follows:

$$\max\{K,\ I_2(x \cdot \eta_T)\} = I_2(x \cdot \eta_T) + \max\{K - I_2(x \cdot \eta_T),\ 0\}.$$

Note that $I_2(x \cdot \eta_T)$ is the payoff from a portfolio strategy without the insurance constraint starting from an "artificial initial wealth" $\overline{W}_0$, where $\overline{W}_0 = \mathrm{E}_0[\eta_T \cdot \mathrm{I}_2(\mathrm{x} \cdot \eta_T)] \le \mathrm{W}_0$. Note also that $\max\{K - I_2(x \cdot \eta_T),\ 0\}$ is the payoff of a put option on $I_2(x \cdot \eta_T)$ with strike K. We use W_T^I to denote the optimal final wealth of an insurer:

$$W_T^I = \max\{K,\ I_2(x \cdot \eta_T)\}. \tag{8}$$

B. Characterizing the Equilibrium Price Process

Recall that Lemma 1 gives the price of the risky asset by using η_T:

$$P_t = \frac{\mathrm{E}_t[\eta_T \cdot D_T]}{\mathrm{E}_t[\eta_T]}. \tag{9}$$

We will sometimes use a formula which follows from equations (7) and (9)

$$P_t = \frac{\mathrm{E}_t[U_1'(W_T^U) \cdot D_T]}{\mathrm{E}_t[U_1'(W_T^U)]}, \tag{10}$$

where W_T^U is the optimal final wealth of the noninsurer. This formula is obtained, intuitively, by equating the expected marginal utility of a dollar invested in the risk free asset with that of a dollar invested in the risky asset at time t. To see the equivalence of the two approaches, we need only observe that for a noninsurer, $U_1'(W_T^U) = U_1'[I_1(y \cdot \eta_T)] = y \cdot \eta_T$, because $I_1(\,\cdot\,)$ is the inverse function of $U_1'(\,\cdot\,)$. That is, the marginal utility of the optimal final wealth is proportional to η_T.

The characterization of P_t requires a characterization of η_T. We can extract η_T from a transformation of the market clearing condition at time T. In particular, since aggregate wealth at T must equal D_T, equations (7)–(8) imply:

$$\lambda \cdot \max\{\alpha \cdot P_0,\ I_2(x \cdot \eta_T)\} + (1 - \lambda) \cdot I_1(y \cdot \eta_T) = D_T. \tag{11}$$

For $\lambda < 1$, it is important to note that given constants P_0, x, and y, the above equation uniquely determines η_T as a function of the exogenous D_T, since $I(\cdot)$ is a strictly decreasing function for both types of managers. On the other hand, given η_T, we have the following three equations for the three constants P_0, x, and y:

$$\mathrm{E}_0[\eta_T] = 1 \tag{12}$$

$$\mathrm{E}_0[\eta_T \cdot D_T] = P_0 \tag{13}$$

$$\mathrm{E}_0[\eta_T \cdot I_1(y \cdot \eta_T)] = P_0. \tag{14}$$

Equation (12) follows from the requirement that $\eta_0 = 1$, and that η_t must be a martingale to be a pricing kernel. Equation (13) follows from (12) and (9). Equation (14) is the budget constraint of a noninsurer, i.e., the present value of the final wealth of a noninsurer is equal to his initial capital W_0. (We use the fact that $W_0 = P_0$, since the risk-free asset is in zero supply and there is one unit of the risky asset.) The budget constraint of an insurer will automatically hold if equations (11)–(14) hold, and Appendix 2 shows that supply equals demand at all dates and states.

In Appendix 2, we prove the following theorem, which summarizes the above remarks.

THEOREM: *Let D_T be square integrable: $\mathrm{E}_0[D_T^2] < \infty$. If η_T, P_0, x, and y satisfy equations (11)–(14), then P_t defined by*

$$P_t = \frac{\mathrm{E}_t[D_T \eta_T]}{\mathrm{E}_t[\eta_T]} \tag{15}$$

is an equilibrium price process. Moreover, P_t is a diffusion process.

It is useful to consider two extreme cases: (a) $\lambda = 1$ and (b) $\lambda = 0$.

(a) $\lambda = 1$. In the case that $\lambda = 1$, that is, the case that everybody is an insurer, Equation (11) becomes: $\max\{\alpha P_0, I_2(x \cdot \eta_T)\} = D_T$, which implies that with probability one, $D_T \geq \alpha P_0$. If with probability one, $D_T > \alpha P_0$, then $I_2(x \cdot \eta_T) = D_T$, which states that the insurance constraint will never be binding, and thus in this scenario, the equilibrium price in the presence of the insurance constraint coincides with the equilibrium price in the absence of the insurance constraint. If, however, $D_T = \alpha P_0$ with positive probability, then $\max\{\alpha P_0, I_2(x \cdot \eta_T)\} = D_T$ does not uniquely determine η_T as a function of D_T: in fact, there will be a continuum of η_T's that satisfy $\max\{\alpha P_0, I_2(x \cdot \eta_T)\} = D_T$, which suggests that there are multiple equilibria in situations when the floor can be hit. We have proved this nonuniqueness of equilibria in a parallel discrete-time setup.

(b) $\lambda = 0$. In this case no one is an insurer, and we will use P_t^* to denote the equilibrium price at time t, and η_T^* to denote the corresponding pricing kernel. Note that P_t^* is given by:

$$P_t^* = \frac{\mathrm{E}_t[U_1'(D_T) \cdot D_T]}{\mathrm{E}_t[U_1'(D_T)]}. \tag{16}$$

To find η_T^*, we note that by Corollary 1, we have $I_1(y^* \cdot \eta_T^*) = D_T$, which gives rise to $\eta_T^* = U_1'(D_T)/y^*$, where y^* is a constant that is determined from the budget constraint:

$$\mathrm{E}_0[\eta_T^* \cdot I_1(y^* \cdot \eta_T^*)] = P_0^*.$$

It may be of some interest to note that, in the presence of portfolio insurers, it is generally futile to try to construct the price equilibrium as if there is a representative agent with a smooth utility function. To see this, let η_{1T} and η_{2T} satisfy:

$$\lambda \cdot \alpha \cdot P_0 + (1 - \lambda) \cdot I_1(y \cdot \eta_{1T}) = D_T, \tag{17}$$

$$\lambda \cdot I_2(x \cdot \eta_{2T}) + (1 - \lambda) \cdot I_1(y \cdot \eta_{2T}) = D_T. \tag{18}$$

It can easily be verified that $\eta_T = \max\{\eta_{1T}, \eta_{2T}\}$. If there is a representative agent with a smooth utility function U for our economy with a mixture of insurers and noninsurers, then η_T has to be proportional to $U'(D_T)$, which is a smooth function of D_T, while the expression $\eta_T = \max\{\eta_{1T}, \eta_{2T}\}$ asserts the contrary.

III. Analysis of Equilibrium and Examples

The previous sections provide a general methodology for finding the market equilibrium by extracting the martingale measure from the market-clearing conditions. The general methodology will now be used to analyze the effects of portfolio insurance on the price level, Sharpe ratio, price volatility, and risk premium. We will also examine the optimal investment strategies of both insurers and noninsurers, and analyze the costs and benefits of using portfolio insurance.

The presence of portfolio insurance strategies will also affect the options market. At the end of this section, we will study the widely observed implied volatility "smile" that exists in the equity index options market; namely, for a given option maturity, the Black-Scholes implied volatilities of out-of-the-money put options are much higher than the implied volatilities of out-of-the-money call options.[5] We will show that the volatility smile is consistent with the presence of portfolio insurance strategies.

[5] See Rubinstein (1994).

In what follows, we provide some analytical results but primarily concentrate on explicit numerical solutions for some examples. The examples used to analyze equilibrium are based upon specific assumptions about the dividend process of the risky security D_T, the utility functions of insurers and noninsurers, and all the parameters in the model. Note that equation (11) implies that $\lambda\alpha P_0 \leq D_T$ for all D_T, since there must be enough aggregate wealth in the worst state for the portfolio insurance to be implemented. Thus, D_T must be bounded away from zero. We therefore assume that there is an $\varepsilon > 0$ such that:

$$D_T = \max\left\{\epsilon,\ \exp\left(\frac{1}{2}\bar{\sigma}^2 T + \bar{\sigma} Z_T\right)\right\}, \tag{19}$$

where ϵ and $\bar{\sigma}$ are positive constants. We assume that $\bar{\sigma} = 15$ percent per year to represent the level of volatility in the S&P 500. We choose $T = 1$ year and ϵ small enough so that D_T is essentially lognormal.[6]

We will essentially concentrate on two examples that are distinguished only by their assumptions on utility functions for insurers.

Example 1: $U_1(W) = U_2(W) = \ln(W)$.

Example 2: $U_1(W) = \ln(W)$ *and* $U_2(W) = W^{1-A}/(1 - A)$, *where* $A = ½$ *is the relative risk aversion of the insurers.*

Example 1 assumes that the insurers and noninsurers have the same utility functions, while Example 2 postulates that the insurers are less risk-averse in their utility function than noninsurers. We have observed that once a risk manager chooses to implement an investment strategy designed to insure against large losses, he chooses a larger exposure to risky assets in the no-loss states than his average exposure to risky assets prior to implementing the insurance strategy. That is, his risk aversion as reflected in his utility function will decrease due to the extra protection embodied in the portfolio insurance constraint. We choose $A = ½$ because it allows us to represent η_T in closed form. Although we believe that Example 2 represents reality better than Example 1, Example 1 is presented to highlight the effects of the portfolio insurance constraint.

For the benchmark case $\lambda = 0$, we will use "*" superscript to denote the equilibrium for an economy without insurers. For example, P_t^* is used to denote the equilibrium price, and η_T^* denotes the pricing kernel. Recall $\eta_t^* = E_t[\eta_T^*]$, and $\eta_0^* = 1$. Note that η_t^* is a martingale, and using equation (6),

[6] For the numerical examples, we choose ε such that the probability that D_T is less than or equal to ϵ is $N(-4)$ ($=0.00003$), where $N(\cdot)$ denotes for the cumulative distribution function of a standard Normal random variable. That is, $\epsilon = \exp(½\bar{\sigma}^2 T - 4\bar{\sigma}\sqrt{T})$. For $\bar{\sigma} = 0.15$, $T = 1$, we obtain $\epsilon = 0.55$. If at time $t \in [0, T)$, there is a significant chance that D_T will be equal to ϵ, then the so-called risky asset becomes essentially riskfree, and there will be little distinction between the equilibrium in the portfolio insurance economy and the equilibrium in the benchmark economy. Analysis of equilibrium in this highly unlikely state (which is an artifact of the choice of ϵ) is not interesting and will not be emphasized in the text.

$d\eta_t^* = -\theta_t^* \cdot \eta_t^* \cdot dZ_t$, where $\theta_t^* \equiv \mu_t^*/\sigma_t^*$ (> 0) is the Sharpe ratio in the benchmark economy. Therefore, good news ($dZ_t > 0$) will lower η_t^*, and bad news ($dZ_t < 0$) will raise η_t^*. As long as Z_t remains around its initial value of zero, the pricing kernel η_t^* will stay around 1. We will find it convenient to use η_t^* as a state variable for analyzing equilibrium.

In order to understand the results in the following subsections, it is important to note that the effective risk aversion of an insurer increases as his wealth level decreases (i.e., for a given investment opportunity set, the fraction of wealth held in the risky asset decreases as wealth decreases), and this sensitivity of the insurer's risk aversion is highest when the wealth level of the insurer is expected to be equal to the floor of the portfolio insurance constraint. Therefore, the contrast between the equilibrium in the portfolio insurance economy and the results in the benchmark economy is largest in states where the insurer's wealth is near the floor.

The stochastic nature of an insurer's risk aversion is best seen in the relationship between the two pricing kernels: η_T and η_T^*, since a pricing kernel is proportional to each person's marginal utility of consumption. If $\eta_T > \eta_T^*$ in a state, then this indicates that the marginal utility of consumption in that state has risen relative to the benchmark economy. That is, the marginal investor is "poorer" and thus wants less risky assets than in the benchmark economy.

We will use the value of η_T^* to describe the state of the economy. Clearly, high values of η_T^* occur only in bad states (where the marginal utility of consumption is high), and low η_T^* values occur in states where the marginal utility of consumption is low. We can use equation (11) to find the relationship between η_T^* and η_T/η_T^*. This relationship will then be used to show that, for example, in bad states (where η_T^* is high), η_T will be high relative to η_T^*, i.e., that the effective risk aversion rises in such states due to the presence of portfolio insurers.

Thus, eliminate D_T from equation (11) by using $I_1(y^* \cdot \eta_T^*) = D_T$, and obtain:

$$\lambda \cdot \max\{\alpha \cdot P_0,\, I_2(x \cdot \eta_T)\} + (1 - \lambda) \cdot I_1(y \cdot \eta_T) = I_1(y^* \cdot \eta_T^*), \qquad (20)$$

which describes η_T as a function of η_T^*. Use this function to define a function g by: $g(\eta_T^*) = \eta_T/\eta_T^*$. Figure 1 is the plot of function g for the examples. Since $g(x)$ can be solved analytically for these examples, it is straightforward to show that (i) $g(x)$ is positive, continuous, and convex in x, (ii) there exists a transitional point $\bar{\eta}$ such that $g'(x)$ is nonpositive for $x < \bar{\eta}$, and strictly positive for $x > \bar{\eta}$. The transition point $\bar{\eta}$ is characterized by the property that when $\eta_T^* = \bar{\eta}$, $\alpha P_0 = I_2(x \cdot \eta_T)$. Recall that the optimal final wealth of an insurer is equal to the sum of payoff $(I_2(x \cdot \eta_T))$ from a portfolio strategy ignoring the insurance constraint and the payoff of a put option on $I_2(x \cdot \eta_T)$ with strike $K = \alpha P_0$. When $\eta_T^* = \bar{\eta}$, the put option expires at the money.

To understand Figure 1, note that for $\eta_T^* > \bar{\eta}$, the insurers' final wealth equals αP_0, and thus low D_T events (i.e., high η_T^* events) will cause the

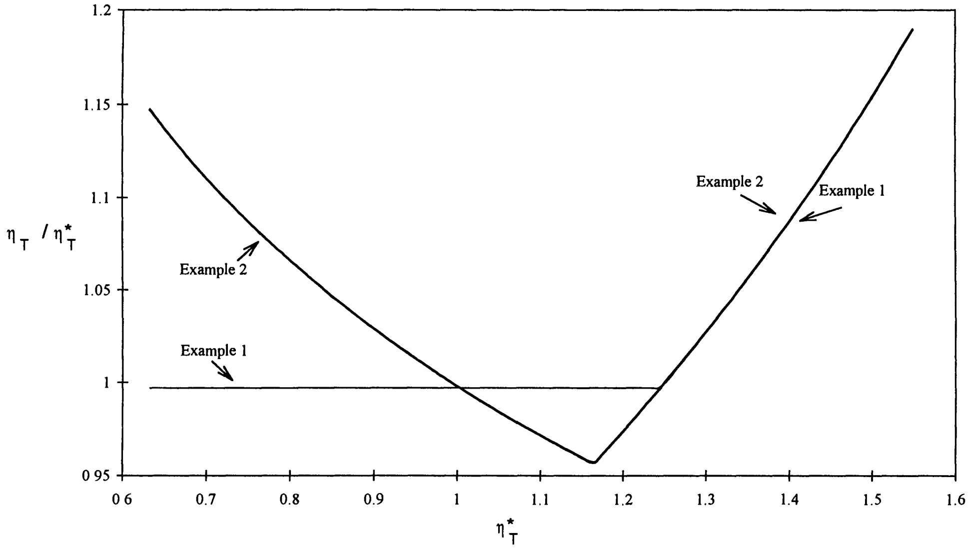
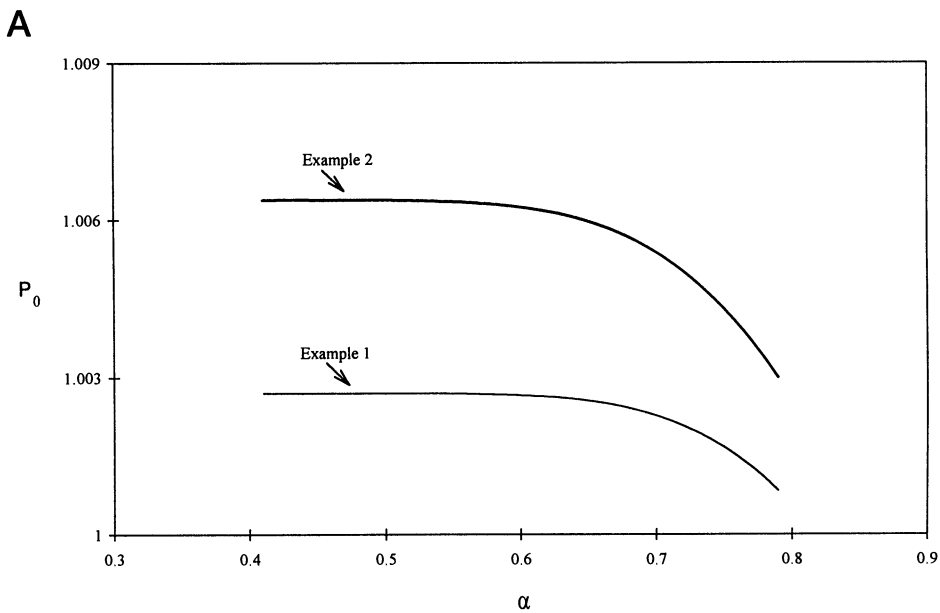


Figure 1. Pricing kernel ratio. $\alpha = 0.8$; $\lambda = 0.4$.

noninsurer to transfer wealth to the insurer, causing each noninsurer to suffer a larger drop in per capita wealth than in the benchmark economy. Thus the marginal utility of consumption of a noninsurer should increase faster than in the benchmark economy, implying that $g(\eta^*_T)$ is an increasing function of η^*_T. As for $\eta^*_T < \bar{\eta}$, as η^*_T goes down, we note that in Example 2, an insurer's marginal utility of consumption decreases less than the noninsurer's marginal utility of consumption, and η_T, representing the marginal utility of consumption averaged over both insurers and noninsurers, has to decrease less than η^*_T, implying that $g(\eta^*_T)$ is a decreasing function of η^*_T. In short, the convex shape of function $g(\eta^*_T)$ is consistent with the intuition that as η^*_T goes up, the risk aversion in the portfolio insurance economy rises relative to the benchmark economy.

A. *Relative Sensitivity of P_t and P^*_t to News*

As a preliminary to analyzing the dependence of the price ratio (P_t/P^*_t) on news, we analyze the dependence of the initial price P_0 on the demand for portfolio insurance. The demand for portfolio insurance varies as either λ or α changes. Figure 2*A* shows P_0 as a function of α. Observe that in both examples, P_0 falls as α increases, since the effective risk aversion of the economy rises as α increases. Figure 2*B* shows P_0 as a function of λ. For a fixed α, an increase in λ increases the weight on portfolio insurers, and this increases the marginal risk aversion of the economy unambiguously when insurers and noninsurers have the same utility function, thus lowering P_0. However, when the insurers have less risk averse utility functions than the noninsurers, an increase in λ, starting from a low λ, will increase P_0, while the reverse happens when λ is



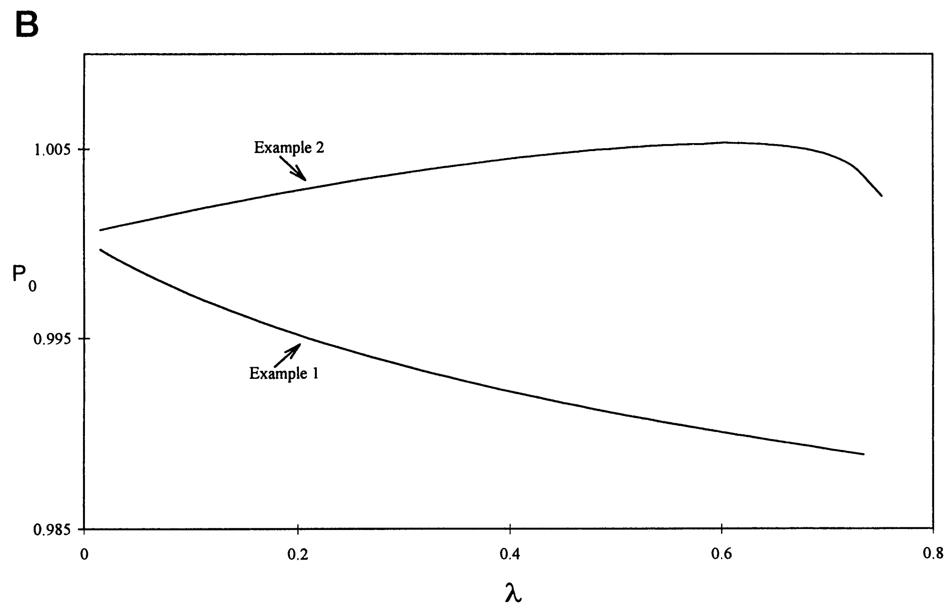


Figure 2A: Initial price as a function of alpha. $\lambda = 0.40$; 2*B*: **Initial price as a function of lambda.** $\alpha = 0.75$.

larger. This occurs because when λ is small, the shift in weight between the two risk aversions has a larger impact than the increased use of portfolio insurance strategies.

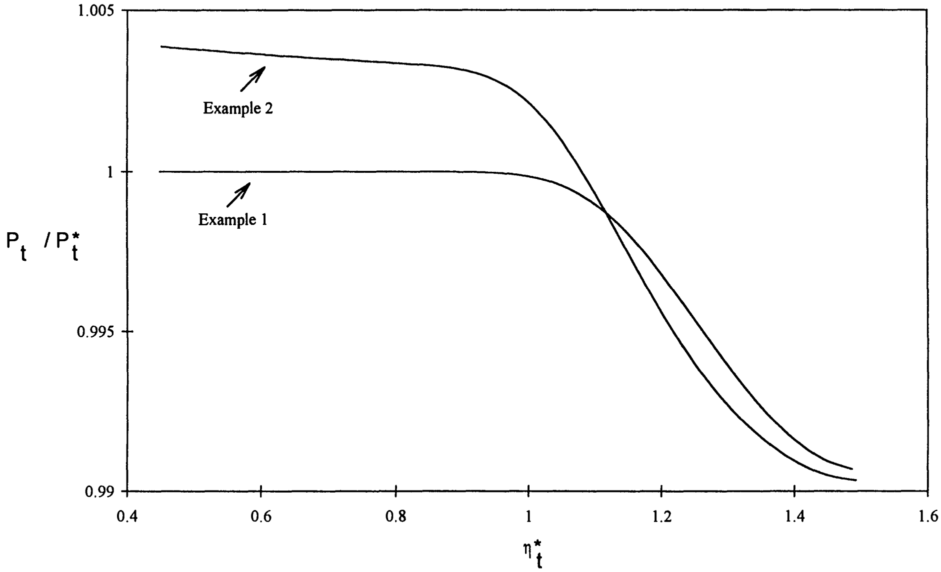


Figure 3. Price ratio. $\alpha = 0.8$; $\lambda = 0.4$; $t = 0.5$.

In order to analyze P_t/P_t^*, we use equations (15) and (20) to define a function h by $h(\eta_t^*, t) = P_t/P_t^*$. Figure 3 graphs h as a function of η_t^* for $t = 1/2$. Figure 3 suggests that h is a decreasing function of η_t^* for both examples, reflecting the fact that the risk aversion in the portfolio insurance economy rises when bad news (i.e., high η_t^* states) comes. Note that in Example 1, $h(\eta_t^*, t)$ is always below 1, since the insurers are always more risk averse than the noninsurers, while in Example 2, $h(\eta_t^*, t)$ can be higher than 1 in good states, since in good states, the effective risk aversion of an insurer can be less than that of a noninsurer. In any case, it is clear that bad news causes P_t to fall by more than P_t^*, and this effect is most pronounced as wealth falls toward the portfolio insurance floor.

The relationship between η_T and η_T^* can help us understand Figure 3. Since our examples assume that the uninsured have log utility, equation (7) implies that in the "*" economy where everyone is uninsured, $\eta_T^* \cdot W_T^U = \eta_T^* \cdot D_T =$ a constant. Therefore:

$$P_t = \frac{\mathrm{E}_t[D_T \cdot \eta_T^* \cdot (\eta_T/\eta_T^*)]}{\mathrm{E}_t(\eta_T)} = \frac{\mathrm{E}_t[D_T \cdot \eta_T^*] \cdot E_t[(\eta_T/\eta_T^*)]}{\mathrm{E}_t(\eta_T)}$$

$$= P_t^* \cdot \frac{\mathrm{E}_t[g(\eta_T^*)] \cdot \mathrm{E}_t(\eta_T^*)}{\mathrm{E}_t(\eta_T)} = P_t^* \cdot \left[1 - \frac{\mathrm{cov}_t(g(\eta_T^*),\ \eta_T^*)}{\mathrm{E}_t(\eta_T)}\right].$$

Dividing by P_t^* in the above equation, we obtain:

$$h(\eta_t^*, t) = 1 - \frac{\mathrm{cov}_t(g(\eta_T^*), \eta_T^*)}{\mathrm{E}_t(\eta_T)}. \tag{21}$$

which implies that $h(\eta_t^*, t)$ is above 1 if and only if $\mathrm{cov}_t(g(\eta_T^*), \eta_T^*)$ is negative, which occurs, for example, when t is close to T, if and only if $g(x)$ is negatively sloped around $x = \mathrm{E}_t(\eta_T^*) = \eta_t^*$. In Example 2 of Figure 1, we have seen that $g(x)$ is positively sloped for large x and negatively sloped for small x, thus we expect $h(\eta_t^*, t)$ to be below 1 for large η_t^*, and above 1 for small η_t^*.

If $h(\eta_t^*, t)$ is a decreasing function of η_t^*, then when good news comes, P_t will rise more than P_t^* does, and when bad news arrives, P_t will fall more than P_t^* does. Although we cannot compute $h(\eta_t^*, t)$ exactly, qualitative properties of $h(\eta_t^*, t)$ can be obtained by linearizing $g(\eta_T^*)$ around η_t^*. Using $g(\eta_T^*) \approx g(\eta_t^*) + g'(\eta_t^*) \cdot (\eta_T^* - \eta_t^*)$, we have

$$\mathrm{cov}_t(g(\eta_T^*), \eta_T^*) \approx g'(\eta_t^*) \cdot \mathrm{var}_t(\eta_T^*), \tag{22}$$

and

$$\begin{aligned} \mathrm{E}_t(\eta_T) &= \mathrm{E}_t[g(\eta_T^*) \cdot \eta_T^*] \\ &\approx g(\eta_t^*) \cdot \eta_t^* + g'(\eta_t^*) \cdot \mathrm{var}_t(\eta_T^*). \end{aligned} \tag{23}$$

By equation (21), we arrive at the following approximation:

$$\begin{aligned} h(\eta_t^*, t) &\approx 1 - \frac{g'(\eta_t^*) \cdot \mathrm{var}_t(\eta_T^*)}{g(\eta_t^*) \cdot \eta_t^* + g'(\eta_t^*) \cdot \mathrm{var}_t(\eta_T^*)} \\ &= \left\{ 1 + \frac{g'(\eta_t^*) \cdot \eta_t^*}{g(\eta_t^*)} \cdot \mathrm{var}_t(\eta_T^*/\eta_t^*) \right\}^{-1}, \end{aligned} \tag{24}$$

which suggests that $h(x, t)$ is a decreasing function of x if and only if $(g'(x) \cdot x)/g(x)$ is an increasing function of x. By straightforward computation, we can show that $(g'(x) \cdot x)/g(x)$ is an increasing function of x, thus we should expect that $h(x, t)$ is a decreasing function of x.

B. Sharpe Ratio Differential

In this subsection, we consider $(\theta_t - \theta_t^*)$, where θ_t is the Sharpe ratio in the portfolio insurance economy, and θ_t^* is the Sharpe ratio in the benchmark economy. Figure 4 presents $(\theta_t - \theta_t^*)$ for $t = 1/2$ as a function of η_t^* for both Example 1 and Example 2. In Example 1, we observe that $(\theta_t - \theta_t^*)$ is always positive because the risk aversion in the portfolio insurance economy is always higher than the risk aversion in the benchmark economy. In Example 2, we observe that $(\theta_t - \theta_t^*)$ is positive in bad states, and negative in good states.

Although the algebraic sign of $(\theta_t - \theta_t^*)$ can be different for the two examples, the sensitivity of the differential is the same: $(\theta_t - \theta_t^*)$ increases as

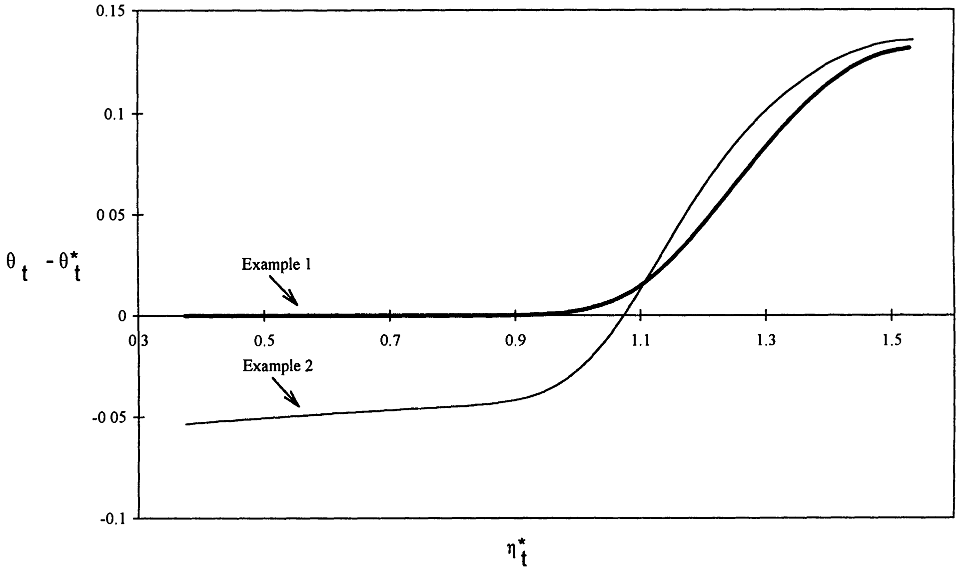


Figure 4. Sharpe ratio differential. $\alpha = 0.8$; $\lambda = 0.4$; $t = 0.5$.

η_t^* increases. The Sharpe ratio, which measures the expected reward per unit of risk, should increase when the effective risk aversion in an economy rises. Since the effective risk aversion in the portfolio insurance economy rises relative to the benchmark economy as η_t^* goes up, we should expect that as η_t^* increases, $(\theta_t - \theta_t^*)$ should increase.

The relationship between the two pricing kernels can be used to study $(\theta_t - \theta_t^*)$. Recall that $\eta_T = \eta_T^* \cdot g(\eta_T^*)$, which, by taking the expectation at time t, gives that $\eta_t = \mathrm{E}_t[\eta_T^* \cdot g(\eta_T^*)]$. Write $\eta_T^* = \eta_t^* \cdot \bar{\epsilon}$, where $\bar{\epsilon}$ can be thought of as being independent of the information filtration at time t with $\mathrm{E}_t(\bar{\epsilon}) = 1$. We can then write $\eta_t = \eta_t^* \cdot q(\eta_t^*, t)$, where we define:

$$q(\eta_t^*, t) = \mathrm{E}_t[\bar{\epsilon} \cdot g(\eta_t^* \cdot \bar{\epsilon})].$$

Applying Ito's Lemma to $\eta_t = \eta_t^* \cdot q(\eta_t^*, t)$, and observing that both η_t and η_t^* are martingale processes, we have

$$\frac{d\eta_t}{\eta_t} = \frac{d\eta_t^*}{\eta_t^*} + \frac{(\partial q/\partial \eta_t^*) \cdot d\eta_t^*}{q(\eta_t^*, t)}.$$

Since $d\eta_t = -\theta_t \cdot \eta_t \cdot dZ_t$, and $d\eta_t^* = -\theta_t^* \cdot \eta_t^* \cdot dZ_t$, the above equation implies

$$\theta_t - \theta_t^* = \frac{(\partial q/\partial \eta_t^*) \cdot \eta_t^* \cdot \theta_t^*}{q(\eta_t^*, t)}.$$

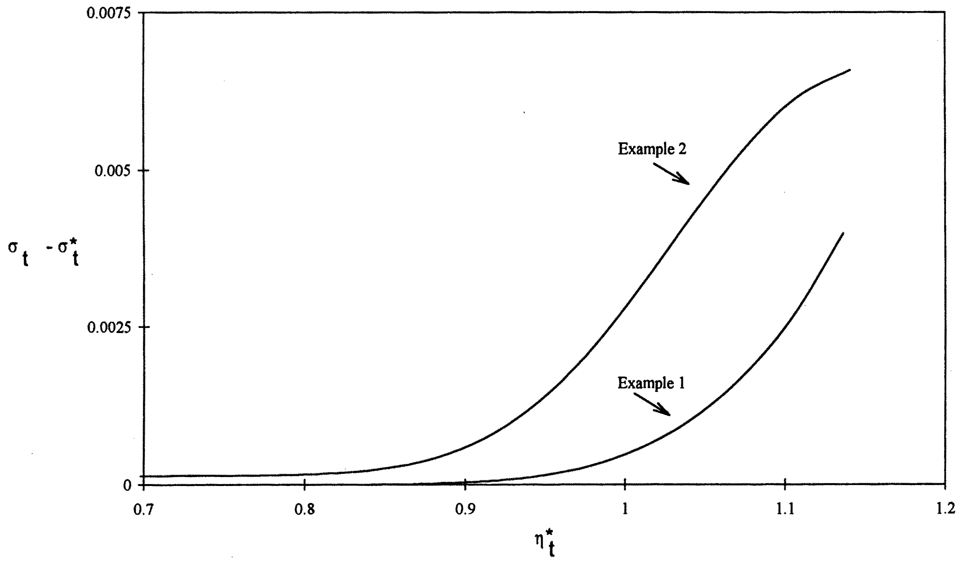


Figure 5. Volatility differential. $\alpha = 0.8$; $\lambda = 0.4$; $t = 0.5$.

Therefore, $(\theta_t - \theta_t^*)$ is positive if and only if $(\partial q/\partial \eta_t^*)$ is positive. From the definition of $q(\eta_t^*, t)$, we have

$$\partial q/\partial \eta_t^* = \mathrm{E}_t[g'(\eta_t^* \cdot \tilde{\epsilon}) \cdot \tilde{\epsilon}^2],$$

which shows that the slope of function g determines the sign of $(\partial q/\partial \eta_t^*)$. In Example 2, $g'(\eta_t^*)$ is positive for large η_t^* and negative for small η_t^*, thus $(\theta_t - \theta_t^*)$ is positive for large η_t^* and negative for small η_t^*. In Example 1, $g'(\eta_t^*)$ is always nonnegative, and thus θ_t is never below θ_t^*.

C. Volatility Differential

Figure 5 shows $(\sigma_t - \sigma_t^*)$ as a function of η_t^* for $t = 1/2$, where σ_t is price volatility in the portfolio insurance economy and σ_t^* is the price volatility in the benchmark economy. We find that $\sigma_t > \sigma_t^*$. Also, good news will bring down the volatility differential, while bad news will raise it.

To understand why price volatility rises with the presence of portfolio insurers, we need to revisit the price ratio relationship: $P_t = P_t^* \cdot h(\eta_t^*, t)$. By Ito's Lemma, we obtain

$$\frac{dP_t}{P_t} = \frac{dP_t^*}{P_t^*} + \frac{dh(\eta_t^*, t)}{h(\eta_t^*, t)} + \frac{dP_t^*}{P_t^*} \cdot \frac{dh(\eta_t^*, t)}{h(\eta_t^*, t)},$$

which, by comparing dZ_t terms, implies:

$$\sigma_t dZ_t = \sigma_t^* dZ_t + \frac{(\partial h/\partial \eta_t^*) \cdot d\eta_t^*}{h(\eta_t^*, t)}.$$

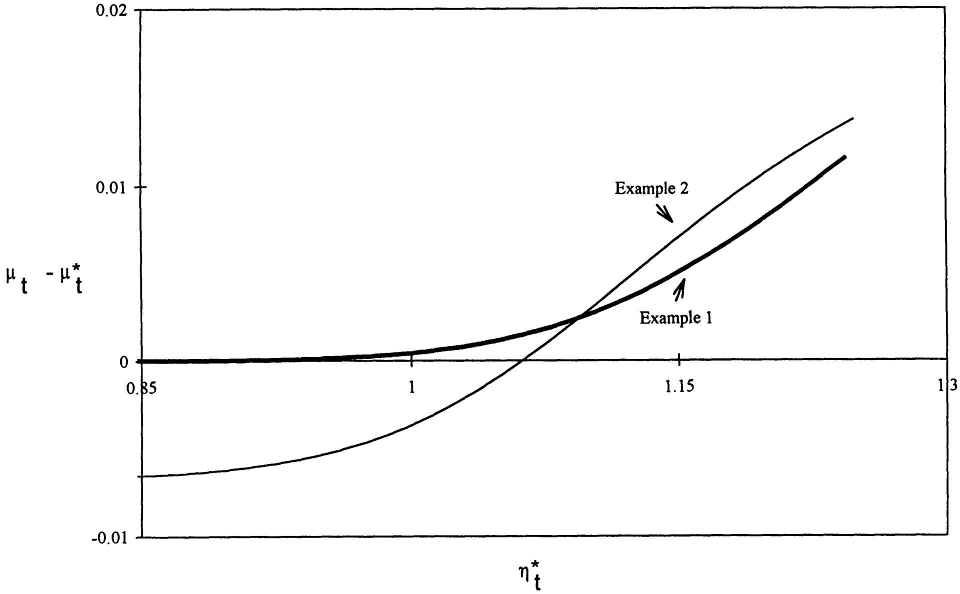


Figure 6. Risk premium differential. $\alpha = 0.8$; $\lambda = 0.4$; $t = 0.5$.

Since $d\eta^*_t = -\theta^*_t \cdot \eta^*_t \cdot dZ_t$, we conclude:

$$\sigma_t = \sigma^*_t - \frac{(\partial h/\partial \eta^*_t) \cdot \eta^*_t \cdot \theta^*_t}{h(\eta^*_t, t)}.$$

Note from the above equation that σ_t is higher than σ^*_t if and only if $(\partial h/\partial \eta^*_t)$ is negative. In Subsection III.A, we have seen that $(\partial h/\partial \eta^*_t)$ is negative, implying that the volatility difference $(\sigma_t - \sigma^*_t)$ is positive.

D. Risk Premium Differential and Mean Reversion of Returns

Based on the identity $\mu_t = \sigma_t \cdot \theta_t$, $\mu^*_t = \sigma^*_t \cdot \theta^*_t$, Figure 6 graphs $(\mu_t - \mu^*_t)$ as a function of η^*_t for $t = 1/2$. Observe that $(\mu_t - \mu^*_t)$ is an increasing function of η^*_t. That is, good news will bring down $(\mu_t - \mu^*_t)$, while bad news will raise it. Note that we have constructed the model of the benchmark economy so that μ^*_t is essentially constant over a large range of values for η^*_t. Therefore μ_t is an increasing function of η^*_t, i.e., there is mean reversion in returns.

E. Optimal Investment Strategy

Figure 7 graphs the optimal investment strategy of an insurer for both Example 1 and Example 2 as a function of η^*_t for $t = 1/2$. The optimal investment strategy of a noninsurer can be easily obtained from the market-clearing condition. We observe that in both examples, when good news comes,

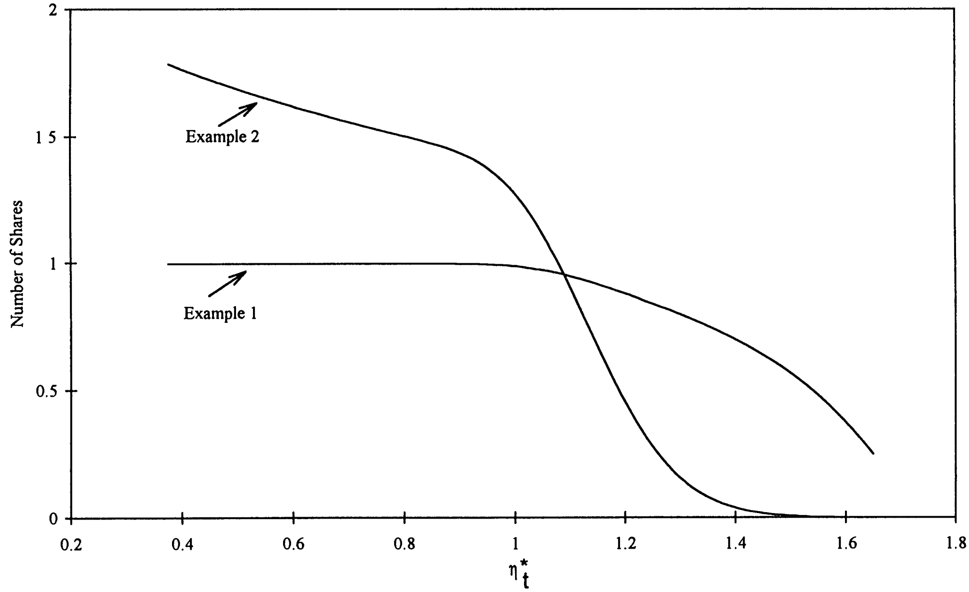


Figure 7. Holdings of risky asset by an insurer. $\alpha = 0.8$; $\lambda = 0.4$; $t = 0.5$.

insurers buy the risky asset, and when bad news arrives, insurers sell risky assets. The insurers' trading behavior is most sensitive to a change in η_t^* when η_t^* is close to $\bar{\eta}$.

To understand the insurers' trading behavior, we need to recall that in Section II.A, it is shown that the insurers' optimal final wealth is the sum of two components: one component is the optimal final wealth without portfolio insurance, and the other component is the payoff of a put option. To achieve the optimal final wealth, and in particular, to mimic the payoff of a put option, insurers have to buy the risky asset if the price of the risky asset goes up, sell the risky asset if the price goes down, since the delta of a put option is more negative when price goes down, and less negative when price goes up. In an equilibrium, by definition, noninsurers have to take the opposite side of the insurers' trades. That is, when good news comes, noninsurers sell the risky asset; when bad news comes, noninsurers buy the risky asset. The risk premium falls when there is good news, and rises when there is bad news in order to induce the noninsurers to engage in these trades.

F. Cost and Benefit of Portfolio Insurance

Recall that the optimal final wealth of an insurer is given by $W_T^I = \max\{K, I_2(x \cdot \eta_T)\}$, where $I_2(\cdot)$ is the inverse function of $U_2'(\cdot)$. The optimal final wealth of a noninsurer can be written as $W_T^U = (y\eta_T)^{-1}$ in both examples. We can analyze the cost and benefit of portfolio insurance by treating both W_T^I and W_T^U as functions of η_T.

In Example 2, we have $W_T^I = \max\{K, (x \cdot \eta_T)^{-2}\}$. Thus, there exists η_L and η_H such that W_T^I is higher than W_T^U when $\eta_T < \eta_L$ or $\eta_T > \eta_H$, and W_T^I is lower than W_T^U when $\eta_T \in (\eta_L, \eta_H)$. Since η_T is a decreasing function of D_T by equation (11), we can say that in either very good states or very bad states, the insurer's wealth is higher than the noninsurer's wealth, and in intermediate states, the insurer suffers losses compared to the noninsurer. In other words, the insurer can capture the upside potential and limit the downside loss at the expense of losing some money when no dramatic event occurs.

In Example 1, $W_T^I = \max\{K, (x \cdot \eta_T)^{-1}\}$. Thus, W_T^I is less than W_T^U unless $W_T^I = K$. In other words, except in very bad states, the insurer will always underperform the noninsurer.

G. The Implied Volatility Smile and Portfolio Insurance

In the equity index options market, out-of-the-money put options command much higher implied volatilities than the out-of-the-money call options (the volatility "smile"). We will now show that the presence of portfolio insurance is consistent with the volatility smile. Thus, the existence of the volatility smile in the options market is evidence that the options market has priced in the equilibrium implications of portfolio insurance.

The pricing of an option is made easy by using the pricing kernel. In an equilibrium, the price of a European put option with strike E and maturity T is given by:

$$P(E)=\mathrm{E}_0[\eta_T \cdot \max\{E - D_T, 0\}].$$

Let us use $\sigma_I(E)$ to denote the Black-Scholes implied volatility of an option with strike price E. That is, $\sigma_I(E)$ is the volatility which, when used in the Black-Scholes Formula, gives the option a price of $P(E)$. Figure 8 plots $\sigma_I(E)$ as a function of E for both examples. Note that the implied volatilities for out-of-the-money puts are much higher than the implied volatilities for at-the-money puts. Also, as out-of-the-money puts go further out of the money, implied volatilities go up even higher. Figure 8 is consistent with what we observe in the U.S. equity index options market.

By numerically computing the pricing kernel from a set of option prices, Rubinstein (1994, p. 796) has observed that the existence of the volatility smile can be explained by a model where the local price volatility σ is a decreasing function of the price of the underlying asset. The analysis in Section III.C shows that local volatility goes up when bad news arrives, that is, local volatility is a decreasing function of P_t.[7] Thus, we have provided a theoretical

[7] In Section III.C it is shown that $\sigma_t - \sigma_t^*$ is decreasing in P_t. However, the examples are constructed such that σ_t^* is essentially constant over a large range of values for P_t.

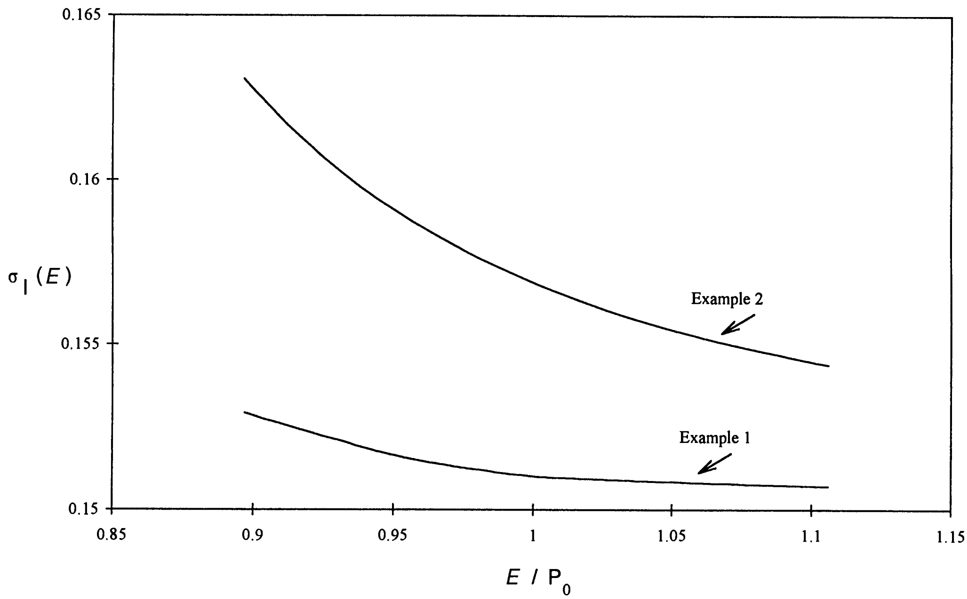


Figure 8. Implied volatility smile. α = 0.8; λ = 0.4.

model that explains the volatility smile as a consequence of the use of portfolio insurance strategies.

H. Volatility and Trading Volume

Numerous studies have found a positive correlation between volatility and trading volume.[8] Our model provides an explanation for this correlation. High volatility states are states in which a large price move is necessary in order to effect a reallocation of the risky asset between the insurers and the noninsurers. This reallocation appears as volume.

To see this, let $X_t^I \equiv X^I(t, \eta_t^*)$ denote an insurer's optimal holdings of the risky asset at time t. Trading volume over a period of time, say Δt, is defined to be $\lambda \cdot |\Delta X_t^I|$, which is the aggregate number of shares transacted between the insurers and the noninsurers. By Ito's Lemma and the fact that $d\eta_t^* = -\theta_t^* \cdot \eta_t^* \cdot dZ_t$, we have

$$dX_t^I = -\frac{\partial X^I}{\partial \eta_t^*} \cdot \theta_t^* \cdot \eta_t^* \cdot dZ_t + \frac{\partial X^I}{\partial t}\, dt + \frac{1}{2} \cdot \frac{\partial^2 X^I}{\partial (\eta_t^*)^2} (\theta_t^* \cdot \eta_t^*)^2 \cdot dt. \qquad (25)$$

[8] See Harris (1986), Karpoff (1987). See also Admati and Pfleiderer (1988) for an explanation based upon asymmetric information.

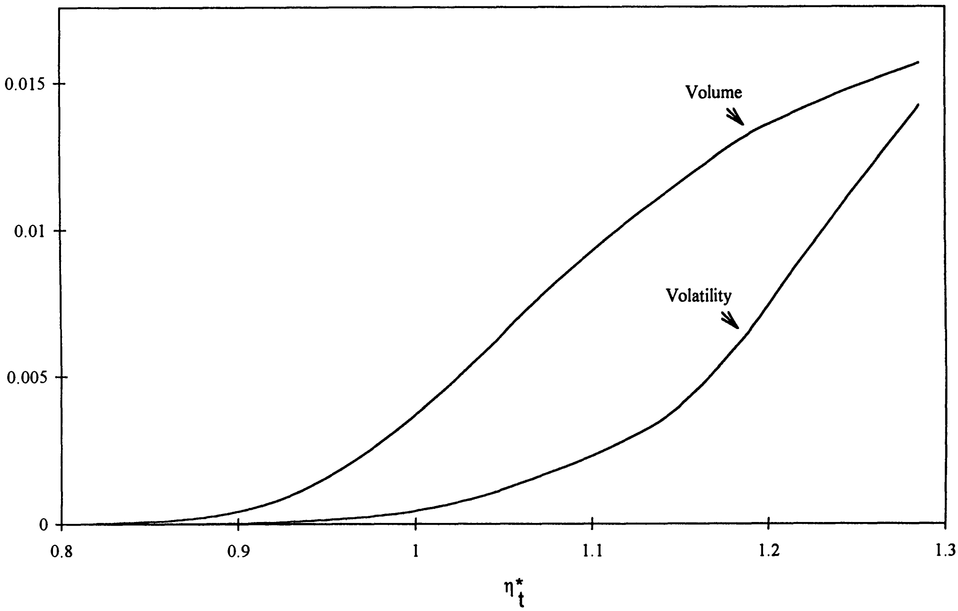


Figure 9. Volatility and trading volume. $\alpha = 0.8$; $\lambda = 0.4$; $\Delta t = 1/12$.

Note that for small dt, the term involving dZ_t dominates the terms involving dt. Thus, it follows from the above equation that

$$\mathrm{E}_t(|\Delta X_t^I|) \approx \left|\frac{\partial X^I}{\partial \eta_t^*}\right| \cdot \theta_t^* \cdot \eta_t^* \mathrm{E}_t(|\Delta Z_t|) \tag{26}$$

$$= \left|\frac{\partial X^I}{\partial \eta_t^*}\right| \cdot \theta_t^* \cdot \eta_t^* \cdot \sqrt{2 \cdot \Delta t/\pi}.$$

Figure 9 graphs $[\lambda \cdot \mathrm{E}_t(|\Delta X_t^I|)]$ and $(\sigma_t - \sigma_t^*)$ as functions of η_t^* for Example 1, when $t = 1/2$, and $\Delta t = 1/12$. Note that $[\lambda \cdot \mathrm{E}_t(|\Delta X_t^I|)]$ is the expected monthly trading volume. From Figure 9, we observe that volume and volatility generally move in tandem: a higher trading volume is associated with a higher price volatility.

IV. Conclusions

We have shown how the martingale approach can be used to analyze a dynamic general equilibrium and yield substantive conclusions. The reader can easily verify that a dynamic programming approach where the state variable is the *history* of the price process would be far more difficult to use in this context.

Our substantive conclusions regarding the impact of portfolio insurance strategies are similar to that of Grossman's (1988) two period example. In particular, we have demonstrated how the volatility of the risky asset price process is determined by the demand for option-like payoffs. It is as if the risky asset is the "derivative" security, and the options are the "underlying" security.

Appendix 1: Proof of Lemma 2

Proof of Lemma 2: Let W_T be any attainable final wealth such that $W_T \geq K$. We have seen that $\mathrm{E}_0[\eta_T \cdot W_T] = W_0$. Our proof is complete if we can show that

$$\mathrm{E}_0[U_2(W_T)] \leq \mathrm{E}_0[U_2(W_T^*)].$$

We will use the standard support argument. Namely, for concave $U_2(\cdot)$, we have that for any a and b, $U_2(a) - U_2(b) \leq U_2'(b) \cdot (a - b)$. Note that

$$\begin{aligned}
&\mathrm{E}_0[U_2(W_T)] - \mathrm{E}_0[U_2(W_T^*)] \\
&= \mathrm{E}_0[U_2(W_T) - U_2(W_T^*)] \\
&\leq \mathrm{E}_0[U_2'(W_T^*) \cdot (W_T - W_T^*)] \\
&= \mathrm{E}_0[U_2'(W_T^*) \cdot (W_T - W_T^*) | W_T^* > K] \cdot \mathcal{P}_0[W_T^* > K] \\
&+ \mathrm{E}_0[U_2'(W_T^*) \cdot (W_T - W_T^*) | W_T^* \leq K] \cdot \mathcal{P}_0[W_T^* \leq K].
\end{aligned}$$

Observe that in the event that $W_T^* > K$, we have $U_2'(W_T^*) = U'(I_2(x \cdot \eta_T)) = x \cdot \eta_T$. In the event that $W_T^* \leq K$, we have that $W_T - W_T^* \geq 0$, and that $U_2'(W_T^*) \leq U'(I_2(x \cdot \eta_T)) = x \cdot \eta_T$, since $U'(\cdot)$ is a decreasing function. Thus, we have

$$\mathrm{E}_0[U_2'(W_T^*) \cdot (W_T - W_T^*) | W_T^* > K] = x \cdot \mathrm{E}_0[\eta_T \cdot (W_T - W_T^*) | W_T^* > K]$$

and

$$\mathrm{E}_0[U_2'(W_T^*) \cdot (W_T - W_T^*) | W_T^* \leq K] \leq x \cdot \mathrm{E}_0[\eta_T \cdot (W_T - W_T^*) | W_T^* \leq K].$$

Therefore, we obtain:

$$\begin{aligned}
&\mathrm{E}_0[U_2'(W_T^*) \cdot (W_T - W_T^*) | W_T^* > K] \cdot \mathcal{P}_0[W_T^* > K] \\
&+ \mathrm{E}_0[U_2'(W_T^*) \cdot (W_T - W_T^*) | W_T^* \leq K] \cdot \mathcal{P}_0[W_T^* \leq K] \\
&\leq x \cdot \mathrm{E}_0[\eta_T \cdot (W_T - W_T^*) | W_T^* > K] \cdot \mathcal{P}_0[W_T^* > K] \\
&+ x \cdot \mathrm{E}_0[\eta_T \cdot (W_T - W_T^*) | W_T^* \leq K] \cdot \mathcal{P}_0[W_T^* \leq K] \\
&= x \cdot \mathrm{E}_0[\eta_T \cdot (W_T - W_T^*)] = 0.
\end{aligned}$$

This completes the proof. Q.E.D.

Appendix 2: Proof of the Theorem

Proof of the Theorem: First let us show that P_t defined by

$$P_t = \frac{\mathrm{E}_t[D_T\eta_T]}{\mathrm{E}_t[\eta_T]}$$

is a diffusion process. According to the corollary in Protter (1990, p. 157), there exists a nonanticipating θ_t such that $\eta_t \equiv \mathrm{E}_t(\eta_T)$ can be written as

$$\eta_t = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 \,\mathrm{d}s - \int_0^t \theta_s \,\mathrm{d}Z_s\right).$$

In other words, $d\eta_t/\eta_t = -\theta_t dZ_t$. Similarly, there exists a nonanticipating a_t such that

$$\frac{d\mathrm{E}_t[D_T\eta_T]}{\mathrm{E}_t[D_T\eta_T]} = a_t dZ_t.$$

Applying Ito's Lemma to $P_t = \mathrm{E}_t[D_T\eta_T]/\eta_t$, we obtain:

$$\frac{dP_t}{P_t} = (a_t + \theta_t)\cdot(\theta_t dt + dZ_t), \tag{A.1}$$

which shows that P_t is a diffusion process.

To show that P_t is an equilibrium price process, we need to show that when the insurers and noninsurers are faced with P_t, the market in the risky asset clears at all times. According to Lemma 1 and equation (12), $\eta_t \equiv \mathrm{E}_t[\eta_T]$ is the pricing kernel for P_t. According to Lemma 2, the optimal final wealth of an insurer can be written as $W_T^I = \max\{K, I_2(x\cdot\eta_T)\}$, provided that W_T^I is attainable. According to Corollary 1, the optimal final wealth of a noninsurer can be written as $W_T^U = I_1(y\cdot\eta_T)$, provided that W_T^U is attainable.

Define $W_t^I = \mathrm{E}_t[\eta_T\cdot W_T^I]/\mathrm{E}_t[\eta_T]$, and $W_t^U = \mathrm{E}_t[\eta_T\cdot W_T^U]/\mathrm{E}_t[\eta_T]$. It follows from equation (11) that

$$\lambda W_t^I + (1-\lambda)W_t^U = P_t.$$

By the same argument used to write (A.1) above, there exist nonanticipating

b_t and c_t such that

$$\frac{dW_t^I}{W_t^I} = (b_t + \theta_t) \cdot (\theta_t dt + dZ_t),$$

$$\frac{dW_t^U}{W_t^U} = (c_t + \theta_t) \cdot (\theta_t dt + dZ_t).$$

Therefore, we may write $dW_t^I = X_t^I dP_t$ and $dW_t^U = X_t^U dP_t$ for nonanticipating X_t^I and X_t^U. Differentiating $\lambda W_t^I + (1 - \lambda) W_t^U = P_t$, we obtain that $\lambda X_t^I dP_t + (1 - \lambda) X_t^U dP_t = dP_t$, that is, $\lambda X_t^I + (1 - \lambda) X_t^U = 1$, which states that the market in the risky asset clears at all times. Q.E.D.

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